Dynamic Hedging of Equity call Options

João L. C. Duque *
Instituto Superior de Economia e Gestão
Dean A. Paxson
Manchester Business School
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I. Introduction

The theory of option pricing assumes generally that options can be replicated through dynamic hedging in the underlying stock. First, we outline the assumptions behind the popular models, such as regarding the distribution of stock returns, and the probability of the terminal stock value reaching certain levels. Then, we define the common "Greeks" of call options, that is the sensitivity of option values to changes in particular variables. Figures show the sensitivity of those Greeks to stock price levels, and time to expiration. Then, we attempt to show that delta and complex hedges, using options with more than one exercise price, are the "solutions" for simultaneous equations establishing delta and gamma (and eventually vega) neutrality, subject to a budget constraint. Finally, we examine the relative profitability and effectiveness (in terms of variance reduction) of delta hedging strategies for three trade positions (in, at and out-of-the-money).

II. The Theory of option replication through dynamic hedging

Suppose that a sequence of stock prices can be modelled as a random continuously compounded rate of return over the time T as:

$$\tilde{s}=\ln(\tilde{S}_T/S_0) \tag{1}$$

that is the random terminal stock price is:

$$\tilde{S}_T=S_0e^{\tilde{s}} \tag{2}$$

If the stock returns are independently and identically distributed in each period, the expected return from the period 0 to T is μ T, where μ is the mean return; and the variance of the return over the period is σ^2 T. If the continuously compounded daily rates of return are normally distributed with mean m and variance σ^2 , then the distribution of stock prices is lognormal with a mean equal to: [1]

$$\tilde{S}_T=S_0e^{(\mu+\sigma^2/2)T} \tag{2}$$

In using the normal distribution, the continuously compounded return may be transformed into a standard normally distributed variable, which has a mean of zero and variance of one, namely

$$z=\frac{\tilde{s}-\mu T}{\sigma\sqrt{T}} \tag{4}$$

with a density function equal to

$$\frac{1}{\sqrt{2\pi}}e^{-z^2/2} \tag{5}$$

The probability that a sample from this unit normal distribution will result in a value of less than a constant d is N(d), which is:

$$N(d)=\int_{-\infty}^{d}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}\,dz \tag{6}$$

For a call option on a stock with an exercise price X, the terminal call value will be:

$$\tilde{C}_T=\begin{pmatrix} \tilde{S}_T-X \text{ for } S_T \geq X \\ 0 \text{ for } S_T < X \end{pmatrix} \tag{7}$$

From the unit normal distribution function and taking the present values of the terminal stock price and exercise price, for a non-dividend paying stock, the solution for a European call option is the Black-Scholes formula:

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$$C=SN(d_1)-Xe^{(-r)T}N(d_2) \quad (8)$$

where

$$d_1=\frac{\ln(S/X)+(r+\sigma^2/2)T}{\sigma\sqrt{T}} \quad (9)$$

and

$$d_2=d_1-\sigma\sqrt{T} \quad (10)$$

This assumes that the interest rate r and the volatility σ are constant.

A similar approach to the value of a call option is to assume that the stock price S follows a generalized Wiener process

$$dS = \mu Sdt + \sigma Sdz_s \quad (11)$$

C is the price of a derivative security contingent on S. From Ito´s lemma it follows that the process followed by a function C, of S and t is:

$$dC=\left(\frac{\partial C}{\partial S}\mu S+\frac{\partial C}{\partial t}+\frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt+\frac{\partial C}{\partial S}\sigma Sdz_s \quad (12)$$

The solution is the Black-Scholes formula, assuming that the volatility is non-stochastic. If the above assumptions hold, a written call option can be hedged by a long position equal to $N(d_1)$ in the stock.

A. OPTION GREEKS

Option "Greeks" (Delta Δ, Theta Θ, Gamma Γ, "Vega" "Λ" and Rho P) are the first and second derivatives of equation 8 with respect to the variabels S, T, Δ, σ and r.

DELTA $\Delta = N(d_1)$ is the first derivative of equation 8, with respect to changes in the stock price. Delta increases from zero as the stock price increases to a maximum of one, (see Figure 1a) and increases/ decreases as time to maturity decreases according to the moneyness of the option (see Figure 1b). "Moneyness" refers to the current stock price divided by the exercise price; for an IN (in-the-money) option, the exercise price is less than the stock price; for an AT (at-the-money) option, the exercise price is equal to the stock price; and for an OUT (out-of-the-money), the exercise price is above the stock price. (Figures 1 and 2 assume S=100, X=100, or 90 for IN, 110 for OUT and unless otherwise is specified, T=.5 year, σ=30%, r=10%.)

THETA is the first derivative of equation 8, with respect to time to expiration.

$$\Theta=-\frac{SN'(d_1)\sigma}{2\sqrt{T}}-rXe^{-r(T)}N(d_2) \quad (13)$$

In Figure 1c, theta is increasingly negative as time to expiration decreases at all exercise prices, until just before expiration the theta for IN and OUT tends to zero. In Figure 1d, the theta decreases as the call becomes more away from-the-money; theta reaches its largest negative value around s=x.

"VEGA" is the first derivative of equation 8, with respect to changes in volatility (also called lambda).

$$\Lambda=S\sqrt{T}N'(d_1)e^{-r(T)} \quad (14)$$

In contrast to theta, in Figure 1d, vega reaches its maximum around s=x, and falls to zero way in or out-of-the-money. In Figure 2a, vega declines as time to expiration decreases.

GAMMA is the second derivative of equation 8, with respect to the stock price change. It is the slope of the delta curve, which as shown in Figure 1a is greatest around s=x.

$$\Gamma=\frac{N'(d_1)}{S\sigma\sqrt{T}} \quad (15)$$

Indeed, in Figure 2b, like vega, gamma is at a maximum around s=x, and falls to zero away-from-the-money. In Figure 2c, generally gamma increases with time to expiration, until just before expiration it decreases for IN and OUT. Gamma is supposed to be a critical consideration in reducing the errors in delta hedging, where there are rapid changes in stock prices.

RHO is the first derivative of equation 8, with respect to the interest rate.

$$\rho = X\,T\,e^{-r(T)}\,N(d_2) \qquad (16)$$

For all exercise prices, in Figure 2d, rho decreases as time to expiration decreases. Rho for options on stocks is often hedged using options on interest-rate futures.

B. DELTA AND COMPLEX HEDGING

Although delta hedging has been commonly discussed and used as a method to test option market efficienty, as in Black and Scholes (1972), Chiras and Manaster (1978), Galai (1983) or Figlewski (1989), more sophisticated strategies have been recently employed.

Hull and White (1987b) found that delta+gamma hedging performs well if traded currrency options are short time-to-maturity and if they have a relatively constant implied volatility. But they also advise that it is possible that given other conditions, complex hedging can underperform strategies such as simple delta hedging. Delta+vega hedging seems to outperform other strategies if volatilities change frequently and time-to-maturity is longer.

Recently, Clewlow et al (1993) studied the effectiveness of various methods of delta, delta+gamma and delta+vega hedging of stock index options, analyzing errors decomposition. They claim that to real improvement was found in delta+gamma or delta+vega hedging, even though delta+gamma hedging seemed to be slightly more effective.

Assuming the time interval Δt is one day (restricted by the data collected), the Taylor expansion in discrete time, similar to equation 12 of a portfolio Π is:

$$\Delta\Pi = \frac{\partial\Pi}{\partial S}\Delta S + \frac{\partial\Pi}{\partial\sigma}\Delta\sigma + \frac{\partial\Pi}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2\Pi}{\partial S^2}\Delta S^2 + \frac{1}{2}\frac{\partial^2\Pi}{\partial\sigma^2}\Delta\sigma^2 + \ldots \qquad (17)$$

where we seek to have the left hand side as small as possible (or profitable, in imperfect option markets). The first term of the ritht hand side includes the option delta, the second term the option vega, the fourth term one half of the option gamma, and the third term is nonstochastic.

Since it is desired that each delta, vega, and gamma term is nil (or neutral), the solution of a series of simultaneous equations results in a complex hedging process. If we take x_i as the number of assets in the portfolio,

$$x_1\frac{\partial C_1}{\partial S} + x_2\frac{\partial C_2}{\partial S} + x_3\frac{\partial C_3}{\partial S} + x_4\frac{\partial S}{\partial S} = 0 \qquad (18)$$

$$x_1\frac{\partial^2 C_1}{\partial S^2} + x_2\frac{\partial^2 C_2}{\partial S^2} + x_3\frac{\partial^2 C_3}{\partial S^2} = 0 \qquad (19)$$

$$x_1\frac{\partial C_1}{\partial\sigma} + x_2\frac{\partial C_2}{\partial\sigma} + x_3\frac{\partial C_3}{\partial\sigma} = 0 \qquad (20)$$

The budget constraint is expressed as:

$$x_1C_1 + x_2C_2 + x_3C_3 + x_4S = K \qquad (21)$$

where K is the total net position (long stock plus bought options less written options). If we set $x_4=+1$, meaning that we go long one share, our unknowns become x_1, x_2, x_3 and K.[2]

The solution for these simultaneous equations can be solved by matrix manipulation. If

$$Ax = b \qquad (22)$$

where A is the coefficient matrix (the delta's, gamma's and vega's for each option and a delta of one and a gamma and vega of 0 for the stock), x is the vector of the unknown number of call options and the total net position, and b is the result vector (-1 for the first equation and -S for the last, once we have set $x_4=+1$, and 0 for each gamma and vega equations, since we aim for neutrality). This matrix equation is solved by multiplying both sides from the left by the inverse of A:

$$A^{-1}Ax = A^{-1}b \qquad (23)$$

Thus the solution is:

$$x = A^{-1}b \qquad (24)$$

We have two primary hypotheses:

- Hypothesis I - Ignoring transaction costs, complex hedging is more effective than simple delta hedging but possibly more costly as a position in more than one option and the stock is required.

- Hypothesis II - However, full cost will be increased with complex positions, as there will be high transaction costs in altering several option positions as required.

III. EMPIRICAL ANALYSIS OF DYNAMIC HEDGING

The sample used in this study consisted of three groups of time series of call options (one close to at-the-money, the others, the next in and out-of-the-money exercise prices) for nine stocks traded on the London International Financial Futures and Options Exchange (LIFFE) for all the month of March 1991. The firms selected were Amstrad, BAirways, BGas, BP, BT, Forte, Gen. Electric, Hanson and Rolls Royce. We selected series expiring in June and September for Amstrad, BGas, Forte and Rolls Royce, expiring in May, August and November for BT and Hanson, expiring in May and August for Gen. Electric, expiring in April, July and October for BAirways and expiring in April and July for BP. We have built 21 time series of three exercise prices for a total of 1281 options (21 of in-the-money, 21 of at-the-money and 21 of out-of-the-money options). The option premiums were collected from Datastream; the underlying stock price simultaneous with the closing quotation was retrieved from the London Stock Exchange Daily Official List and has been adjusted for the present value of expected dividends. The option premiums are average closing quoted prices (bid+ask)/2.

We tried first no hedging, then simple delta hedging for each exercise price, then delta+gamma hedging as well as delta+vega hedging (using two exercise prices). We restrict complex hedging to using options in the same series, rather than options from different series on the same underlying stock. Since different series will have different gamma and vega characteristic, possibly useful in hedging with a minimum number of options, this restriction is limiting.

A. IGNORING TRANSACTION COSTS

Assuming no transaction costs, we hedged a long position on a stock by taking a short delta neutral position in the option. Both positions were marked-to-market daily and the returns were computed on a daily continuously compounded basis. Simultaneously the interest income on the portfolio (assuming the full option premiums and the full option proceeds from short positions are paid/received immediately, and differences settled each day) was computed as well as capital gains/losses. The risk free return on the daily net investment position was used as a benchmark to compare the results of different strategies. All returns either exceeding or failing to exceed that return, were taken as abnormal returns and added to the end of the sample period. This can be considered as the total abnormal return of the strategy (that theoretical should be zero). This total abnormal return was retained and the process was repeated 21 times (each for a different time series). Each time the stock price moves and the at-the-money series changes, we also change the exercise price of the selected options in order to match the original requirement (to get the closest in, at and out-of-the-money options). [3]

Some descriptive statistics for the 21 total abnormal returns, were calculated and are summarised in Table 1. The first conclusion drawn from the first 4 columns of Table 1 is the reduction in the standard deviation of the abnormal returns, when compared with the no hedge strategy. However, there is some reduction in the average abnormal return. There appear to be some differences among the delta hedging strategies, namely the least effective performance was in delta hedging the stock position with dynamic positions in out-of-the-money options. Also all delta strategies had negative skewness and substantial kurtosis compared to a straight stock position.

The next step was to compare delta+gamma hedging strategies, using different pairs of option series: in and out-of-the-money series, in and at-the-money series and lastly at and out-of-the-money series (columns 5 to 7 of Table 1). We observe a decrease in the average abnormal return especially in the at + out-of-the-money strategies (column 7). For all these strategies, a reduction in average abnormal returns was accompanied by an increase in the riskness of the portfolio (the standard deviation increased significantly when compared with the simple delta hedging strategy) and negative skewness was significant.

An errors decomposition of the simple delta strategies showed that vega changes appear to be the main problem in achieving near perfection, so we tried delta+vega hedging with the same pairs of series. The results are presented in columns 8 to 10 of Table 1. Contrary to Hull and White (1987a), who were not studying equities, delta+vega hedging strategies result in higher abnormal returns and sometimes higher standard deviations compared to delta+gamma strategies.

The results (averages and standard deviations) of delta, gamma, delta and gamma and delta and vega strategies are plotted in Figure 3. If we start from no hedging in the stock, and if the real concern is to eliminate risk (replication error), the "best strategy" seemed to be delta hedging using in or at-the-money options. However, all the strategies (namely delta+vega with in and out-of-the-money options, delta+gamma with in and out-of-the-money options and all of the simple delta strategies) within the marked area in Figure 3 achieved a reduction in standard deviation compared to no hedging.

B. INTRODUCING TRANSACTION COSTS

The introduction of transaction costs is a natural extension of the previous work. As Figlewski (1989) points out, transaction costs vary according to the type of cost and the type of investor. The costs can be divided into two different categories: the bid-ask spread and the other costs. We were particularly concerned with the bid-ask spread that becomes the most important cost when hedging strategies are executed. To see the extent of the bid-ask spread, we have analyzed one week of bid and ask prices in order to gauge their influence on our sample. From that week of data we got a typical set of spreads for each price for a large range of quotes. For instance, an average price of 6.5p is a typical result of 5.5p - 7.5p bid-ask prices, and for an average price of 72p the typical bid and ask prices are 71p - 73p.

We have used this structure as a proxy for the bid and ask prices for the data used previously. Thus we have assumed purchases at the asked prices and sales at the bid prices, initially, and for every re-balancing. The results indicate naturally that all the hedging strategies perform worse than those in Table 1 (see Figure 4 which include the delta and delta and gamma strategies). Simple delta strategies still outperform all the others. However, complex strategies with transaction costs involving in + out-of-the-money series, result in a better outcome than those using in + at-the-money series. There may be mis-pricing of options as indicated by significantly high or low implied volatilities, but there do not appear to be any hedging strategies that produce positive returns, when transaction costs are considered.

IV CONCLUSION

We examined the profitability and effectiveness of delta hedging strategies. Then, we attemped more complex hedges, using options with more than one exercise price, as the "solution" for simultaneous equations establishing delta, gamma and vega neutrality, subject to a budget constraint. Assuming no transaction costs, we found that the most successful strategies involved writing in-the-money options.

The first hypothesis, that more complex hedging considering gamma and/or vega might be more effective, in terms of reduced risk, was not supported. The second hypothesis, that when transaction costs were introduced hedging strategies would be significantly altered was supported. The most successful strategies were still the simple delta strategies, but the generally negative average abnormal returns raise doubts about the practicality of those strategies. Before transaction costs, some of the delta + vega strategies involving writing IN options were successfful; but after transaction costs, the abnormal returns were eliminated.

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NOTES

1. A nice proof of this is in Stoll & Whaley (1993), page 239.
2. We use the approach of long one share for ease of calculation, although it contrasts with the convention of a position in a multiple of one option hedged by delta positions in the stock. Thus our transaction costs in options are high relative to the conventional approach.
3. Maintaining this correctness in "moneyness" resulted in increased transaction costs.

TABLE 1
ABNORMAL RETURNS FROM OPTION STRATEGIES
(Continuously Compounded Rate of Return)

SUMMARY HEDGING	NO HEDGE	DELGA HEDGING			DELGA+GAMMA HEDGING			DELTA+VEGA HEDGING		
		with In-Money	with At-Money	with Out-Money	with In+Out Money	with In+At Money	with At+Out Money	with In+Out Money	with In+At Money	with At+Out Money
	1	2	3	4	5	6	7	8	9	10
Average	0.0009	-0.0003	-0.0002	-0.0003	-0.0008	-0.0014	-0.0041	0.0000	0.0017	-0.0003
Std	0.0208	0.0105	0.0106	0.0156	0.0176	0.0257	0.0403	0.0189	0.0393	0.0328
Max	0.0950	0.0837	0.0689	0.1060	0.0900	0.1299	0.1393	0.1132	0.4631	0.1795
Min	-0.0968	-0.0857	-0.0736	-0.1152	-0.1232	-0.1968	-0.3909	-0.0938	-0.1371	-0.1750
Range	0.1918	0.1694	0.1425	0.2212	0.2232	0.3267	0.5302	0.2115	0.6003	0.3545
Kurtosis	5.1133	21.3470	11.2253	17.0519	12.0400	15.1717	25.4994	12.5104	60.6698	7.5439
Skewness	0.0342	-0.2226	-0.1678	-0.3114	-0.9794	-0.8631	-3.1482	0.5877	5.9358	0.0146
N. Observations	420	420	420	420	420	420	420	420	420	420

Figure 1

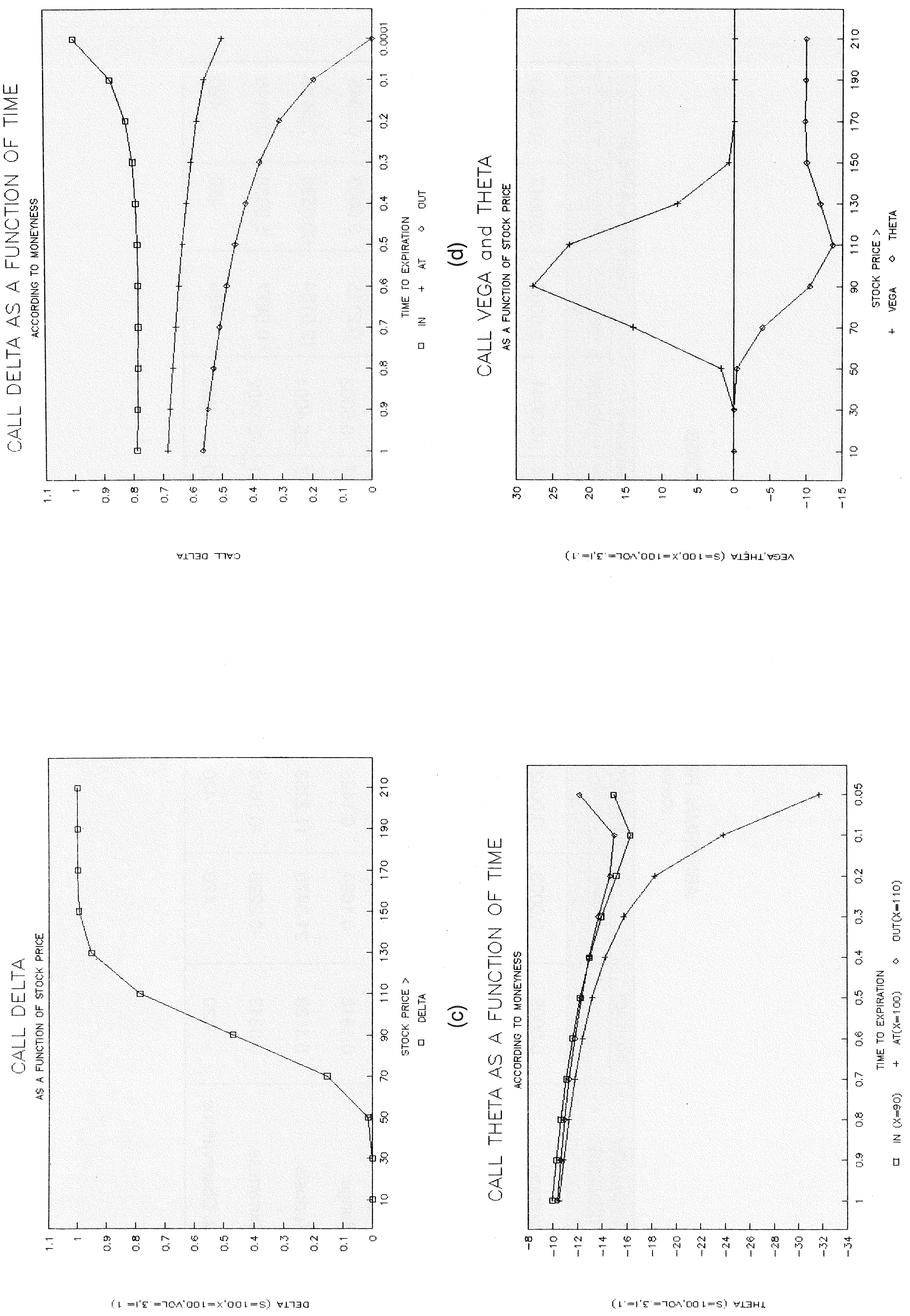
(a)
CALL DELTA
AS A FUNCTION OF STOCK PRICE
DELTA (S=100,X=100,VOL=.3,I=.1)
STOCK PRICE >
□ DELTA
(b)
CALL DELTA AS A FUNCTION OF TIME
ACCORDING TO MONEYNESS
CALL DELTA
TIME TO EXPIRATION
□ IN + AT ◇ OUT
(c)
CALL THETA AS A FUNCTION OF TIME
ACCORDING TO MONEYNESS
THETA (S=100,VOL=.3,I=.1)
TIME TO EXPIRATION
□ IN (X=90) + AT(X=100) ◇ OUT(X=110)
(d)
CALL VEGA and THETA
AS A FUNCTION OF STOCK PRICE
VEGA,THETA (S=100,X=100,VOL=.3,I=.1)
STOCK PRICE >
+ VEGA ◇ THETA

Figure 2

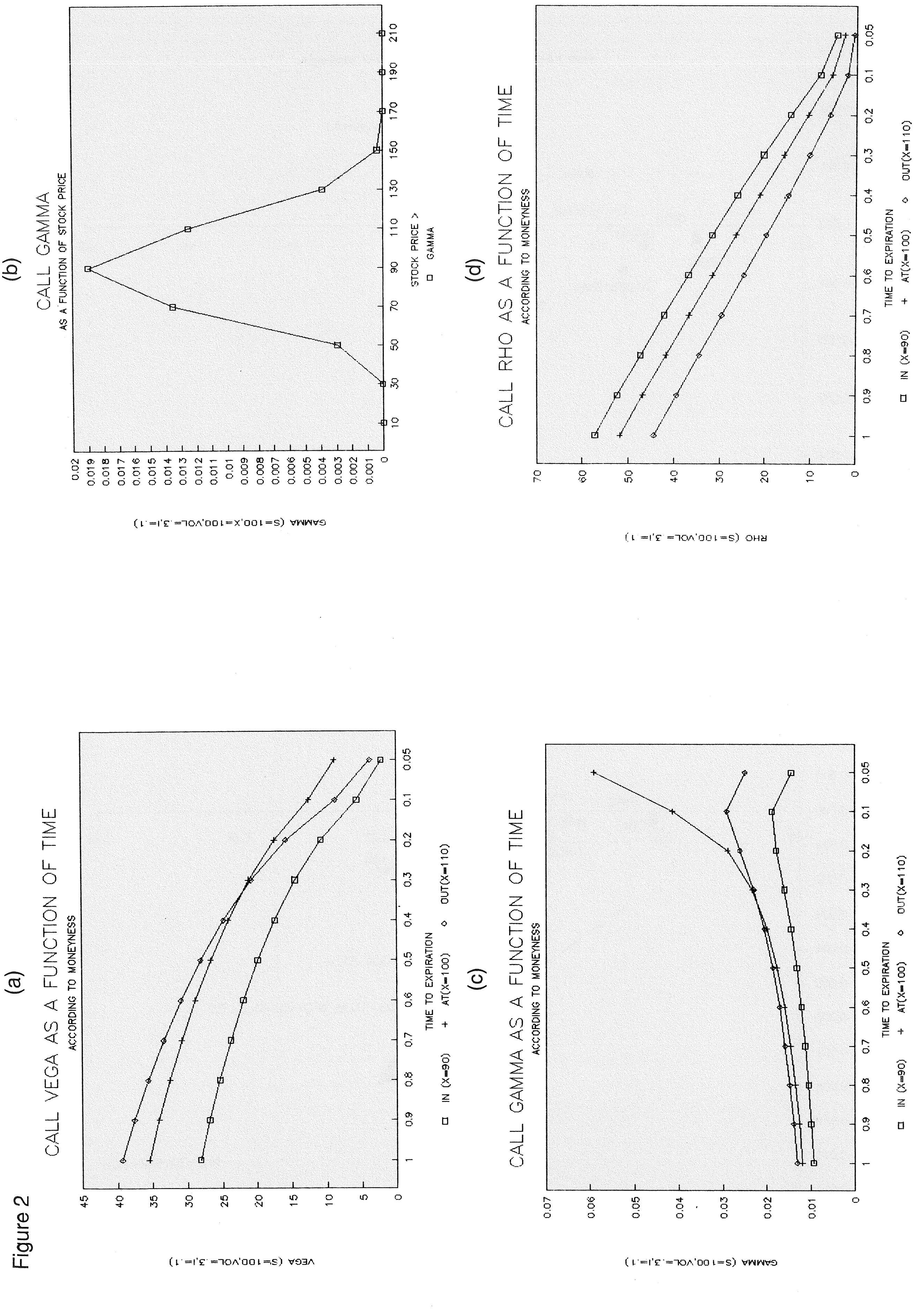
(a)
CALL VEGA AS A FUNCTION OF TIME
ACCORDING TO MONEYNESS
VEGA (S=100,VOL=.3,I=.1)
TIME TO EXPIRATION
IN (X=90)
AT(X=100)
OUT(X=110)
(b)
CALL GAMMA
AS A FUNCTION OF STOCK PRICE
GAMMA (S=100,X=100,VOL=.3,I=.1)
STOCK PRICE >
GAMMA
(c)
CALL GAMMA AS A FUNCTION OF TIME
ACCORDING TO MONEYNESS
GAMMA (S=100,VOL=.3,I=.1)
TIME TO EXPIRATION
IN (X=90)
AT(X=100)
OUT(X=110)
(d)
CALL RHO AS A FUNCTION OF TIME
ACCORDING TO MONEYNESS
RHO (S=100,VOL=.3,I=.1)
TIME TO EXPIRATION
IN (X=90)
AT(X=100)
OUT(X=110)

Figure 3

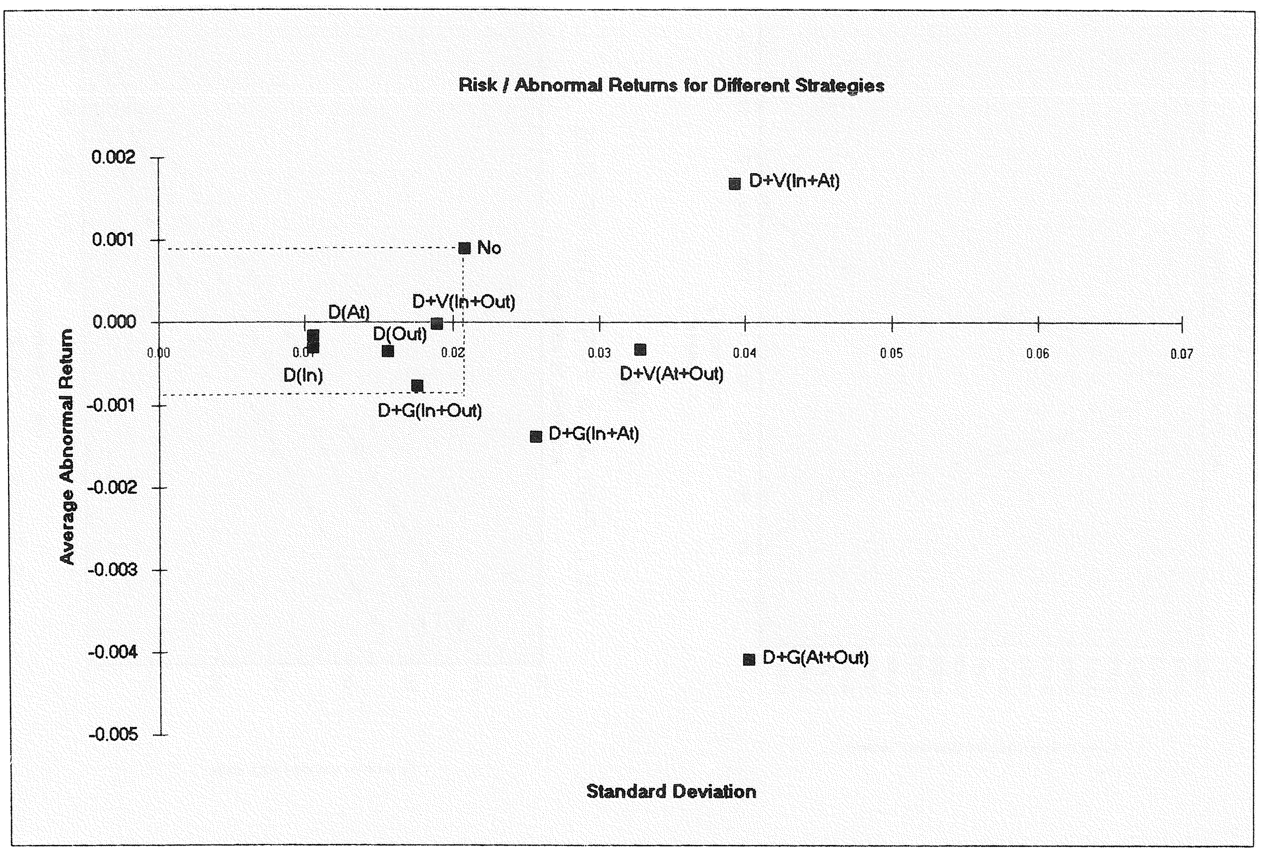

Figure 4