Using Tail Conditional Expectation for capital requirement calculation of a general insurance undertaking

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Abstract:

In this paper we develop a solvency model to estimate the necessary economic capital of a real insurance undertaking operating solely in the Automobile branch, applying the Tail Conditional Expectation risk measure. The model assumes a one year time horizon static approach with an unchanged asset and liability structure for the company.

After discussing the main factors affecting the whole of the insurance activity and their influence on the assets and liabilities on that real insurance undertaking used in the study, we calculate its necessary economic capital, by using the Monte Carlo simulation technique to generate the probability distribution of the possible future profit and losses with impact on the company's fair value.

This paper introduces an application of a set of techniques that are usually applied to manage asset and liability risks to capital requirements. With a simulated exercise applied to a real insurance undertaking we show its feasibility, its advantages and how useful it may be for investors, regulators and remaining stakeholders when the technique is explored in depth.

Key words: Tail Conditional Expectation, Value-at-Risk, capital requirement, resampling, Monte Carlo simulation, risk management.

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1. Introduction

The determination of the economic capital requirement to ensure, with high probability, the development of operations, even in adverse environments, is a primer question, due to the role of insurance undertakings in the economy.

In this article we develop a solvency model to estimate the necessary economic capital for a real portfolio of a particular insurer, using a set of specific risk analysis tools that have been widely used for different purposes and aims. It allows us to calculate the economic capital requirement for an insurance undertaking, in order to face adverse situations with a chosen high probability, given his current asset and liabilities structure and considering a one year time horizon.

We limit our study to the automobile branch, identifying the main assets, liabilities and operations that cause uncertainty on the economic value of the insurer under study. We measure the causes of uncertainty by the impact on the economic results and enhance the interest and practical applications of the model to the insurance industry.

Here we only consider stocks and bonds as manageable assets, while for liabilities we will account for premium and claims reserves. The stocks and bonds considered are those that the insurance undertaking actually shows in the balance sheet and the simulations are based on the assumption that they are kept constant within the time interval under study. As far as the reserves are concerned, they relate to the underwriting of insurance contracts and respective claims settling.

There are several types of risks in the insurance activity that affect assets, liabilities or both. In this paper we model the equity risk, interest rate risk, credit risk, reserve risk and the premium risk, determining the exposure of the assets and liabilities to the different types of risks.

We suggest the capital requirement to be determined based upon the Tail Conditional Expectation (simply, *TCE*) risk measure, assuming a (future) simulated profit and loss distribution for the company, which in turn, is estimated by means of Monte Carlo simulation.

We assume that in the case of ruin before the end of the period, the insurance undertaking has the ability to provide additional capital to ensure the continuity of its activities. We consider a static approach, stressing the need for a periodical re-evaluation, since it is not expectable that, within a reasonable time horizon, neither the asset and liabilities structure nor the future profit and loss distribution remains unchanged. Furthermore, for the sake of simplicity, we assume that the fair value of the insurer's liabilities equals the *best estimat*e, *i.e.*, the market value risk margin is null.

Thus, we present *VaR* and *TCE* risk measures, define the risk factors affecting the whole of the insurance industry and the particular insurance undertaking studied, model their individual and aggregate behaviour and detail the simulation procedures. Finally, these procedures are applied to a non life insurer operating in the motor branch, and are used to calculate his economic capital requirement. The importance of these sort of risk measures to compute capital requirement is enhanced by the newly proposed regulations for the insurance industry in the European market, under the programme *Solvency II*. For more details, please see Linder and Ronkainen (2004).

The body of this article is divided into five sections. In the next section we formulate the solvency model, introducing the individual risk factors involved, their modelling and simulation procedures.

In Section 3 we develop our application under the assumptions considered. In Section 4 we show the results of our application and calculate the capital requirement for the period, risk by risk and for the aggregate. Finally, in the last section we present the main conclusions.

2. The model

2.1. *VaR* and *TCE*

The Value-at-Risk (simply, $VaR_\alpha(X)$ or VaR_α) is defined as the quantile of order α of the probability distribution of the random variable X that represents the future results, profit and losses, of the insurer. That is, $VaR_\alpha = \inf\left[x \in \Re \mid F_X(x) \geq \alpha\right]$, where $F_X(x)$ is the distribution function of *X*. The Tail Conditional Expectation, denoted as $TCE_\alpha(X)$ or simply TCE_α, is defined as $TCE_\alpha(X) = E[X \mid X < VaR_\alpha(X)]$. That is, while with the *VaR* we are interested in knowing how much can a firm lose within a certain time horizon, under certain set of considerations, the *TCE* allow us to estimate the expected loss whenever the occurred loss is greater than the *VaR* (assuming that the *VaR* is negative). Therefore, it seems that *TCE* is a more conservative but safer risk measure to adequately protect the insurance undertaking industry, their shareholders and remaining stakeholders.

Considering the advantages of the *TCE* over the *VaR* as presented by Artzner (1999) and Lynn Wirch and Hardy (1999), this will be the chosen risk measure to determine the necessary economic capital. Nevertheless, since *VaR* is nowadays the most commonly used risk measure and that we need the *VaR* for the computation of the *TCE*, we will present both risk measures for comparative purposes.

2.2. Modelling the individual risks

As far as **equity risk** is concerned, it is defined as the risk associated with stock price returns fluctuation, assuming a well diversified insurer's portfolio. If this is the case, we may then consider the Sharpe (1964) and Lintner's (1965) *Capital Asset Pricing Model* (*CAPM*). Assuming a capital market in equilibrium, and that a set of assumptions is fulfilled (see for instance in Elton et al, 2007), the expected rate of return of the portfolio will then be given by

$$E(R_p) = R_f + \beta_p \left(E(R_m) - R_f\right), \qquad \textbf{(Eq. 1)}$$

where R_f stands for the risk free rate of return, R_m is the market rate of return, and β_p represents the *N* assets portfolio beta that equals $\beta_p = \sum_{i=1}^{N} w_i \beta_i$, with each individual beta component determined by $\beta_i = \sigma_{im} / \sigma_m^2$, denoting $\sigma_{i,m}$ the covariance between the rate of return of stock *i* and the market rate of return; σ_m^2 stands for the equity market variance and w_i is the weight of stock *i* in the portfolio.

Considering a portfolio with a large number of assets we can assume, without loss of accuracy that $\sigma_p \cong \beta_p \sigma_m$ (see for instance in Elton et al, 2007). In order to estimate the parameters β_i we use the Market Model.

For simulation purposes we assumed that the simulated instantaneous market rate of return follows a Geometric Brownian motion, whose dynamics are given by the equation $d\ln S_t = (\mu - \sigma^2/2)dt + \sigma dW_t$ where S_t is the stock market price level at time t, W_t a standard Brownian motion, μ is the *drift* constant, σ is the volatility, and $dW_t = Z(dt)^{1/2}$ is the increment of W_t , $W_t \sim Normal(0;dt)$; Z is a standard normal random variable. For practical purposes the above stochastic process is discretized in short time intervals, say $\Delta_t = k/p$, where p is the number of increments and k the chosen time horizon. Thus, the simulated stock price level at $t+\Delta t$ will be given by

$$S_{t+\Delta t} = S_t \cdot \exp\{(\mu - \sigma^2/2)\Delta t + \sigma z\sqrt{\Delta t}\}. \quad \textbf{(Eq. 2)}$$

In order to model the pricing behaviour of the debt instruments that are significant for the assets side of the balance sheet of the insurance undertaking, we need to consider both the credit risk and the interest rate risk.

Starting with the **credit risk**, we use the J.P. Morgan's (1997) *CreditMetrics* to model the credit risk of the debt instruments under consideration.

The credit spreads are extracted from the companies *rating* scores. The better the issuer's rating, the lesser the credit spread required and, consequently, the larger the discounted value of the debt. The model assumes that if the market value of the issuer's debt follows beyond a given threshold, the entity will enter in default. This reasoning is extended in a way that will allow us to determine a relationship between the assets value and the issuer's rating. We assume that the rate of return of the issuer's asset follows a Normal distribution in the case of a single credit, or a Multivariate Normal distribution for a portfolio of credits. For the simulation procedure of the assets rate of returns we generate a set of correlated (pseudo-) Normal random values and a new rating and its respective credit spread is assigned, if necessary.

The analysis is even refined considering scenarios for default. The credit rates of recovery are highly volatile, which means that, for each scenario of default, we simulate a value for the credit recovery rate assuming a Beta distribution with parameters in accordance with the credit's level of the subordination. If the debt is simulated to default, then the credit value will equal the simulated rate of recovery times the nominal value of the credit. Otherwise, the issuer's credit risk considering the simulated rating is added up to the respective risk free discount rate in order to estimate the value of the debt instrument.

In order to model the debt **interest rate risk** we use simulated zero coupon bonds with maturity equal to the duration of each bond portfolio, as suggested by J.P. Morgan and Reuters (1996).

We assume that the daily short term interest rate behaviour follows a modified one period one factor, short term interest rate model of Cox, Ingersol and Ross (1985) (CIR model) suggested by Fisher, May and Walther (2002), and whose parameter estimation method is easier to implement. The notation, the parameter estimation and the simulation procedure is according to those proposed by these authors.

The short term interest rate behaviour is assumed to follow the stochastic dynamics given by

$$dr(t) = \left(b - a \cdot r(t)\right)dt + \sigma\sqrt{r(t)}dW(t) \ , \qquad \textbf{(Eq. 3)}$$

where b, a and σ are positive constants and *W(t)* is a standard Brownian motion process. As far as bond prices for the several maturities are concerned, these are assumed to value $\{p(t,r(t),T)\}$ defined by

$$p(t,r(t),T)=A(t,T)e^{-B(t,T)r(t)}\ , \qquad \textbf{(Eq. 4)}$$

where $A(t,T)=\left[\frac{2he^{(a+h)(T-t)/2}}{2h+(a+h)\left(e^{(T-t)h}-1\right)}\right]^{2b/\sigma^2}$; $B(t,T)=\left[\frac{2\left(e^{(T-t)h}-1\right)}{2h+(a+h)\left(e^{(T-t)h}-1\right)}\right]$; $h=\sqrt{a^2+2\sigma^2}$,

t is the time moment when the value is determined and T is the maturity.

The implicit *yield* curve of the CIR model is estimated from

$$r(t,T)=-\frac{\log(p(t,r(t),T))}{T-t}. \qquad \textbf{(Eq. 5)}$$

If equation (3) holds, it means that the process is not directly observed, since it is developed under the risk neutral probability measure. However, as we need to simulate the stochastic interest rate process under the real world probability measure, an acceptable market estimate for a can be $\tilde{a}$ (see Fisher, May and Walther, 2002). The parameters b, $\tilde{a}$ and σ can be empirically determined using market data. The estimation method for the parameters b, $\tilde{a}$ and σ is based upon the Martigales Estimation Functions as presented in Fisher, May and Walther (2002). The actual short term interest rate, $r(0)$, and the estimates of $\hat{b}$ and $\hat{\sigma}$ are then used to estimate a, assuming that market prices $\left(p^M{}_i\right)$ equal the theoretical prices (p_i) for i, $i=1,\ldots,n$ zero coupon bonds, with maturity T_i at time zero ($t = 0$). The estimate for a is then obtained by

$$\underset{a}{Min}\quad \sum_{i=1}^{n}\left(p_i-p_i^M\right)^2 .$$

Short term interest rates have to be simulated using the estimate $\tilde{a}$, instead of a, since we are interested in generating real world scenarios for $r(t)$, as in Fisher, May and Walther (2002). Given a time discretization into equally time spaced instants we split the time interval $[0,T]$ into N equal time intervals: $\Delta = T/N$. Then, we simulate the future values of the short term interest rate, r_n, using the recursion method, for $n = 1,...,N$ with the starting point $r(0)$ according to

$$r_n = r_{n-1} + \left(\hat{b} - \hat{\tilde{a}} r_{n-1}\right)\Delta + \hat{\sigma}\sqrt{r_{n-1}}\,\Delta\widetilde{W}_n \qquad \textbf{(Eq. 6)}$$

With the yield curve, considering the credit risk, and using the inverted version of equation (5) it is then possible to determine a simulated future value of each bond, given both credit and interest rate risks. The difference between the simulated future value of each bond and its present (discounted) value corresponds to the simulated result (gain or loss) of holding each bond for the one year time period.

The **reserve risk** is related to the risk of adverse development of the claims reserve. It corresponds to an estimate of the total cost that the insurer will have to bear in order to settle all claims occurred until the end of the year, whether they have been reported or not. This is a net value, after the deduction of all payments already done concerning those claims.

Define I_{ij} as the incremental payments made in the development year j regarding claims occurred in the year i and $R = \sum_{(i,j)\in\Delta} I_{i,j}$ is the total reserve. Here, Δ represents the set of indexes associated to the total future incremental payments displayed in the usual development matrix, i.e., $\Delta = \{(i,j) : 0 \leq i \leq N; N - i + 1 \leq j \leq N + 1\}$, N is the observed development period, we assume the claims development stops at N+1. For more details please see Taylor (2000). We will use a Generalised Linear Model (simply, *GLM*) where the variables I_{ij} are considered to be independent and identically distributed (*iid*), whose distribution belong to the *Exponential family*, described in McCullagh and Nelder (1989).

A *GLM* is characterized by a random component and a systematic component. Regarding the first component, consider a set of independent *r.v.*'s Y_i, $i = 1, 2, \ldots, n$ with density $f(y_i \mid \theta_i, \phi)$, where θ_j is the canonical shape of a location parameter and ϕ is a scale parameter. As for the random component, consider a matriz $\boldsymbol{X}_{(n \times p)}$, whose elements, x_{ij}, are the n observations of p explanatory variables $X_j, j=1,\ldots p$. The i-th observation of these variables generates a linear predictor (linear combinations of the explanatory variables) η_i, given by $\eta_i = \sum_{j=1}^{p} x_{ij}\beta_j$, $i = 1,\ldots,n$, where the β_j, $j=1,\ldots,p$, are unknown parameters, to be estimated from the data.

The two components relate each other through $\mu_i = h(\eta_i) = h(z_i^T \beta)$ and $\eta_i = g(\mu_i)$, where h is a monotonous and differentiable function; $g = h^{-1}$ is the link function; z_i is a vector of dimension p, function of the vector of explanatory variables, say, x_i; $\mu_i = E(Y_i)$ and $Var(Y_i) = \phi V\left(\mu_i\right) / w_i$, where w_i is a constant and $V(\mu_i)$ the variance function.

Then, consider a triangle of development of incremental payments I_{ij}, with $0 \le i \le N$, $0 \le j \le N+1$. Suppose $w_{ij} = 1, \forall i, j$, thus we have $V\left(I_{i,j}\right) = \phi \cdot V\left(\mu_{i,j}\right)$. The variance function has the following shape $V\left(\mu_{i,j}\right) = \mu_{i,j}^{\zeta}$, $\zeta \ge 0$. The link function and the linear predictor are given by

$$\eta_{i,j} = \ln \mu_{ij} = \mu + \alpha_i + \beta_j \ , \qquad \textbf{(Eq. 7)}$$

where α_i denotes the effect caused by the occurrence period *i*, β_j the effect caused by the development period *j* and μ the global average.

The estimates for the future incremental payments $(I_{i,j}, j \geq n-i+1)$ are given by $\hat{\mu}_{i,j} = \exp\{\hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j\}$, where $\hat{\mu}$, $\hat{\alpha}_i$, $\hat{\beta}_j$ are estimates of the maximum *quasi*-likelihood. In order to avoid the over parameterisation the constraints $\alpha_0 = \beta_0 = 0$ are introduced. Estimate $\hat{R}$ is obtained through $\hat{R} = \sum_{(i,j)\in\Delta} \hat{\mu}_{ij}$. The standard error (*SE*) of $\hat{R}$ will be,

$$SE(\hat{R}) = \sqrt{E\left\{(R-\hat{R})^2\right\}} = \sqrt{\sum_{(i,j)\in\Delta} A\hat{Q}E(\hat{\mu}_{i,j}) + \sum_{\substack{(i,j)\\(x,y)\}\in\Delta\\(i,j)\neq(x,y)}} \left\{\hat{\mu}_{i,j}\hat{\mu}_{x,y} Cov(\hat{\eta}_{i,j}, \hat{\eta}_{x,y})\right\}} \ ,$$

$$A\hat{Q}E(\hat{\mu}_{i,j}) = \hat{\phi}\cdot\hat{\mu}_{i,j}^{\zeta} + \left\{V(\hat{\mu}) + V(\hat{\alpha}_i) + V(\hat{\beta}_j) + 2\left\{Cov(\hat{\mu},\hat{\alpha}_i) + Cov(\hat{\mu},\hat{\beta}_j) + Cov(\hat{\alpha}_i,\hat{\beta}_j)\right\}\right\}\cdot\hat{\mu}_{i,j}^2 .$$

Renshaw and Verrall (1998) proposed a stochastic version of the *Chain Ladder* method, which assumes that the incremental payments follow an *over-dispersed Poisson* distribution, and a linear predictor with shape as in Equation (7) and a logarithmic link function. (In this *GLM* the parameter ζ of the variance function assumes the value *1*, but the scale parameter ϕ is estimated instead of being pre-determined. The model has the following assumption: $E(I_{i,j}) = \mu_{i,j}$, $V(I_{ij}) = \varphi V(\mu_{ij}) = \varphi\mu_{ij}$, $\zeta = 1$, $\phi > 0$, $i = 0,...,n$; $\sum_{i=0}^{N-j} I_{ij} \geq 0$ and $0 \leq j \leq N$.

In the model the scale parameter is estimated using the approximation of the Generalised Pearson's statistic proposed by McCullagh and Nelder (1989). The model fit is done with two tests: (1) Wald's nullity test of the linear predictor parameter and (2) the global significance test using the scale *quasi*-deviance statistic. In addition, we will analyse the graphical representation of Person residuals using a Normal probability plot and the graphical representation of the residuals against the adjusted values $\hat{\mu}_i$ and against each of explanatory variables of the linear predictor.

We will simulate the possible values of *R*, using a *bootstrap* method in association with the over-dispersed Poisson *GLM*. The Bootstrap method requires the existence of a set of observations of *iid* random variables. However, the $I_{i,j}$'s do not satisfy this assumption since they depend on the parameters, therefore we will use the Pearson's residuals of the model, $r_{i,j}$, $0 \le i \le N$, $0 \le j \le N-i$, since they can be considered as *observations* of the random variables. The residuals $r_{0,N}$, $r_{N,0}$ and $r_{0,N+1}$ will be dropped since by definition they are equal to zero, as exposed by Pinheiro, Andrade e Silva and Centeno (2003). The new triangle of residuals will be converted in a pseudo-data triangle $I_{i,j}^{bs}$ using $I_{i,j}^{bs} = r_{i,j}^{bs}\sqrt{\hat{\mu}_{i,j}} + \hat{\mu}_{i,j}$ with $r_{i,j}^{bs}$, satisfying $\{(i,j): 0 \le i \le N, 0 \le j \le N-i\} \cup \{(i,j) = (0, N+1)\}$ and $\hat{\mu}_{i,j}$ as the estimated values.

We will apply the over-dispersed Poisson *GLM* to the pseudo-data triangle in order to obtain the reserve estimate, called *pseudo-reserve*. We use the notation $\hat{R}_{(b)}^{bs}$, $1 \le b \le B$, for the pseudo-reserves and $\hat{R}$ for the original estimate. This process is repeated a large number of *B* times. As far as the computation of the *SE* is concerned, we'll have to add to the standard deviation of the Bootstrap results, say $\hat{\sigma}_{bs}(\hat{R})$, a volatility measure of the stochastic process inherent to the over-dispersed Poisson *GLM*. According to England

and Verral (1999) this is $\hat{\phi} \cdot \hat{R}$, where $\hat{\phi}$ is an estimate of the scale parameter. Finally, we obtain the *SE* of the *bootstrap* estimates for $\hat{R}$, $S\hat{E}_{bs}\left(\hat{R}\right) = \sqrt{\hat{\phi} \cdot \hat{R} + \frac{n}{n+p}\hat{\sigma}_{bs}^2\left(\hat{R}\right)}$.

Given that the total reserve is a sum of the random future payments, its estimate should equal the discounted value of the incremental future payments, discounted with an appropriate rate, therefore the claims reserve is also subject to interest rate risk. In this paper we will use the risk-free interest rate as an approximation to the appropriate discount rate for liabilities. We will simulate the risk-free interest rate term structure in one year's time using the CIR model as explained in the previous section. The difference between the expected value of the discounted reserve today and the discounted value of the simulated reserve within one year will be the result associated of the development of the claims reserve (including the corresponding interest rate risk).

Premium Risk is associated with the premium reserves. In motor insurance contracts are usually done on a annual basis and premiums are received upfront. Insurers are required to build premium reserves to cover future claims of the set of policies in force. Premium risk is the risk that those reserves are not sufficient to face these future payments.

To calculate the risk it is necessary to model the future annual claim payments. To find the distribution of the aggregate claims cost in the time interval, say (0, *t*], we use the well known Collective Risk Model, for details please see, for instance, Bowers *et al.* (1997). Under this model the aggregate claims cost is written as a random sum of individual claims, denoted *S*(*t*):

$$S(t) = \sum_{i=0}^{N(t)} X_i \ , \text{ where } X_0 \equiv 0, \qquad \textbf{(Eq. 8)}$$

where X_i is the *i*-th individual random claim and $N(t)$ is the number of claims in (0, *t*]. $\left\{N(t), t \geq 0\right\}$ is a stochastic counting process, $\left\{X_i, i = 1,2,...\right\}$ is a sequence of *iid* random variables, with common distribution function $G(x)$, and independent of $N(t)$, therefore $\left\{S(t), t \geq 0\right\}$ is a compound process.

In the classical model $N(t)$ follows a Poisson distribution. In the application we test both a Poisson and a negative binomial distributions using the classical χ^2 test. For the claim amount distribution we use both the χ^2 and the Kolmogorov tests, for Gama, Pareto and Lognormal distributions. For details see Klugman, Panjer and Wilmot (1998),

Once the distributions chosen, we simulate the process given by (8). For each simulated path we first generate a number $N(t) = n$, then generate the n values $X_i = x_i, i = 1,...,n$. The premium risk for the set of policies in force, considering its remaining time, is calculated from the difference between the aggregate claims cost and the expected premium reserve. For the sake of simplicity, and since the duration of the premium reserves in the automobile insurance is usually less than six months, we will neglect the interest rate risk of premium reserve.

For the **Aggregation of Risks** we assume that the joint distribution of risks follows a multivariate distribution belonging to the Elliptic Distribution Family, as in Embrechts, McNeil and Straumann (1999) and Embrechts, Lindskog and McNeil (2003). Thus, the dependence between risk factors is measured by their linear correlation coefficients and it will be so in the simulation procedures, whenever applicable.

3. Application

3.1 General considerations

Based on the statistical and financial information of an insurance undertaking at December 31, 2002, we modelled the five risk factors earlier explained. All the necessary technical information, the prospectus related to the insurer bond portfolio, the historical bond and stock prices, stock index figures and interest rates were collected from Bloomberg delivery information system. Given the lack of information to study the joint behaviour of the major risk factors we assumed that all risks were independent, with the exception of the equity and interest rate risks for which we studied their correlation. For confidential issues all monetary values related to the real insurance undertaking used in this study are *masked*.

3.2 Equity risk

The stock portfolio held by the insurer under study is composed by 11 listed companies from a single Euro Zone country. Therefore, we chose the main representative stock index for that country as a *proxy* of its relevant market portfolio with impact on the stocks' systematic risk. When estimating the stocks *beta* coefficients using the Market Model, we used daily closing prices from January 2, 1998 to December 31, 2002 adjusted for dividends, stock splits and other price factors. We tested all regression equations for their global significance (*F* test) and for all the individual parameters (*t* test) rejecting the null hypothesis at 5% significance level for all the cases. The results are presented in Table 1.

Table 1 - Estimates of the individual and portfolio's *betas*

Stock	#1	#2	#3	#4	#5	#6	#7	#8	#9	#10	#11	Port. *Beta*
Beta	0.43	0.83	0.75	0.85	0.48	0.72	0.95	1.76	1.24	1.12	1.30	0.81

The average and standard deviation of the stock index instantaneous rate of return for the same time interval were, respectively, 3.59% and 18.46% per annum, and the linear correlation coefficient between the instantaneous short-term interest rate and the instantaneous stock index rate of return was positive but small, and not statically significant.

Then, using the process given by equation 2 we ran 5,000 simulations for the one year stock index daily figures, assuming a *time step* = 1/260 per year. From the stock index simulated paths we could then estimate the simulated one year portfolio returns for the real insurance undertaking stock portfolio, according to the estimated *CAPM* parameters. The random component of the simulation process was based on the generation of standard *Normal* independent *r.v.*'s. As a *proxy* for the risk free interest rate we used a one year maturity German Treasury Bill yield, observed at December 31, 2002.

3.3 Interest rate and credit risks

We started by dividing the insurer's bond portfolio (entirely composed by Euro Zone bonds) into three sub-portfolios regarding the issuer's type: government bonds; bonds issued by banks and other financial institutions; and a single bond issued by one telecommunications company. These three groups were those actually observed within the real portfolio of the insurance undertaking that we are studying. Then, we estimated the weighted Fisher-Weil duration for each sub-portfolio. The risk free yield curve was extracted from a series of German zero coupon bonds (coupon strips) with different maturities, ranging from 1 to 27 years, and whose prices were observed at December 31, 2002. Any intermediate maturity yield was estimated by linear interpolation. For the 3 and 6-months maturities we used the German Treasury Bill yields observed at December 31, 2002 for these maturities.

Bond cash-flows were discounted by using a discount rate that adds the corresponding risk free maturity to the relevant credit spread, estimated in accordance to the industry sector and the issuers' rating of the bond. Credit spreads are regularly supplied by J.P. Morgan and could be found in www.riskmetrics.com. The credit spread for German Treasury Bill and Bonds was assumed to be negligible and, therefore, null.

In order to simplify the simulation of the bond portfolio, we assumed that the interest rate and the credit risk of holding any of the mentioned sub-portfolios was similar to the risk of holding an equivalent zero coupon bond with analogous duration.

Then we simulated the three zero coupon bond prices using the interest rate model explained in Section 2. As a *proxy* for the risk free short term interest rate we used the German Treasury Bill yield with 3-months maturity. In the estimation process of the parameter a we used the market prices of German Treasury Bills maturing in 3 and 6 months' time and the market prices of Coupon Strips of German Treasury Bonds maturing in 1, 2, 3, 4, 5, 6, 7, 8 e 9 years' time. Historical parameters were estimated using historical data from January 1, 1998 to at December 31, 2002 and the results were: $\hat{\hat{a}}$=3.0411, $\hat{b}$=0.1068, $\hat{a}$=2.9210 and $\hat{\sigma}$=0.0947.

We started by using equation (6) to simulate the daily behaviour of the risk free short term interest rate. The random component of the stochastic process was based on the generation of standard *Normal* independent variables.

After, we simulated the prices of the three zero coupon bonds using equation (4) considering no credit risk. Then, from equation (5), we calculated the corresponding risk free yield and the whole process was repeated 5,000 times, having generated 5,000 values for the one year risk free yield.

Afterwards, we applied the *Credit Metrics* model considering each actual sub-portfolio average credit rating (Table 2) in order to incorporate the credit risk spread into the simulations. Credit ratings were collected from Standard and Poor's and based upon the JP Morgan (www.riskmetrics.com) we built a rating transition probability matrix for the one year time frame.

Table 2 - Rating and duration of the bond's portfolio

Portfolio	Rating	Duration
Government Bonds	AAA	2.71
Bonds issued by banks and other financial institutions	AA	5.08
Bonds issued by telecommunication firms	A	2.05

In order to estimate the correlation coefficient matrix for the 3 sub-portfolios we used several *proxies*: the Dow Jones Euro Stoxx Bank Index rate of return as proxy for the banks and financial institutions bond portfolio; the bond itself for the telecom company bond; and the German coupon strip with 2.5 years maturity for the government bonds portfolio, whose duration was 2.71. The correlation matrix among the three sub-portfolios of bonds is shown in Table 3 and all the figures are significant at a 5% level.

As a result, we manage to generate, a set of correlated Standard Normal *r.v.*'s from a set of independent Standard Normal *r.v.*'s by applying the Cholesky decomposition, as in Horn and Johnson (1985).

Table 3 - Variance-covariance matrix of asset's rate of return proxies

	Governmental	Bank & fin. Inst.	Telecom. Company
Governmental	**0.00052**	-0.00253	-0.00197
Bank and financial inst.	-0.00253	**0.06446**	0.04020
Telecom. company.	-0.00197	0.04020	**0.16080**

Adding up the simulated credit spread to the simulated risk free yield for all zero coupon bonds we found the appropriate yield considering the credit risk. From this latter risky yield and inverting (5) we got the future value of each zero coupon yield, taking into account both interest rate and credit risks. Whenever the simulated rating was considered a default, we assumed the bond value to equal the credit recovery rate (simulated by a Beta distribution) times the face value of the bond. The process ends up by comparing each simulated value to its initial price in order to calculate the annual rate of return for each zero coupon bond and then by multiplying this rate of return by its respective market value at December 31, 2002.

3.4 Reserve risk

Our insurer' portfolio was recent and not yet stable. Thus, in order to apply any stochastic methods to the claims payments we had to exclude the occurrences for 1997 and 1998, because we know that the payment pattern of those years was significantly different.

We applied the Renshaw and Verral's (1998) model to the claims payments matrix, occurred between 1999 and 2002. We did the *quasi-deviance* scale test and concluded that the model was globally significant. We also did the individual parameter test, and concluded they were significant with one exception, the parameter associated with the effect of the occurrence year of 1999. One possible economic reason might be the fact that in this year claims were almost fully developed. Nevertheless, given that the matrix is not fully stable and that we only considered four occurrence years, we decided to keep this parameter in the model. The above results are presented in Tables 4 and 5.

Table 4 - Estimates of the parameters of the over-dispersed Poisson's model

Parameter	Estimate	Standard Error	Test of the nullity of the parameters		
			W	$\chi^2_{(1)}$ at 5%	Conclusion
U	15.4555	0.0737	44,025.29	3.84	Statistically significant
β_1	-0.8152	0.0828	97.01	3.84	Statistically significant
α_1	0.1298	0.0932	1.94	3.84	Not Statistically significant
β_2	-2.2827	0.1890	145.84	3.84	Statistically significant.
α_2	0.2337	0.0933	6.27	3.84	Statistically significant
β_3	-2.8761	0.3637	62.54	3.84	Statistically significant
α_3	0.5434	0.0979	30.83	3.84	Statistically significant
β_4	-2.3506	0.2836	68.70	3.84	Statistically significant

Table 5 - Scale deviance test

H_0: The model is adequate	
Scale deviance (D^*)	3.02
n	16
p	8

$\chi^2_{(n-p)}$ at 2.5%		15.50
$D^* <$ $\chi^2_{(n-p)}$ at 2.5%:	⇨	Accept H0

The *SE* of the total reserve was 14%, which is a value that we find acceptable since the matrix is not fully stable. The graphical representations of the residuals against each of the explanatory variables do not seem to show any systematic standards. As a conclusion, the over-dispersed Poisson model has an acceptable fit to the data.

We then applied a Bootstrap procedure associated with the validated model to the paid claims matrix. We simulated 5,000 paid claims matrices and calculated 5,000 values for undiscounted value of the claims reserve. We simulated 5,000 times the risk-free interest rate term structure in a one year period and determined the discounted value of the simulated claims reserve in December 31, 2003. We got the reserve risk results subtracting the simulated values at December 31, 2003 from the expected value of the discounted claims reserve at December 31, 2002 (for claims occurred 1999 and 2002). Regarding the *bootstrap* results, the *SE* of the estimated reserve is 15%, in line the results of the analytic model. The graphical representations of the residuals against each of the explanatory variables did not evidence any systematic pattern.

3.5 Premium risk

First we fit the distribution of the number of claims per year of the whole portfolio, based on data consisting the number of claims occurred per policy in the last year. We applied the Chi-square test (with a 5% significance level) to the mentioned Poisson and negative binomial distributions. The parameters of the distributions were estimated by maximum likelihood estimation (MLE).

From Table 6 we can see that the Poisson distribution was clearly rejected. The negative binomial was accepted, with a *p-value* of 24,3%. Assuming that the number of claims follows a negative binomial we determined the parameters of the distribution of the number of claims for the set of policies in force at December 31, 2002, $\alpha = 58,211$ and $p = 0.925$, corresponding to the sum *m idd* negative binomial, where *m* is the number of policies in force at that date.

Table 6 - Distributions for the number of claims per policy

No. of claims per policy				
Distribuition	**Parameters**	**Estimates**	**p-value**	**Degrees of freedom**
Poisson (λ)	λ	0.066	0.000	2
Negative binomial (α, p)	α, p	0.809, 0.925	0.244	3

Next, we studied, using a Chi-square test, at 5% level, the fit for the individual claim amount distribution, based on a list of total cost, claim by claim, of all the claims occurred and reported in 2002. We tested a Lognormal, Pareto and a Gamma distribution. We use the MLE for the Lognormal and both the moment and ML estimates for the Pareto and Gamma.

From Table 7 we see that the Gamma distribution was clearly rejected as well as the Pareto with ML estimation. The distribution that better fits the data is the Pareto, with parameters estimated by the moments method (MME), however with just one degree of freedom. The Lognormal was rejected, but if we exclude its tail (claim amounts above € 30,000) one observes that this distribution has a better fit than the Pareto. Hence, we also performed the Chi-square test to a Lognormal distribution truncated at 30,000 with a Pareto tail. The latter distribution gives a *p-value* of 20% and the χ^2 has two degrees of freedom.

Table 7 - Distributions for modeling the individual claim amount

Distribuition	Parameters	Estimates	p-value	Degrees of freedom
Lognormal, MLE	μ σ	6.702 1.346	0.007	2
Pareto, MME	α k	2.051 2,357.180	0.192	1
Pareto, MLE	α k	0.194 4.670	0	2
Gamma, MME	α β	0.025 89,374.516	0	2
Gamma, MLE	α β	7.483 299.604	0	2
Lognormal, MLE, truncated at € 30.000 with Pareto tail, MME	μ σ α β	6.702 1.346 2.051 2,357.180	0.199	2

In addition, we performed Kolmogorov tests, at 5% level, for the distributions Lognormal, Pareto and truncated Lognormal with Pareto tail. All distributions were accepted. Based on the results of both tests we chose the Lognormal distribution with Pareto tail to model the individual claim amounts. Results are shown in Table 8

Assuming that the number of claims followed a Negative Binomial distribution and that individuals claims amounts followed a Lognormal distribution with the “Pareto tail”, we simulated, using equation (8), 5,000 values for the aggregate total cost of claims of the policies in force at December 31, 2002, considering the remaining time that they will be in force. We deducted from the expected value of premium provision, the simulated values and achieved the results of the premium risk (suffiency/insuffiency of premium provision).

Table 8 - Results of the Kolmogorov test

Distribuition	**Test statistic**	**Critical value (5%)**	**Conclusion**
Lognormal, MLE	20.75%	33.84%	Accepted
Pareto, MME	16.33%	33.84%	accepted
Lognormal, MME, truncated at € 30,000, with Pareto tail, MME	20.75%	33.84%	Accepted

3.6 Aggregation of risks

For the aggregation of Risks and calculation of the ***VaR*** and ***TCE***, the global results for the insurer will be the result of the aggregation of all individual risk factors. We considered, risk by risk, each of the simulated values, obtaining 5,000 possible global results for the next year. Results are shown in the next section.

4. Results

In this section, we present the results from our application, considering the *VaR* and the *TCE* measures for one year time horizon and different alpha levels. We start by presenting the individual analysis for each risk factor and we conclude with the aggregated results for the entire portfolio of assets and liabilities of the company.

4.1 Equity risk

Table 9 - Equity risk results (euros)

Level	95.0%	97.5%	99.0%	99.5%

VaR	-21,437	-29,686	-39,189	-43,119
TCE	-31,885	-38,571	-45,432	-50,258

Starting with the equity risk we see from Table 9 that the results are in line with the expectations, given the model in use and the parameters estimates. Taking into account the reduced exposure to the stock market and the simulated results, the equity risk does not seem to be a menace to this particular insurer's solvency. The worst *VaR* loss scenario in a one year time period with a 99.5% level is €43,119 and the corresponding *TCE* is an expected loss of €50,258. This is quite small in relative terms as will we see later in this section after comparing these figures with the Reserve and Premium risks.

4.2 Interest rate and credit risks

Table 10 - Interest rate and credit risk results (euros)

Statistics	Governmental Bonds	Bonds issued by Banks and other Financial Institutions	Bonds issued by the Telecommunications company
VaR			
95.0%	429,139	84,444	29,883
97.5%	428,076	83,523	29,604
99.0%	426,668	82,561	29,206
99.5%	425,892	25,619	18,346
TCE			
95.0%	427,684	67,355	20,294
97.5%	426,694	50,734	10,846
99.0%	425,490	2,164	-17,093
99.5%	424,638	-75,536	-54,387

Table 10 shows that both interest rate and credit risks are also small for the modelled bond portfolio. In the worst *VaR* loss scenario in a one year time period with a 99.5% level the figures are all positive expressing no defaults, nor significant losses in the bond portfolio. Additionally, as the governmental bond portfolio (a high rating and low duration portfolio) was of much higher significance the total bond portfolio doesn't seem strongly affected by interest rate or credit risk.

However, when the *TCE* is computed, we experience a potential capital loss in terms of both corporate bonds portfolio. As we see, the *TCE* measure for the simulated figures in a one year time period with a 99.5% level is negative either for the portfolio of bonds issued by the banks and other financial institutions (-€75,536) either for the bond portfolio issue by the telecommunications company (-€54,387). Even though, the total bond portfolio is still positive in this scenario, as a result of the strong weight of the governmental bond portfolio. The negative results shown in the *TCE* measure are the result of the simulated bond ratings downgrading with a consequent raise in the required credit spreads.

4.3 Reserve risk

Table 11 - Reserve risk results (euros)

Level	95.0%	97.5%	99.0%	99.5%
VaR	-1,944,351	-2,409,572	-2,771,884	-2,931,341
TCE	-2,488,839	-2,743,515	-3,005,122	-3,158,818

Analysing Table 11, we can observe that there is some reserve risk arising from the most adverse development scenarios. The reserve risk follows approximately a Normal distribution and even tough it is the second more severe single risk factor; it does not have a heavy tail.

4.4 Premium risk

Table 12 - Premium risk Results (euros)

Level	95.0%	97.5%	99.0%	99.5%
VaR	-1,510,020	-1,916,298	-2,658,329	-3,146,273

TCE	-2,503,282	-3,319,229	-4,950,018	-7,020,996

From Table 12 we see that the premium risk is the single risk factor that presents future results more severe to the insurer. This is due to the heavy tail of the estimated distribution.

4.5 Determination of the aggregate *VaR* and *TCE*

Table 13 - Insurance undertaking's aggregate results (euros)

Level	95.0%	97.5%	99.0%	99.5%
VaR	-1,988,506	-2,488,230	-3,493,925	-3,981,377
TCE	-3,061,498	-3,920,409	-5,471,164	-7,201,006

From Table 13 we observe that the *TCE* is expectedly more conservative as a risk measure than *VaR*, by presenting capital requirements (clearly) higher. Taking *TCE* as risk measure at December 31, 2002, the insurance undertaking would need, at the confidence level of 99.5%, an economic capital of € 7,201,006, to be solvent.

The difference between *VaR* and *TCE* becomes more significant as the confidence level increases, this is due to the heavy tail of the global profit and loss distribution, and very influenced by the premium risk heavy tail. *TCE* is much more sensitive to heavy tailed distributions. The study of such distributions requires very special techniques and care, which discussion is beyond the scope of this work.

5. Conclusions

The model presented had the objective of showing that it is possible to build up a solvency model that determines the economic capital requirement using the risk measure

TCE, based upon the main risk factors that affect the insurance activity and the balance sheet structure of an insurance undertaking at a given time, for a selected time horizon.

In order to build up the model it is necessary to identify the assets, liabilities and operations that generate value, the risks that affect them, as well as the dependences among them. This procedure will lead to a more sound knowledge of the whole activity and structure of an insurance undertaking.

The construction and application of a solvency model for the automobile branch allowed observing that the results obtained depend heavily on the estimates of the involving parameters and on the data used in the estimation. The construction of a solvency model like this one will force insurance companies to invest considerably in human resources training, information technology, and on the access to databases with relevant and accurate information. Also important and sensitive are the correlations between risks for the calculation of capital requirements, since the benefits of an increased diversification might result into a considerable lower capital need.

As far as the model practical results are concerned, we conclude that the insurer had a conservative investment portfolio with limited interest rate and credit risks. Given the estimates of the parameters and the reduced exposure to the stock markets, the equity risk did not seem to influence the insurer's solvency. Nevertheless, by considering the volatility observed in stock markets in recent years, it is possible that the actual future results became less favourable than the simulated ones.

From the joint application of the over-dispersed Poisson model with the *bootstrap* procedure we concluded that the reserve risk is material to solvency of the insurance undertaking. Individually, the more potentially demanding risk is the premium risk, since the total aggregate claims cost distribution has a heavy tail, as we have already underlined at the end of the preceding section. As we noticed, heavy tail distributions need special care and an accurate estimation is not an easy task. An improper fit can lead to *unfair* capital requirement calculation, either excessive or defective.

Finally, we remark, that this is a static approach that assumes the maintenance of the current asset and liabilities structure, not taking into account, namely, new business underwritings. Thus, this analysis should be conducted periodically. In addition, we should point out that the *TCE* risk measure can be shown to be a lot more conservative than the *VaR* risk measure.

References

Artzner, P. "Application of Coherent Risk Measures to Capital Requirements in Insurance", *North American Actuarial Journal*, Vol.2 No.2 (1999), 11-25.

Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A., and Nesbitt, C.J. (1997); *Actuarial Mathematics*, 2^{n} Ed., Shaumburg, Il: The Society of Actuaries

Cox, J., J. Ingersoll and S. Ross, "A Theory of the Term Structure of Interest Rates", *Econometrica*, Vol.53, No.2 (1985), 373-407.

Elton, E., Gruber, M., Brown, S. and Goetzmann, W. (2007), *Modern Portfolio Theory and Investment Analysis*, 7th Ed., New York: John Wiley & Sons.

Embrechts, P., A. McNeil and D. Straumann, "Correlation: Pitfalls and Alternatives", *RISK*, Vol.12, No. 5 (1999), 69-71.

Embrechts, P., Lindskog, F. and McNeil, A. (2003), Modelling Dependence with Copulas and Applications to Risk Management. In: S. Rachev (Ed.) Handbook of Heavy Tailed Distributions in Finance. Elsevier.

England, P. and R. Verral, "Analytic and Bootstrap Estimates of Prediction Errors in Claims Reserving", *Insurance: Mathematics and Economics*, Vol.25, No.3 (1999), 281-293.

Fischer, T., A. May and B. Walther. "Simulation of the Yield Curve: Checking a Cox-Ingersoll-Ross Model", Pre-print No. 2226, Stochastik und Operations Research, Technische Universität Darmstadt, 2002.

Horn, R. and Johnson, C. (1990), *Matrix Analysis,* Cambridge University Press.

J.P. Morgan and Reuters, "Risk Metrics^{TM}", Technical Document, 4th Ed., Morgan Guaranty Trust Company, New York, 1996.

J.P. Morgan, CreditMetrics^{TM}, Technical Document, J. P. Morgan & Co Inc., New York, 1997,

Klugman, S., Panjer, H. and Wilmot G. (2008), *Loss Models, From Data to Decisions*, 3rd edn, Hoboke, NJ: John Wiley & Sons.

Linder, U. and V. Ronkainen "Solvency II, Towards a New Insurance Supervisory System in the EU", *Scandinavian Actuarial Journal*, Vol.104, No.6 (2004), pp. 462-474.

Lintner, J. "Security Prices, Risk and Maximal Gains from Diversification", *The Journal of Finance,* Vol.20, No. 4 (1965), 587-615.

Lynn Wirch, J. and M. Hardy. "A Synthesis of Risk Measures for Capital Adequacy", *Insurance Mathematics and Economics*, Vol.25, No.11 (1999), 337-347

McCullagh, P. and Nelder, J. (1989), *Generalised Linear Models*, 2.nd Ed., London: Chapman and Hall.

Pinheiro, P., J. Andrade e Silva and M. Centeno. "Bootstrap Methodology in Claims Reserving", *The Journal of Risk and Insurance*, Vol.70, No.4 (2003), 701-714.

Renshaw, A. and R. Verrall, "A Stochastic Model Underlying the Chain Ladder Technique, *British Actuarial Journal*, Vol.4, No.19 (1998), 903-923.

Sharpe, W. "Capital Asset Prices: A Theory of Market Equilibrium Under Condition of Risk", *The Journal of Finance*, Vol.19, No.3 (1964), 425-442.

Taylor, G. (2000) Loss Reserving: An Actuarial Perspective, Kluwer Academic.