ERGODIC PROPERTIES OF POLYGONAL BILLIARDS WITH STRONGLY CONTRACTING REFLECTION LAWS

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Abstract. We consider convex polygonal billiards with a reflection law that contracts the reflection angle towards the normal. Polygonal billiards with orthogonal reflections are called slap maps. For polygons without parallel sides facing each other, the slap map has a finite number of ergodic absolutely continuous invariant probability measures. We show that any billiard with a strongly contracting reflection law has the same number of ergodic SRB measures with the same mixing periods as the ergodic absolutely continuous invariant probabilities of its corresponding slap map. The case of billiards in regular polygons and triangles is studied in detail.

1. Introduction

The dynamics of polygonal billiards has been studied in much detail for the specular reflection law (cf. [10, 18, 19]). More recent work deals with a new class of billiards for which the reflection angle measured with respect to the normal is a contraction of the incidence angle. These new billiards differ greatly from the ones with specular reflection law in that they typically have uniformly hyperbolic attractors with Sinai-Ruelle-Bowen (SRB) measures when the contraction is strong [2, 3, 7, 8, 13]. In this paper we show that the ergodic properties of polygonal billiards with strongly contracting reflection laws are related to the ergodic properties of their slap maps, and present several examples.

The slap map can be considered as the degenerate billiard map corresponding to a zero reflection angle, i.e. the billiard ball is always reflected in the direction orthogonal to the billiard table. According to our main result below, the information obtained for the slap map is of crucial importance to understanding the ergodic properties of billiards with strongly contracting reflection laws.

Slap maps on d-gons reduce naturally to one-dimensional piecewise affine maps of the interval. If a d-gon does not have periodic points of period 2, then its slap map is a piecewise smooth expanding map of the interval, and admits not more than d ergodic absolutely continuous invariant measures (acips) [9]. In Theorem 1.1 we prove that billiards

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with strongly contracting reflection laws inherit the ergodic properties of the corresponding slap map.

Theorem 1.1. *For almost every convex d-gon P there is a neighborhood $\mathcal{U}$ of P such that the following holds. For any billiard on $P' \in \mathcal{U}$ with a strongly contracting reflection law, the number of ergodic SRB measures is the same as the number of ergodic absolutely continuous invariant probability measures of the slap map of P.*

The next results regard billiards in regular polygons.

Corollary 1.2. *For any regular d-gon P with $d \geq 3$ there is a neighborhood $\mathcal{U}$ of P such that any billiard on $P' \in \mathcal{U}$ with a strongly contracting reflection law has a unique ergodic SRB measure if $d = 3$ or $d = 5$, and exactly d ergodic SRB measures if $d \geq 7$.*

Corollary 1.3. *Any billiard on an acute triangle with a strongly contracting reflection law has a unique ergodic SRB measure. Moreover, this measure is mixing.*

Corollary 1.4. *Billiards on almost every non-acute triangle with strongly contracting reflection laws have unique ergodic SRB measure with an even number of mixing components.*

The paper is organized as follows. In Section 2, we review definitions and the general properties of polygonal billiards with contracting reflection laws. In Section 3, we prove several results for piecewise expanding maps used in the the proof of Theorem 1.1. These results may be of independent interest. For example, in Proposition 3.11, we show that periodic points of a piecewise expanding map are dense in the support of every acip of the map. Although this may come as no surprising for experts, nevertheless we were not able to find a reference to it in the literature. In Section 4, we introduce the polygonal slap maps and derive their properties from those of 1-dimensional piecewise expanding maps. Section 5 is devoted to the study of several important properties of polygonal billiards with strongly contracting reflection laws. In Section 6, we prove Theorem 1.1. In Section 7, we prove Corollaries 1.2-1.4 and other results, using previous work on polygonal slap maps in [9]. In Appendix A, we review results of Pesin and Sataev on general hyperbolic maps with singularities.

2. Polygonal billiards with contracting reflection laws

2.1. **Definitions and general properties.**

Denote by $\mathcal{P}_d$ the set of convex polygons with d sides and perimeter equal to one[1]. The billiard

[1]This is required so that different polygons have the same phase space. In fact, we can rescale a polygon with perimeter L by multiplying every side by the factor $1/L$, thus obtaining a new polygon with perimeter one. Notice that the dynamics does not change under rescaling of a polygonal table.

in $P \in \mathcal{P}_d$ with the specular reflection law is the flow on the unit tangent bundle of P generated by the motion of a free point-particle in the interior of P with specular reflection at the boundary ∂P (i.e., the angle of reflection equals the angle of incidence). The corresponding billiard map Φ_P is the first return map on M, the set of unit vectors attached to ∂P and pointing inside P.

Each element of $x \in M$ can be identified with a pair (s, θ), where s is the arclength parameter of ∂P of the base point of x, and θ is the angle formed by x with the positively oriented tangent to ∂P at s. Accordingly, we can write

$$M = [0,1] \times [-\pi/2, \pi/2].$$

Let $0 = s_0 < s_1 < \cdots < s_d = 1$ be the values of the arc-length parameter corresponding to the vertices of P. Let

$$V = \{s_0, \dots, s_d\} \times [-\pi/2, \pi/2].$$

Denote by S_1^+ the subset of M consisting of elements whose forward trajectories hit a vertex of P, i.e. $S_1^+ = \Phi_P^{-1}(V)$. Let

$$N = \partial M \cup V \cup S_1^+.$$

The set S_1^+ is a union of finitely many curves that are graphs of analytic strictly monotone functions from intervals of $(0,1)$ to $(-\pi/2, \pi/2)$ (see Proposition 2.3).

The map Φ_P is defined on $M \setminus N$, and $\Phi_P(x)$ is the unit vector corresponding the next collision of x with ∂P for every $x \in M \setminus N$. It turns out that Φ_P is a smooth piecewise map on $M \setminus N$ with singular set N in the sense of Definition A.1. For a detailed definition of the billiard map, we refer the reader to [8] for polygonal billiard tables, and to [6] for general billiard tables.

A reflection law is a function $f\colon [-\pi/2, \pi/2] \to [-\pi/2, \pi/2]$. For example, the specular reflection law corresponds to the function $f(\theta) = \theta$. Let $R_f\colon M \to M$ be the map $R_f(s, \theta) = (s, f(\theta))$. The billiard map on P with reflection law f is the map $\Phi_{f,P}\colon M \setminus N \to M$ given by

$$\Phi_{f,P} = R_f \circ \Phi_P.$$

Notice that this map is injective if and only if f is injective. For a more detailed introduction of $\Phi_{f,P}$, and a discussion of its main properties, see [8].

As always in the analysis of piecewise smooth maps, we need to study the negative iterates of S_1^+ under the map $\Phi_{f,P}$. For this reason, for $n \geq 1$, we define

$$S_n^+ = S_n^+(f, P) = S_1^+ \cup \Phi_{f,P}^{-1}(S_1^+) \cup \cdots \cup \Phi_{f,P}^{-n+1}(S_1^+).$$

Definition 2.1. Given a differentiable reflection law f, we define

$$\lambda(f) := \max_{\theta \in [-\pi/2, \pi/2]} |f'(\theta)|.$$

A differentiable reflection law is called *contracting* if $\lambda(f) < 1$. The simplest example of a contracting reflection law is $f(\theta) = \sigma\theta$ with $0 < \sigma < 1$ [2, 3, 7, 13].

Definition 2.2. We denote by $\mathcal{B}$ the set of all contracting reflection laws that are C^2 embeddings from the closed interval $[-\pi/2, \pi/2]$ to itself.

Unless otherwise stated, we will always assume throughout this paper that $f \in \mathcal{B}$. This assumption makes the map $\Phi_{f,P}$ a C^2 diffeomorphism from $M \setminus N$ to $f(M \setminus N)$, i.e., a piecewise smooth map on $M \setminus N$ with singular set N (c.f. Definition A.1).

We say that a curve γ in M is *strictly decreasing* (resp. *strictly increasing*), if it is the graph of a strictly decreasing (resp. *strictly increasing*) function $h \colon I \to [-\pi/2, \pi/2]$, where I is an interval of $[0, 1]$.

Proposition 2.3. *Suppose that $f \in \mathcal{B}$ and $f' > 0$. Then S_n^+ is a union of finitely many strictly decreasing C^2-curves for every $n \geq 1$. Moreover, the curves forming S_1^+ are analytic, and do not depend on f.*

Proof. The proof is by induction. The case $n = 1$ is proved in [8, Proposition 2.1]. Assume now that the conclusion of the proposition is true for some $n > 0$. By [8, Proposition 2.1], we know that S_1^+ is formed by finitely many strictly decreasing curves, and N_1^- is formed by finitely many strictly increasing curves and finitely many vertical and horizontal segments. It follows that $S_n^+ \cap N_1^-$ is finite. Since $\Phi_{f,P}$ is a C^k embedding, $\Phi_{f,P}^{-1}(S_n^+)$ consists of finitely many curves of class C^k. If we write $(s', \theta') = \Phi_P(s, \theta)$, then the derivative of $\Phi_{f,P}$ is given by [13]

$$D\Phi_{f,P}(s,\theta) = -\begin{pmatrix} \alpha(s,\theta) & \gamma(s,\theta) \\ 0 & \beta(s,\theta) \end{pmatrix},$$

where $\alpha(s,\theta) = \cos\theta/\cos\theta'$, $\beta(s,\theta) = f'(\theta')$, $\gamma(s,\theta) = t(s,\theta)/\cos\theta'$, and $t(s,\theta)$ denotes the euclidean distance of $\mathbb{R}^2$ between the points in ∂P with coordinates s and s'. We then see that $D\Phi_{f,P}^{-1}$ preserves the negative cone of $\mathbb{R}^2$ when $f' > 0$. This fact combined with the induction hypothesis implies that the curves $\Phi_{f,P}^{-1}(S_n^+)$ are strictly decreasing. The same is true for S_{n+1}^+. □

For every $n \geq 1$ the set S_n^+ is a union of finitely many strictly decreasing analytic curves for $f \equiv 0$.

In this paper, we study polygonal billiards with contracting reflection laws f with small $\lambda(f)$. We call these laws *strongly contracting.* The degenerate case $f \equiv 0$ plays a key role in our analysis. Later in Section 4, we introduce a 1-dimensional map ψ_P associated to the polygon P, which is strictly related to the billiard map $\Phi_{0,P}$. Notice that, even though $\Phi_{0,P}$ is a 2-dimensional map on M, the restriction of $\Phi_{0,P}$ to the circle $\{\theta = 0\} \subset M$ coincide with ψ_P.

Remark 2.4. All the points of S_1^+ are discontinuities points of $\Phi_{f,P}$ when $f \in \mathcal{B}$, and continuity points when $f \equiv 0$. In fact, the discontinuity points of $\Phi_{0,P}$ are contained in V.

2.2. **Attractors and SRB measures.** Let

$$M^+ = \{x \in M : \Phi_{f,P}^n(x) \notin N \ \forall n \geq 0\}$$

be the set of all elements of M with infinite positive semi-orbit. The set

$$A := \overline{\bigcap_{n\geq 0} \Phi_{f,P}^n(M^+)}$$

is the (generalized) attractor of $\Phi_{f,P}$. It is easy to check that

$$A \subset \left\{(s,\theta) \in M : |\theta| \leq \frac{\pi}{2}\lambda(f)\right\}.$$

In [13], it was proved that polygonal billiards with contracting reflection laws have always dominated splitting (for a more direct proof, see also [8, Proposition 3.1]). In these billiards, parabolic and hyperbolic attractors may coexist. For examples of rectangular billiards exhibiting such a coexistence, see [8, Corollary 8.9 and Remark 8.14].

The proofs of our main results rely on a theory of hyperbolic piecewise smooth maps developed by Pesin and Satev independently [14, 15, 16]. For the convenience of the reader, a summary of this theory together with the main results used in this paper is provided in Appendix A. The theory has four standing hypotheses called Conditions H1-H4. We added the auxiliary condition H5, which implies H3 and H4. The results that we need are Theorems A.5, A.7 and A.8.

Definition 2.5. Suppose $f \in \mathcal{B}$ or $f \equiv 0$. For every $n \geq 1$, the *branching number of* S_n^+ is given by

$$p(S_n^+) = p(S_n^+, f, P) = 1 + \inf_{\varepsilon>0} \sup_{\Gamma\in\mathcal{H}(\varepsilon)} \#(\Gamma \cap S_n^+),$$

where $\mathcal{H}(\varepsilon)$ is the set of horizontal segments Γ ($\theta = const$) contained in $M \setminus V$ of length ε.

The number $p(S_n^+) - 1$ is the maximum number of smooth components of S_n^+ intersected by short horizontal segments. Notice that $p(S_1^+) = 2$ for a convex polygon, and $p(S_1^+)$ is always less than $d - 1$ for a general d-gon.

The unstable direction of $\Phi_{f,P}$ is always the horizontal one.

Definition 2.6. Suppose $f \in \mathcal{B}$ or $f \equiv 0$. The least expansion rate of $D\Phi_{f,P}^n$ along the unstable direction is given by

$$\alpha(\Phi_{f,P}^n) = \inf_{x\in M^+} \left\| D\Phi_{f,P}^n(x)\,(1,0)\right\|, \qquad n \in \mathbb{N}.$$

Proposition 2.7. *Let $P \in \mathcal{P}'_d$, and let $f \in \mathcal{B}$. If $p(S_n^+) < \alpha(\Phi_{f,P}^n)$ for some $n \geq 1$, then Condition H5 holds for horizontal curves ($\theta = 0$). In particular, the conclusions of Theorems A.5, A.7 and A.8 hold for $\Phi_{f,P}$.*

Proof. We will prove that under the hypotheses of the proposition, the map $\Phi_{f,P}$ satisfies Conditions H1-H3 and a weaker version of H4. In the course of the proof, we will explain that this is enough to obtain the conclusions of Theorems A.5, A.7 and A.8.

The map $\Phi_{f,P}$ always satisfies H1. This is a direct consequence of the fact that the standard billiard map Φ_P satisfies H1 (see [11, Theorem 7.2]), and a reflection law $f \in \mathcal{B}$ together with its inverse has bounded second derivatives. In [8, Corollary 3.4], we proved that $\Phi_{f,P}$ satisfies H2 if and only if P does not have parallel sides facing each other.

Horizontal segments ($\theta = const$) are always mapped by Φ_P to finitely many horizontal segments. It turns out that the unstable subspace E^u of $\Phi_{f,P}$ coincides, when it is defined, with the subspace spanned by the vector $(1, 0)$ in coordinates (s, θ) [8, Proposition 3.1]. For the billiard map $\Phi_{f,P}$, Theorems A.5, A.7 and A.8 remain valid if H4 is proved not for all u-manifolds, but only for horizontal segments. This weaker version of H4 follows from the analogous weaker version of H5 (c.f. Lemma A.9), which is proved in a growth lemma (see Lemma 2.8). □

The branching number $p(S_n^+)$ can be interpreted as a measure of local complexity generated by the singularities. Thus, the condition $p(S_n^+) < \alpha(\Phi_f^n)$ amounts to saying that the least expansion of $\Phi_{f,P}^n$ is larger than the local complexity of $\Phi_{f,P}^n$.

2.3. **Growth Lemma.**

The proof of the next lemma is adapted from the one of [8, Proposition 4.3]. This type of results are often called 'growth lemmas' (see e.g. [6]). Recall that N_ε is the ε-neighbourhood of N_1^+ for $\varepsilon > 0$.

Lemma 2.8. *Let $P \in \mathcal{P}'_d$, and let $f \in \mathcal{B}$. If there is $m \in \mathbb{N}$ such that $a := p(S_m^+)/\alpha(\Phi_{f,P}^m) < 1$, then there exists $\varepsilon_0 > 0$ and $c > 0$ such that for any horizontal curve Γ, $n \geq 0$ and $0 < \varepsilon < \varepsilon_0$,*

$$\ell\left(\Gamma \cap \Phi_{f,P}^{-n}(N_\varepsilon)\right) \leq c\,\varepsilon\,(a^{n/m-1} + \ell(\Gamma)).$$

Proof. We set $\varepsilon_0 < (1-\lambda)\pi/2$ so that the orbits do not get into the ε_0-neighborhood of ∂M. So N_ε can be replaced by the ε-neighborhood of $V \cup S_1^+$. The sets N_1^+ and N_ε are thus taken to be $V \cup S_1^+$ and the ε-neighborhood of $V \cup S_1^+$, respectively. Moreover, we write Φ to mean $\Phi_{f,P}$.

Define $N_r^+ = M \setminus (V \cup S_r^+)$. The set $\Gamma \setminus N_r^+$ has finitely many connected components, say u_r components for any $r \geq 0$, where $N_0^+ =$

$\emptyset$. So,

$$\begin{aligned}\ell\left(\Gamma \cap \Phi^{-r}(N_\varepsilon)\right) &= \ell\left((\Gamma \setminus N_r^+) \cap \Phi^{-r}(N_\varepsilon)\right)\\ &= \ell\left(\Phi^{-r}\left(\Phi^r(\Gamma \setminus N_r^+) \cap N_\varepsilon\right)\right).\end{aligned}$$

Furthermore, $\Phi^r(\Gamma \setminus N_r^+)$ is the union of disjoint horizontal curves

$$\Phi^r(\Gamma \setminus N_r^+) = \bigcup_{i=1}^{u_r} \Gamma_{r,i}.$$

Notice that $u_r \le u_{r+1}$ and $u_0 = 1$.

Writting $\widetilde{\Gamma}_{n,i} = \Gamma_{n,i} \cap N_\varepsilon$, we aim for an upper bound on the sum

$$\ell\left(\Gamma \cap \Phi^{-n}(N_\varepsilon)\right) = \sum_{i=1}^{u_n} \ell\left(\Phi^{-n}(\widetilde{\Gamma}_{n,i})\right).$$

Fix $n = sm + s'$ where $s \ge 0$ and $s' \in \{0, \dots, m-1\}$. To start we will derive upper bounds on the Lebesgue measures of the pre-images $\Phi^{-s'}(\widetilde{\Gamma}_{n,i})$. Since the sets $\widetilde{\Gamma}_{n,i}$ are disjoint,

$$\begin{aligned}\ell\left(\Gamma \cap \Phi^{-n}(N_\varepsilon)\right) &= \ell\left(\Phi^{-sm} \circ \Phi^{-s'}\left(\bigcup_{i=1}^{u_n} \widetilde{\Gamma}_{n,i}\right)\right)\\ &= \sum_{i=1}^{u_n} \ell\left(\Phi^{-sm} \circ \Phi^{-s'}(\widetilde{\Gamma}_{n,i})\right).\end{aligned}$$

From the transversality between N_1^+ and the horizontal direction, there is a constant $C' > 0$ independent of n and Γ such that

$$\ell(\widetilde{\Gamma}_{n,i}) \le C'\varepsilon.$$

Let η be the maximum number of intersections between a horizontal line in the phase space M and N_1^+. Given $j \in \{1, \dots, u_{sm}\}$ define

$$J_j = \{i \in \{1, \dots, u_n\} \colon \Phi^{-s'}(\widetilde{\Gamma}_{n,i}) \subset \Gamma_{sm,j}\}.$$

Thus, $\#J_j \le (\eta+1)^{s'} \le (\eta+1)^m$,

$$\bigcup_{j=1}^{u_{sm}} J_j = \{1, \dots, u_n\}$$

and

$$\sum_{i \in J_j} \ell(\Phi^{-s'}(\widetilde{\Gamma}_{n,i})) \le \sum_{i \in J_j} \frac{\ell(\widetilde{\Gamma}_{n,i})}{\alpha(\Phi^{s'})} \le (\eta+1)^m C'\,\varepsilon =: C'_m \varepsilon. \tag{2.1}$$

Next step is the core argument of this proof. We will consider pre-images (of the previous pre-images) under the map $\psi = \Phi^m$. Alternatively these pre-images are forward iterates of Γ by the map ψ. These image intervals of Γ can be pictured as a directed tree, rooted at the original interval Γ, and described by the descendance relation introduced below.

By its definition $p = p(S_m^+)$ is the smallest positive integer for which there is $C_m > 0$ such that the following holds: if Γ is a horizontal curve with $\ell(\Gamma) < C_m$, then $\Gamma \setminus S_m^+$ has at most p components with positive length. We name these curves Γ short curves, and all other horizontal ones we call long curves.

Write $\Upsilon_{r,i} = \Gamma_{rm,i}$ with $i \in \{1, \dots, u_{rm}\}$ and $0 \le r \le s$. Let $\mathcal{I}$ denote the set of all such pairs (r,i). Given $(r,i),(r',i') \in \mathcal{I}$, we say that (r',i') *descends from* (r,i), and write $(r,i) \succ (r',i')$, when $r' > r$ and $\psi^{-(r'-r)}(\Upsilon_{r',i'}) \subset \Upsilon_{r,i}$.

Let $\mathcal{L}$ to be the set of pairs $(k,l) \in \mathcal{I}$ such that $\Upsilon_{k,l}$ is a long interval, i.e., $\ell(\Upsilon_{k,l}) \ge C_m$. Given $(k,l) \in \mathcal{L}$ we define $I_{k,l}$ to be the set of all $j \in \{1, \dots, u_{sm}\}$ such that $(k,l) \succ (s,j)$ or $(k,l) = (s,j)$, but for which there is no $(k',l') \in \mathcal{L}$ such that $(k,l) \succ (k',l') \succ (s,j)$.

It follows that $\#I_{k,l} \le p^{s-k-1}$ if $(k,l) \in \mathcal{L}$. By definition these sets are pairwise disjoint, i.e., $I_{k,l} \cap I_{k',l'} = \emptyset$ for $(k,l) \ne (k',l')$ and $(k,l),(k',l') \in \mathcal{L}$.

We also define

$$\widetilde{I} = \{1, \dots, u_{sm}\} \setminus \bigcup_{(k,l)\in\mathcal{L}} I_{k,l}$$

which corresponds to the situation that Γ is a short curve and the indices $j \in \widetilde{I}$ identify curves $\Upsilon_{s,j}$ whose pre-images up to the sm-th iterate are all short. Therefore, $\#\widetilde{I} \le p^{s-1}$ and

$$\bigcup_{(k,l)\in\mathcal{L}} \bigcup_{j\in I_{k,l}} J_j \cup \bigcup_{j\in\widetilde{I}} J_j = \{1, \dots, u_n\}.$$

For each $(k,l) \in \mathcal{L}$ define

$$\Lambda_{k,l} = \bigcup_{j\in I_{k,l}} \bigcup_{i\in J_j} \psi^{-(s-k)}(\Phi^{-s'}(\widetilde{\Gamma}_{n,i})) \subset \Upsilon_{k,l}.$$

Notice that this is a disjoint union, hence

$$\begin{aligned}
\sum_{i=1}^{u_n} \ell\left(\Phi^{-sm}(\Phi^{-s'}(\widetilde{\Gamma}_{n,i}))\right) &\le \sum_{j\in\widetilde{I}} \sum_{i\in J_j} \ell\left(\psi^{-s}(\Phi^{-s'}(\widetilde{\Gamma}_{n,i}))\right) \\
&\quad + \sum_{(k,l)\in\mathcal{L}} \sum_{j\in I_{k,l}} \sum_{i\in J_j} \ell\left(\psi^{-s}(\Phi^{-s'}(\widetilde{\Gamma}_{n,i}))\right) \\
&= \sum_{j\in\widetilde{I}} \sum_{i\in J_j} \ell\left(\psi^{-s}(\Phi^{-s'}(\widetilde{\Gamma}_{n,i}))\right) \\
&\quad + \sum_{(k,l)\in\mathcal{L}} \ell\left(\psi^{-k}(\Lambda_{k,l})\right).
\end{aligned}$$

Using the fact that the maps are affine,

$$\frac{\ell\left(\psi^{-k}(\Lambda_{k,l})\right)}{\ell\left(\psi^{-k}(\Upsilon_{k,l})\right)} = \frac{\ell(\Lambda_{k,l})}{\ell(\Upsilon_{k,l})}.$$

In addition, writing $\alpha = \alpha(\psi)$, for $(k,l) \in \mathcal{L}$,

$$\ell(\Lambda_{k,l}) \leq \sum_{j \in I_{k,l}} \sum_{i \in J_j} \frac{\ell(\Phi^{-s'}(\widetilde{\Gamma}_{n,i}))}{\alpha^{s-k}} \leq C'_m \varepsilon \left(\frac{p}{\alpha}\right)^{s-k}.$$

Finally, recalling that a long curve satisfies $\ell(\Upsilon_{k,l}) \geq C_m$, i.e. when $(k,l) \in \mathcal{L}$,

$$\sum_{(k,l) \in \mathcal{L}} \ell\left(\psi^{-k}(\Lambda_{k,l})\right) \leq \frac{C'_m \varepsilon}{C_m} \sum_{k=0}^{s} \left(\frac{p}{\alpha}\right)^{s-k} \sum_l \ell\left(\psi^{-k}(\Upsilon_{k,l})\right).$$

Furthermore, $\sum_l \ell\left(\psi^{-k}(\Upsilon_{k,l})\right) = \ell\left(\psi^{-k}(\cup_l \Upsilon_{k,l})\right) \leq \ell(\Gamma)$.

On the other hand, similarly we obtain

$$\sum_{j \in \widetilde{I}} \sum_{i \in J_j} \ell\left(\psi^{-s}(\Phi^{-s'}(\widetilde{\Gamma}_{n,i}))\right) \leq C'_m \varepsilon \left(\frac{p}{\alpha}\right)^{s}.$$

All the above estimates imply that

$$\ell\left(\Gamma \cap \Phi^{-n}(N_\varepsilon)\right) \leq C'_m \varepsilon (p/\alpha)^s + \frac{C'_m \ell(\Gamma)}{C_m(1 - p/\alpha)} \varepsilon.$$

□

3. Piecewise expanding maps

In this section, we collect a series of results on piecewise expanding maps on the interval, which play an essential role in the proof of Theorem 1.1. The motivation here is to find sufficient conditions guaranteeing the separation of the supports of distinct ergodic acips of a piecewise expanding map. With the exception of Parts (1)-(3) of Theorem 3.3, all the other results are new.

A map $f \colon [0,1] \to [0,1]$ is called *piecewise expanding* if there exist a constant $\sigma > 1$ and intervals $I_0, \ldots, I_m$ such that

(1) $[0,1] = \bigcup_{i=0}^m I_i$ and $\operatorname{int}(I_i) \cap \operatorname{int}(I_j) = \emptyset$ for $i \neq j$,
(2) f is C^1 and $|f'| \geq \sigma$ on each I_i,
(3) $1/|f'|$ is a function of bounded variation on each I_i.

Define $f_\pm(x) = \lim_{y \to x^\pm} f(y)$ and $f'_\pm(x) = \lim_{y \to x^\pm} f'(y)$ for every $x \in [0,1]$. Let D be the set of all the discontinuities points and the turning points of f:

$$D = \{x \in [0,1] \colon f_-(x) \neq f_+(x) \text{ or } f'_-(x)\, f'_+(x) < 0\}. \tag{3.1}$$

Definition 3.1. Define the set-valued map $F(x) = \{f_-(x), f_+(x)\}$, and its iterates recursively by $F^{n+1}(x) = \bigcup_{y \in F^n(x)} F(y)$ for $n \in \mathbb{N}$. Also, define $F^n(X) = \bigcup_{x \in X} F^n(x)$ for every $X \subset [0,1]$

Definition 3.2. A sequence $\{x_n\}_{n \geq 0}$ of $[0,1]$ is called a *forward itinerary* of x_0 under f if $x_{n+1} \in F(x_n)$ for every $n \geq 0$.

Notice that $x \in [0,1]$ has more than one forward itinerary if and only if x is a point of discontinuity of f^n for some $n \in \mathbb{N}$.

Throughout the paper, we will use the standard abbreviation *acip* for an invariant probability measure of f that is absolutely continuous with respect to the Lebesgue measure of $[0,1]$. We will also write '$(\mathrm{mod}\ 0)$' to specify that an equality holds up to a set of zero Lebesgue measure.

Given a Borel measure ν, we denote by $\mathrm{supp}(\nu)$ the smallest closed sets of full ν-measure.

Theorem 3.3. *Let f be a piecewise expanding map. Then,*

(1) *there exists $1 \le k \le m$ such that f has exactly k ergodic acip's $\mu_1, \ldots, \mu_k$ with bounded variation densities,*
(2) *for every $1 \le i \le k$, there exist $k_i \in \mathbb{N}$ and an absolutely continuous probability measure ν_i with respect to the Lebesgue measure such that*
   (a)
$$\mu_i = \frac{1}{k_i} \sum_{j=0}^{k_i - 1} f_*^j \nu_i$$
   (b) $(f^{k_i}, f_*^j \nu_i)$ *is exact for all j,*
(3) $\mathrm{supp}(f_*^j \nu_i)$ *and* $\mathrm{supp}(\mu_i)$ *are both unions of finitely many intervals, for all j,*
(4) *the union of the basins of $\mu_1, \ldots, \mu_k$ is equal* $(\mathrm{mod}\ 0)$ *to* $[0,1]$,
(5) *the periodic points of f are dense in* $\mathrm{supp}(\mu_i)$.

Proof. Parts (1) and (2) are proved in [5, Theorems 7.2.1 and 8.2.1]. To prove Part (3), it suffices to prove it for $\mathrm{supp}(f_*^j \nu_i)$. The claim in this case can is demonstrated by applying [5, Theorem 8.2.2] to the acip $f_*^j \nu_i$ (by Part (2b)) of the piecewise expanding map f^{k_i}. Part (4) is proved in [20, Theorem 3.1, Corollary 3.14]. Part (5) follows from Proposition 3.11. □

By the previous theorem, each set

$$\Lambda_{i,j} := \mathrm{supp}(f_*^j \nu_i)$$

is a union of pairwise disjoint closed intervals. We call $\Lambda_{i,1}, \ldots, \Lambda_{i,k_i}$ and k_i in Theorem 3.3 the mixing[2] components and the mixing period of μ_i, respectively.

Definition 3.4. Given an ergodic acip μ of f, a sequence $(x_0, \ldots, x_n)$ is called a boundary segment of μ if

(1) $x_0 \in D \cap \mathrm{int}(\mathrm{supp}\, \mu)$,
(2) $x_i \in \partial\, \mathrm{supp}\, \mu$, for all $i = 1, \ldots, n-1$,
(3) one of the following alternatives holds

[2]In view of Theorem Part (2) of Theorem 3.3, one may be tempted to call these components exact rather than mixing. However, mixing and exactness are equivalent concepts for a piecewise expanding map [5, Corollary 7.2.1].

(a) $x_n = x_i$ for some $1 \leq i \leq n-1$,
(b) $x_n \in \operatorname{int}(\operatorname{supp}\mu)$.

In case (a) we say that the boundary segment is pre-periodic, while in case (b) we say that boundary segment is open.

Lemma 3.5. *Let μ be an acip of f. Then any $x \in \partial \operatorname{supp}\mu$ is in a boundary segment.*

Proof. A sequence $\{y_n\}_{n\leq 0}$ is called a negative orbit of $x \in [0,1]$ if $y_0 = x$, and $y_{n+1} \in F(y_n)$ for every $n < 0$. We claim that every $x \in \partial \operatorname{supp}(\mu)$ has a negative orbit $\{y_n\}_{n\leq 0}$ such that for some $m < 0$,

(1) $y_i \in \partial \operatorname{supp}\mu$ for all $m < i \leq 0$,
(2) $y_m \in D \cap \operatorname{int}(\operatorname{supp}\mu)$.

Consider now a forward itinerary $\{z_n\}_{n\geq 0}$ of x under f contained in $\operatorname{supp}\mu$. Notice that this is possible because $f(\operatorname{supp}\mu) = \operatorname{supp}\mu$ (mod 0). If $z_n \in \partial \operatorname{supp}\mu$ for all $n \geq 0$, since $\partial \operatorname{supp}\mu$ is finite, there exists $1 \leq i < n$ such that $z_n = z_i$. In this case we obtain a pre-periodic boundary segment containing x. Otherwise we obtain an open boundary segment containing x.

Finally we prove the claim. By Part (3) of Theorem 3.3, the set $J := \operatorname{int}\operatorname{supp}(\mu)$ is a union of finitely many open intervals with disjoint closures. Define $J' = J \setminus D$. By [5, Lemma 8.2.1], we have $f(J') = J$ (mod 0). The sets J and $f(J')$ are both unions of finitely many open intervals (those of J have pairwise disjoint closures). Thus, $J \setminus f(J')$ consists of finitely many points. This implies that for every $x \in \partial J$, there exists $y \in (D \cap J) \cup \partial J$ such that $x \in F(y)$. As a consequence, if $\{y_n\}_{n\leq 0}$ is a negative orbit of x, then either $y_n \in D \cap J$ for some $n < 0$, or $y_n \in \partial J$ for every $n < 0$. Let us call regular the negative orbits of the later type.

We show that x can not have only regular negative orbits, thus proving the claim. Suppose instead that all negative orbits of x are regular. Denote by g the continuous extension of $f|_J$ to $\operatorname{supp}\mu$. Since y_n is a continuity point of g for every $n < 0$, the condition $y_{n+1} \in F(y_n)$ is equivalent to $y_{n+1} = g(y_n) \in \partial J$. This combined with finiteness of ∂J implies that $\{y_n\}_{n\leq 0}$ is periodic, i.e., there exists $m \in \mathbb{N}$ such that $y_{n-m} = y_n$ for every $n \leq 0$. In particular, x is a periodic point of g, and has a unique negative orbit such that $y_n \in \partial J$ for every $n < 0$. By this last observation and the fact that f is expanding, we can find an open interval $I \subset J$ having x as an endpoint such that $J \cap f^{-m}(I)$ is an interval contained in I. Since μ is an acip, we obtain $\mu(I) = \mu(f^{-m}(I)) < \mu(I)$, a contradiction. Hence, x must have a negative orbit $\{y_n\}_{n\leq 0}$ which is not regular.

□

Lemma 3.6. *Suppose that μ_1 and μ_2 are two distinct ergodic acip's of f. If $x \in \partial \operatorname{supp}(\mu_1) \cap \partial \operatorname{supp}(\mu_2)$, then $f^n(x) \in D$ for some $n \geq 0$, or x is pre-periodic.*

Proof. Let μ be an acip of f, and let $J = \operatorname{int}\operatorname{supp}(\mu)$. As explained in the proof of Lemma 3.5, we have $f(J \setminus D) = J$ up to a set of finitely many points contained in J. It follows that if $x \in \partial\operatorname{supp}(\mu) \setminus D$, then $f(x) \in \partial\operatorname{supp}(\mu)$.

Now, let x be as in the hypothesis of the lemma. From the previous conclusion, it follows that $f^n(x) \in D$ for some $n \geq 0$, or $f^n(x) \in \partial\operatorname{supp}(\mu_1) \cap \partial\operatorname{supp}(\mu_2)$ for every $n \geq 0$. Since $\partial\operatorname{supp}(\mu_1) \cap \partial\operatorname{supp}(\mu_2)$ is finite, x must be pre-periodic in the second case. This completes the proof. □

Corollary 3.7. *Suppose that $D \cap F^n(D) = \emptyset$ for every $n \in \mathbb{N}$, and no point of D is pre-periodic. Then*

1. $\Lambda_{i,j} \cap \Lambda_{i',j'} = \emptyset$ *whenever* $i \neq i'$ *or* $j \neq j'$,
2. *if μ is an acip of f, then $\partial\operatorname{supp}(\mu)$ does not contain pre-periodic points of f.*

Proof. To prove the first claim, apply Lemmas 3.5 and 3.6 to the ergodic acip's $f_*^j \nu_i$ and $f_*^{j'} \nu_{i'}$ of $f^{k_i k_{i'}}$. To prove the second claim, notice that if $x \in \operatorname{supp}(\mu)$ is pre-periodic, then by Lemma 3.5 there would exist a pre-periodic point in D, contradicting the hypotheses of the corollary. □

Lemma 3.8. *Let μ be an acip of f that satisfies that hypotheses of Corollary 3.7. If $x \in \partial\operatorname{supp}(\mu)$, then x is contained in an open boundary segment and $x \notin D$.*

Proof. The first part is an immediate consequence of Lemma 3.5 because no point in D can be pre-periodic. For the second part, $x \notin D$, notice that if $x \in D \cap \partial\operatorname{supp}(\mu)$, by Lemma 3.5 there is a boundary segment through x, which contradicts the assumption that $D \cap F^n(D) = \emptyset$ for all $n \in \mathbb{N}$. □

Recall that $[0,1] = \bigcup_j^m I_j$ for a piecewise expanding map f. Define $\ell(f) = \min_j |I_j|$, where $|I_j|$ denotes the length of I_j.

Lemma 3.9. *Let f be a piecewise expanding map with least expansion coefficient $\sigma > 2$. Then for every interval $I \subset [0,1]$, there exist $i \in \mathbb{N}$ and an open subinterval $W \subset I$ such that*

1. $f^i|_W : W \to I_j$ *is a diffeomorphism, for some* $1 \leq j \leq m$,
2. $f^{i+1}(W)$ *is an open interval and* $|f^{i+1}(W)| \geq \sigma \ell(f)$.

Proof. Part (1). Let $B = \bigcup_{j=1}^m \partial I_j$. We claim that given any interval $I \subset [0,1]$, there exist $i \in \mathbb{N}$ and a subinterval $W = (a,b) \subset I$ with $a, b \in f^{-i}(B)$ such that

$$W \cap f^{-k}(B) = \emptyset \qquad \text{for } 0 \leq k \leq i.$$

Indeed, if this was not the case, then for every $i \in \mathbb{N}$, no two consecutive points of $I \cap (B \cup f^{-1}(B) \cup \cdots \cup f^{-i}(B))$ would belong to $f^{-i}(B)$. It is not difficult to see that this would imply that $f^i(I)$ consists of at most 2^i intervals. But $\sigma > 2$, and so the length of one of these intervals would be not less than $(\sigma/2)^i \to +\infty$, as $i \to +\infty$, giving a contradiction. By the definition of B, we have $f^i(W) = \operatorname{int} I_j$ for some j.

Part (2). From Part (1), it follows that $f^{i+1}(W) = f(\operatorname{int} I_j)$ is an open interval, and so

$$|f^{i+1}(W)| = |f(I_j)| \geq \sigma |I_j| \geq \sigma \ell(f).$$

□

Corollary 3.10. *Let f be a piecewise expanding map. Then there exists a constant $\eta = \eta(f) > 0$ such that for every interval $I \subset [0,1]$, there exists $n \in \mathbb{N}$ for which $f^n(I)$ contains an open interval of length greater than 2η.*

Proof. Apply Lemma 3.9 to f^k with $k > 0$ being the smallest integer such that the least expansion of f^k is greater than 2. Then $n = ik$, where i is as in Lemma 3.9 (applied to f^k), and $\eta(f) = \ell(f^k)$. □

In the next proposition, we prove that the periodic points of a piecewise expanding map f are dense in the support of every acip of f. This results is probably known to experts, but we did not find any reference to it in the literature.

Proposition 3.11. *Let f be a piecewise expanding map of $[0,1]$, and suppose that ν is an acip of f. Then the periodic points of f are dense in $\operatorname{supp}(\nu)$.*

Proof. By Theorem 3.3, it suffices to prove the proposition for every exact component of (f,ν). So we can assume without loss of generality that (f,ν) is exact. Moreover, since f^k is also exact for every $k \in \mathbb{N}$, we can further assume without loss of generality that the least expansion σ of f is greater than 2.

Recall that $[0,1] = \bigcup_{j=1}^m I_j$, where I_j is an interval and $f|_{\operatorname{int} I_j}$ is diffeomorphism. Let $I'_j = \operatorname{int} I_j \cap \operatorname{supp}(\nu)$ for every $1 \leq j \leq m$. Notice that some set I'_j might be empty.

To obtain the wanted conclusion, we show that every open interval $U \subset \operatorname{supp}(\nu)$ contains a periodic point of f. By the ergodicity of f, for every j, there exist two subintervals $J_j \subset I'_j$ and $V_j \subset U$ and a constant $l_j \in \mathbb{N}$ such that $f^{l_j} \colon J_j \to V_j$ is a diffeomorphism. By Lemma 3.9, for every j, there exist a subinterval $W_j \subset V_j$ and two constants $n_j \in \mathbb{N}$ and $1 \leq \sigma(j) \leq m$ such that $f^{l_j} \colon W_j \to \operatorname{int} I_{\sigma(j)}$ is a diffeomorphism.

In particular, the previous claim implies that $I_{\sigma(j)} \subset \operatorname{supp}(\nu)$, and so $I'_{\sigma(j)} = \operatorname{int} I_{\sigma(j)}$. Putting all together, we conclude that for every j, there exists a subinterval $K_j \subset J_j \subset I'_j$ such that $f^{l_j+n_j} \colon K_j \to \operatorname{int} I_{\sigma(j)}$ is a diffeomorphism.

The correspondence $j \mapsto \sigma(j)$ defines a function on $\{1,\dots,m\}$, and we immediately see that $\sigma^k(j) = j$ for some j and some $k \in \mathbb{N}$. Hence, there exists a subinterval $K'_j \subset K_j$ and $\bar{n} \in \mathbb{N}$ such that $f^{\bar{n}} \colon K'_j \to \operatorname{int} I_j$ is a diffeomorphism. Since this transformation is expanding, using the

Banach Fixed Point Theorem, we conclude that there exists $x \in K'_j \subset I'_j$ such that $f^{\bar{n}}(x) = x$. It follows that $f^{l_j}(x) \in U$ is a periodic point of f. □

4. Polygonal slap maps

Consider $P \in \mathcal{P}_d$, and let $s \in [0,1]$ be the arclength parameter of ∂P computed with respect to a fixed vertex of P. Denote by $\widehat{V} = \{s_0, s_1, \ldots, s_d\}$ the set of vertices of P. We say that a vertex $v \in \widehat{V}$ is acute (resp. obtuse, right) if the interior angle formed by the sides meeting at v is acute (resp. obtuse, right).

Given a point $s \notin \widehat{V}$, consider the line orthogonal to ∂P at s, and intersect it with P. The endpoints of the resulting segment are $s, s' \in [0,1)$. The *slap map of* P is the map $\psi_P \colon [0,1] \setminus \widehat{V} \to [0,1]$ given by $\psi_P(s) = s'$. Since the values of ψ_P at $\widehat{V}$ are not relevant, we may consider a left continuous extension to $[0,1]$, also denoted by ψ_P.

The map ψ_P is piecewise linear and decreasing. If we identify the endpoints of the interval $[0,1]$, to get the unit circle S^1, the slap map ψ_P, seen as a circle map, has at most d discontinuity points, corresponding to the non-acute vertices of P. Notice also that the acute vertices of P are fixed points of ψ_P.

Definition 4.1. We say that $P \in \mathcal{P}_d$ has *no parallel sides facing each other* if the endpoints of every straight segment contained inside P and joining orthogonally two sides of P are vertices of P.

Denote by $\mathcal{P}'_d$ the subset of all polygons of $\mathcal{P}_d$ without parallel sides facing each other. It is not difficult to see that $P \in \mathcal{P}'_d$ iff ψ_P is piecewise expanding. Furthermore, $P \notin \mathcal{P}'_d$ if and only if ψ_P has periodic points of period 2 that are not vertices of P.

Definition 4.2. We say that $P \in \mathcal{P}_d$ has an *orthogonal vertex connection* if ψ_P has a forward itinerary containing two vertices of P, possibly the same, one being a non-acute vertex. The number of elements forming such an itinerary is called the length of the orthogonal connection.

Since the existence of a right vertex gives always rise to an orthogonal vertex connection, a given vertex of a polygon without orthogonal vertex connections can only be either acute or obtuse. It follows that if a polygon does not have orthogonal vertex connections, then one can find a neighborhood $\mathcal{V}$ of P in $\mathcal{P}_d$ such that $P' \in \mathcal{V}$ has exactly the same number of obtuse vertices as P.

Definition 4.3. We say that $P \in \mathcal{P}_d$ has a pre-periodic vertex if a non-acute vertex of P has a pre-periodic forward itinerary.

Denote by $\widehat{\mathcal{P}}_d$ the subset of all polygons in $\mathcal{P}'_d$ without orthogonal vertex connections and pre-periodic vertices.

We observe that the condition $P \in \widehat{\mathcal{P}}_d$ is just a reformulation of the hypothesis of Corollary 3.7 for polygonal slap maps. It follows that all the results proved in Section 3 hold for ψ_P when $P \in \widehat{\mathcal{P}}_d$.

Since Theorem 3.3 (1) holds for expanding circle maps as well (see [20, Theorem 3.1]), and the number of discontinuities of ψ_P, seen as a circle map, is at most d, we have the following result.

Proposition 4.4. *If $P \in \widehat{\mathcal{P}}_d$, then*

(1) *ψ_P has at most d ergodic acips,*
(2) *each mixing component of ψ_P is a union of finitely many closed intervals,*
(3) *two distinct mixing components of ψ_P are disjoint,*
(4) *the periodic points of ψ_P are dense in the support of every acip.*

Recall that the space $\mathcal{P}_d$ carries a natural topology and measure inherited from the Euclidean space, see [8, Section 5.1]. Clearly, $\mathcal{P}'_d$ is residual and full measure in $\mathcal{P}_d$.

Lemma 4.5. *$\widehat{\mathcal{P}}_d$ is a residual and full measure subset of $\mathcal{P}_d$.*

Proof. A simple adaptation of the proof of [8, Proposition 5.3]. □

5. Strongly contracting reflections laws

Theorem 5.1. *Let $P \in \mathcal{P}'_d$ with no orthogonal vertex connections of length less or equal than n and*

$$n > \frac{\log 2}{\log \alpha(\Phi_{0,P})}.$$

Then there is $\lambda_0 > 0$ such that if $\lambda(f) < \lambda_0$ and $f \in \mathcal{B}$, $\Phi_{f,P}$ is hyperbolic and has finitely many ergodic SRB measures.

Proof. The proof is similar to the one of [8, Theorem 4.10] with [8, Theorem 4.4] replaced by Theorem 2.7. □

Observe that if $\alpha(\Phi_{0,P}) > c$, then $\alpha(\Phi_{0,P'}) > c$ for any polygon P' sufficiently close to P.

Proposition 5.2. *Let $P \in \mathcal{P}'{}_d$ without orthogonal vertex connections. There exist a neighborhood $\mathcal{V}$ of P in $\mathcal{P}_d$ and $\lambda_0 > 0$ such that for every $P' \in \mathcal{V}$ and every $f \in \mathcal{B}$ with $\lambda(f) < \lambda_0$, the map $\Phi_{f,P'}$ has finitely many ergodic SRB measures.*

Proof. If $\lambda(f)$ is sufficiently small, then $\alpha(\Phi_{f,P}) > c$. Notice that the branching number of S_1^+ is constant in a small neighborhood of P. For $\lambda(f)$ small, we choose

$$n \geq \frac{\log 2}{\log c} > \frac{\log p(S_1^+)}{\log \alpha(\Phi_{f,P})}.$$

Finally, we know that the set of polygons with no parallel sides facing each other and with no orthogonal vertex connections of length less or equal than n is open ([8, Corollary 5.4]). To obtain the wanted conclusion, we apply Theorem 5.1 to P' in a neighborhood of P and $f \in \mathcal{B}$ with small $\lambda(f)$. □

Lemma 5.3. *Given $P \in \widehat{\mathcal{P}}_d$, there exist $\lambda_0, \eta > 0$ and a neighborhood $\mathcal{V}$ of P in $\mathcal{P}_d$ such that if $P' \in \mathcal{V}$, $f \in \mathcal{B}$ with $\lambda(f) < \lambda_0$ and Γ is a horizontal segment, then we can find a segment $\Gamma' \subset \Gamma$ and $k \geq 1$ such that $\Phi^k_{f,P'}(\Gamma')$ is a segment, and $\ell(\Phi^k_{f,P'}(\Gamma')) \geq \eta$, where ℓ is the length of a curve.*

Proof. Choose $n \in \mathbb{N}$ large enough so that $\alpha(\Phi^n_{f,P'}) \geq ad$ with $a > 1$, for every P' and f in neighborhoods of P and 0, respectively. Notice that $p(S_1^+) = 2$, and [8, Corollary 5.4] yields that the set of polygons with no parallel sides facing each other and with no orthogonal vertex connections of length less or equal than n is open. In particular, $\Phi^n_{0,P'}(S_1^+) \cap S_1^+ = \emptyset$ for any P' close to P. By [8, Proposition 4.8]

$$p(S_n^+(\Phi^n_{f,P'})) = p(S_1^+) = 2 \leq a^{-1}\alpha(\Phi^n_{f,P'})$$

for $\lambda(f)$ small. Thus, there exists a uniform lower bound

$$\frac{\alpha(\Phi^n_{f,P'})}{p(S_n^+)} \geq a > 1.$$

Any horizontal segment $\Gamma \subset M$ such that $\#(\Gamma \cap S_n^+) \geq p(S_n^+)$ will be called *long curve*. All other horizontal segments $\Gamma \subset M$ will be referred as *short curves*. Define $\eta > 0$ to be the minimum length of a long curve, and consider the following procedure.

We start with any horizontal curve Γ_0 contained in one of the connected components of the domain of $\Phi^n_{f,P'}$. At step $j \geq 0$ iterate the given curve Γ_j by $\Phi^n_{f,P'}$. If $\Phi^n_{f,P'}(\Gamma_j)$ is long we have proven the lemma. If not, split $\Phi^n_{f,P'}(\Gamma_j)$ in at most $p(S_n^+)$ pieces (each contained in a different component of the domain of $\Phi^n_{f,P'}$) and pick Γ_{j+1} as the longest one. Then,

$$\ell(\Gamma_{j+1}) \geq \frac{\ell(\Phi^n_{f,P'}(\Gamma_j))}{p(S_n^+)} \geq \frac{\alpha(\Phi^n_{f,P'})\,\ell(\Gamma_j)}{p(S_n^+)} \geq a\,\ell(\Gamma_j)\ ,$$

which proves that the length of Γ_j grows geometrically. Hence, after a finite number of iterates we must obtain a long curve, whose length is at least η. □

Lemma 5.4. *Given $P \in \mathcal{P}'_d$ and a periodic point x of $\Phi_{0,P}$, there are a constant $\lambda_0 > 0$, a neighborhood $\mathcal{V} \subset \mathcal{P}_d$ of P and a continuous map $(f, P') \mapsto x'_{f,P'}$ on $\{f \in \{0\} \cup \mathcal{B} \colon \lambda(f) < \lambda_0\} \times \mathcal{V}$ such that*

(1) *$x'_{0,P} = x$, and $x'_{f,P'}$ is a hyperbolic periodic point of $\Phi_{f,P'}$,*
(2) *the stable direction at $x'_{f,P'}$ is uniformly transversal to the horizontal direction in (f, P').*

Proof. Let $k > 0$ be the period of the point x. Consider the map $F_k \colon C^2([-\pi/2, \pi/2], \mathbb{R}) \times \mathcal{P}_d \times M \to \mathbb{R}^2$ defined by $F_k(f, P, s, \theta) = \Phi^k_{f,P}(s, \theta) - (s, \theta)$. This map is C^1, and since $P \in \mathcal{P}'_d$, the point x is hyperbolic, which in turn implies that the matrix $D_{(s,\theta)}F_k(f, P, x) = D\Phi^k_{f,P}(x) - I$ is invertible.

Conclusion (1) follows by applying the implicit function theorem (Banach version) to F_k. The proof of [8, Proposition 3.1] yields the necessary computations to obtain Conclusion (2). □

The points x and $x'_{f,P'}$ have the same period, and visit the same sequence of sides in the same order.

Definition 5.5. The periodic point $x'_{f,P'}$ is called the *continuation* of x (for $\Phi_{f,P'}$).

6. Ergodic Properties

In this section, we prove Theorem 6.1. Theorem 1.1 presented in Section 1 follows directly from Theorem 6.1 and Lemma 4.5.

Let $\mathcal{E}(\Phi_{f,P})$ denote the set of ergodic SRB measures of $\Phi_{f,P}$ and $\mathcal{E}(\psi_P)$ the set of ergodic acip's of the slap map of P.

Theorem 6.1. *For every polygon $P \in \widehat{\mathcal{P}}_d$ there exists $\lambda_0 > 0$ and a neighborhood $\mathcal{V}$ of P in $\mathcal{P}_d$ such that for every $f \in \mathcal{B}$ with $\lambda(f) < \lambda_0$ and every $P' \in \mathcal{V}$, there is a bijection*

$$\Theta \colon \mathcal{E}(\psi_P) \to \mathcal{E}(\Phi_{f,P'})$$

such that μ and $\Theta(\mu)$ have the same mixing period for all $\mu \in \mathcal{E}(\psi_P)$.

Corollary 6.2. *The number of ergodic SRB measures of $\Phi_{f,P'}$ is equal to the number of ergodic acip's of ψ_P. In particular, $\#\mathcal{E}(\Phi_{f,P'}) \leq d$.*

Proof. The statement is immediate from the previous theorem. By item 1 of Theorem 3.3 we know that the number of ergodic acip's is bounded by the number of intervals where the slap map is piecewise expanding, which is bounded from above by d. □

Theorem 6.1 is proved in Section 6.2.

6.1. **Trapping regions.** For each $\mu \in \mathcal{E}(\psi_P)$, we will construct a forward invariant set $R(\mu)$ for $\Phi_{f,P}$ containing $\operatorname{supp}(\mu) \times \{0\}$ so that $R(\mu_1)$ and $R(\mu_2)$ are disjoint for any two distinct $\mu_1, \mu_2 \in \mathcal{E}(\psi_P)$.

If $\operatorname{supp}(\mu) = [0, 1]$, then we choose $R(\mu) = M$. So, let us consider the case $\operatorname{supp}(\mu) \neq [0, 1]$.

Recall that the slap map ψ_P of a polygon P without parallel sides facing each other is a piecewise linear expanding map. Let a be the maximum of $|\psi'_P|$.

By Lemma 3.8 each point $z \in \partial \operatorname{supp} \mu$ belongs to an open boundary segment (see definition 3.4) $(z_0, \ldots, z_m)$ with $z_k = z$ for some $k = 1, \ldots, m-1$. The order of the boundary point z is defined to the number

$\kappa(z) = k$. If there is more than one boundary segment through z, we take the order to be the largest possible. Notice that $\kappa(z_i) \geq \kappa(z_{i-1})+1$ for all $i = 1, \dots, m$ and any boundary segment $(z_0, \dots, z_m)$. Moreover, again by Lemma 3.8, $\{z_1, \dots, z_{m-1}\} \subset \partial \operatorname{supp} \mu \setminus D$.

By Part (3) of Theorem 3.3, we have $\operatorname{supp}(\mu) = \bigcup_j A_j$, where $A_j = [\alpha_j, \beta_j]$ are pairwise disjoint closed intervals. Let $z \in \partial A_j$.

We want to cover the intervals A_j by larger disjoint intervals B_j. Let $B_j = [\alpha'_j, \beta'_j]$ where

$$\alpha'_j = \begin{cases} 0 & \text{if } \alpha_j = 0 \\ \alpha_j - (2a)^{\kappa(\alpha_j)}\varepsilon & \text{otherwise} \end{cases}$$

and

$$\beta'_j = \begin{cases} 1 & \text{if } \beta_j = 1 \\ \beta_j + (2a)^{\kappa(\beta_j)}\varepsilon & \text{otherwise} \end{cases} .$$

The fixed constant $\varepsilon > 0$ is chosen such that all the B_j's are disjoint and $\partial B_j \times \{0\} \cap S_1^+ = \emptyset$. By Corollary 3.7, we can also assume that

$$\overline{B_j \setminus A_j} \cap \widehat{V} = \emptyset.$$

Given $\delta > 0$, a curve $\Gamma \subset M$ is *δ-long* if it is the graph of a function $s = h(\theta)$ whose domain contains the interval $(-\delta, \delta)$. That is, the curve crosses the strip $\{|\theta| < \delta\}$.

Recall that Φ_P is the billiard map with standard reflection law for the polygon P, and write $\Phi_P(s, \theta) = (s_1(s, \theta), \theta_1(s, \theta))$.

Suppose that $s \in [0, 1]$ is not a discontinuity point of ψ_P. Let I be the open subinterval of $[0, 1]$ corresponding to the side of P containing s. Define

$$\gamma_s = \left\{(\bar{s}, \bar{\theta}) \in M \colon \bar{s} \in I \text{ and } s_1(\bar{s}, \bar{\theta}) = \psi_P(s)\right\}.$$

Lemma 6.3. *The set γ_s is a curve with the following properties:*

1. *$\partial\gamma_s \subset V$ (the endpoints of γ_s lie on V),*
2. *γ_s is strictly decreasing,*
3. *γ_s is δ-long with $\delta \geq C d(s, \partial I)$, where C is a constant depending only on P.*

Proof. If h_s denotes the length of the segment of the plane connecting the points s and $\psi_P(s)$, then we immediately see that γ_s is the graph of the function given by $\bar{\theta}(\bar{s}) = \arctan((s-\bar{s})/h_s)$ for $\bar{s} \in I$. This function is analytic and strictly decreasing. Let $s_1 < s_2$ be the endpoints of I. Clearly, $\bar{\theta}(s_2) < 0 < \bar{\theta}(s_1)$, and $\delta = \min\{\bar{\theta}(s_1), -\bar{\theta}(s_2)\} \geq C \cdot d(s, \partial I)$, where C is a constant depending only on P. □

Let

$$\delta = C \min_j d(\partial B_j, \widehat{V} \cap (0, 1)),$$

where C is the constant in Lemma 6.3. By that lemma, we can construct curves γ_y for the endpoints y of B_j that are interior to $[0, 1]$. If any endpoint y of B_j is either 0 or 1, then we define γ_y to be a vertical segment $\{y\} \times [-\delta, \delta]$. In this way we obtain a curvilinear rectangle R_j in M containing $B_j \times \{0\}$ bounded by the curves γ_y and inside the invariant horizontal strip $\{|\theta| \leq \delta\}$ for $\lambda(f) \leq 2\delta/\pi$. In particular, $A_j \times \{0\} \subset R_j$.

Define the union of the curvilinear rectangles by

$$R(\mu) = \bigcup_j R_j \setminus (V \cup S_1^+). \tag{6.1}$$

Lemma 6.4. *Given $P \in \widehat{\mathcal{P}}_d$ and $\mu \in \mathcal{E}(\psi_P)$, there exist a neighborhood $\mathcal{V}$ of P and $\lambda_0 > 0$ such that for any $P' \in \mathcal{V}$, any $f \in \mathcal{B}$ with $\lambda(f) < \lambda_0$, the set $R(\mu)$ is forward invariant under $\Phi_{f,P'}$.*

Proof. By Lemma 3.8, it follows that $\psi_P(\partial \operatorname{supp}(\mu)) \subset \operatorname{supp}(\mu)$. Moreover, by Lemma 6.3, the vertical boundary of ∂R_j are mapped by $\Phi_{f,P}$ to vertical segments. On the other hand, horizontal segments in ∂R_j are mapped by $\Phi_{f,P}$ into horizontal lines.

Consider $z \in \partial A_j$ and the nearby curve γ at the endpoint of the corresponding B_j such that their distance $d(z, \gamma)$ is around $(2a)^i \varepsilon$. When we apply $\Phi_{f,P'}$ to γ, by continuity (as we have previously chosen ε small enough such that there are no discontinuity points in $\overline{B_j \setminus A_j}$) we obtain a curve whose distance from $\psi_P(z)$ is less than $a'\, d(z, \gamma)$, where a' is the largest expansion rate of $\Phi_{f,P'}$. We take f sufficiently small and P' in a small neighborhood of P such that $a'\, d(z, \gamma) < (2a)^{i+1}\varepsilon$. Thus, by taking a sufficiently small λ_0, a curvilinear rectangle R_j that is not intersecting V is mapped by $\Phi_{f,P'}$ inside some other $R_{j'}$.

The case when R_j intersects S_1^+ is the same as the previous one with the particularity that $\Phi_{f,P'}(R_j)$ is a union of a finite number of curvilinear rectangles.

It remains to consider the case that $\operatorname{int}(R_j)$ intersects V. In this case the set $\Phi_{f,P'}(R_j)$ is no longer connected. Take one connected component of $R_j \setminus V$ and z is the point in $R_j \cap V \cap \{\theta = 0\}$. In order to see that its image is inside $R(\mu)$ we just need to show that $d(\Phi_{0,P}(z), \Phi_{f,P'}(V')) < (2a)^i \varepsilon$. This holds as long as the maps $\Phi_{0,P}$ and $\Phi_{f,P'}$ are sufficiently close (because the vertices V' of P' are close to the vertices V of P). □

The construction of $R(\mu)$ depends on ε. We can further choose a smaller ε such that

$$R(\mu_1) \cap R(\mu_2) = \emptyset \tag{6.2}$$

for any two distinct $\mu_1, \mu_2 \in \mathcal{E}(\psi_P)$.

6.2. **Proof of Theorem 6.1.** Let x be a periodic point of the slap map ψ_P and $x'_{f,P'}$ its periodic point continuation for $\Phi_{f,P'}$ (see Definition 5.5) for $\lambda(f)$ sufficiently small and P' sufficiently close to P. We recall that the index $\bar{\omega} = \bar{\omega}(\Phi_{f',P})$ of $D^-_{\bar{\omega}}$ is defined after Condition H4 in the appendix.

Given $\mu \in \mathcal{E}(\psi_P)$, let F be a finite set of periodic points of ψ_P contained in the interior of $\operatorname{supp}(\mu)$ and being η-dense in $\operatorname{supp}(\mu)$, where η is as in Lemma 5.3. The set of the continuation of all the periodic points in F is denoted by F'. Also, given $\mu' \in \mathcal{E}(\Phi_{f,P'})$ denote by $E(\mu')$ be the full μ'-measure set corresponding to μ' as in Theorem A.5.

Definition 6.5. We say that μ' is related to μ if there is $y \in E(\mu') \cap D^-_{\bar{\omega}}$ and $n \geq 0$ such that $W^s_{loc}(F') \cap \Phi^n_{f,P'}(W^u_{loc}(y)) \neq \emptyset$. We also write

$$\mathcal{H}(\mu) = \{\mu' \in \mathcal{E}(\Phi_{f,P'})\colon \mu' \text{ is related to } \mu\}.$$

Remark 6.6. From Lemmas 6.7-6.10 below it follows that the above definition is equivalent to the condition $\operatorname{supp}(\mu') \subset R(\mu)$.

Let $P \in \widehat{\mathcal{P}}_d$, $\lambda_0 > 0$, $f \in \mathcal{B}$ with $\lambda(f) < \lambda_0$, $\mathcal{V}$ a neighbourhood of P in $\mathcal{P}_d$ and $P' \in \mathcal{V}$. Throughout the proof of Theorem 6.1, λ_0 and $\mathcal{V}$ are taken to be sufficiently small so that the results of previous sections apply. We will point out in the course of the proof when we need further restrictions on λ_0 and $\mathcal{V}$ is needed.

Notice that, by using Lemma 5.3 and the ergodicity of μ, for every $x_1, x_2 \in F$ with neighborhoods U_1, U_2 in $\operatorname{supp}(\mu)$, there exist $n_1, n_2 \in \mathbb{N}$ and $x_0 \in F$ such that

$$x_0 \in \psi_P^{n_1}(U_1) \cap \psi_P^{n_2}(U_2). \tag{6.3}$$

Lemma 6.7. *For every $\mu \in \mathcal{E}(\psi_P)$, $\#\mathcal{H}(\mu) \leq 1$.*

Proof. We want to show that if μ'_1 and μ'_2 are in $\mathcal{E}(\Phi_{f,P'})$ and both related to μ (for the same F as we have fixed it), then they have to be the same. This follows from proving that $E(\mu'_1) = E(\mu'_2)$.

So, there are $y_i \in E(\mu'_i) \cap D^-_{\bar{\omega}}$ and $n_i \geq 0$ such that

$$W^s_{loc}(x'_i) \cap \Phi^{n_i}_{f,P'}(W^u_{loc}(y_i)) \neq \emptyset \tag{6.4}$$

for some $x'_i \in F'$, $i = 1, 2$.

First we extend (6.3) to F' for sufficiently small f and P' sufficiently close to P. This follows from the fact that the stable manifolds of points of F' are uniformly long because of Lemma 5.4. Therefore, there exist $m_i \in \mathbb{N}$ and $x'_0 \in F'$ such that

$$\Phi^{m_i}_{f,P'}(W^u_{loc}(x'_i)) \cap W^s_{loc}(x'_0) \neq \emptyset, \quad i = 1, 2. \tag{6.5}$$

Next, by (6.4) and Theorem A.8, it follows that $x'_i \in E(\mu'_i)$. Now, from (6.5), we obtain $x'_0 \in E(\mu'_1) \cap E(\mu'_2)$ using again Theorem A.8. Since distinct ergodic components are disjoint, we conclude that $E(\mu'_1) = E(\mu'_2)$. □

Lemma 6.8. *For every $\mu \in \mathcal{E}(\psi_P)$, $\#\mathcal{H}(\mu) = 1$.*

Proof. Write $R = R(\mu)$. Since R is forward invariant, if λ_0 and $\mathcal{V}$ are sufficiently small, then conclusion (2) of Theorem A.7 applies to the periodic point $x_i' \in R \cap F'$, and implies that there exists $\mu' \in \mathcal{E}(\Phi_{f,P'})$ such that $\operatorname{supp}(\mu') \subset R$.

By continuity, the stable manifold of F' is η-dense in $\operatorname{supp}(\mu)$ and uniformly transversal to the horizontal direction for sufficiently small λ_0 and $\mathcal{V}$. Take $y \in E(\mu') \cap D^-_{\bar{\omega}}$. By Lemma 5.3 some forward iterate of $W^u_{loc}(y)$ will intersect one of the stable manifolds of the periodic points F'. Hence μ' is related to μ. □

Lemmas 6.7 and 6.8 show that we have a well-defined map

$$\Theta\colon \mathcal{E}(\psi_P) \to \mathcal{E}(\Phi_{f,P'}), \qquad \mu \mapsto \mu',$$

where μ' is related to μ. In the following two lemmas we show that Θ is a bijection as stated in Theorem 6.1.

Lemma 6.9. Θ *is one-to-one.*

Proof. Let μ_1 and μ_2 be two ergodic acip's of ψ_P such that $\mu' = \Theta(\mu_1) = \Theta(\mu_2)$. Since $\operatorname{supp}(\mu') \subset R(\mu_1) \cap R(\mu_2)$ is non-empty, we have $\mu_1 = \mu_2$ as argued in (6.2). □

Lemma 6.10. Θ *is onto.*

Proof. By Part (3) of Theorem 3.3, the union B of the basins of the elements of $\mathcal{E}(\psi_P)$ coincides with $\{\theta = 0\}$ up to a zero Lebesgue measure set. Let $\eta > 0$ be the constant in Lemma 5.3. We claim that ψ_P has the following property denoted by (*): there exists an $\eta/2$-dense set $\{z_1, \ldots, z_r\} \subset B$ of $[0,1]$ such that for every z_i, we can find $k \in \mathbb{N}$ and $\mu \in \mathcal{E}(\psi_P)$ for which $\psi_P^k(z_i) \in \operatorname{int}\operatorname{supp}(\mu)$. Indeed, let $\mu \in \mathcal{E}(\psi_P)$, and pick $y \in \operatorname{int}\operatorname{supp}(\mu)$. There exists $\alpha > 0$ such that $I := (y-\alpha, y+\alpha) \subset \operatorname{int}\operatorname{supp}(\mu)$. Let φ be the characteristic function of I. The set of the discontinuities of φ is given by $\Delta(f) = \{y-\alpha, y+\alpha\}$. Since $\mu(\Delta(f)) = 0$, it follows that $n^{-1}\sum_{k=0}^{n-1}\varphi(\psi_P^k(x)) \to \mu(I) > 0$, as $n \to +\infty$ for every $x \in B(\mu)$.

Now, let $\mu' \in \mathcal{E}(\Phi_{f,P'})$ and take $y \in E(\mu') \cap D^-_\epsilon$ for $\Phi_{f,P'}$. By Lemma 5.3, there is $m \in \mathbb{N}$ such that $\Phi^m_{f,P'}(W^u_{loc}(y))$ contains an interval of length greater than or equal to η. The vertical projection of this interval onto $\{\theta = 0\}$ contains in its interior a point z_i.

By continuity, if λ_0 and $\mathcal{V}$ are sufficiently small, then the intersection $\Phi^{m+t_i}_{f,P'}(W^u_{loc}(y)) \cap R(\mu_{j_i})$ contains an interval. Now, arguing as in the second part of the proof of Lemma 6.8, we obtain $\mu' \in \mathcal{H}(\mu)$, which in turn implies $\Theta(\mu_{j_i}) = \mu'$. □

6.2.1. *Mixing components.* To complete the proof of Theorem 6.1, it remains to show that ψ_P and $\Phi_{f,P'}$ have the same number of mixing components.

Consider an ergodic acip μ for ψ_P. As above, let $R(\mu)$ be the union of the rectangles of R_j containing $\Lambda_j \times \{0\}$, where Λ_j are the mixing components of ψ_P. Let μ' be the unique ergodic SRB measure for $\Phi_{f,P'}$ such that $\operatorname{supp}(\mu') \subset R(\mu)$, and assume that μ and μ' have mixing periods k and k', respectively.

We first notice that k divides k' because $\Phi^k_{f,P'}(R_j) \subset R_j$ for all $j = 0, \dots, k-1$. This follows as in the proof of Lemma 6.4.

To prove that $k = k'$ we will assume without loss of generality that $k = 1$. For the general case we can replace $\Phi_{f,P'}$ by $\Phi^k_{f,P'}$, resp. ψ_P by ψ^k_P. Hence, the slap map has a unique mixing component Λ which is a finite union of intervals. Denote by D' the set of discontinuities of ψ_P in $\operatorname{int}\Lambda$, and let $\mathcal{I}$ be the collection of connected components of $\Lambda \setminus D'$. By Lemma 3.9 applied to the restriction of ψ_P to Λ, for any arbitrarily small open interval J there exists $n \geq 1$ such that $\psi^n_P(J)$ contains an interval in $\mathcal{I}$.

For each $I \in \mathcal{I}$, consider the open set $\Omega(I) := \bigcup_{n\geq 1} \psi^n_P(I)$.

Lemma 6.11. *The set $\Omega(I)$ is a finite union of intervals.*

Proof. For each $I \in \mathcal{I}$, define $\Omega_n(I) := \bigcup_{j=0}^n \psi^i_P(I)$. These sets can be defined by the recursive relation $\Omega_{n+1}(I) = \Omega_n(I) \cup \psi_P(\Omega_n(I) \setminus D)$. Of course each $\Omega_n(I)$ is a finite union of intervals. The proof's strategy will be to establish a lower bound on the size of the connected components of these sets. This will imply a similar lower bound on the size of the connected components of $\Omega(I)$, thus proving the lemma.

For $x \in \mathbb{R}$ and $\delta > 0$ set $V_\delta(x,+) := (x, x+\delta)$ and $V_\delta(x,-) := (x-\delta, x)$. Given $d \in D$, and a sign $\sigma = \pm$, we say that (d,σ) is interior to $\Omega_n(I)$ if there exists $\delta > 0$ such that $V_\delta(d,\sigma) \subset \Omega_n(I)$. Let D_I denote the set of pairs $(d,\sigma) \in D \times \{-,+\}$ such that for some $n \geq 1$, (d,σ) is interior to $\Omega_n(I)$. Since D is a finite set, there exists $n_0 \in \mathbb{N}$ such that for all $(d,\sigma) \in D_I$ and $n \geq n_0$, (d,σ) is interior to $\Omega_n(I)$. By the same reason there exists $\delta > 0$ such that for all $(d,\sigma) \in D_I$ and $n \geq n_0$, $V_\delta(d,\sigma) \subset \Omega_n(I)$. Make $\delta > 0$ small enough so that every connected component of $\Omega_{n_0}(I)$ has length $\geq \delta$.

We claim that for any $n \geq n_0$ every connected component of $\Omega_n(I)$ has length $\geq \delta$. This implies that the same holds for the connected components of $\Omega(I)$, thus proving the lemma.

To finish, we prove the claim by induction. For $n = n_0$ the claim is clear. Given $n \geq n_0$, let $J_1, \dots, J_r$ be the connected components of $\Omega_n(I)$, and assume (induction hypothesis) that $|J_i| \geq \delta$ for all $i = 1, \dots, r$. For each i, let $J_{i,1}, \dots, J_{i,k_i}$ be the connected components of $J_i \setminus D$. Then

$$\Omega_{n+1}(I) = \bigcup_{i=1}^r \bigcup_{j=1}^{k_i} \psi_P(J_{i,j}) \cup \Omega_n(I) \ ,$$

and it is enough to show that $\psi_P(J_{i,j})$ has length $\geq \delta$, for all $i = 1, \dots, r$, $j = 1, \dots, k_i$.

If $k_i = 1$ then $J_i = J_{i,1}$ and since ψ_P is expanding, by the induction hypothesis

$$|\psi_P(J_{i,j})| = |\psi_P(J_i)| \geq |J_i| \geq \delta \ .$$

Otherwise, if $k_i > 1$, every interval $J_{i,j}$ has at least one endpoint $d \in D$. Let $\sigma = -$, resp. $\sigma = +$, if d is a left, resp. right, boundary point of $J_{i,j}$. Then (d, σ) is interior to $\Omega_n(I)$, which implies that $(d, \sigma) \in D_I$. By the choice of $\delta > 0$, $V_\delta(d, \sigma) \subset J_{i,j}$. Therefore $|J_{i,j}| \geq \delta$, and since ψ_P is expanding,

$$|\psi_P(J_{i,j})| \geq |J_{i,j}| \geq \delta \ .$$

□

Since $\Lambda = \Omega(I)$ mod 0 and $\Omega(I)$ is a finite union of intervals, it follows that $\#(\Lambda \setminus \Omega(I)) < +\infty$. Set $\Omega := \bigcap_{I \in \mathcal{I}} \Omega(I)$. Then $\Lambda \setminus \Omega = \bigcup_{I \in \mathcal{I}} \Lambda \setminus \Omega(I)$ is also a finite set.

Fix $\delta > 0$ arbitrarily small. Let K be the complement in Ω of a small δ'-neighborhood of the points in $\partial\Omega$. If $\delta' > 0$ is small enough there exists a δ-dense set $\{x_1, \ldots, x_r\}$ consisting of periodic orbits of ψ_P completely contained in K.

By exactness of $(\psi_P|_\Lambda, \mu)$ there exists $m \geq 1$ such that the open set $\psi^m(I)$ contains the compact set K, for all $I \in \mathcal{I}$.

Let now x be any of the periodic points x_i. Then, for any $\eta > 0$ there is $n \geq 1$ such that $\psi^n(x - \eta, x + \eta)$ contains an interval $I \in \mathcal{I}$. Thus $\mathcal{O}(x) \subset K \subset \psi^m(I) \subset \psi^{n+m}(x - \eta, x + \eta)$. Let $m' = m + n$ denote this integer, so that $\mathcal{O}(x) \subset \psi^{m'}(x - \eta, x + \eta)$.

We claim that the continuation x' of x for $\Phi_{f,P'}$ lies in the support Λ' of the ergodic SRB measure μ'. The periodic points $x_1, \ldots, x_r$ above satisfy $x \in W^u(x_i)$ for all $i = 1, \ldots, r$, where the relation $x \in W^u(x_i)$ means that for any $\eta > 0$ there exists $n_i \geq 1$ such that $x \in \psi^{n_i}(x_i - \eta, x_i + \eta)$. In fact, for some $n \geq 1$ the iterate $\psi^n(x_i - \eta, x_i + \eta)$ contains an interval $I \in \mathcal{I}$, which implies that $x \in K \subset \psi^m(I) \subset \psi^{n+m}(x_i - \eta, x_i + \eta)$. By continuity, possibly making $\lambda(f)$ weaker and taking a smaller neighborhood of P, we have $\Phi^{n_i}_{f,P'} W^u_\eta(x'_i) \cap W^s(x') \neq \emptyset$ for all $i = 1, \ldots, r$. Take any point $x'_1 \in \Lambda' \cap D^-_{\bar{\omega}}$. It follows from Lemma 3.11 that there exist $n \geq 1$ and $i = 1, \ldots, r$ such that $\Phi^n_{f,P'} W^u_{loc}(x'_1) \cap W^s_{loc}(x'_i) \neq \emptyset$. Hence, by transitivity, there exists $n' \geq 1$ such that $\Phi^{n'}_{f,P'} W^u_{loc}(x'_1) \cap W^s_{loc}(x') \neq \emptyset$. Therefore, by Theorem A.8, $x' \in \Lambda'$.

By Lemma 5.4, we have $\Phi^{m'}_{f,P'}(W^u_\delta(x')) \cap W^s(y) \neq \emptyset$ for all $y \in \mathcal{O}(x')$. This implies that $(\Phi_{f,P'}|_{R(\mu)}, \mu')$ is also exact. In fact, if $(\Phi_{f,P'}|_{R(\mu)}, \mu')$ was not mixing, denoting by $\Lambda'_0, \ldots, \Lambda'_{k'-1}$ its mixing components and assuming $x' \in \Lambda'_i$ we would have $\Phi^n_{f,P'}(W^u_\delta(x')) \subset \Lambda'_{(n+i) \bmod k'}$ for all $n \geq 0$. Taking $n = m$ this contradicts the heteroclinic connections above. Hence $k = k' = 1$.

7. Examples

7.1. **Regular Polygons.** In this subsection we characterize the ergodicity of billiards with strongly contracting reflection laws on regular polygons with an odd number of sides.

Let P_d be a regular polygon with $d \geq 3$ and odd.

Lemma 7.1. $P_d \in \widehat{\mathcal{P}}_d$.

Proof. By [9, Lemma 3.1] the slap map ψ_{P_d} is conjugated to the skew-product map $F_d : [0,1] \times \mathbb{Z}_d \to [0,1] \times \mathbb{Z}_d$ defined by $F_d(x,y) = (\phi_d(x), \sigma_x(y))$ where

$$\phi_d(x) = -\frac{1}{\cos(\frac{\pi}{d})} \left(x - \frac{1}{2}\right) \pmod 1$$

and

$$\sigma_x(y) = y + \left[\frac{d}{2}\right] \delta(x) \quad \text{and} \quad \delta(x) = \begin{cases} -1 & x < 1/2 \\ 1 & x > 1/2 \end{cases}.$$

It was proved in [8, Proposition 6.2] that the discontinuity point of ϕ_d is not pre-periodic. This shows that P_d has no orthogonal vertex connections and no pre-periodic vertices. □

Since P_d has no parallel sides facing each other and no orthogonal vertex connections, we get the following result that is a consequence of Proposition 5.2.

Theorem 7.2. *For any regular polygon P_d with $d \geq 3$ and odd, there exist a neighborhood $\mathcal{V}$ of P_d in $\mathcal{P}_d$ and $\lambda_0 > 0$ such that for every $P \in \mathcal{V}$ and every $f \in \mathcal{B}$ with $\lambda(f) < \lambda_0$, $\Phi_{f,P}$ has finitely many ergodic SRB measures.*

Knowing the ergodic properties of the slap map of P_d we obtain the following characterization of the ergodic and mixing components of billiards with strongly contracting reflection laws on polygons close to P_d.

Theorem 7.3. *For any regular polygon P_d with $d \geq 3$ and odd the following holds:*

(1) *Given any triangle Δ sufficiently close to P_3 and contracting reflection law $f \in \mathcal{B}$ such that $\lambda(f)$ is sufficiently small, the billiard map $\Phi_{f,\Delta}$ has a unique mixing SRB measure.*
(2) *Given any convex pentagon P sufficiently close to P_5 and contracting reflection law $f \in \mathcal{B}$ such that $\lambda(f)$ is sufficiently small, the billiard map $\Phi_{f,P}$ has a unique ergodic SRB measure with two mixing components.*
(3) *Given any odd integer $d \geq 7$, any convex d-gon P sufficiently close to P_d and contracting reflection law $f \in \mathcal{B}$ such that $\lambda(f)$ is sufficiently small, the billiard map $\Phi_{f,P}$ has exactly d ergodic*

SRB measures with $2^{m(d)}$ mixing components, where $m(d)$ is the integer part of $-\log_2(-\log_2\cos(\pi/d))$.

Proof. By [9, Theorem 1.1], the slap map ψ_{P_d} has a unique ergodic acip if $d=3$ or $d=5$, and exactly d ergodic acip's if $d \geq 7$. Moreover, every acip has $2^{m(d)}$ mixing components, where $m(d)$ is the integer part of $-\log_2(-\log_2\cos(\pi/d))$. In particular, $m(3)=0$ and $m(5)=1$. The theorem follows from Theorem 6.1 and Lemma 7.1. □

Remark 7.4. Theorems 7.5 and 7.6 below, generalize (1) of Theorem 7.3 to any triangle Δ.

7.2. **Triangles.** In this subsection we prove that almost every triangular billiard with strongly contracting reflection laws has a unique ergodic SRB measure.

Let Δ denote a triangle.

Theorem 7.5. *There exists a residual and full measure subset $\mathcal{R}$ of the set of triangles $\mathcal{P}_3$ such that for any triangle $\Delta \in \mathcal{R}$ and any contracting reflection law $f \in \mathcal{B}$ such that $\lambda(f)$ is sufficiently small, the billiard map $\Phi_{f,\Delta}$ has a unique ergodic SRB measure. Moreover, this measure is mixing if Δ is acute. Otherwise, it has an even number of mixing components.*

Proof. By Lemma 4.5, the set of triangles that satisfy the assumptions of Theorems 6.1 is residual and full measure. The number of ergodic and mixing components follows from [9, Theorem 1.2]. □

If we restrict to acute triangles we get the following result.

Theorem 7.6. *The billiard map $\Phi_{f,\Delta}$ of any acute triangle Δ has a unique mixing SRB measure provided $\lambda(f)$ is sufficiently small.*

Proof. By [9, Theorem 1.2], the slap map of any acute triangle has a unique mixing acip. Since acute triangles have no orthogonal vertex connections and no pre-periodic vertices, the claim follows from Theorem 6.1. □

Based on these results we conjecture that,

Conjecture 7.7. *The billiard map $\Phi_{f,\Delta}$ of any obtuse triangle Δ has a unique ergodic SRB measure with an even number of mixing components provided $\lambda(f)$ is sufficiently small.*

7.3. **More non-ergodic examples.** We have seen that a billiard on a generic triangle with a strongly contracting reflection law is ergodic. Moreover, billiards with strongly contracting reflection laws whose billiard table is close to a regular d-gon with $d \geq 7$ and odd has exactly d ergodic SRB measures. A natural question is whether billiards on quadrilaterals with strongly contracting reflection laws are ergodic. The following result answers this question in the negative. We define a kite to be a quadrilateral with a symmetry axis along a diagonal.

Proposition 7.8. *There is a kite Q and $\lambda_0 > 0$, such that for every quadrilateral Q' sufficiently close to Q and any contracting reflection law $f \in \mathcal{B}$ with $\lambda(f) < \lambda_0$, the billiard map $\Phi_{f,Q'}$ has at least two ergodic SRB measures.*

Proof. Follows from [9, Section 5.1] and Theorem 6.1. □

Besides regular polygons with an odd number of sides there is another interesting class of polygons whose billiard with a strongly contracting reflection law is non-ergodic and has an arbitrary number of ergodic SRB measures. In the following we describe this class of polygons and also construct an infinite convex polygon whose slap map has infinitely many ergodic acips.

7.3.1. *Chambers.*

Definition 7.9. A *chamber* is a cyclic pentagon C, i.e. a pentagon inscribed in a circle with vertices t, u, q, p and v ordered anticlockwise such that:

(1) t, u and v belong to the same semicircle;
(2) u and v are at the same positive distance of t;
(3) p and q are antipodal to u and v, respectively.

The vertex t is called the *top* of the chamber.

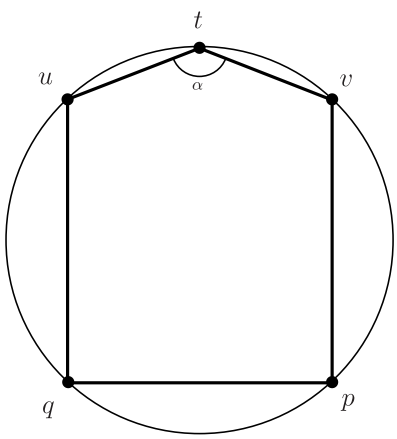

FIGURE 1. Chamber.

The sides of a chamber C adjacent to the top t are called the *ceiling* of the chamber and the side joining q and p is called the *floor* of the chamber. Associated to C we define its *chamber angle* $\alpha(C)$ to be the interior angle of C at the top vertex t. Notice that $\frac{\pi}{2} \leq \alpha(P) < \pi$. Up to similarity, the chamber is uniquely defined by its chamber angle. So we shall write C_α to highlight the dependency on α. The following result is easy to derive.

Lemma 7.10. *The line passing through t and p (resp. q) is perpendicular to the line passing through t and u (resp. v).*

Let $\psi_C : [0,1] \to [0,1]$ be the slap map of a chamber C. Let I denote the closed interval in $[0,1]$ corresponding to the floor of the chamber. Lemma 7.10 shows that $\psi_C^2(I) = I$. Let h be the affine transformation that maps $[0,1]$ to I. When restricted to the interval I, the second iterate of the slap map $\phi_\alpha := h^{-1} \circ \psi_C^2|_I \circ h$ has the analytic expression

$$\phi_\alpha(x) = \frac{2}{1-\cos\alpha}\left(x - \frac{1}{2}\right) \pmod 1.$$

Notice that ϕ_α is an expanding symmetric Lorenz map for $\alpha \in [\pi/2, \pi)$.

Lemma 7.11. *$\phi_\alpha : I \to I$ has a unique ergodic acip with 2^m mixing components where $m = m(\alpha)$ is the integer part of $-\log_2(1 - \log_2(1 - \cos\alpha))$. Moreover, the discontinuity point of ϕ_α is not pre-periodic for almost every $\alpha \in [\pi/2, \pi)$.*

Proof. See for example [9, Theorem 2.3] for the existence of the acip and the number of mixing components. The second claim follows because the set of parameters α such that $\phi_\alpha^n(0) = 1/2$ for some $n > 0$ is countable. □

Given a chamber C we denote by $\Delta = \Delta(C)$ the isosceles triangle obtained by extending the ceiling and the floor of C. Also denote by $\Lambda(C)$ the union of the ceiling and floor of the chamber C. Lemma 7.11 shows that ψ_Δ has a unique ergodic acip and 2^m mixing components. The support of the acip is contained in $\Lambda(C)$. Moreover, by the second part of Lemma 7.11, $\Delta(C_\alpha) \in \widehat{\mathcal{P}}_3$ for almost every $\alpha \in [\pi/2, \pi)$. These observations together with Theorem 6.1 show that

Proposition 7.12. *For almost every $\alpha \in [\pi/2, \pi)$ there exist a neighborhood $\mathcal{V}$ of the $\Delta(C_\alpha)$ in $\mathcal{P}_3$ and $\lambda_0 > 0$ such that for every $P \in \mathcal{V}$ and $f \in \mathcal{B}$ with $\lambda(f) < \lambda_0$, $\Phi_{f,P}$ has a unique ergodic SRB measure and 2^m mixing components where m is the integer of Lemma 7.11.*

Using the chambers as building blocks we can construct convex polygons with arbitrary number of ergodic components. Let us explain the details of the construction. We call *ceiling arc* (resp. *floor arc*) the arc of the circle that join the vertices of the ceiling (resp. floor) of the chamber. Denote by $\gamma(C)$ the union of the ceiling and floor arcs of C. We say that two chambers C and C' are *separated* if they are inscribed in the same circle and $\gamma(C) \cap \gamma(C') = \emptyset$. This definition guarantees that the slap map dynamics of $\Delta(C)$ and $\Delta(C')$ is separated.

Let $P = \Delta(C) \cap \Delta(C')$, the intersection of the isosceles triangles of two separated chambers. The polygon P is a convex hexagon and the slap map ψ_P has two ergodic acips. In fact, since the restriction of the maps ψ_P and $\psi_{\Delta(C)}$ to the set $\Lambda(C)$ coincide, we conclude from

Lemma 7.11 that $\psi_{\Delta(C)}$ and hence ψ_P have a unique ergodic acip whose support is contained in $\Lambda(C)$. The same conclusion applies to C'. Thus, ψ_P has two ergodic acips with disjoint supports. Notice that we can always choose C and C' separated so that $\Delta(C)$ and $\Delta(C')$ have no orthogonal vertex connections and no pre-periodic vertices. Thus, applying Theorem 6.1 to both isosceles triangles, we conclude that the contracting billiard map $\Phi_{f,P}$ has at least two ergodic SRB measures for $\lambda(f)$ sufficiently small.

This construction can be easily generalized to any finite number of separated chambers. A finite number of chambers $C_1, \ldots, C_n$ are separated if they are inscribed in the same circle and $\gamma(C_i) \cap \gamma(C_j) = \emptyset$ whenever $i \neq j$. Let $\mathcal{C}_m$ be the set of all polygons $P = \bigcap_{i=1}^{m} \Delta(C_i)$ such that $C_1, \ldots, C_m$ are separated chambers and $\Delta(C_i) \in \widehat{\mathcal{P}}_3$ for $i = 1, \ldots, m$. Notice that, $\mathcal{C}_m \subset \mathcal{P}_{3m}$.

Proposition 7.13. *For any $m \geq 1$ and $P \in \mathcal{C}_m$ there exist a neighborhood $\mathcal{V}$ of P in $\mathcal{P}_{3m}$ and $\lambda_0 > 0$ such that for every $P' \in \mathcal{V}$ and $f \in \mathcal{B}$ with $\lambda(f) < \lambda_0$, $\Phi_{f,P'}$ has m ergodic SRB measures.*

Proof. Applying Proposition 7.12 to each isosceles triangle $\Delta(C_i)$ we obtain the result. □

7.3.2. *Convex infinite polygon with infinitely many ergodic acips.* We do not know any example of a convex billiard table whose billiard with a strongly contracting reflection law has an infinite number of ergodic SRB measures. However, we can construct a convex infinite polygon P whose slap map has an infinite number of ergodic acips. Since the support of the acips accumulate we cannot apply the main theorem of this paper to obtain such example.

In the following we outline the construction of P. Let $\{C_n\}_{n\in\mathbb{N}}$ be a family of separated chambers. We can construct this family as follows. Suppose that we have $C_1, \ldots, C_n$ separated chambers. Pick a point t_{n+1} on the circle which is not contained in the union $\bigcup_{i=1}^{n} \gamma(C_i)$. That point will be the top of the chamber C_{n+1}. Now we just need to increase the chamber angle $\alpha(C_{n+1})$ so that $\gamma(C_{n+1}) \cap \gamma(C_i) = \emptyset$ for $i = 1, \ldots, n$. In this way we construct a countable family of separated chambers. Let P denote the intersection of all chambers in the family,

$$P = \bigcap_{n=1}^{\infty} C_n.$$

Notice that the polygon P is convex and has an infinite number of sides with arbitrary small length. By Lemma 7.11, $\psi_P|_{\Lambda(C_n)}$ has a unique ergodic acip. Also notice that $\Lambda(C_i) \cap \Lambda(C_j) = \emptyset$ whenever $i \neq j$. This shows the following proposition.

Proposition 7.14. *The slap map of P has countably many ergodic acips.*

Appendix A. Hyperbolic piecewise smooth maps

Let M be a compact smooth manifold (possibly with boundary and corners) with Riemannian metric ρ, and let U be an open subset of M.

Definition A.1. A map $g\colon U \to M$ is called a *piecewise smooth map* on U if g is a C^2 diffeomorphism from U to $g(U)$. The compact set $N := M \setminus U$ is called the *singular set* of g.

Definition A.2. Let $U^+ = \{x \in U\colon g^n(x) \notin N\ \forall n \geq 0\}$ be the set of all elements of U with infinite positive semi-orbit. Define

$$D = \bigcap_{n\geq 0} g^n(U^+).$$

The set $A = \overline{D}$ is called the *generalized attractor* of g.

Condition H1: There exist positive constants $\widetilde{A}$ and a such that for every $x \in U$,

$$\|D^2 g(x)\| \leq \widetilde{A}\rho(x, N)^{-a} \quad \text{and} \quad \|D^2 g^{-1}(g(x))\| \leq \widetilde{A}\rho(x, N)^{-a}.$$

A cone in T_xU, $x \in U$ with an axial linear subspace $P \subset T_xU$ and angle $\alpha > 0$ is the set given by

$$C_\alpha(x, P) = \{v \in T_xU\colon \angle(v, P) \leq \alpha\}.$$

Condition H2: The map g is uniformly hyperbolic on U. Namely, there exist two constants $c > 0$, $\lambda > 1$ and cones

$$C^s(x) = C_{\alpha^s(x)}(x, P^s(x)) \quad \text{and} \quad C^u(x) = C_{\alpha^u(x)}(x, P^u(x))\,,$$

with axial subspaces $P^u(x), P^s(x)$ and positive angles $\alpha^u(x), \alpha^s(x)$ for every $x \in U$ such that

1. $T_xU = P^u(x) \oplus P^s(x)$,
2. $\dim P^u(x)$ and $\dim P^s(x)$ are constant,
3. the minimum angle between $C^u(x)$ and $C^s(x)$ is uniformly bounded away from zero,
4. $Dg(x)(C^u(x)) \subset C^u(g(x))$ and $Dg^{-1}(g(x))(C^s(g(x))) \subset C^s(x)$ whenever $x \in U$,
5. if $x \in U^+$, then $\|Dg^n(x)v\| \geq c\lambda^n\|v\|$ for $v \in C^u(x)$, and $\|Dg^n(x)v\| \leq c\lambda^{-n}\|v\|$ for $v \in C^s(x)$.

Definition A.3. We say that a piecewise smooth map g is *hyperbolic* or that g has a *hyperbolic attractor* if it satisfies Conditions H1 and H2.

Notable examples of hyperbolic piecewise maps are Lozi maps, Belykh maps and geometrical Lorenz maps [4, 12, 21]. The maps associated to the billiards studied in this paper are also always hyperbolic piecewise smooth.

The Lebesgue measure of M generated by the Riemanninan metric ρ is denoted by ν. Similarly, given a submanifold $W \subset M$, the Lebesgue measure of W is denoted by ν_W. The set $N_\epsilon \subset M$ denotes the ϵ-neighbourhood of N with $\epsilon > 0$.

Condition H3: There exist positive constants B, β, ϵ_0 such that

$$\nu(g^{-n}(N_\epsilon)) < B\epsilon^\beta \qquad \text{for } n \geq 1 \text{ and } \epsilon \in (0, \epsilon_0).$$

A smooth submanifold $W \subset M$ is a *u-manifold* if the dimension of W is equal to the one of the unstable subspaces of g, and the tangent space of W is contained in $C^u(x)$ for every $x \in W$.

Condition H4: There exist positive constants β and ϵ_0 such that for every u-manifold W, there exist an integer n_0 and a constant $B_0 > 0$ such that for every $0 < \epsilon < \epsilon_0$,

(1) $\nu_W(W \cap g^{-n}(N_\epsilon)) < \epsilon^\beta \nu_W(W)$ for $n > n_0$,
(2) $\nu_W(W \cap g^{-n}(N_\epsilon)) < B_0\epsilon^\beta \nu_W(W)$ for $n \geq 1$.

For $\omega > 0$ and $l \geq 1$, define

$$D^\pm_{\omega,l} = \left\{x \in D \colon \rho(g^{\pm n}(x), N) \geq l^{-1}e^{-n\omega} \ \ \forall n \geq 0\right\},$$

and

$$D^\pm_\omega = \bigcup_{l \geq 1} D^\pm_{\omega,l}, \qquad D^0_\omega = D^-_\omega \cap D^+_\omega.$$

If g is a hyperbolic piecewise smooth map, then the generalization of Pesin's theory to maps with singularities [11, 14, 15] implies that there is $\bar\omega = \bar\omega(g) > 0$ such that a local stable manifold $W^s_{loc}(x)$ (resp. local unstable manifold $W^u_{loc}(x)$) exists for every $x \in D^+_{\bar\omega,l}$ (resp. $x \in D^-_{\bar\omega,l}$). The local manifolds $W^s_{loc}(x)$ and $W^u_{loc}(x)$ are C^1 embedded submanifolds whose tangent subspaces at x are equal to the stable subspace $E^s(x)$ and the unstable subspaces $E^u(x)$, respectively. These manifolds form two transversal invariant laminations. For convenience, in the rest of this section, we will always write $\bar\omega$ instead of $\bar\omega(g)$.

Definition A.4. Let $g \colon U \to M$ be a hyperbolic piecewise smooth map. An invariant probability measure μ on the attractor A is called SRB if $\mu(D^0_{\bar\omega}) = \mu(A) = 1$, and the conditional measures of μ on the unstable local manifolds of g are absolutely continuous.

The next theorem was proved by Sataev [15, Theorems 5.12 and 5.15]. See also [1].

Theorem A.5. *Let g be a hyperbolic piecewise smooth map satisfying Conditions H3 and H4. Then there exist m ergodic SRB measures $\mu_1, \dots, \mu_m$ supported on the attractor A and m pairwise disjoint subsets $E_1, \dots, E_m$ of A such that*

(1) $\mu_i(E_i) = 1$ *for* $i = 1, \dots, m$,
(2) *for every SRB measure μ on A, there exist $\alpha_1, \dots, \alpha_m \geq 0$ with $\sum_{i=1}^m \alpha_i = 1$ such that $\mu = \sum_i^m \alpha_i \mu_i$,*
(3) $E_i \cap D^-_{\bar\omega,l}$ *is closed for every* $l > 0$,

(4) *if* $O \subset U$ *is open and* $O \cap E_i \cap D^-_{\bar{\omega},l} \neq \emptyset$ *for some* $i = 1, \dots, m$, $l > 0$, *then* $\mu_i(O \cap E_i \cap D^-_{\bar{\omega},l}) > 0$,
(5) $W^u_{loc}(x) \subset E_i$ *for every* $x \in E_i \cap D^-_{\bar{\omega}}$.

Remark A.6. Under conditions weaker than H3 and H4, Pesin proved that the map g admits countably many ergodic SRB measures, and derived several properties of these measures [14].

The next theorem is consequence of Theorem A.5 and results from [14]. It will be used in Section 6.

Theorem A.7. *Suppose that* g *is a hyperbolic piecewise smooth map satisfying Conditions H3 and H4. Let* $\mu_1, \dots, \mu_m$ *and* $E_1, \dots, E_m$ *be the measures and the corresponding sets as in Theorem A.5. Then*

(1) *for each* $i = 1, \dots, m$, *there exist disjoint subsets* $M_{i,1}, \dots, M_{i,k_i}$ *with* $k_i \in \mathbb{N}$ *such that* $E_i = \bigcup_{j=1}^{k_i} M_{i,j}$ *(mod 0),* $g(M_{i,j}) = M_{i,j+1}$ *where* $M_{i,k_i+1} := M_{i,1}$, *and the induced system* $(g^{k_i}|_{M_{i,1}}, \mu_i|_{M_{i,1}})$ *is Bernoulli,*
(2) *if* $x \in D^-_{\bar{\omega}}$ *and* ν *is a probability measure of* U *supported on* $W = W^u_{loc}(x)$ *and absolutely continuous with respect to* ν_W, *then every weak-* limit point of the sequence*

$$\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} g^k_* \nu$$

is an SRB measure of g,
(3) *the set of periodic points of* g *is dense in* A.

We call the sets E_i and $M_{i,j}$ the *ergodic components* of g and the *mixing components* of g, respectively.

The next result of Sataev permits to characterize the ergodic components of a hyperbolic piecewise smooth map in terms of the periodic orbits of the map. Our formulation differs slightly from the original one (see [16, Theorem 5.1]). We include its proof, for the convenience of the reader.

Theorem A.8. *Let* g *be a hyperbolic piecewise smooth map satisfying H3 and H4. Let* x_0 *be a periodic point of* g, *and suppose that* $x_1 \in E_i \cap D^-_{\bar{\omega}}$, *where* $E_1, \dots, E_m$ *are the sets in Theorem A.5. If there exist* $n_1 \geq 0$ *such that* $W^s_{loc}(x_0) \cap g^{n_1}(W^u_{loc}(x_1)) \neq \emptyset$, *then* $x_0 \in E_i$.

Proof. Let $y \in W^u_{loc}(x_1)$ such that $g^{n_1}(y) \in W^s_{loc}(x_0)$. Since x_0 is a hyperbolic, there exists $l \geq 1$ such that $x_0 \in D^-_{\bar{\omega},l}$. Hence, if k is the period of x_0, then $g^{n_1+kn}(y) \in D^-_{\bar{\omega},2l}$ for n sufficiently large. Since $x_1 \in E_i$, conclusion (5) of Theorem A.5 implies that $g^{n_1+kn}(y) \in E_i$. But $g^{n_1+kn}(y)$ converges to x_0 as $n \to +\infty$, and $E_i \cap D^-_{\bar{\omega},2l}$ is closed by conclusion (3) of Theorem A.5. Therefore $x_0 \in E_i$. □

Finally, we introduce a new condition which implies H3 and H4.

Condition H5: There exist constants $C > 0$, $a \in (0,1)$ and $\epsilon_1 > 0$ such that

$$\nu_W(W \cap g^{-n}(N_\epsilon)) \leq C\epsilon\,(a^n + \nu_W(W))$$

for every u-curve W, every $n \geq 1$ and every $\epsilon \in (0, \epsilon_1)$.

Lemma A.9. *H5 implies H3 and H4.*

Proof. Since $\nu_W(W)$ is bounded uniformly in the u-manifold W and $a \in (0,1)$, there exists a constant B such that $C(a^n + \nu_W(W)) < B$ for every $n \geq 1$ and every u-manifold W. Thus, from H5, it follows that $\nu_W(W \cap g^{-n}(N_\epsilon)) < B\epsilon$ for every $n \geq 1$ and every u-manifold W. Condition H3 with $\beta = 1$ follows from this inequality by covering properly the set U with u-manifolds, and using Fubini's Theorem.

Conditions H4 follows directly from H5 by taking

$$\beta \in (0,1), \qquad \epsilon_0 = \min\left\{\epsilon_1, (2C)^{\frac{1}{\beta-1}}\right\},$$

and

$$B_0(W) = \frac{1 + a/\nu_W(W)}{2}, \qquad n_0(W) = \left\lfloor \frac{\log \nu_W(W)}{\log a} \right\rfloor,$$

where $\lfloor x \rfloor$ is the integer part of x. Indeed, let $\epsilon < \epsilon_0$. If $n > n_0(W)$, then

$$\begin{aligned}\nu_W(W \cap g^{-n}(N_\epsilon)) &\leq C\epsilon\,(a^n + \nu_W(W)) \\ &\leq 2C\epsilon\nu_W(W) \leq \epsilon^\beta \nu_W(W).\end{aligned}$$

Moreover, for every $n \geq 1$,

$$\begin{aligned}\nu_W(W \cap g^{-n}(N_\epsilon)) &\leq C\epsilon\,(a^n + \nu_W(W)) \leq C\epsilon(a + \nu_W(W)) \\ &\leq 2C\epsilon B_0(W)\nu_W(W) \leq \epsilon^\beta B_0(W)\nu_W(W).\end{aligned}$$

□

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