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Optimal reinsurance policy: The adjustment coefficient and the expected utility criteria☆

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Abstract

This paper is concerned with the optimal form of reinsurance from the ceding company point of view, when the cedent seeks to maximize the adjustment coefficient of the retained risk. We deal with the problem by exploring the relationship between maximizing the adjustment coefficient and maximizing the expected utility of wealth for the exponential utility function, both with respect to the retained risk of the insurer.

Assuming that the premium calculation principle is a convex functional and that some other quite general conditions are fulfilled, we prove the existence and uniqueness of solutions and provide a necessary optimal condition. These results are used to find the optimal reinsurance policy when the reinsurance premium calculation principle is the expected value principle or the reinsurance loading is an increasing function of the variance. In the expected value case the optimal form of reinsurance is a stop-loss contract. In the other cases, it is described by a nonlinear function.

Keywords: Optimal reinsurance; Risk; Stop loss; Ruin probability; Adjustment coefficient; Premium principles; Exponential utility function

1. Introduction

The main reason for an insurer to reinsure part of its risks (and by risk we mean a policy, a set of policies or a portfolio) is the protection against losses that might create financial embarrassment or even insolvency. When rating a reinsurance contract the reinsurer will use some premium calculation principle. The cedent company will seek the form of reinsurance that gives him the best risk protection for some defined risk measure. This paper deals with optimal reinsurance structures when the insurer seeks to maximize the adjustment coefficient of the retained risk (see Gerber (1979)). The maximization of the adjustment coefficient is equivalent to the minimization of the upper bound of the probability of ruin provided by the Lundberg inequality. The behaviour of the ruin probability as a function of the retention is very similar to the behaviour of Lundberg's upper bound, which makes it acceptable as an approximation to the ultimate probability of ruin (see Centeno (1997)).

There are several typical forms of reinsurance, namely quota share, surplus, excess of loss and stop-loss reinsurance. Theoretical results in favour of the stop-loss contract go back to Borch (1960). He proved that stop loss minimizes the variance of the retained risk if the reinsurer charges the pure premium only. Similar results in favour of the stop-loss contract were developed later (see Kahn (1961), Vajda (1962), Olhin (1969), and Lemaire (1973)). Taking the maximization of the expected utility as the optimality criterion, Arrow (1963) proved a similar result in favour of the stop-loss contract.

Hesselager (1990) proved that the stop-loss contract maximizes the adjustment coefficient, under the assumption that the reinsurance premium is calculated according to the expected value principle and the value of the reinsurance premium is fixed a priori.

We study the problem under quite general assumptions, namely that the premium calculation principle is a convex

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functional and that some other quite general conditions are fulfilled.

Part of the difficulty in studying the problem lies in the fact that the adjustment coefficient is defined in an implicit form and its domain does not have a structure appropriate for the use of arguments based on classical implicit function theorems (see Section 3, below). We overcome this difficulty by showing that maximizing the adjustment coefficient is equivalent to solving a two-step problem. The first step in this new problem consists in maximizing the expected utility of wealth of the retained risk for an exponential utility function, for all positive values of the coefficient of risk aversion. The second step consists in solving a single-variable equation. The optimal adjustment coefficient equals the coefficient of risk aversion for which the maximal expected value of the utility function is -1. The reinsurance policy that maximizes the adjustment coefficient is the treaty that maximizes the expected utility of wealth for that particular value of the risk-aversion coefficient. It turns out that the maximization step in the two-step problem is easier to deal with from the mathematical point of view than the original one. Thus, we are able to solve both problems. We show that one optimal reinsurance policy always exists and provide a necessary condition for a policy to be optimal. We prove that stop loss is indeed the optimal form of reinsurance if the reinsurer rates the contracts by the expected value principle; when the reinsurance loading is an increasing function of the variance (for example, in the variance or standard deviation premium principles), then the optimal form will be of a nonlinear type (not an already known typical form), very easily constructed.

This paper is organized as follows. Section 2 contains a rigorous formulation of the problem, the basic notation and the blanket assumptions that will be used. Section 3 contains some essentially technical elements that will be used to obtain the main results. In Section 4 we analyse the relationship between the maximization of the adjustment coefficient of the retained risk and the maximization of the expected value of the utility of the insurer's wealth. In Section 5 we prove existence and uniqueness of optimal policies for the expected utility criterion. This result is used in Section 6 to prove the existence and uniqueness of a policy which maximizes the adjustment coefficient. A necessary condition for optimality is obtained in Section 7. In Sections 8 and 9 we present the solutions to the problem in the particular cases when the reinsurance premium is computed by the expected value principle or the loading is a function of the variance. Section 10 contains a numerical example.

2. Preliminaries

In this section we formulate the problem and outline the notation and the main assumptions that will be used throughout the text.

The value of the aggregate claims for a given period of time is a non-negative random variable denoted by Y. Aggregate claims over consecutive periods are assumed to be i.i.d.

A reinsurance policy is a function $Z : [0, +\infty[\mapsto [0, +\infty[$, mapping each possible aggregate value of the claims in a given period into the corresponding value refunded under the reinsurance contract. The set of all possible reinsurance policies is:

$$\mathcal{Z} = \{Z : [0, +\infty[\mapsto \mathbb{R} \mid \ Z \text{ is measurable and } 0 \le Z(y) \le y,\ \forall y \ge 0\}.$$

We do not distinguish between functions which differ only on a set of zero probability. i.e., two measurable functions, ϕ and ϕ' are considered to be the same whenever $\Pr\{\phi(Y) = \phi'(Y)\} = 1$. Similarly, a measurable function Z is an element of $\mathcal{Z}$ whenever $\Pr\{0 \le Z(Y) \le Y\} = 1$. For each period of time, the premium charged for a reinsurance policy is computed by a real functional $P : \mathcal{Z} \mapsto [0, +\infty]$. Assuming the insurer receives a given amount of premiums, c, per unit of time, with $c > E[Y]$, and acquires a given reinsurance policy $Z \in \mathcal{Z}$ for the same period, the profit he obtains, per unit of time, is the random variable

$$L_Z = c - P(Z) - (Y - Z(Y))$$

(meaning a loss when its value is negative). We assume that all the payments related to premiums and claims, both for the insurer and reinsurer, are made at the same point of time, so that investments and interest do not have to be taken into consideration. Concerning the random variable Y and the premium, we make the following assumptions:

Assumption 1. Y is a continuous random variable with density function f, and $E[Y^2] < +\infty$. □

Assumption 2. No reinsurance policy exists that guarantees a non-negative profit, i.e., $\Pr\{L_Z < 0\} > 0$ holds for every $Z \in \mathcal{Z}$. □

Assumption 3. The reinsurance premium is a convex, non-negative functional, such that $P(0) = 0$. It is continuous in the mean-squared sense, i.e., $\lim_{k\to\infty} P(Z_k) = P(Z')$ holds for every sequence $\{Z_k \in \mathcal{Z}\}_{k=1,2,\ldots}$ such that

$$\lim_{k\to\infty} \int_0^{+\infty} (Z_k(y) - Z'(y))^2 f(y)\mathrm{d}y = 0. \quad \square$$

Throughout this paper we will always suppose that Assumptions 1–3 hold and no further reference to them will be included in the formulation of definitions or propositions.

Though some regularity of the probability measure is necessary, the requirement that Y is a continuous random variable can be much weakened. We provide this assumption in order to simplify the technical content of our proofs, so we focus on the general features of the approach we propose. In contrast, the requirements that $E[Y^2] < +\infty$ and that the premium principle is continuous in the mean-squared sense cannot be lifted. This is because our approach depends in a fundamental way on Hilbert space arguments.

Notice that Assumption 2 is required in order to make the problem non-trivial: If there exists some policy satisfying

$\Pr\{L_Z < 0\} = 0$, then the risk of ruin under this policy is obviously zero.

Consider the map $G : \mathbb{R} \times \mathcal{Z} \mapsto [0, +\infty]$, defined by

$$G(R, Z) = \int_0^{+\infty} \mathrm{e}^{-RL_Z(y)} f(y)\mathrm{d}y, \quad R \in \mathbb{R},\ Z \in \mathcal{Z}.$$

Let R_Z denote the adjustment coefficient of the retained risk for a particular reinsurance policy, $Z \in \mathcal{Z}$. R_Z is defined as the strictly positive value of R which solves the equation

$$G(R, Z) = 1, \tag{1}$$

for that particular Z, when such a root exists. It comes as a Corollary of Lemma 1 below that Eq. (1) cannot have more than one positive solution. This means that the map $Z \mapsto R_Z$ is a well defined functional in the set

$\mathcal{Z}^+ = \{Z \in \mathcal{Z} :$ (1) admits a positive solution$\}$.

Now, suppose that the insurance company detains a certain amount of reserves, $u > 0$, to cover eventual losses. If a reinsurance policy $Z \in \mathcal{Z}$ is in force year after year, then the probability of ultimate ruin is

$$\psi_Z(u) = \Pr\left\{u + \sum_{k=1}^{n} L_{Z_k}(w) < 0,\ \text{for some } n = 1, 2, \ldots\right\}.$$

It can be shown (see for example Gerber (1979)) that the probability of ruin satisfies the Lundberg inequality:

$$\psi_Z(u) \le \exp(-uR_Z).$$

Further, the behaviour of $\psi_Z(u)$ as a function of u is quite similar to the behaviour of $\exp(-uR_Z)$ for most common reinsurance forms (see Centeno (1997)). Therefore, the established practice of seeking to maximize R_Z instead of minimizing $\psi_Z(u)$ itself is acceptable.

Thus, we deal with the following optimization problem:

Problem 1. Find $(\hat{R}, \hat{Z}) \in]0, +\infty[\times \mathcal{Z}^+$ such that

$\hat{R} = R_{\hat{Z}} = \max\{R_Z : Z \in \mathcal{Z}^+\}$. □

A policy $\hat{Z} \in \mathcal{Z}$ is said to be *optimal for the adjustment coefficient criterion* if $(R_{\hat{Z}}, \hat{Z})$ solves this problem.

3. Approximation by policies of bounded retained risk

In this section we discuss some regularity properties of the map $(R, Z) \mapsto G(R, Z)$. This is mainly technical work which we will use to prove our results. Lemma 1 shows that for any fixed $Z \in \mathcal{Z}$, the map $R \mapsto G(R, Z)$ is smooth and convex on the interior of the domain where its value is finite, but it can exhibit undesirable properties at the boundary of this domain. Lemma 2 below shows that any reinsurance policy can be approximated in an adequate sense by a sequence of policies with bounded (but not uniformly bounded) retained risk, and the functional G exhibits good properties in this set of policies.

Lemma 1. *Fix $Z \in \mathcal{Z}$, and suppose that a $R > 0$ exists such that $G(R, Z) < +\infty$.*

Then, there is a constant $\eta_Z \in]0, +\infty]$ such that map $R \mapsto G(R, Z)$ is smooth in $[0, \eta_Z[$, and $G(R, Z) = +\infty$ for all $R > \eta_Z$. Further,

$$\lim_{R \to \eta_Z^-} G(R, Z) = G(\eta_Z, Z).$$

For $R \in [0, \eta_Z[$, we have

$$\frac{\partial^k G(R, Z)}{\partial R^k} = \int_0^{+\infty} (-L_Z(y))^k \mathrm{e}^{-RL_Z(y)} f(y)\, \mathrm{d}y, \quad k \ge 0. \quad \square \tag{2}$$

Proof. Let $\eta_Z = \sup\{R : G(R, Z) < +\infty\}$. Then,

$$\lim_{R \to \eta_Z^-} G(R, Z) = \lim_{R \to \eta_Z^-} \left(\int_{\{y \ge 0: L_Z(y) \ge 0\}} \mathrm{e}^{-RL_Z(y)} f(y)\, \mathrm{d}y + \int_{\{y \ge 0: L_Z(y) < 0\}} \mathrm{e}^{-RL_Z(y)} f(y)\, \mathrm{d}y \right).$$

The dominated convergence theorem guarantees that

$$\lim_{R \to \eta_Z^-} \int_{\{L_Z \ge 0\}} \mathrm{e}^{-RL_Z(y)} f(y)\, \mathrm{d}y = \int_{\{L_Z \ge 0\}} \mathrm{e}^{-\eta_Z L_Z(y)} f(y)\, \mathrm{d}y \in [0, +\infty[,$$

while the monotone convergence theorem guarantees that

$$\lim_{R \to \eta_Z^-} \int_{\{L_Z < 0\}} \mathrm{e}^{-RL_Z(y)} f(y)\, \mathrm{d}y = \int_{\{L_Z < 0\}} \mathrm{e}^{-\eta_Z L_Z(y)} f(y)\, \mathrm{d}y \in [0, +\infty].$$

i.e., $\lim_{R \to \eta_Z^-} G(R, Z) = G(\eta_Z, Z)$.

Now, fix $R < \eta_Z$. Suppose that, for some $k \ge 0$,

$$\frac{\partial^k G(R, Z)}{\partial R^k} = \int_0^{+\infty} (-L_Z(y))^k \mathrm{e}^{-RL_Z(y)} f(y)\, \mathrm{d}y$$

holds. Then,

$$\begin{aligned} &\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\frac{\partial^k G(R + \varepsilon, Z)}{\partial R^k} - \frac{\partial^k G(R, Z)}{\partial R^k} \right) \\ &= \lim_{\varepsilon \to 0} \int_0^{+\infty} \frac{(-L_Z(y))^k \mathrm{e}^{-(R+\varepsilon)L_Z(y)} - (-L_Z(y))^k \mathrm{e}^{-RL_Z(y)}}{\varepsilon} f(y)\mathrm{d}y \\ &= \lim_{\varepsilon \to 0} \int_0^{+\infty} (-L_Z(y))^k \frac{\mathrm{e}^{-(R+\varepsilon)L_Z(y)} - \mathrm{e}^{-RL_Z(y)}}{\varepsilon} f(y)\mathrm{d}y \\ &= \lim_{\varepsilon \to 0} \int_0^{+\infty} \frac{1}{\varepsilon} \int_0^{\varepsilon} (-L_Z(y))^{k+1} \mathrm{e}^{-(R+\theta)L_Z(y)} \mathrm{d}\theta\ f(y)\mathrm{d}y. \end{aligned}$$

Fix $\hat{L} < +\infty$, an upper bound for L_Z, and fix $\hat{R} \in]R, \eta_Z[$. Then, there exists $q \in [\hat{L}, +\infty[$ such that $\mathrm{e}^{\hat{R}x} \ge x^{k+1}\mathrm{e}^{Rx}$, $\forall x > q$. Fix q satisfying these conditions. In this case, we have

$$\begin{aligned} &\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\frac{\partial^k G(R + \varepsilon, Z)}{\partial R^k} - \frac{\partial^k G(R, Z)}{\partial R^k} \right) \\ &= \lim_{\varepsilon \to 0} \left(\int_{\{L_Z \ge -q\}} \frac{1}{\varepsilon} \int_0^{\varepsilon} (-L_Z(y))^{k+1} \mathrm{e}^{-(R+\theta)L_Z(y)} \mathrm{d}\theta\ f(y)\mathrm{d}y \right. \\ &\quad \left. + \int_{\{L_Z < -q\}} \frac{1}{\varepsilon} \int_0^{\varepsilon} (-L_Z(y))^{k+1} \mathrm{e}^{-(R+\theta)L_Z(y)} \mathrm{d}\theta\ f(y)\mathrm{d}y \right). \end{aligned}$$

When $L_Z \geq -q$, we have

$$|(-L_Z)^{k+1}\mathrm{e}^{-(R+\theta)L_Z}| \leq q^{k+1}\mathrm{e}^{\hat{R}q}.$$

Hence, the dominated convergence theorem implies that

$$\lim_{\varepsilon\to 0}\int_{\{L_Z\geq -q\}}\frac{1}{\varepsilon}\int_0^{\varepsilon}(-L_Z(y))^{k+1}\mathrm{e}^{-(R+\theta)L_Z(y)}\mathrm{d}\theta\ f(y)\mathrm{d}y = \int_{\{L_Z\geq -q\}}(-L_Z(y))^{k+1}\mathrm{e}^{-RL_Z(y)}f(y)\mathrm{d}y.$$

When $L_Z < -q$, we have

$$0 \leq (-L_Z)^{k+1}\mathrm{e}^{-(R+\theta)L_Z} \leq \mathrm{e}^{-\hat{R}L_Z},$$

and again the dominated convergence theorem guarantees that

$$\lim_{\varepsilon\to 0}\int_{\{L_Z< -q\}}\frac{1}{\varepsilon}\int_0^{\varepsilon}(-L_Z(y))^{k+1}\mathrm{e}^{-(R+\theta)L_Z(y)}\mathrm{d}\theta\ f(y)\mathrm{d}y = \int_{\{L_Z< -q\}}(-L_Z(y))^{k+1}\mathrm{e}^{-RL_Z(y)}f(y)\mathrm{d}y.$$

This proves that $\frac{\partial^{k+1}G(R,Z)}{\partial R^{k+1}}$ exists and is as given in (2). □

Lemma 1 has the following immediate corollary:

Corollary 1. *Eq. (1) has at most one positive solution. It has no positive solutions if $E[L_Z] \leq 0$.* □

Proof. Due to (2), Lemma 1 implies that the map $R \mapsto G(R, Z)$ is strictly convex in the interval $[0, \eta_Z]$. Since $G(0, Z) = 1$, it follows that Eq. (1) cannot have more than one positive solution. Indeed, it has one positive solution if and only if $\frac{\partial G}{\partial R}(0, Z) < 0$ and $G(\eta_Z, Z) \geq 1$. Since (2) implies $\frac{\partial G}{\partial R}(0, Z) = -E[L_Z]$, it follows that (1) has no positive solution when $E[L_Z] \leq 0$. □

The proof of Corollary 1 shows that the set $\mathcal{Z}^+$ is, in general, a strict subset of $\{Z \in \mathcal{Z} : E[L_Z] > 0\}$. Indeed, the existence of a reinsurance policy such that $E[L_Z] > 0$ and $G(\eta_Z, Z) < 1$ cannot be ruled out. However, Lemma 2 below shows that, for any such Z, there is a "similar" $\tilde{Z} \in \mathcal{Z}^+$ such that $R_{\tilde{Z}}$ exists and is greater than η_Z. Here a policy "similar" to Z means a policy $\tilde{Z} \in \mathcal{Z}^+$ such that $G(R, \tilde{Z})$ is arbitrarily close to $G(R, Z)$ at any point $R \in]0, +\infty[$. Such policies can be constructed using the following family of maps:

Notation 1. *For each $k > 0$, consider the map $T_k : \mathcal{Z} \mapsto \mathcal{Z}$, defined by*

$$T_kZ(y) = \begin{cases} Z(y), & \text{if } y - Z(y) \leq k; \\ y - k, & \text{if } y - Z(y) > k. \end{cases}$$

Lemma 2. *For each $Z \in \mathcal{Z}$ (fixed) the sequence $\{T_kZ,\ k \in \mathbb{N}\}$ satisfies the following properties:*

$$G(R, T_kZ) < +\infty, \quad \forall R > 0,\ k \in \mathbb{N};$$

$$\lim_{k\to\infty} G(R, T_kZ) = G(R, Z), \quad \forall R > 0. \quad \square$$

Proof. The sequence $\{T_kZ,\ k \in \mathbb{N}\}$ converges pointwise to Z, and $0 \leq T_kZ \leq Y$. Therefore, the dominated convergence theorem guarantees that $\lim_{k\to\infty} E[(T_kZ - Z)^2] = 0$. By continuity of P, this implies $\lim_{k\to\infty} P(T_kZ) = P(Z)$. Also, $\mathrm{e}^{R(P(T_kZ)-c+y-T_kZ(y))} \leq \mathrm{e}^{R(P(T_kZ)-c+k)}$, hence

$$G(R, T_kZ) \leq \mathrm{e}^{R(P(T_kZ)-c+k)} < +\infty, \quad \forall R > 0,\ k \in \mathbb{N}.$$

Finally,

$$G(R, T_kZ) = \mathrm{e}^{R(P(T_kZ)-c)}\int_0^{+\infty}\mathrm{e}^{R(y-T_kZ(y))}f(y)\mathrm{d}y.$$

By continuity of P, we have $\lim_{k\to\infty}\mathrm{e}^{R(P(T_kZ)-c)} = \mathrm{e}^{R(P(Z)-c)}$. By the monotone convergence theorem, we have $\lim_{k\to\infty}\int_0^{+\infty}\mathrm{e}^{R(y-T_kZ(y))}f(y)\mathrm{d}y = \int_0^{+\infty}\mathrm{e}^{R(y-Z(y))}f(y)\mathrm{d}y$. Therefore, $\lim_{k\to\infty} G(R, T_kZ) = G(R, Z)$ holds for every $R > 0$. □

4. Maximization of expected utility of wealth

The discussion in the previous section shows that the map $Z \mapsto R_Z$ is a functional defined in an implicit form whose domain, $\mathcal{Z}^+$, lacks a convenient structure to allow for optimization methods based on the implicit function theorem. It turns out that this difficulty can be overcome by exploiting the close relationship between maximizing the adjustment coefficient of the retained risk (Problem 1) and maximizing the expected utility of wealth with an arbitrary coefficient of risk aversion (Problem 2, below). In this section we discuss in detail the relationship between the two problems.

Consider the exponential utility function with coefficient of risk aversion $R > 0$,

$$U_R(w) = -\mathrm{e}^{-Rw}.$$

For any given coefficient of risk aversion, $R > 0$, the expected utility of the profit obtained by the insurance company in a given unit of time is

$$E[U_R(L_Z)] = -G(R, Z). \tag{3}$$

We consider the maximization problem:

Problem 2. Find $\hat{Z} \in \mathcal{Z}$, such that

$$E[U_R(L_{\hat{Z}})] = \max\{E[U_R(L_Z)] : Z \in \mathcal{Z}\}.$$

Here $R > 0$ is a given constant (fixed). □

A policy $Z \in \mathcal{Z}$ is said to be *optimal for the expected utility criterion* with coefficient of risk aversion R if it solves Problem 2 for that particular R. When it is clear from the context which coefficient of risk aversion is being considered, we will just say that the policy is optimal for the expected utility criterion.

It follows immediately from (3) that a policy is optimal for the expected utility criterion if and only if it is a minimizer of the functional $Z \mapsto G(R, Z)$, with the same (fixed) value of R being considered. The following relationship between Problems 1 and 2 is the key to our approach:

Proposition 1. *A pair $(\hat{R}, \hat{Z}) \in]0, +\infty[\times \mathcal{Z}$ solves Problem 1 (i.e., $\hat{Z}$ is optimal for the adjustment coefficient criterion) if and only if it satisfies the following conditions:*

1. *$\hat{Z}$ is optimal for the expected utility criterion (i.e., it solves Problem 2) with coefficient of risk aversion $R = \hat{R}$;*
2. *$G(\hat{R}, \hat{Z}) = 1$.* □

Proof. Condition 2 can be written $\hat{R} = R_{\hat{Z}}$; hence it is necessary. Assume that condition 2 holds but condition 1 fails, i.e., that there is $Z \in \mathcal{Z}$ such that $G(\hat{R}, Z) < G(\hat{R}, \hat{Z}) = 1$. Then, Lemma 2 guarantees the existence of $\tilde{Z} \in \mathcal{Z}^+$ such that $G(\hat{R}, \tilde{Z}) < G(\hat{R}, \hat{Z}) = 1$. This implies $R_{\tilde{Z}} > \hat{R}$, and hence $\hat{R}$ is not the greatest adjustment coefficient. Thus, conditions 1 and 2 are both necessary.

Conversely, if conditions 1 and 2 hold, then, for any $Z \in \mathcal{Z}^+$ we have $G(\hat{R}, Z) \geq G(\hat{R}, \hat{Z}) = 1$. Taking into account that $G(0, Z) = 1$ and that Lemma 1 guarantees that the map $R \mapsto G(R, Z)$ is convex, this implies $R_Z \leq \hat{R}$. Since this holds for every $Z \in \mathcal{Z}^+$, it follows that $\hat{R}$ is the greatest possible adjustment coefficient. □

Proposition 1 shows that Problem 1 can be solved in two steps:

1. For each $R \in]0, +\infty[$ find Z_R, the respective optimal policy for the expected utility criterion. Equivalently, find

   $Z_R = \arg\min\{G(R, Z) : Z \in \mathcal{Z}\}$;
2. Solve the equation with one single real variable

   $G(R, Z_R) = 1$.

Notation 2. *We will always adhere to the notation used above:*

For each $Z \in \mathcal{Z}^+$, R_Z denotes the positive solution of the equation $G(R, Z) = 1$, for the particular (fixed) Z being considered.

For each $R > 0$, Z_R denotes the optimal policy for the expected utility criterion with the particular coefficient of risk aversion R.

Below we show that the map $R \mapsto Z_R$ is well defined for $R \in]0, +\infty[$.

5. Existence and uniqueness of optimal policies for the expected utility criterion

From the economic theory point of view, Proposition 1 presents an important relationship between the adjustment coefficient criterion and the expected utility criterion: The maximal adjustment coefficient equals the coefficient of risk aversion for which the maximal expected utility that can be attained is -1. The optimal policy for the adjustment coefficient criterion coincides with the optimal policy for the expected utility criterion for this particular value of the coefficient of risk aversion. From the mathematical point of view, the main issue is that Problem 2 is much easier to study than Problem 1.

In this section we prove that there is always an optimal policy for the expected utility criterion and that all the optimal policies are equivalent from the economic point of view, in the sense that the net result, and hence the profit, is the same with probability one.

Definition 1. Two strategies $Z_1, Z_2 \in \mathcal{Z}$ are said to be economically equivalent if

$$\Pr\{Z_1 - P(Z_1) = Z_2 - P(Z_2)\} = 1. \quad \square \tag{4}$$

Notice that (4) implies that two economically equivalent policies differ (up to null sets) only by an additive constant and this constant must be the difference between the two premiums. That is, Z_1 and Z_2 are economically equivalent if and only if there exists a constant x such that

$$Z_2 = Z_1 + x \quad \text{and} \quad P(Z_2) = P(Z_1) + x. \tag{5}$$

The equalities (5) mean that by treaty Z_2 the reinsurer agrees to pay an extra x, independently of whatever claims occur, provided the cedent pays him that same extra x. When $P(Z + x) \neq P(Z) + x$ holds for every $x \neq 0$, as is the case of the expected value principle, Z is the unique element of $\mathcal{Z}$ which is economically equivalent to itself. However, there are premiums, like the standard deviation or the variance principle, for which the equality

$$P(Z + x) = P(Z) + x$$

holds for any $Z \in \mathcal{Z}$ and any $x \in \mathbb{R}$. In Section 9 we will show that economically equivalent treaties have relevance for such premiums.

When the infimum of the support of the distribution of Y is zero the concept is not relevant too. Let ν be that number, i.e.,

$$\nu = \sup\{y \geq 0 : \Pr\{Y < y\} = 0\}.$$

$\nu = 0$ implies that if $Z \in \mathcal{Z}$ and $x \neq 0$, then $(Z + x) \notin \mathcal{Z}$ must hold. Hence the existence of optimal equivalent policies, against a unique optimal policy, can only happen when $\nu > 0$ and $P(Z + x) = P(Z) + x$ holds for some $x \neq 0$.

Theorem 1. *For each $R \in]0, +\infty[$ there is an optimal policy for the expected utility criterion. All optimal policies for the same R are economically equivalent.* □

Proof. In our proof we consider the equivalent problem of minimizing the functional $Z \mapsto G(R, Z)$, for the particular value of R being considered.

A simple computation shows that convexity of the functional P implies that the functional $Z \mapsto G(R, Z)$ is convex, the map $Z \mapsto P(Z) - Z$ is convex and the functional $(P(Z) - Z) \mapsto G(R, Z)$ is strictly convex. Since $\mathcal{Z}$ is convex, it follows that all minimizers of the functional $Z \mapsto G(R, Z)$ must be economically equivalent.

Existence of a minimizer is a consequence of the classical Banach–Alaoglu Theorem from functional analysis (see, e.g., Rudin (1991)). Below we present only the key points of the argument.

Let L_2 denote the set of all random variables of type $\phi = \phi(Y)$ with finite variance (i.e., the set of all measurable functions, $\phi : [0, +\infty[\mapsto \mathbb{R}$, such that $\int_0^{+\infty} \phi(y)^2 f(y) \mathrm{d}y < +\infty$). A sequence $\{\phi_k \in L_2,\ k \in \mathbb{N}\}$ is said to *converge weakly* to a random variable $\phi \in L_2$ if and only if

$$\lim_{k\to\infty} \int_0^{+\infty} \phi_k(y)\psi(y) f(y) \mathrm{d}y = \int_0^{+\infty} \phi(y)\psi(y) f(y) \mathrm{d}y$$

holds for all $\psi \in L_2$. In our context, the Banach–Alaoglu Theorem states that if a subset of L_2 is bounded in the mean-squared sense and it is closed with respect to weak convergence, then it is compact with respect to weak convergence. We will also use Theorem 3.12 in Rudin (1991), p. 66, which states that if a convex subset of L_2 is closed with respect to mean-squared convergence, then it is also closed with respect to weak convergence.

Since $E[Z^2] \leq E[Y^2] < +\infty$ holds for all $Z \in \mathcal{Z}$, we see that the set of all reinsurance policies is a subset of L_2, and it is bounded in the mean-squared sense. $\mathcal{Z}$ is convex and it can be checked that it is closed with respect to mean-squared convergence. Hence it is closed with respect to weak convergence and the Banach–Alaoglu Theorem guarantees that any sequence $\{Z_k \in \mathcal{Z},\ k \in \mathbb{N}\}$ contains a subsequence which converges weakly to some $Z \in \mathcal{Z}$.

Convexity of the reinsurance premium implies convexity of the functional $Z \mapsto G(R, Z)$. This means that the epigraph of this functional (i.e., the set $\{(Z, \gamma) \in \mathcal{Z} \times \mathbb{R} : \gamma \geq G(R, Z)\}$) is a convex closed set (with respect to mean-squared convergence of $\{Z_k \in \mathcal{Z}\}$ and to usual convergence of $\{\gamma_k \in \mathbb{R}\}$). Therefore, Theorem 3.12 in Rudin (1991) guarantees that it is also closed with respect to weak convergence of $\{Z_k \in \mathcal{Z}\}$ and to usual convergence of $\{\gamma_k \in \mathbb{R}\}$. This implies that any sequence $\{Z_k \in \mathcal{Z}\}$, converging weakly to some $Z \in \mathcal{Z}$, satisfies

$$\lim_{k\to\infty} G(R, Z_k) \geq G(R, Z). \tag{6}$$

Now, consider a sequence $\{Z_k \in \mathcal{Z}\}$ such that $\lim_{k\to\infty} G(R, Z_k) = \inf\{G(R, Z),\ Z \in \mathcal{Z}\}$. As stated above, $\{Z_k \in \mathcal{Z}\}$ admits a subsequence which converges weakly to some $\hat{Z} \in \mathcal{Z}$. Inequality (6) states that $G(R, \hat{Z}) \leq \inf\{G(R, Z),\ Z \in \mathcal{Z}\}$; hence $\hat{Z}$ is a minimizer. □

6. Existence and uniqueness of the optimal policy for the adjustment coefficient criterion

In this section we use the results above to prove the existence and uniqueness of solutions to Problem 1.

Theorem 2. *There is an optimal policy for the adjustment coefficient criterion. All optimal policies are economically equivalent.* □

Proof. The fact that all optimal policies are economically equivalent is a straightforward consequence of Proposition 1 and Theorem 1. To see this, suppose the adjustment coefficient admits two different global maximizers, $\hat{Z}, \tilde{Z} \in \mathcal{Z}$. Proposition 1 states that $\hat{Z}$ and $\tilde{Z}$ are both optimal policies for the expected utility criterion with the particular coefficient of risk aversion $R = \hat{R} = \widetilde{R}$. Then, Theorem 1 states that the two treaties must be economically equivalent.

In order to prove existence, we will proceed in the following way: first we prove that the set $\{R \in]0, +\infty[\colon G(R, Z_R) \geq 1\}$ is non-empty. Then, we prove that the infimum of this set solves equation $G(R, Z_R) = 1$.

Suppose that $G(R, Z_R) < 1$ holds for all $R \in]0, +\infty[$. Consider a sequence $\{R_k\} \to +\infty$, and the corresponding sequence $\{Z_{R_k}\}$. In the proof of Theorem 1 we showed that $\{R_k\}$ can be chosen in such a way that $\{Z_{R_k}\}$ converges in the weak sense towards some $\tilde{Z} \in \mathcal{Z}$, and

$$G(R, \tilde{Z}) \leq \lim_{k\to\infty} G(R, Z_{R_k}) \tag{7}$$

holds for every $R \in]0, +\infty[$. Since the map $R \mapsto G(R, Z)$ cannot cross the line $G = 1$ more than once in the positive semiaxis, the hypothesis $G(R_k, Z_{R_k}) < 1$ implies that $G(R, Z_{R_{k+m}}) < 1$ holds for every $k, m \in \mathbb{N}$, $R \leq R_k$. Therefore, (7) implies $G(R, \tilde{Z}) \leq 1$, $\forall R \in]0, +\infty[$, which implies $\Pr\{L_{\tilde{Z}} < 0\} = 0$. Since this contradicts Assumption 2, we conclude that $G(R, Z_R) \geq 1$ must hold for some finite R.

Let $\hat{R} = \inf\{R > 0 : G(R, Z_R) \geq 1\}$. By the definition of the map $R \mapsto Z_R$, $G(\hat{R} + \varepsilon, Z_{\hat{R}+\varepsilon}) \leq G(\hat{R} + \varepsilon, Z_{\hat{R}})$ holds for all $\varepsilon > 0$. Since the map $R \mapsto G(R, Z_{\hat{R}})$ is continuous, this implies $G(\hat{R}, Z_{\hat{R}}) = \lim_{R\to\hat{R}^+} G(R, Z_{\hat{R}}) \geq \limsup_{R\to\hat{R}^+} G(R, Z_R) \geq 1$. Now, chose a sequence $\{R_k < \hat{R}\}$, converging to $\hat{R}$. Again, this sequence can be chosen in such a way that the corresponding sequence $\{Z_{R_k}\}$ converges weakly to some $\tilde{Z} \in \mathcal{Z}$, and

$$G(R, \tilde{Z}) \leq \lim_{k\to\infty} G(R, Z_{R_k}) \leq 1$$

holds for every $R < \hat{R}$. Now, consider the maps $T_k : \mathcal{Z} \mapsto \mathcal{Z}$, $k \in \mathbb{N}$, defined in Section 3. Then,

$$\begin{aligned}\frac{\partial G}{\partial R}(R, T_k\tilde{Z}) &= \int_0^{+\infty} (P(T_k\tilde{Z}) - c + y - T_k\tilde{Z}(y)) \\ &\quad \times \mathrm{e}^{R(P(T_k\tilde{Z})-c+y-T_k\tilde{Z}(y))} f(y)\mathrm{d}y \\ &\leq (P(T_k\tilde{Z}) - c + k)\mathrm{e}^{\hat{R}(P(T_k\tilde{Z})-c+k)},\end{aligned}$$

holds for all $R < \hat{R}$. Therefore, the mean value theorem guarantees that

$$\begin{aligned}G(\hat{R}, T_k\tilde{Z}) &= G(\hat{R} - \varepsilon, T_k\tilde{Z}) + \varepsilon\frac{\partial G}{\partial R}(\rho, T_k\tilde{Z}) \\ &\leq \mathrm{e}^{(\hat{R}-\varepsilon)(P(T_k\tilde{Z})-P(\tilde{Z}))} G(\hat{R} - \varepsilon, \tilde{Z}) \\ &\quad + \varepsilon(P(T_k\tilde{Z}) - c + k)\mathrm{e}^{\hat{R}(P(T_k\tilde{Z})-c+k)}.\end{aligned}$$

By taking the limit when $\varepsilon \to 0$, this implies

$$G(\hat{R}, Z_{\hat{R}}) \leq G(\hat{R}, T_k\tilde{Z}) \leq \mathrm{e}^{\hat{R}(P(T_k\tilde{Z})-P(\tilde{Z}))}.$$

By taking the limit when $k \to +\infty$, this implies $G(\hat{R}, Z_{\hat{R}}) \leq 1$. □

7. Necessary condition for optimality

Since the optimal policy for the adjustment coefficient criterion must also be optimal for the expected utility criterion, necessary conditions for optimality with respect to the latter of these criteria are also necessary for optimality with respect to the first.

At least in some important cases (see Sections 8 and 9, below), it turns out that our necessary optimality conditions can be used in a straightforward way to show that optimal policies for the expected utility criterion must have a certain structure which depends on the premium calculation principle, but does not depend on the particular coefficient of risk aversion being considered. Therefore, the optimal policy for the adjustment coefficient criterion must have the same structure. Hence, provided this structure can be smoothly parameterized by a small number of variables, we achieve reduction of both problems into finite-dimensional optimization problems which can be solved by standard mathematical programming techniques.

In order to obtain a necessary optimality condition, we use the so-called "needle-like perturbations". This type of perturbation is widely used in optimal control, at least since the pioneering work of Pontryagin and Gamkrelidze (see e.g. Gamkrelidze (1978)). The main feature of this type of perturbation is that infinitesimal analysis is done by making the support of the perturbation, rather than its magnitude, go to zero.

Fix a reinsurance policy, $Z \in \mathcal{Z}$. For each $v > 0$, $\varepsilon > 0$, $\alpha \in [0, 1]$, we consider the perturbed reinsurance policy

$$Z_{v,\alpha,\varepsilon}(y) = \begin{cases} Z(y), & \text{if } y \notin [v, v+\varepsilon]; \\ \alpha y, & \text{if } y \in [v, v+\varepsilon]. \end{cases}$$

In what follows we suppose the expression

$$\Delta P_Z(y) = \lim_{\alpha \to \frac{Z(y)}{y}} \lim_{\varepsilon \to 0^+} \frac{P(Z_{y,\alpha,\varepsilon}) - P(Z)}{\varepsilon(\alpha y - Z(y))}$$

defines a function $y \mapsto \Delta P_Z(y)$ in a domain having probability equal to one.

Theorem 3. *Suppose $Z \in \mathcal{Z}$ is optimal for the expected utility criterion with the particular coefficient of risk aversion $R > 0$. Then, Z satisfies the following conditions:*

$$\begin{cases} e^{-RL_Z(y)} f(y) \geq G(R, Z)\Delta P_Z(y), & \textit{if } Z(y) = y; \\ e^{-RL_Z(y)} f(y) = G(R, Z)\Delta P_Z(y), & \textit{if } 0 < Z(y) < y; \\ e^{-RL_Z(y)} f(y) \leq G(R, Z)\Delta P_Z(y), & \textit{if } Z(y) = 0, \end{cases}$$

with probability equal to one. □

Proof. Let $Z \in \mathcal{Z}$ be optimal for the expected utility criterion with the particular (fixed) coefficient of risk aversion R. Since Z is optimal,

$$G(R, Z_{v,\alpha,\varepsilon}) - G(R, Z) \geq 0 \tag{8}$$

must hold for every $v > 0$, $\alpha \in [0, 1]$, $\varepsilon > 0$ But,

$$\begin{aligned} &G(R, Z_{v,\alpha,\varepsilon}) - G(R, Z) \\ &\quad = \int_0^{+\infty} e^{-RL_Z(y)}(e^{-R(L_{Z_{v,\alpha,\varepsilon}}(y) - L_Z(y))} - 1) f(y)dy \\ &\quad = \int_0^{+\infty} e^{-RL_Z}(e^{R(P(Z_{v,\alpha,\varepsilon}) - P(Z)) - R(Z_{v,\alpha,\varepsilon} - Z)} - 1) f(y)dy \\ &\quad = \int_0^{+\infty} e^{-RL_Z}(e^{R(P(Z_{v,\alpha,\varepsilon}) - P(Z)) - R(Z_{v,\alpha,\varepsilon} - Z)} \\ &\qquad - e^{R(P(Z_{v,\alpha,\varepsilon}) - P(Z))}) f(y)dy \\ &\qquad + \int_0^{+\infty} e^{-RL_Z}(e^{R(P(Z_{v,\alpha,\varepsilon}) - P(Z))} - 1) f(y)dy \\ &\quad = e^{R(P(Z_{v,\alpha,\varepsilon}) - P(Z))} \int_0^{+\infty} e^{-RL_Z}(e^{-R(Z_{v,\alpha,\varepsilon} - Z)} - 1) f(y)dy \\ &\qquad + (e^{R(P(Z_{v,\alpha,\varepsilon}) - P(Z))} - 1) G(R, Z). \end{aligned}$$

Therefore, inequality (8) reduces to

$$\begin{aligned} &\frac{1}{\varepsilon}\int_v^{v+\varepsilon} e^{-RL_Z(y)}(e^{-R(\alpha y - Z(y))} - 1) f(y)dy \\ &\quad \geq G(R, Z)\frac{e^{-R(P(Z_{v,\alpha,\varepsilon}) - P(Z))} - 1}{\varepsilon}. \end{aligned}$$

If $v > 0$ is a Lebesgue point of both the functions $y \mapsto f(y)$, $y \mapsto e^{-RL_Z(y)} f(y)$, this implies that

$$\begin{aligned} &e^{-RL_Z(v)}(e^{-R(\alpha v - Z(v))} - 1) f(v) \\ &\quad \geq -RG(R, Z) \lim_{\varepsilon \to 0^+} \frac{P(Z_{v,\alpha,\varepsilon}) - P(Z)}{\varepsilon} \end{aligned} \tag{9}$$

must hold, provided $\Delta P(v)$ is well defined. Now, consider the case when $Z(v) < v$. Then, inequality (9) implies

$$\begin{aligned} &\lim_{\alpha \to \frac{Z(v)}{v}^+} e^{-RL_Z(v)} \frac{e^{-R(\alpha v - Z(v))} - 1}{\alpha v - Z(v)} f(v) \\ &\quad \geq -RG(R, Z) \lim_{\alpha \to \frac{Z(v)}{v}^+} \lim_{\varepsilon \to 0^+} \frac{P(Z_{v,\alpha,\varepsilon}) - P(Z)}{\varepsilon(\alpha v - Z(v))}, \end{aligned}$$

i.e., $e^{-RL_Z(v)} f(v) \leq G(R, Z)\Delta P(v)$. In the case when $Z(v) > 0$, inequality (9) implies

$$\begin{aligned} &\lim_{\alpha \to \frac{Z(v)}{v}^-} e^{-RL_Z(v)} \frac{e^{-R(\alpha v - Z(v))} - 1}{\alpha v - Z(v)} f(v) \\ &\quad \leq -RG(R, Z) \lim_{\alpha \to \frac{Z(v)}{v}^+} \lim_{\varepsilon \to 0^+} \frac{P(Z_{v,\alpha,\varepsilon}) - P(Z)}{\varepsilon(\alpha v - Z(v))}, \end{aligned}$$

i.e., $e^{-RL_Z(v)} f(v) \geq G(R, Z)\Delta P_Z(v)$. Since this holds for every v in the domain of ΔP_Z except in a set of points that are not Lebesgue points of the functions $y \mapsto f(y)$, $y \mapsto e^{-RL_Z(y)} f(y)$, we conclude that it holds almost certainly. □

One important class of functionals for which ΔP_Z is defined with probability one for each $Z \in \mathcal{Z}$ is the class of functionals of the type

$$P(Z) = \gamma\left(\int_0^{+\infty} Q(y, Z(y)) f(y)dy\right), \quad Z \in \mathcal{Z}, \tag{10}$$

where $Q : \mathbb{R}^2 \mapsto \mathbb{R}^n$, $\gamma : \mathbb{R}^n \mapsto \mathbb{R}$ are smooth functions. Indeed, for functionals of this class, we have:

$$\Delta P_Z(v) = D\gamma \cdot \frac{\partial Q}{\partial z}(v, Z(v)) f(v), \quad \text{a.e. } v > 0,$$

for any $Z \in \mathcal{Z}$. Here, $D\gamma$ denotes the differential of $\gamma(x)$, evaluated at the point $x = \int_0^{+\infty} Q(y, Z(y)) f(y)dy$. For this particular class of functionals, Theorem 3 takes the form of the following corollary.

Corollary 2. *Suppose P is of type (10) and $Z \in \mathcal{Z}$ is optimal for the expected utility criterion with the particular coefficient of risk aversion $R > 0$. Then, Z satisfies the following conditions.*

$$\begin{cases} e^{-RL_Z(y)} \geq G(R, Z) D\gamma \cdot \frac{\partial Q}{\partial z}(y, Z(y)), & \textit{if } Z(y) = y; \\ e^{-RL_Z(y)} = G(R, Z) D\gamma \cdot \frac{\partial Q}{\partial z}(y, Z(y)), & \textit{if } 0 < Z(y) < y; \\ e^{-RL_Z(y)} \leq G(R, Z) D\gamma \cdot \frac{\partial Q}{\partial z}(y, Z(y)), & \textit{if } Z(y) = 0, \end{cases}$$

with probability equal to one. □

Proof. Follows immediately from Theorem 3 and from the considerations that precede the corollary. □

8. The expected value principle

In this section we use the results from previous sections to solve Problems 1 and 2 in the case when the reinsurance premium is

$$P(Z) = (1+\beta)E[Z],$$

where β is a positive constant.

It is straightforward to check that a premium of this type satisfies Assumption 3. In order to satisfy Assumption 2 too, β cannot be too small. Indeed, if $\beta \le \frac{c-E[Y]}{E[Y]}$, then reinsurance of the totality of risks (i.e., $Z(y) = y,\ \forall y$) satisfies $\Pr\{L_Z < 0\} = 0$.

For a premium computed according to the expected value principle satisfying Assumption 2, we can prove the following:

Theorem 4. *Assume the reinsurance premium is computed by the expected value principle. For each positive value of the coefficient of risk aversion, there is an optimal policy for the expected utility criterion. There is an optimal policy for the adjustment coefficient criterion. The optimal policy for any of the above criteria is unique and it is a stop-loss contract.* □

Theorem 4 is a generalization of the result obtained by Hesselager (1990). In his paper he proved a similar result under the constraint that the feasible reinsurance solutions had a fixed expected value.

Proof of Theorem 4. Theorems 1 and 2 guarantee existence of solutions and that all optimal solutions must be economically equivalent. The considerations that follow Definition 1 show that, under the expected value principle, any strategy is economically equivalent only to itself. Hence the optimal solution must be unique. Concerning the structure of the solution, the expected value principle is a premium of type (10), with $Q(y,z) = z$, $\gamma(x) = (1+\beta)x$. Therefore, Corollary 2 states that the optimal policy for the expected utility criterion with any coefficient of risk aversion, $R > 0$, must satisfy

$$\begin{cases} e^{R(P(Z)-c)} \ge G(R,Z)(1+\beta), & \text{if } Z(y) = y; \\ e^{R(P(Z)-c+y-Z(y))} = G(R,Z)(1+\beta), & \text{if } 0 < Z(y) < y; \\ e^{R(P(Z)-c+y)} \le G(R,Z)(1+\beta), & \text{if } Z(y) = 0. \end{cases}$$

Notice that

$$\frac{e^{R(P(Z)-c)}}{G(R,Z)} = \frac{1}{\int_0^{+\infty} e^{R(y-Z(y))} f(y)\mathrm{d}y} \le 1.$$

Therefore, the first condition cannot be satisfied by any pair $(R,Z) \in\ ]0,+\infty[\ \times \mathcal{Z}$. Hence, optimal policies for the expected utility criterion are stop-loss contracts

$$Z^M(y) = \begin{cases} 0, & \text{if } y \le M; \\ y-M, & \text{if } y > M, \end{cases}$$

with

$$M = c - (1+\beta)E[Z^M] + \frac{1}{R}\ln((1+\beta)G(R,Z^M)). \qquad (11)$$

Then, Proposition 1 shows that the solution of Problem 1 must be a pair (R, Z^M), satisfying condition (11) and $G(R,Z^M) = 1$. □

Notice that Theorem 4 implies that the retained risk must be bounded in order to achieve the maximum expected utility of wealth (and, therefore, also to achieve the maximum adjustment coefficient). If the support of the distribution of Y is unbounded, then, the stop-loss threshold must always be finite, no matter how large the reinsurance charge is. In order to see this, suppose that the support of the distribution of Y is unbounded and $Z = 0$ maximizes the expected utility of wealth for a particular coefficient of risk aversion, $R > 0$. Then, Corollary 2 implies that there exists a sequence $y_k \to +\infty$, such that $e^{R(-c+y_k)} \le 1+\beta$, which is clearly impossible.

If the support of the distribution of Y is bounded, the argument above does not apply because the sequence y_k lies outside the support of the distribution. From the mathematical point of view, there are examples in which the optimal stop-loss threshold for moderately high coefficients of risk aversion is large enough for the support of the distribution of Y to be contained in the interval $[0, M[$ and hence for no risk to be effectively reinsured. However, in order to obtain this situation for values of the coefficient of risk aversion close to the maximum adjustment coefficient, it seems necessary to consider extremely high premium loadings, making these cases unrealistic compared to real-world practice.

9. Variance-related principles

In this section we present the solutions for the case when the reinsurance premium is computed according to a principle of the type

$$P(Z) = E[Z] + g(\mathrm{Var}(Z)), \qquad (12)$$

where $g : [0,+\infty[\ \mapsto [0,+\infty[$ is a function smooth in $]0,+\infty[$ such that $g(0) = 0$ and $g'(x) > 0,\ \forall x \in\ ]0,+\infty[$. In order to see that there are relevant principles of this type satisfying Assumptions 2 and 3, notice that the standard deviation principle (i.e., $P(Z) = E[Z] + \beta\sqrt{\mathrm{Var}(Z)}$, with $\beta > 0$) and the variance principle ($P(Z) = E[Z] + \beta\mathrm{Var}(Z)$, with $\beta > 0$) are principles of this type, with $g(x) = \beta\sqrt{x}$ and $g(x) = \beta x$, respectively. In order to check convexity of these principles, see Deprez (1985). Similar to the expected value principle (Section 8, above) the loading coefficient, β, of the standard deviation or the variance principle must be sufficiently large in order for Assumption 2 to hold.

Theorem 5. *Assume that the reinsurance premium is computed by a functional of type (12). For each positive value of the coefficient of risk aversion, there is an optimal policy for the expected utility criterion. There is an optimal policy for the adjustment coefficient criterion. The optimal policy for any of the above criteria must be economically equivalent to one of the following policies:*

(a) $Z \equiv 0$, *(no risk is reinsured);*

(b) *a contract satisfying*

$$y = Z(y) + \frac{1}{R}\ln\frac{Z(y)+\alpha}{\alpha}, \quad a.e.\ y \geq 0, \tag{13}$$

where $\alpha > 0$ is a constant such that

$$\alpha = \frac{1}{2g'(\mathrm{Var}(Z))} - E[Z], \tag{14}$$

and R is the risk-aversion coefficient or the maximal adjustment coefficient, according to which optimality criterion is being considered.

If g' is bounded in a neighbourhood of zero, then $Z \equiv 0$ cannot be optimal for any of the two criteria. □

Proof. Existence and uniqueness up to economical equivalence are guaranteed by Theorems 1 and 2.

A simple computation shows that when g' is unbounded in any neighbourhood of zero, then $\Delta P_{Z\equiv 0}(y) = +\infty$ with probability equal to one may hold. In this case, Theorem 3 does not exclude the possibility of $Z \equiv 0$ being optimal. Indeed, it can be shown that $Z \equiv 0$ is optimal when Y follows an exponential distribution and the reinsurance premium is computed by the standard deviation principle with sufficiently high loading coefficient. If g' is bounded in a neighbourhood of zero, then $\Delta P_{Z\equiv 0}(y) = 1$ almost certainly and optimality of $Z \equiv 0$ is excluded by Theorem 3.

Let Z denote the optimal contract and assume that $\Pr\{Z(Y) > 0\} > 0$. Take $\gamma(x_1, x_2) = x_1 + g(x_2 - x_1^2)$, $Q(y, z) = (z, z^2)$. This yields $\Delta P_Z(y) = D\gamma \cdot \frac{\partial Q}{\partial z}(y, Z(y)) = 1 + 2g'(\mathrm{Var}(Z))(Z(y) - E[Z])$, and Corollary 2 states that a maximizer of the expected utility criterion must satisfy

$$\begin{cases} e^{R(P(Z)-c)} \geq G(R,Z) \\ \quad \times (1 + 2g'(\mathrm{Var}(Z))(y - E[Z])), & \text{when } Z(y) = y; \\ e^{R(P(Z)-c+y-Z(y))} = G(R,Z) \\ \quad \times (1 + 2g'(\mathrm{Var}(Z))(Z(y) - E[Z])), & \text{when } 0 < Z(y) < y; \\ e^{R(P(Z)-c+y)} \leq G(R,Z) \\ \quad \times (1 - 2g'(\mathrm{Var}(Z))E[Z]), & \text{when } Z(y) = 0, \end{cases}$$

with probability equal to one. Making

$$\alpha_1 = \frac{e^{R(P(Z)-c)}}{2g'(\mathrm{Var}(Z))G(R,Z)}, \tag{15}$$

$$\alpha_2 = E[Z] - \frac{1}{2g'(\mathrm{Var}(Z))}, \tag{16}$$

this reduces to

$$\begin{cases} y \leq \alpha_1 + \alpha_2, & \text{when } Z(y) = y; \\ Z(y) = \alpha_1 e^{R(y-Z(y))} + \alpha_2, & \text{when } 0 < Z(y) < y; \\ \alpha_1 e^{Ry} + \alpha_2 \leq 0, & \text{when } Z(y) = 0. \end{cases}$$

Clearly, the third condition cannot be satisfied by any $y \geq 0$ if the first condition holds for some $y > 0$, and vice versa. Therefore, the maximizer must be of one of the following forms:

$$Z(y) = \begin{cases} 0, & \text{if } y \leq \frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}; \\ \alpha_1 e^{R(y-Z(y))} + \alpha_2, & \text{if } y > \frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}, \end{cases} \tag{17}$$

or

$$Z(y) = \begin{cases} y, & \text{if } y \leq \alpha_1 + \alpha_2; \\ \alpha_1 e^{R(y-Z(y))} + \alpha_2, & \text{if } y > \alpha_1 + \alpha_2, \end{cases} \tag{18}$$

where (17) holds in the case $\alpha_1 + \alpha_2 \leq 0$ (the third condition holds for some $y \geq 0$), and (18) holds in the case $\alpha_1 + \alpha_2 > 0$ (the first condition holds for some $y > 0$).

We conclude the proof by showing that if any of the solutions (17) or (18) is optimal, then there is an economically equivalent policy with parameters $(\tilde{\alpha}_1, \tilde{\alpha}_2)$ satisfying $\tilde{\alpha}_2 = -\tilde{\alpha}_1$.

In what follows, Z denotes one (fixed) optimal policy.

First suppose that $\alpha_{1+}\alpha_2 < 0$. Using equation (17), we obtain

$$\begin{aligned} G(R,Z) &= e^{R(P(Z)-c)}\left(\int_0^{\frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}} e^{Ry} f(y)\mathrm{d}y \right. \\ &\quad \left. + \int_{\frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}}^{+\infty} \frac{Z(y)-\alpha_2}{\alpha_1} f(y)\mathrm{d}y\right) \\ &= e^{R(P(Z)-c)}\left(\int_0^{\frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}} e^{Ry} f(y)\mathrm{d}y \right. \\ &\quad \left. + \frac{E[Z]}{\alpha_1} - \frac{\alpha_2}{\alpha_1}\int_{\frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}}^{+\infty} f(y)\mathrm{d}y\right). \end{aligned}$$

Therefore, equality (15) reduces to

$$\alpha_1 = \frac{1}{2g'(\mathrm{Var}(Z))\left(\int_0^{\frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}} e^{Ry} f(y)\mathrm{d}y + \frac{E[Z]}{\alpha_1} - \frac{\alpha_2}{\alpha_1}\int_{\frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}}^{+\infty} f(y)\mathrm{d}y\right)}.$$

This implies

$$\begin{aligned} &\int_0^{\frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}} e^{Ry} f(y)\mathrm{d}y - \frac{\alpha_2}{\alpha_1}\int_{\frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}}^{+\infty} f(y)\mathrm{d}y \\ &\quad = \left(\frac{1}{2g'(\mathrm{Var}(Z))} - E[Z]\right)\frac{1}{\alpha_1}. \end{aligned} \tag{19}$$

But equality (16) implies that

$$\left(\frac{1}{2g'(\mathrm{Var}(Z))} - E[Z]\right)\frac{1}{\alpha_1} = \frac{-\alpha_2}{\alpha_1}.$$

Substituting in (19) we obtain

$$\int_0^{\frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}} \left(e^{Ry} - \frac{-\alpha_2}{\alpha_1}\right) f(y)\mathrm{d}y = 0,$$

which holds if and only if $\Pr\{Y \leq \frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}\} = 0$, this is to say if and only if $\frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1} \leq \nu$.

Now, we suppose that $\alpha_1 + \alpha_2 > 0$. An analogous argument shows that optimality implies

$$\int_0^{\alpha_1+\alpha_2} (\alpha_1 + \alpha_2 - y) f(y)\mathrm{d}y = 0,$$

which holds if and only if $\Pr\{Y \leq \alpha_1 + \alpha_2\} = 0$, that is to say if and only if $\alpha_1 + \alpha_2 \leq \nu$.

Hence $\alpha_1 + \alpha_2 \neq 0$ can be optimal only if $0 \leq \alpha_1 + \alpha_2 \leq \nu$ or $\alpha_1 + \alpha_2 < 0$ and $\frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1} \leq \nu$.

Consider the function $y \mapsto Z^*(y)$ defined implicitly by

$$y = Z^*(y) + \frac{1}{R} \ln\left(\frac{Z^*(y) - \alpha_2}{\alpha_1}\right), \quad y \geq 0. \tag{20}$$

The considerations above show that $\Pr\{Z = Z^*\} = 1$, and hence $Z^* \in \mathcal{Z}$.

Fix $x \in \mathbb{R}$ and consider the function $\tilde{Z}(y) = Z^*(y) + x$. By adding and subtracting x to $Z^*(y)$ in the Eq. (20), we obtain:

$$\begin{aligned} y &= Z^*(y) + x + \frac{1}{R} \ln(\mathrm{e}^{-Rx}) \\ &\quad + \frac{1}{R} \ln\left(\frac{Z^*(y) + x - (\alpha_2 + x)}{\alpha_1}\right) \\ &= \tilde{Z}(y) + \frac{1}{R} \ln\left(\frac{\tilde{Z}(y) - (\alpha_2 + x)}{\alpha_1 \mathrm{e}^{Rx}}\right). \end{aligned}$$

This shows that $\tilde{Z}$ solves Eq. (20) with parameters

$$\tilde{\alpha}_1 = \alpha_1 \mathrm{e}^{Rx} > 0, \qquad \tilde{\alpha}_2 = \alpha_2 + x.$$

As it is clear that there is a $x \in \mathbb{R}$ which solves

$$-\alpha_2 - x = \alpha_1 \mathrm{e}^{Rx},$$

then, for that particular x, $\tilde{Z}$ solves (13) with $\alpha = -\alpha_2 - x = \alpha_1 \mathrm{e}^{Rx}$. A simple computation shows that $0 < \tilde{Z}'(y) < 1$, $\forall y > 0$ and $\tilde{Z}(0) = 0$. Therefore, $\tilde{Z} \in \mathcal{Z}$ and it is economically equivalent to Z. □

Notice that, unlike what happens with the expected value principle, the optimal reinsurance policy leaves the insurance company with an unbounded retained risk whenever the support of the distribution of Y is unbounded. However, the distribution of the net claims will always have moment generating function, which may not happen with the original distribution.

10. A numerical example

In this section we provide a numerical example for the standard deviation principle. We skip the computations since they involve non-trivial numerical issues whose discussion largely exceeds the scope of the present paper. Basically, we combine a minimization algorithm to find Z_R among the reinsurance treaties of the types described in Theorem 5, with a bisection algorithm to solve the equation $G(R, Z_R) = 1$. A thorough numerical study of optimal policies will be published elsewhere.

The data for the present example is as follows. We assume the claims follow a distribution with density

$$f(y) = \frac{3}{(1+y)^4}, \quad \forall y > 0.$$

Therefore, $E[Y] = \frac{1}{2}$ and $\mathrm{Var}[Y] = \frac{3}{4}$. The amount of premiums received by the direct insurer is assumed to be $c = 0.6$ per unit of time, and the reinsurance premium is supposed to be computed by the principle

$$P(Z) = E[Z] + 0.3\sqrt{\mathrm{Var}[Z]}.$$

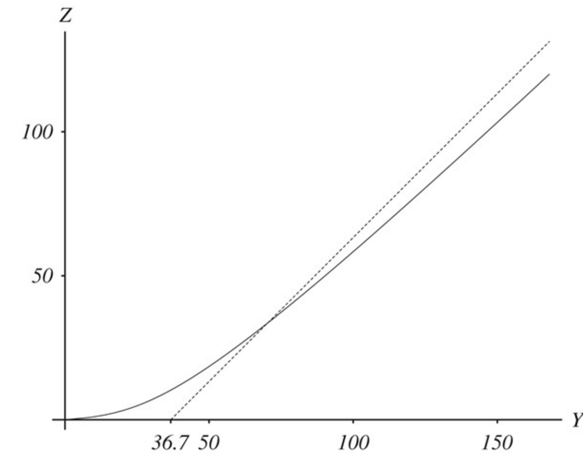

Fig. 1. Optimal treaty (solid line) compared to the best stop-loss treaty (dashed line).

The optimal policy is the following:

$$Z(y) = 0.570513\mathrm{e}^{0.11136(y - Z(y))} - 0.570513.$$

It is interesting to compare this with the best stop-loss contract for the same data. The retention limit for this contract is $M = 36.7248$. The following table summarizes the main data for the optimal contract and for the best stop-loss treaty.

	Optimal treaty	Best stop loss
Adjustment coefficient	0.1113600	0.0988751
$E[Z]$	0.0334840	0.0003513
$\mathrm{Var}[Z]$	0.0328331	0.0265077
Reinsurance premium ($P(Z)$)	0.0878437	0.0491949

The optimal policy improves the adjustment coefficient by 12.6% with respect to the best stop-loss treaty, at the cost of an increase of 78.6% in the reinsurance premium.

Fig. 1 shows the graphs of the optimal and the best stop-loss policies on the same plot. It shows that the improved performance of the optimal policy is achieved partly by compensating a lower level of reinsurance against very high losses (which occur rarely) by reinsuring a substantial part of moderate losses, which occur more frequently but are inadequately covered or not covered at all by the stop-loss treaty.

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