Lisbon School of Economics & Management
Universidade de Lisboa

MASTER'S IN MONETARY AND FINANCIAL ECONOMICS

MASTER'S FINAL WORK

DISSERTATION

RISK REALLOCATION UNDER CENTRAL BANKS' LARGE-SCALE ASSET PURCHASES

ALBERTO LÓPEZ MARTÍN

JANUARY - 2022

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ABSTRACT, KEYWORDS, AND JEL CODES

Crises have some common features: increases in risk premia, decrease of interest rates, and flight-to-quality, among others. We propose a continuous-time model with heterogeneous agents to study the effects of large-scale asset purchases on the market price of risk and the risk-free rate. We observe how, when the central bank buys risky assets using risk-less debt, there is a reduction of risk-taking in the economy, as the risk is transferred to non-market participants. Quantitative easing reduces the exposure of intermediaries' balance sheets to capital shocks, leading to a reduction in the risk premium and an increase in the risk-free rate.

KEYWORDS: Monetary Policy; Risk premium; Quantitative Easing; Continuous-time Models; Heterogeneous Agents.

JEL CODES: E43; E44; E52; E58.

RESUMO, PALAVRAS-CHAVE, E CÓDIGOS JEL

Todas as crises têm alguns pontos em comum: aumentos no prémio de risco, diminuição das taxas de juro e fuga para a qualidade, entre outros. Neste estudo propomos um modelo em tempo contínuo com agentes heterogéneos para estudar os efeitos das compras de ativos em larga escala no preço de mercado da taxa com e sem risco. Observamos como, quando o banco central compra ativos com risco financiando-se com dívida sem risco, há uma redução de risco na economia, uma vez que este é transferido para participantes sem acesso ao mercado. A flexibilização quantitativa reduz a exposição das folhas de balanço dos intermediários aos choques no capital, levando a uma redução no prémio de risco e o aumento da taxa de juro sem risco.

PALAVRAS-CHAVE: Política Monetária; Prémio de risco; Flexibilização quantitativa; Tempo contínuo; Agentes heterogéneos.

CÓDIGOS JEL: E43; E44; E52; E58.

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Obrigado.

Risk reallocation under central banks' large-scale asset purchases

By Alberto López Martín

Abstract: Crises have some common features: increases in risk premia, decrease of interest rates, and flight to quality, among others. This thesis proposes a continuous-time model with heterogeneous agents to study the effects of large-scale asset purchases on the market price of risk and the risk-free rate. We observe how, when the central bank buys risky assets using risk-less debt, there is a reduction of risk-taking in the economy, as the risk is transferred to non-market participants. Quantitative easing reduces the exposure of intermediaries' balance sheets to capital shocks, leading to a reduction in the risk premium and an increase in the risk-free rate.

1 Introduction

The crisis of 2007-08 required central banks around the globe to expand their monetary policy toolbox in an attempt to ease credit conditions and compress risk premia. The shock in 2007-08, coupled with the sovereign debt crisis that followed, drove economies to situations in which conventional monetary policy instruments were unable to support a sustained economic recovery. Taylor rules would recommend cutting nominal interest rates well below zero and the previously well-functioning relationship between changes in official interest rates and market interest rates was no longer reliable.

Quantitative easing, or the purchase of large quantities of financial assets, became the tool of choice of monetary authorities facing the zero lower bound of policy rates as a means to stimulate lending. By reducing the supply to the private sector of risk-bearing assets (for example, due to their long maturity), and increasing the supply of less risky assets (for example, bank reserves) central banks expected to lower longer-term market interest rates, increase the money supply, and hence channel more lending to consumers and businesses. The pattern described above is what is usually referred to as the *portfolio balance effect*, and was described in Tobin (1958, 1969). All central banks in advanced economies implemented this kind of measures to alleviate the effects of the global financial crisis.

Crises have some common features: increases in risk premia, a decrease of interest rates, and flights to quality, among others. In this thesis, we study a simple model that relies on the model with heterogeneous agents introduced by Brunnermeier and Sannikov

(2014), which, in turn follows Basak and Cuoco (1998), and that allows us to unravel some of the macro effects of unconventional monetary policy measures and, in particular, their countercyclical behavior.

We begin by considering an economy with two heterogeneous rational agents: a financial expert, with the required know-how to accumulate a risky asset (capital), and a household, that finances the expert's purchase of risky assets by holding expert-issued risk-less debt. In addition, we introduce a central bank. The monetary authority redistributes risk in the economy by issuing risk-less bonds to purchase capital, then transferring profits to households. In this model we do not consider a dynamic intervention of the central bank—its intervention is decided exogenously.

When solving this model, we obtain closed expressions for the risk-free interest rate and the evolution of the expert sector's relative wealth. From here, we are able to reason how, after a negative shock to the capital of intermediaries, their ability to continue holding risky assets decreases and, as a consequence, risk premia increase. We observe how, when the central bank buys risky assets using risk-less debt, there is a reduction of risk taking in the economy, as this risk is transferred to non-market participants. The asset purchases performed by the central bank alter the equilibrium in the economy, leading to a decrease of the market price of risk and an increase in the risk-free interest rate.

This paper is organized as follows. Section 2 reviews some of the relevant existing literature. Section 3 presents and solves the reference continuous-time model. Section 4 concludes.

2 LITERATURE REVIEW

Our model builds on the model of Brunnermeier and Sannikov (2014). The original model studies the equilibrium dynamics of an economy with financial frictions. This work belongs to a thread of papers describing and examining the effect of financial disintermediation in times of crises.

There is a wealth of papers that study the effects of large-scale asset purchases empirically and for different jurisdictions. Krishnamurthy and Vissing-Jorgensen (2011) evaluate the effect of the Federal Reserve's quantitative easing programs on interest rates. According to the authors' findings, the Fed's influence was greater on the risk premia of assets being purchased. For example, QE1 (2008-09) had a large effect on the reduction of mortgage rates, partly due to the fact that QE1 involved large purchases of agency backed mortgage-backed securities (MBS). In turn, QE2 (2010-11), which involved only Treasury purchases, impacted mainly Treasury and agency bond rates, and less so MBS and corporate rates. The authors attribute other effects due to a signaling channel (c.f. Eggertsson and Woodford (2003); Clouse et al. (2003))

Krishnamurthy et al. (2017) focus on the effect of the European Central Bank (ECB) programs, namely the Securities Markets Programme (SMP), the Outright Monetary Transactions (OMT) framework, and the three-year long-term refinancing operations (LTROs). Their interest is on the channels through which the SMP and OMT affected sovereign bond yields of so-called GIIPS countries, as well as stock returns. A number of other authors have also studied the effect of LSAP programmes on bank lending to the private sector (Andrade et al. (2015); Carpinelli and Crosignani (2017); Fonseca et al. (2015); Garcia-Posada and Marchetti (2016)). Empirical studies have also been performed to assess the effect of QE in different asset types (De Santis and Zaghini (2021); Albertazzi et al. (2021); Balcilar et al. (2020); Farinha and Vidrago (2021); Lewis and Roth (2019); Guo et al. (2020)).

Work has also been done in identifying different channels through which LSAPs transmit their effects to the real economy (Koijen et al. (2017); Krishnamurthy and Vissing-Jorgensen (2011); Vayanos and Vila (2021)). However, to our knowledge, there is limited literature on the theoretical study of the risk channel of unconventional monetary policy. In the finance literature, Silva (2020) appears to be the first to study the effects of quantitative easing measures on risk redistribution. As expected, and consistent with the empirical evidence, he finds that asset purchases reduce the risk premium and increase asset prices. The novelty of our work is the simplicity of the approach. While conclusions are similar to Silva (2020), our model has a more direct formulation. For example, we do not consider heterogeneity within the sectors, we assume all agents have log preferences, and our central bank's decision on the amount of capital held is decided exogenously.

3 THE MODEL

Our model builds on the model with heterogeneous agents described in Brunnermeier and Sannikov (2014). We introduce a central bank in their framework. Whenever appropriate we adopt their notation.

In our economy there is one risky asset, capital, and a risk-free asset, non-contingent bonds. In this model, we consider two different types of rational agents: financial experts and households, and a central bank. There is a mass-one continuum of experts and households. Only experts and the central bank can hold the risky asset. Households lend to the expert at the risk-free rate, but cannot hold capital. Experts cannot issue outside equity, but can partially finance their holdings of the risky asset by issuing bonds to households. Non-conventional monetary policy consists in buying capital with the revenue obtained from the issuance of bonds, and transferring the profits associated with this operation, as a lump-sum, to households. Together with market clearing, our assumptions imply that, at the aggregate level, there is a positive net supply of capital, while bonds are in zero net supply.

3.1 Agents

3.1.1 Financial experts

Experts have a linear constant returns to scale production function of the form $Y_t = AK_t, A \in \mathbb{R}^+$. An individual expert's capital stock follows a linear stochastic process

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta)dt + \sigma dZ_t, \tag{1}$$

where ι is the re-investment rate of capital, and the investment function Φ satisfies $\Phi(0) = 0, \Phi'(0) = 1, \Phi' > 0$, and $\Phi'' < 0$. The concavity of Φ reflects adjustment costs derived from the transformation it requires for output to turn into capital. δ is the depreciation rate of capital. The last term is a growth rate shock which follows a Brownian motion with volatility σ. This term can be interpreted as the risk of holding capital.

To partially finance this holding of the risky asset, the expert issues non-contingent bonds at a rate r_t to households. Let us denote by θ_t, $\theta_t < 0$, the expert's short position on bonds.

The expert's wealth evolves following the process

$$\frac{dn_t}{n_t} = \left(-\frac{c_t}{n_t} + \theta_t r_t\right) dt + (1-\theta_t) dr_t^K(\iota_t), \tag{2}$$

where r_t^K denotes the rate of return on capital, whose dynamics are described in (6). The representative expert has standard preferences over utility flows from future consumption c_t discounted at rate $\rho \geq 0$:

$$\mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} u(c_t) dt\right]. \tag{3}$$

We assume the expert has logarithmic utility, so that $u(c_t) = \log(c_t)$. Then the expert's problem amounts to deciding their reinvestment rate ι_t, consumption flows c_t, and their short position on bonds θ_t, so as to maximize (3) subject to the evolution of their wealth in (2) and an initial value n_0.

Return rate on capital. Let q_t denote the price of capital and let us assume that the following stochastic process describes its law of motion:

$$\frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^q dZ_t. \tag{4}$$

The instantaneous rate of return on capital is given by $dr_t^K = \frac{A-\iota_t}{q_t} dt + \frac{d(q_t k_t)}{q_t k_t}$, where the first term represents the profit rate of capital after investment. We can simplify this expression using (1), (4), and Itô's product rule (47) to obtain the following capital gains rate:

$$\frac{d(q_t k_t)}{q_t k_t} = \left(\mu_t^q + \Phi(\iota_t) - \delta + \sigma_t^q \sigma\right) dt + \left(\sigma_t^q + \sigma\right) dZ_t, \tag{5}$$

which can be used to get the return on capital

$$dr_t^K = \left(\frac{A-\iota_t}{q_t} + \mu_t^q + \Phi(\iota_t) - \delta + \sigma_t^q \sigma\right) dt + \left(\sigma_t^q + \sigma\right) dZ_t. \tag{6}$$

The volatility term is the total risk, composed by the fundamental risk σ and the price risk σ_t^q. For convenience in later calculations, we denote the drift by $\mu_t^{r^K}$ and the volatility by $\sigma_t^{r^K}$.

Note that we can write the expert's wealth process in (2) as

$$\frac{dn_t}{n_t} = \left(-\frac{c_t}{n_t} + \theta_t r_t + (1-\theta_t)\mu_t^{r^K}\right) dt + (1-\theta_t)\left(\sigma_t^q + \sigma\right) dZ_t. \tag{7}$$

We denote aggregate consumption of the expert sector by C_t.

3.1.2 Central bank

The central bank invests in capital $\breve{K}_t$ by issuing risk-less assets in the form of bonds $\breve{B}_t$, so that $q_t\breve{K}_t = \breve{B}_t$. We will assume that $\breve{B}_t$ is determined exogenously. The capital bought by the central bank represents a portion of that available in the economy, $\breve{K}_t = \varepsilon\bar{K}_t$, $0 \leq \varepsilon \leq 1$. These bonds $\breve{B}_t$ pay the same rate r_t than those issued by the financial expert. Proceeds obtained from this portfolio are rebated to households in the form of a lump-sum T_t so that

$$\breve{K}_t r_t^K - \breve{B}_t r_t = \varepsilon K_t(r_t^K - r_t q_t) = T_t. \tag{8}$$

The central bank has net wealth equal to zero.

3.1.3 Households

Households do not have the necessary expertise to accumulate capital, but may hold bonds. The financial frictions in the model arise precisely from the impossibility of households to purchase any state-contingent claims issued by the expert against capital.

The representative household of this type can save an amount $\widehat{n}_t$ of the risk-less debt issued by the financial expert at interest rate r_t and receives a transfer T_t from the central bank. A household's wealth process can be described as

$$\frac{d\widehat{n}_t}{\widehat{n}_t} = \left(r_t + \frac{T_t}{\widehat{n}_t} - \frac{\widehat{c}_t}{\widehat{n}_t}\right) dt, \tag{9}$$

where $\widehat{c}_t$ denotes the household's consumption. As a means to avoid short-selling of bonds, we impose $\widehat{n}_t \geq 0$.

Households' preferences are also represented by a logarithmic utility function

$$\mathbb{E}_0\left[\int_0^\infty e^{-\widehat{\rho}t}\log(\widehat{c}_t)dt\right], \tag{10}$$

with $\widehat{\rho} \geq 0$. We will assume households to be more patient than financial experts, i.e. $\rho > \widehat{\rho}$.

The household's problem amounts to choosing their consumption so as to maximize (10) subject to (9), the condition $\widehat{n}_t \geq 0$, and an initial value $\widehat{n}_0$.

3.1.4 Aggregates

Let us denote by $\bar{K}_t$ the economy's capital. If we denote by K_t the experts' aggregate capital, then

$$K_t = \int_0^1 k_{n,t}dn, \tag{11}$$

where k_n denotes the n-th individual expert's capital stock. As the only risk in the model is at an aggregate level, we have that $K_t = k_t$. Capital is held by the expert sector and the central bank, so

$$\bar{K}_t = K_t + \breve{K}_t = (1-\varepsilon)\bar{K}_t + \varepsilon\bar{K}_t.$$

Let us denote by $\bar{N}_t$ the economy's aggregate net worth. Then $\bar{N}_t = q_t\bar{K}_t$. If we denote by N_t and $\widehat{N}_t$ the aggregate wealths of the expert and the household sectors, respectively, then $\bar{N}_t = N_t + \widehat{N}_t$ (remember that we assume the central bank's wealth $\breve{N}_t$ to be equal to zero). Note that, for the reason above, $N_t = n_t$ and $\widehat{N}_t = \widehat{n}_t$. Finally, let η_t be the (relative) wealth of the expert sector, i.e.

$$\eta_t = \frac{N_t}{\bar{N}_t} = \frac{N_t}{q_t\bar{K}_t}. \tag{12}$$

3.2 Equilibrium

An equilibrium in this economy is defined as paths for price $\{q_t\}$, expert decisions $\{\iota_t, \theta_t, c_t\}$, household decisions $\{\widehat{c}_t\}$, net worths $\{n_t, \widehat{n}_t\}$, and aggregate quantities so that: (i) both agents maximize their objective functions, given relevant information, and (ii) all markets clear.

3.2.1 The financial expert's problem

First, the expert wants to decide on a capital reinvestment strategy that maximizes their rate of return on capital (6). This problem's first-order condition amounts to satisfying

$$\Phi'(\iota_t) = \frac{1}{q_t}, \tag{13}$$

hence establishing a relationship between price and the replacement cost of capital—Tobin's q formula. The expert's problem at the end of Subsection 3.1.1 is a simple stochastic optimal control problem with infinite horizon and two control variables. Let

$$V(n_t, \eta_t) = \max_{\{c_t, \theta_t\}_{t \geq 0}} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \log(c_t) dt \right], \tag{14}$$

then the Hamilton-Jacobi-Bellman equation associated to the expert's problem is

$$\begin{aligned} \rho V(n_t, \eta_t) = \max_{\{c_t, \theta_t\}_{t \geq 0}} \big\{ & \log(c_t) + (n_t \mu_t^n) \partial_n V_t + (\eta_t \mu_t^\eta) \partial_\eta V_t + \tfrac{1}{2} (n_t \sigma_t^n)^2 \partial_{nn} V_t \\ & + \tfrac{1}{2} (\eta_t \sigma_t^\eta)^2 \partial_{\eta\eta} V_t + (n_t \sigma_t^n \eta_t \sigma_t^\eta) \partial_{n\eta} V \big\} . \end{aligned} \tag{15}$$

In this case, the equation can be solved analytically, yielding optimal policies for the expert's consumption and their position on the risky asset. The first order conditions for the problem are

$$c_t^{-1} = \partial_n V, \tag{16}$$

$$\mu_t^{r^K} - r_t = - \left((n_t \sigma_t^n) \frac{\partial_{nn} V_t}{\partial_n V_t} + (\eta_t \sigma_t^\eta) \frac{\partial_{n\eta} V_t}{\partial_n V_t} \right) (\sigma + \sigma_t^q), \tag{17}$$

Note that a single expert's change in their portfolio composition has virtually no impact on the aggregate wealth of the expert sector, for that reason, some of the terms in (15) vanish when calculating their first-order derivatives with respect to θ_t.

We can use a "guess and verify" strategy to solve these conditions. Let us assume that the value function is of the form $V(n_t, \eta_t) = \psi \log(n_t) + \Psi(\eta_t)$, where $\psi \in \mathbb{R}$ and $\Psi \in \mathcal{C}^2(\mathbb{R})$. Then, from (16), we derive the choice of optimal consumption

$$c_t^{-1} = \frac{\psi}{n_t} \tag{16'}$$

and from (17), obtain a condition for the choice of an optimal portfolio

$$\mu_t^{r^K} - r_t = \sigma_t^n \left(\sigma_t^q + \sigma\right). \tag{17'}$$

If we substitute these optimal values in (15), we can fully characterize the value function, in particular, $\Psi(\eta_t)$ is a solution of the following differential equation:

$$(\psi\rho - 1)\log n_t = -\log\psi - 1 - \rho\Psi(\eta_t) + \psi\left(\mu_t^{r^K} - \theta_t(\mu_t^{r^K} + r_t) - \frac{1}{2}(1-\theta_t)^2\left(\sigma_t^q + \sigma\right)^2\right) \\ + \eta_t\mu_t^\eta\Psi'(\eta_t) + \frac{1}{2}(\eta_t\sigma_t^\eta)^2\Psi''(\eta_t). \tag{18}$$

From here, we can also conclude that $\psi = \frac{1}{\rho}$, as the expression on the left-hand side does not depend on the state variable η_t. As a result, the expert's consumption equals

$$c_t = \rho n_t. \tag{16''}$$

All these conditions, namely: the choice of an optimal investment strategy in (13), and the choices for optimal consumption and portfolio composition in (16″) and (17′), respectively, solve the expert's problem.

We observe in (17′) how the optimal risk exposure of an agent with a log utility depends on how attractive the risk investment is, measured by the Sharpe ratio, which equals the volatility of the expert's net worth:

$$\frac{\mu_t^{r^K} - r_t}{\sigma_t^{r^K}} = \frac{\frac{A-\iota_t}{q_t} + \mu_t^q + \Phi(\iota_t) - \delta + \sigma_t^q\sigma - r_t}{\sigma_t^q + \sigma} = \sigma_t^n. \tag{19}$$

3.2.2 The household's problem

The value function for the household's problem

$$\widehat{V}(\widehat{n}_t, \eta_t) = \max_{\{\widehat{c}_t\}_{t\geq 0}} \mathbb{E}_0\left[\int_0^\infty e^{-\widehat{\rho}t}\log(\widehat{c}_t)dt\right] \tag{20}$$

has the following associated Hamilton-Jacobi-Bellman equation:

$$\rho\widehat{V}(\widehat{n}_t,\eta_t) = \max_{\{\widehat{c}_t\}_{t\geq 0}} \left\{\log(\widehat{c}_t) + (\widehat{n}_t\mu_t^{\widehat{n}})\partial_{\widehat{n}}\widehat{V}_t + (\eta_t\mu_t^{\eta})\partial_{\eta}\widehat{V}_t + \tfrac{1}{2}(\eta_t\sigma_t^{\eta})^2\partial_{\eta\eta}\widehat{V}_t\right\}. \tag{21}$$

As before, the equation can be solved analytically. The first order condition for the problem is

$$c_t^{-1} = \partial_{\widehat{n}}\widehat{V}. \tag{22}$$

Applying the same strategy as earlier, let us assume that the value function is of the form $V(\widehat{n}_t,\eta_t) = \widehat{\psi}\log(\widehat{n}_t) + \widehat{\Psi}(\eta_t)$, where $\widehat{\psi} \in \mathbb{R}$ and $\widehat{\Psi} \in \mathcal{C}^2(\mathbb{R})$. From (22), we derive the choice of optimal consumption for the household as

$$\widehat{c}_t = \frac{1}{\widehat{\psi}}\widehat{n}_t. \tag{21$'$}$$

Substituting the guess for $\widehat{V}$ and this condition in equation (21), we obtain $\widehat{\psi} = \frac{1}{\hat{\rho}}$, hence solving the household's problem.

3.2.3 The stochastic discount factor

Let us now assume that experts have a stochastic discount factor following the stochastic process given by

$$\frac{d\xi_t}{\xi_t} = -r_t dt - \lambda_t dZ_t, \tag{23}$$

where λ_t is the market price of risk, which is commonly interpreted as the risk premium per unit of volatility, and r_t is the risk-free rate.

Let us consider the following trading strategy: at $t = 0$, the expert invests one unit in capital and, at each period $t > 0$, reinvests the returns of the previous investment again in capital. The gains rate of this strategy is $dr_t^K = \frac{dv_t}{v_t}$, where v_t is the value of the strategy at time t (for example, $v_0 = 1$). Let us now consider a two-period portfolio problem for the expert, in which the expert can only trade at two given times, say at t and $t+\delta$. We have the following Euler equation

$$v_t = \mathbb{E}_t\left[\frac{\xi_{t+\Delta}}{\xi_t}v_{t+\Delta}\right], \tag{24}$$

which is equivalent to

$$\mathbb{E}_t\left[\xi_{t+\Delta}v_{t+\Delta}\right] = \xi_t v_t. \tag{25}$$

In words, the agent's expectation at any given time t on the discounted value of capital at time $t+\Delta$, for $\Delta > 0$, equals the current discounted value of capital. A stochastic process satisfying this property is a *martingale*.

Now, using (6) and (23) with Itô's product rule (47), we have

$$\frac{d\xi_t v_t}{\xi_t v_t} = \left(-r_t + \mu_t^{r^K} - \lambda_t \sigma_t^{r^K}\right) dt + \left(\sigma_t^{r^K} - \lambda_t\right) dZ_t. \tag{26}$$

As $\frac{d\xi_t v_t}{\xi_t v_t}$ is a martingale, its drift must be equal to zero, and so $\mu_t^{r^K} = r_t + \lambda_t \sigma_t^{r^K}$. Substituting in the value of the volatility for the stochastic process (6) that describes the return on capital, we obtain

$$\mu_t^{r^K} - r_t = \lambda_t\left(\sigma_t^q + \sigma\right). \tag{27}$$

3.2.4 Law of motion of the experts' wealth

At this point, we have now all the information needed to derive the dynamics of the share of experts' wealth in the economy, namely $\eta_t = \frac{N_t}{q_t K_t}$, with $0 \leq \eta_t \leq 1$. For that, we make use of Itô's quotient rule (48). Let us first make apparent the role of the price of risk λ_t in (7) and (5), using (27).

We mentioned above how, given the absence of risks other than aggregate, we have $N_t = n_t$. Then,

$$\begin{aligned}\frac{dN_t}{N_t} = \frac{dn_t}{n_t} &= \left(-\frac{c_t}{n_t} + \theta_t r_t + (1-\theta_t)\mu_t^{r^K}\right) dt + (1-\theta_t)\left(\sigma_t^q + \sigma\right) dZ_t \\ &= \left(-\frac{c_t}{n_t} + r_t + (1-\theta_t)\lambda_t\left(\sigma_t^q + \sigma\right)\right) dt + (1-\theta_t)\left(\sigma_t^q + \sigma\right) dZ_t, \end{aligned} \tag{28}$$

and

$$\begin{aligned}\frac{d(q_t k_t)}{q_t k_t} &= \left(\mu_t^q + \Phi(\iota_t) - \delta + \sigma_t^q \sigma\right) dt + \left(\sigma_t^q + \sigma\right) dZ_t \\ &= \left[\left(r_t - \frac{A - \iota_t}{q_t}\right) + \lambda_t\left(\sigma_t^q + \sigma\right)\right] dt + \left(\sigma_t^q + \sigma\right) dZ_t. \end{aligned} \tag{29}$$

Using Itô's quotient rule, we obtain

$$\frac{d\eta_t}{\eta_t} = \left(-\frac{c_t}{n_t} + \frac{A - \iota_t}{q_t} - \theta_t \left(\sigma_t^q + \sigma\right)\left(\lambda_t - \left(\sigma_t^q + \sigma\right)\right)\right) dt - \theta_t \left(\sigma_t^q + \sigma\right) dZ_t. \tag{30}$$

Let us consider the stochastic processes for consumption c_t and net worth n_t. Following the standard notation, we can conclude that, as in this case, the ratio $\frac{c_t}{n_t}$ is constant, the volatilities of the respective consumption and net worth processes are equal, $\sigma_t^c = \sigma_t^n$.

In (23), we assumed a form for the dynamics of the stochastic discount factor ξ_t, where the volatility was given by (minus) the market price of risk λ_t. Log utility gives $\xi_t = \frac{e^{-\rho t}}{c_t}$. If we apply Itô's product rule (47), we obtain

$$\frac{d\xi_t}{\xi_t} = \left(-\rho - \mu_t^c + (\sigma_t^c)^2\right) dt - \sigma_t^c dZ_t, \tag{31}$$

and then, by (28), $\sigma_t^n = \sigma_t^c = \lambda_t = (1 - \theta_t)\left(\sigma_t^q + \sigma\right)$.

We then obtain an alternate expression for the law of motion of η_t:

$$\frac{d\eta_t}{\eta_t} = \left(\frac{A - \iota_t}{q_t} - \rho + \theta_t^2 \left(\sigma_t^q + \sigma\right)^2\right) dt - \theta_t \left(\sigma_t^q + \sigma\right) dZ_t. \tag{32}$$

3.2.5 Law of motion of the price of capital

After the calculation of the dynamics of the experts' share of wealth in the economy, we can now proceed to the calculation of the law of motion of the price of capital $q_t = q(\eta_t)$, which we assumed, in (4), to be given by a stochastic process with drift μ_t^q and volatility σ_t^q. For convenience, we will use the notation $f(x) := \rho x + \hat{\rho}(1 - x)$. By Itô's lemma (46), we have

$$\frac{dq(\eta_t)}{q(\eta_t)} = \frac{q'(\eta_t)\mu^\eta \eta_t + \frac{1}{2} q''(\eta_t)(\sigma^\eta \eta_t)^2}{q(\eta_t)}\, dt + \frac{q'(\eta_t)}{q(\eta_t)} \eta_t \sigma^\eta \, dZ_t. \tag{33}$$

First, we calculate the volatility of the process. From

$$\sigma^q(\eta_t) = \frac{q'(\eta_t)}{q(\eta_t)} \eta_t \sigma^\eta = \frac{q'(\eta_t)}{q(\eta_t)} \left(1 - \varepsilon - \eta_t\right)\left(\sigma_t^q + \sigma\right),$$

we derive

$$\sigma^q(\eta_t) = \frac{(1-\varepsilon-\eta_t)\,\frac{q'(\eta_t)}{q(\eta_t)}}{1-(1-\varepsilon-\eta_t)\,\frac{q'(\eta_t)}{q(\eta_t)}}\sigma, \tag{34}$$

which, after substituting the expressions for $q(\eta_t)$ in (40) (and its first derivative), leads us to

$$\sigma^q(\eta_t) = \frac{\phi(\hat{\rho}-\rho)\,(1-\varepsilon-\eta_t)}{1+\phi f(1-\varepsilon)}\sigma. \tag{35}$$

We proceed now with the calculation of the drift $\mu^q = \mu^q(\eta_t)$, which, after simplification, equals

$$\mu^q(\eta_t) = \phi(\hat{\rho}-\rho)\left[\frac{\eta_t(1-\eta_t)}{1+\phi f(\eta_t)}(\hat{\rho}-\rho) + \frac{(1+\phi\hat{\rho})}{\eta_t}\left(\frac{(1-\varepsilon-\eta_t)\sigma}{1+\phi f(1-\varepsilon)}\right)^2\right] \tag{36}$$

3.2.6 Market clearing

Prior to calculating the market clearing conditions for the model, let us specify the form of the investment function $\Phi(\iota_t)$. Let us assume

$$\Phi(\iota_t) = \frac{1}{\phi}\log(\phi\iota_t + 1), \tag{37}$$

for ϕ capital's adjustment cost parameter.[1] From the solution to the expert's problem (13), in this case we can calculate the optimal investment rate as

$$\iota_t = \frac{q_t - 1}{\phi}. \tag{38}$$

From our discussion on agents with log utility in section 3.2.4, we have that $\bar{C}_t = C_t + \hat{C}_t = \rho N_t + \hat{\rho}(q_t\bar{K}_t - N_t)$. In turn, aggregate output equals $(A-\iota_t)K_t$. We can conclude that the market for consumption goods clears if

$$\rho\eta_t + \hat{\rho}(1-\eta_t) = \frac{A-\iota_t}{q_t} \tag{39}$$

[1]With this functional form, if ϕ tends to 0, then $\Phi(\iota_t)$ tends to ι_t, so that investment carries no adjustment costs.

(or $f(\eta_t) = \frac{A-\iota_t}{q_t}$). From here, we obtain the value for the price of capital,

$$q(\eta_t) = \frac{1+\phi A}{1+\phi f(\eta_t)}. \tag{40}$$

Substituting this expression back in (38), we have the following expression for investment:

$$\iota(\eta_t) = \frac{A - f(\eta_t)}{1+\phi f(\eta_t)}. \tag{41}$$

Experts hold a share η_t of total wealth in the economy. The market for capital clears if $N_t(1-\theta_t) = (1-\varepsilon)q_t K_t$ or, equivalently, if $(1-\theta_t) = (1-\varepsilon)\frac{1}{\eta_t}$.

The risk-free rate is determined from the choice of an optimal portfolio (17′) in the solution to the expert's problem. The following expression for the risk-free interest rate is only dependent on the model's parameters and the state variable,

$$r_t = f(\eta_t) + \frac{1}{\phi}\log\left(\frac{1+\phi A}{1+\phi f(\eta_t)}\right) - \delta + \frac{\phi(\hat{\rho}-\rho)\sigma^2}{1+\phi f(1-\varepsilon)}\left((1-\varepsilon)-\eta_t\right) - (1-\varepsilon)\frac{1}{\eta_t}\left(\frac{1+\phi f(\eta_t)}{1+\phi f(1-\varepsilon)}\right)^2\sigma^2$$
$$+\,\phi(\hat{\rho}-\rho)\left[\frac{\eta_t(1-\eta_t)(\hat{\rho}-\rho)}{1+\phi f(\eta_t)} + \frac{(1+\phi\hat{\rho})}{\eta_t}\left(\frac{(1-\varepsilon)-\eta_t}{1+\phi f(1-\varepsilon)}\sigma\right)^2\right]. \tag{42}$$

Finally, substituting the different conditions in the expression (32) for the law of motion of η_t, we obtain

$$\frac{d\eta_t}{\eta_t} = \left((\hat{\rho}-\rho)(1-\eta_t) + \left(\frac{(1-\varepsilon)-\eta_t}{\eta_t}\frac{1+\phi f(\eta_t)}{1+\phi f(1-\varepsilon)}\right)^2\sigma^2\right)dt + \left(\frac{1+\phi f(\eta_t)}{1+\phi f(1-\varepsilon)}\right)\frac{(1-\varepsilon)-\eta_t}{\eta_t}\sigma dZ_t. \tag{43}$$

The Sharpe ratio of risky investment is

$$\begin{aligned}\frac{\frac{A-\iota_t}{q_t} + \mu_t^q + \Phi(\iota_t) - \delta + \sigma_t^q\sigma - r_t}{\sigma_t^q + \sigma} &= (1-\varepsilon)\frac{1}{\eta_t}\left(\sigma_t^q + \sigma\right) \\ &= (1-\varepsilon)\frac{1}{\eta_t}\left(\frac{1+\phi f(\eta_t)}{1+\phi f(1-\varepsilon)}\right)\sigma. \end{aligned}\tag{44}$$

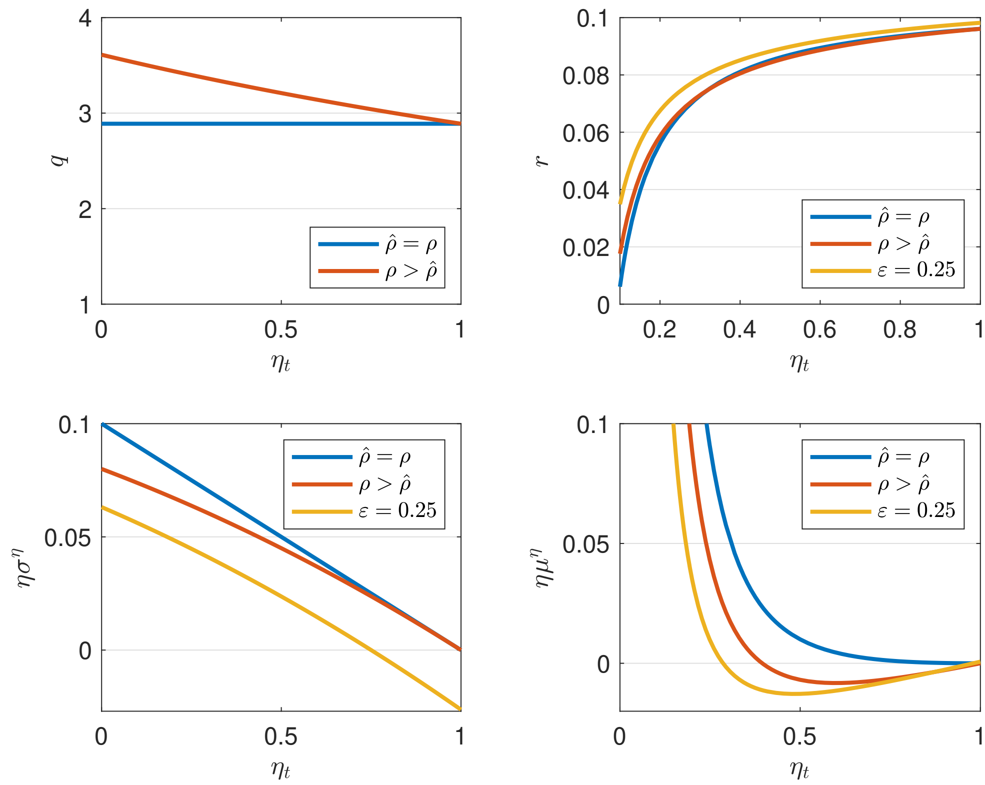

FIGURE 1: Price of capital, risk-free rate, drift and volatility of η_t.

3.3 *Numerical example*

We adopt values as presented in related literature, for example Silva (2020) and Brunnermeier and Sannikov (2014). We set the level of technology $A = \frac{1}{3}$, which corresponds to a capital-output ratio of 3. The depreciation rate δ equals 0.05. We will assume households to be more patient than financial experts, and so $\rho = 0.05 > 0.02 = \widehat{\rho}$. Finally, the fundamental risk $\sigma = 10\%$ and the capital cost parameter $\phi = 10$.

In Figure 1, we observe the behaviour of the price of capital, the risk-free rate, and the risk and volatility of the wealth share of experts η_t for different values of the parameters. Let us start with the case in which households are as patients as experts, and so $\widehat{\rho} = \rho = 0.05$, and there is no intervention of the central bank ($\varepsilon = 0$). Our results in this case agree with the description in Brunnermeier and Sannikov (2014). We observe in the top left panel how the price of capital is constant and equal to $q = \frac{1+\phi a}{1+\phi\rho}$. Experts' balance sheet concentrates all risk in the economy, as in this case they are the only ones able to hold capital. In the event of a negative shock on capital, experts' relative wealth would decrease. To counter this effect, experts demand a higher risk premium, and so the risk-free rate must adjust and decrease, as observed in the top right panel of Figure 1. With a lower risk-free rate, experts pay less to households on their loans, and so experts' relative wealth moves quickly back to higher values, as can be seen in the blue line in the bottom right panel.

Let us look now at the case when households are more patient than financial experts, $\rho > \widehat{\rho}$, but still with no central bank intervention. In this case, the price of capital depends on η_t and is no longer constant. In fact, in the top left panel of Figure 1, we observe $q'(\eta_t) < 0$: in the case of a negative shock, as experts' relative wealth decreases, the price of capital increases. Differences in discount rates of the agents lead to an increase in the path of the risk-free rate, as observed when we compare the blue and red lines in the top right panel of Figure 1 and the left panel of Figure 2. As we now have differences in the consumption patterns of agents, if experts' relative wealth increases substantially, their higher consumption rate, when compared to households, will make them lose wealth over time, on average (see bottom right panel in Fig. 1). As a result, and contrary to the case above, the economy is no longer dominated by the experts' sector. Finally, in the bottom left panel, we observe how, in this case, we have a mitigation of a negative shock on capital. Total risk equals

$$\sigma_t^q + \sigma \overset{(34)}{=} \frac{1}{1 - (1 - \eta_t)\frac{q'(\eta_t)}{q(\eta_t)}}\sigma, \tag{45}$$

but as $q'(\eta_t) < 0$, we have $\sigma_t^q + \sigma < \sigma$.

Finally, we look at the full model, in which the central bank sector owns part of the capital in the economy. In this case, the price of capital would follow the same function as in the previous case. For this example, we assume the proportion of capital owned by the central bank to be $\varepsilon = 0.25$. As we can observe in the bottom left panel, this results in a lower volatility in the relative wealth of experts. As in the previous cases, as experts' wealth decrease, their net-worth risk increases. However, as the expert sector is no longer the only agent holding capital, the increase in risk is lower than otherwise. As a result, the risk-free rate is higher and presents a more convex profile. The reduction

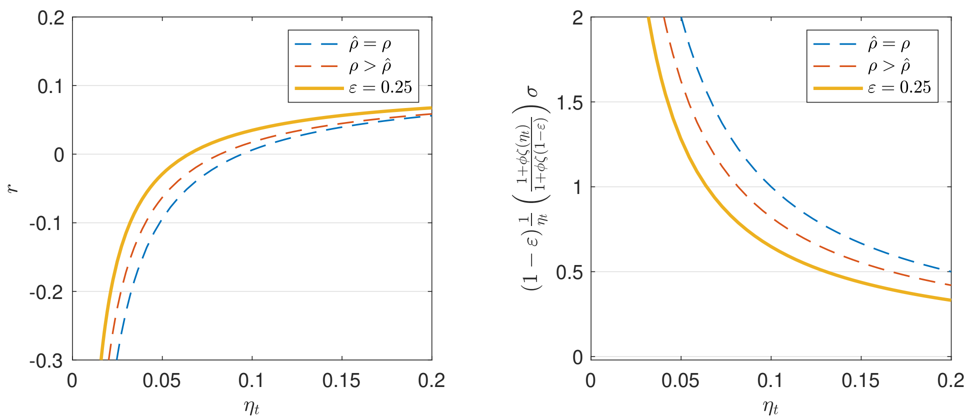

FIGURE 2: Risk-free rate and Sharpe ratio (detail, $0 < \eta_t \leq 0.2$).

in risk for experts has a direct impact on the risk premium. In Figure 2, we can observe how a negative shock on capital leads to an increase in the risk premium, measured by the Sharpe ratio (right panel). However, the market price of risk is lower than in the case without central bank intervention, hence demonstrating the positive effect of capital purchases. In the left panel, we observe how this dampening effect on the risk premium translates into a risk-free rate higher than in cases without the purchases by the central bank.

4 CONCLUSION

The model above intends to be the stepping stone toward a resolution of our question: can we quantify the effects of unconventional monetary policy and, in particular, their countercyclical behavior? The simplicity of our model allows us to solve it, finding closed formulas for the economy's variables, and develop some intuition about the behavior of risk premia.

In contrast with the model in Brunnermeier and Sannikov (2014), our economy is not dominated by the expert sector. By assuming the household sector has a smaller discount rate than experts, their higher consumption rate prevents them from growing too much. In times of crisis, empirical evidence has shown how, after a shock on the capital of intermediaries, their ability to hold risky assets decreases and, as a consequence the market price of risk (or risk premium) increases. In turn, the rate of risk-free assets decreases. In our model, we unravel what is the effect of large-scale asset purchases by the central bank. We find that quantitative easing redistributes risk in the economy, reducing the exposure of intermediaries' balance sheets to capital shocks, and leading to a reduction in the risk premium.

Our work has focused on the effects of central bank's purchases on the risk premium and the risk-free rate. However, in order to fully understand other macro effects, further work will be done to study the influence of these policy measures on aggregate output and capital, as well as in the evolution of agents' consumption. Welfare effects of the intervention of the central bank, and the study of an optimal policy, would require the endogenization of the central bank policy decision. Another possibility would be to consider the framework of heterogeneous agents of Gârleanu and Panageas (2015).

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A Itô's Lemma

The concepts in this appendix are standard to any introductory text on stochastic differential equations, e.g. Oksendal (2003) or Pham (2009). We include them here for ease of reference and as a means to uniformize notation.

Let Z_t be a one-dimensional Brownian motion. Let X_t be a one-dimensional geometric Itô process such that $\frac{dX_t}{X_t} = \mu^X dt + \sigma^X dZ_t$. Let $g(X) \in C^2(\mathbb{R})$. Then $g(X_t)$ is again an Itô process and

$$dg(X_t) = \left(g'(X)(\mu^X X_t) + \frac{1}{2}g''(X)((\sigma^X)^2 X_t)^2\right)dt + g'(X)(\sigma^X X_t)dZ_t. \tag{46}$$

This result is usually known as *Itô's lemma*. Additional formulas can be derived from this one by considering different functions. Let Y_t be a geometric Itô process, with $\frac{dY_t}{Y_t} = \mu_Y dt + \sigma_Y dZ_t$. Throughout this article, we use repeatedly *Itô's product rule*:

$$\frac{dX_tY_t}{X_tY_t} = \left(\mu^X + \mu^Y + \sigma^X\sigma^Y\right)dt + (\sigma^X + \sigma^Y)dZ_t \tag{47}$$

and *Itô's quotient rule*:

$$\frac{d(X_t/Y_t)}{X_t/Y_t} = \left(\mu^X - \mu^Y + \sigma^Y\left(\sigma^Y - \sigma^X\right)\right)dt + (\sigma^X - \sigma^Y)dZ_t. \tag{48}$$