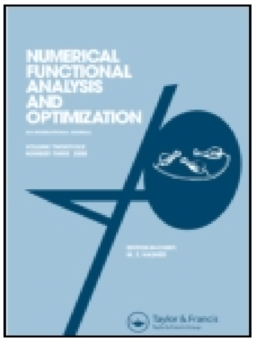

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# THE LAMBERT FUNCTION ON THE SOLUTION OF A DELAY DIFFERENTIAL EQUATION

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□ *Recently, a new method for computing the analytical solution of a delay differential equation was developed considering a constant initial function. It is based on the existence of a specific class of polynomials in the delay. In this article, we extend this new method to the case of a continuous initial function. We also show the relationship between the new solution's method and the solution expressed in terms of the Lambert function.*

**Keywords** Delay differential equation; Lambert function; Polynomials in the delay.

**AMS Subject Classification** 34K06; 33E20.

## 1. INTRODUCTION

Many examples of mathematical models describing population growth deal with time delays because in the growth process there exists some kind of lags that originate oscillatory behaviour. The classical logistic equation, an ordinary differential equation (ODE), and the delay logistic equation, a delay differential equation (DDE), are very good examples between different stability behaviors. The solutions of the logistic ODE converge monotonically to the *carrying capacity* value, while the solutions of the logistic DDE can exhibit three different kinds of behaviour: positive solutions monotonically decreasing to zero, oscillatory solutions tending to zero and oscillatory solutions tending to a periodic solution [3]. All these different patterns are conditioned with negative values of $rB$ in $x'(t) = Bx(t-r)$, where $r > 0$ is the delay of the DDE. This equation

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is obtained after linearization at the equilibrium of the logistic delay equation.

Delay differential equations are a special class of a differential equations called functional differential equations. The most important results about existence, uniqueness and properties of linear and nonlinear DDEs's solutions can be seen in [2, 5, 8].

Recently the Lambert $W$ function has been used to obtain closed form solutions of first order delay differential equations [1, 4].

In [6], we developed a new method to obtain the analytical solution of $x'(t) = Bx(t-r)$, based on the concept of generating function. The main results achieved are related to the existence of a special class of polynomials in the delay, $P_j^n(rB)$.

In the present work, we extend this new method for a given continuous function as an initial condition. We also show how the Lambert W function used in the solution of $x'(t) = Bx(t-r)$, through $x(t) = C\exp((W(rB)/r)t)$, [4], is related to the polynomials $P_j^n(rB)$.

This article is organized in the following way. Section 2 presents the theorem in which the polynomials $P_j^n(rB)$ are defined as well as the general result. In Section 3, the main result, which concerns with the Lambert function, is stated. In the last section, some conclusions are drawn.

## 2. A GENERATING FUNCTION APPROACH ON THE SOLUTION OF A DELAY DIFFERENTIAL EQUATION

Consider the first order delay differential equation with constant coefficients

$$\begin{cases} x'(t) = Bx(t-r), & t \geq 0 \\ x(t) = \phi(t), & t \in [-r, 0] \end{cases}, \tag{1}$$

where $B$ and $r$ are constants, $r > 0$ is the delay, and $\phi(t)$ is a given continuous function on $[-r, 0]$.

In [6] it was assumed that $\phi(t)$ was constant on $[-r, 0]$. When we apply the Method of Step Algorithm (MSA), the solutions $x_n(t)$ defined on $A_n = [(n-1)r, nr]$, $n \geq 1$, reveal a kind of a tree structure for the general solution $x(t)$ of the problem. This led to the statement that $x(t)$ is the generating function for a sequence of polynomials in the delay, $P_j^n(rB)$.

**Theorem 1.** *The solution of Problem* (1) *with* $\phi(t) = C$ *if* $t \in [-r, 0]$, *can be written as*

$$X(r, t) = \sum_{j \geq 0} v_j(r) t^j,$$

*for $t \geq 0$. The sequence $v_j(r)$ is defined by*

$$v_j(r) = C\frac{B^j}{j!}P_j^n(rB),$$

*where the polynomials $P_j^n(rB)$ are defined by*

$$P_j^n(rB) = \begin{cases} 1 + \sum_{i=0}^{n-(j+1)} \frac{(-rB)^{i+1}}{(i+1)!}(i+j)^{i+1} & \text{if } j \leq n-1 \\ 1 & \text{if } j = n \\ 0 & \text{if } j \geq n+1. \end{cases}$$

The complete version with detailed proofs can be seen in the forthcoming article [7].

There is a significant advantage of this theorem when compared to the traditional MSA. It provides, through the definition of $P_j^n(rB)$, an explicit formula for $x_n(t)$ on the interval $A_n$, without the need of computing all the solutions $x_{n-1}(t)$ defined on the previous intervals $A_{n-1}$.

In this article, we extend the previous method by considering a given continuous function $\phi(t)$ on $[-r, 0]$. If we apply the MSA we obtain the following solutions for $x_n(t)$, defined over $t \in A_n = ](n-1)r, nr]$: if $t \in ]0, r]$, we have

$$x_1(t) = \phi(0) + B\int_0^t \phi(s-r)ds, \tag{2}$$

if $t \in ]r, 2r]$ we have

$$\begin{aligned} x_2(t) = \phi(0)[1 + B(t-r)] + B\int_0^r \phi(s-r)ds \\ + B^2\int_r^t \int_0^{s_1 - r} \phi(s-r)dsds_1, \end{aligned} \tag{3}$$

and, if $t \in A_n = ](n-1)r, nr]$ for $n \geq 2$, we obtain

$$\begin{aligned} x_n(t) = \phi(0)\sum_{j=0}^{n-1}\frac{B^j}{j!}(t-jr)^j + B\int_0^r \phi(u-r)du\sum_{j=0}^{n-2}\frac{B^j}{j!}(t-(j+1)r)^j \\ + B^2\int_r^{2r}\int_0^{u-r}\phi(s-r)dsdu\sum_{j=0}^{n-3}\frac{B^j}{j!}(t-(j+2)r)^j + \cdots \\ + B^{n-1}\int_{(n-2)r}^{(n-1)r}\int_{(n-3)r}^{u_{n-2}-r}\cdots\int_r^{u_2-r}\int_0^{u_1-r}\phi(s-r)dsdu_1du_2\ldots du_{n-2} \end{aligned}$$

$$+ B^n \int_{(n-1)r}^{t} \int_{(n-2)r}^{u_{n-2}-r} \cdots \int_{(n-(n-1))r}^{u_{n-(n-1)}-r} \int_0^{u_0-r} \phi(s-r)\, ds du_0 \ldots du_{n-3} du_{n-2}.$$

We observe that a similar tree structure emerges, as for the constant initial condition. This fact allows us to state the following proposition, whose proof is easily derived by induction over $n$.

**Proposition 2.** *The solution of Problem* (1) *on each interval* $A_n = ((n-1)r, nr]$, $n \geq 1$, *is given by*

$$x_n(t) = \phi(0)\sum_{j=0}^{n-1} \frac{B^j}{j!}(t-jr)^j + \sum_{j=1}^{n} I_j^n[\phi(s-r)],$$

*where the integrals* $I_j^n[\phi(s-r)]$ *are defined by*

$$I_j^n[\phi(s-r)] = \begin{cases} \displaystyle\sum_{i=1}^{n-1} \frac{B^i}{(i-1)!}(t-ir)^{i-1} \int_0^r \phi(s-r)\, ds & \text{if } j = 1 \\ \displaystyle B^j \sum_{i=1}^{n-j} \frac{B^{i-1}}{(i-1)!}(t-(i+(j-1)r))^{i-1} \int_{(j-1)r}^{jr} \int_{(j-2)r}^{s_{j-1}-r} \cdots \int_0^{s_1-r} \phi(s-r)\, ds ds_1 \ldots ds_{j-1} & \text{if } 1 < j \leq n-1 \\ \displaystyle B^j \int_{(j-1)r}^{t} \int_{(j-2)r}^{s_{j-1}-r} \cdots \int_r^{s_2-r} \int_0^{s_1-r} \phi(s-r)\, ds ds_1 \ldots ds_{j-1} & \text{if } j = n \end{cases}$$

*for* $n > 1$, *and at* $x_1(t)$, $I_1^1[\phi(s-r)] = B\int_0^t \phi(s-r)\, ds$.

We notice that the definition of the integrals $I_j^n[\phi(s-r)]$ is similar to the definition of the polynomials $P_j^n(rB)$. Again, even if $\phi(t)$ is not a constant, it is possible to obtain any solution $x_n(t)$ independently of the previous $x_j(t)$, $1 \leq j < n$. We point out that if we combine the structure of a DDE with the method we used to solve it, we can find a new expression for the solution. This is connected with the recent research of the Lambert function and DDEs.

## 3. MAIN RESULT

The Lambert W function, denoted by $W(z)$, is defined as the inverse of the function $f(z) = ze^z$, satisfying

$$W(z)e^{W(z)} = z.$$

The function $W(z)$ is a multivalued function defined in general for $z$ complex and assuming complex values. The function is also represented by the power series

$$W(z) = \sum_{n\geq 1} \frac{(-n)^{n-1}}{n!} z^n \tag{4}$$

which is convergent for $|z| < 1/e$.

If we assume a solution of type $e^{\lambda t}$ for the equation $x' = Bx(t-r)$, then we have

$$\lambda - Be^{-\lambda r} = 0 \Leftrightarrow \lambda r e^{\lambda r} = rB \Leftrightarrow \lambda r = W(rB).$$

The solution of this DDE can then be expressed as

$$x(t) = Ce^{\frac{W(rB)}{r}t}.$$

The relevant question is how to obtain a unique solution involving the Lambert function, when we introduce an initial condition $\phi(t)$ on $[-r, 0]$.

Next, we prove the two main results of this article: First, a unique solution for problem (1), when we introduce an initial continuous function $\phi(t)$ on $[-r, 0]$, can be obtained in terms of the Lambert function. Second, this solution involves a differentiable function $M(t)$, over $t \geq 0$, which allows for an equivalence with the solution obtained by using the MSA method.

**Theorem 3.** *Consider the problem*

$$\begin{cases} x'(t) = Bx(t-r), & t \geq 0 \\ x(t) = \phi(t), & t \in [-r, 0], \end{cases}$$

*where $\phi(t)$ is a given continuous function, $B \in \mathbb{R}$ and $r > 0$. Let $M(t)$ be a differentiable function defined for $t \geq 0$, that satisfies:*

$$M(0) = \phi(0) \tag{5}$$

$$M'(t) = \begin{cases} B\phi(t-r)e^{-\frac{W(rB)}{r}t} - \dfrac{W(rB)}{r}M(t) & \text{if } t \in [0, r] \\ -Be^{-W(rB)}(M(t) - M(t-r)) & \text{if } t > r. \end{cases} \tag{6}$$

*Then the solution of the problem is given by*

$$X(t) = \begin{cases} \phi(t), & \text{if } t \in [-r, 0] \\ M(t)e^{\frac{W(rB)}{r}t}, & \text{if } t \geq 0. \end{cases} \tag{7}$$

*At $t = 0$, the derivative $M'(0)$ represents the right-hand side derivative and $W(rB)$ is the Lambert function satisfying $W(rB)e^{W(rB)} = rB$.*

*Proof.* To prove that $M(t)$ is a solution of the problem, we start by showing that $M'(t)$ is a continuous function on $\mathbb{R}_0^+$. From the properties of the continuous functions it follows that $M'(t)$ is continuous for $t \neq r$.

For $t = r$

$$\begin{aligned} M'(r) &= B\phi(0)e^{-W(rB)} - \frac{W(rB)}{r}M(r) \\ \lim_{t\to r^+} M'(t) &= -Be^{-W(rB)}M(r) + Be^{-W(rB)}M(0) \\ &= B\phi(0)e^{-W(rB)} - \frac{W(rB)}{r}M(r) = M'(r). \end{aligned}$$

Next we show that $X(t)$ in Equation (7) verifies the DDE.

For $t = 0$

$$\begin{aligned} X'(0) &= \lim_{h\to 0^+} \frac{X(0+h) - X(0)}{h} = \lim_{h\to 0^+} \frac{M(h)e^{\frac{W(rB)}{r}h} - \phi(0)}{h} = \\ &= \lim_{h\to 0^+} \left( M'(h)e^{\frac{W(rB)}{r}h} + M(h)\frac{W(rB)}{r}e^{\frac{W(rB)}{r}h} \right) \\ &= \lim_{h\to 0^+} \left( B\phi(h-r) - \frac{W(rB)}{r}M(h)e^{\frac{W(rB)}{r}h} + M(h)\frac{W(rB)}{r}e^{\frac{W(rB)}{r}h} \right) \\ &= B\phi(-r) = BX(-r), \end{aligned}$$

which means that $X'(t) = BX(t-r)$ at $t = 0$.

For $t \in ]0, r[$

$$\begin{aligned} X'(t) &= M'(t)e^{\frac{W(rB)}{r}t} + M(t)\frac{W(rB)}{r}e^{\frac{W(rB)}{r}t} \\ &= B\phi(t-r) - \frac{W(rB)}{r}M(t)e^{\frac{W(rB)}{r}t} + M(t)\frac{W(rB)}{r}e^{\frac{W(rB)}{r}t} \\ &= B\phi(t-r) = BX(t-r), \end{aligned}$$

therefore, $X'(t) = BX(t-r)$ holds for $t \in ]0, r[$.

For $t = r$

$$\begin{aligned} X'(r) &= M'(r)e^{W(rB)} + M(r)\frac{W(rB)}{r}e^{W(rB)} \\ &= B\phi(0) - \frac{W(rB)}{r}M(r)e^{W(rB)} + M(r)\frac{W(rB)}{r}e^{W(rB)} \\ &= B\phi(0) = BX(0), \end{aligned}$$

again $X'(t) = BX(t-r)$ at $t = r$.

For $t > r$

$$\begin{aligned} X'(t) &= M'(t)e^{\frac{W(rB)}{r}t} + M(t)\frac{W(rB)}{r}e^{\frac{W(rB)}{r}t} \\ &= -Be^{-W(rB)}(M(t) - M(t-r))e^{\frac{W(rB)}{r}t} + M(t)\frac{W(rB)}{r}e^{\frac{W(rB)}{r}t} \\ &= M(t-r)Be^{-W(rB)}e^{\frac{W(rB)}{r}t} = BM(t-r)e^{\frac{W(rB)}{r}(t-r)} = BX(t-r), \end{aligned}$$

again $X'(t) = BX(t-r)$ for $t > r$. □

Next we deal with the determination of an explicit form of $M(t)$.

For $t \in [0, r]$, we have an Initial Value Problem (IVP) consisting in Equation (5) and the first branch of Equation (6) yielding

$$\begin{aligned} M(t) &= e^{-\int_0^t \frac{W(rB)}{r}ds}\left\{\phi(0) + \int_0^t e^{\frac{W(rB)}{r}s}B\phi(s-r)e^{-\frac{W(rB)}{r}s}ds\right\} \\ &= e^{-\frac{W(rB)}{r}t}\left\{\phi(0) + B\int_0^t \phi(s-r)ds\right\}. \end{aligned}$$

Therefore,

$$M(t) = e^{-\frac{W(rB)}{r}t}\left\{\phi(0) + B\int_0^t \phi(s-r)ds\right\} \quad \text{for } t \in [0, r]. \tag{8}$$

Taking into account (2), we then have

$$M(t) = e^{-\frac{W(rB)}{r}t}x_1(t), \quad \text{for } t \in ]0, r]. \tag{9}$$

For $t > r$, we also have an IVP considering the second branch of Equation (6) and the initial condition given by (9)

$$M(r) = e^{-W(rB)}\left\{\phi(0) + B\int_0^r \phi(s-r)ds\right\}.$$

Therefore, for $t > r$

$$M(t) = e^{\int_r^t -Be^{-W(rB)}ds}\left\{M(r) + \int_r^t e^{Be^{-W(rB)}(s-r)}Be^{-W(rB)}M(s-r)\,ds\right\},$$

using integration by parts and simplifying we get

$$M(t) = M(t-r) + e^{-\frac{W(rB)}{r}t}\left\{\phi(0)[1 - e^{W(rB)}] + B\int_0^r \phi(s-r)\,ds - \int_r^t e^{\frac{W(rB)}{r}s}M'(s-r)\,ds\right\}, \quad \text{for } t > r. \quad (10)$$

We can obtain the exact form of $M(t)$ in the interval $]r, 2r]$, computing $M(t-r)$ by using (8)

$$M(t-r) = e^{-\frac{W(rB)}{r}(t-r)}\left\{\phi(0) + B\int_0^{t-r} \phi(s-r)\,ds\right\},$$

and then we have

$$M(t) = e^{-\frac{W(rB)}{r}t}\left\{\phi(0)[1 - e^{W(rB)}] + B\int_0^r \phi(s-r)\,ds - \int_r^t e^{\frac{W(rB)}{r}s}M'(s-r)\,ds + e^{W(rB)}[\phi(0) + B\int_0^{t-r}\phi(s-r)\,ds]\right\}.$$

Making $s - r = u$ in the first integral we obtain

$$M(t) = e^{-\frac{W(rB)}{r}t}x_2(t), \quad \text{for } t \in ]r, 2r]. \quad (11)$$

The successive application of the previous procedure, yielding Equations (9) and (11), defines the function $M(t)$.

**Proposition 4.** *Let $M(t)$ be a function defined on every interval $A_n = ](n-1)r, nr]$, with $n \geq 1$ by*

$$M(t)|_{t\in A_n} = e^{-\frac{W(rB)}{r}t}x_n(t),$$

*and $M(0) = \phi(0)$ at $t = 0$. Then $M(t)$ verifies conditions* (5) *and* (6) *of Theorem* 3.

*Proof.* At $t = 0$ we have

$$M'(0) = \lim_{h\to 0^+}\frac{M(0+h) - M(0)}{h} = \lim_{h\to 0^+}\frac{e^{-\frac{W(rB)}{r}h}x_1(h) - \phi(0)}{h}$$

$$= \lim_{h\to 0^+}\left[-\frac{W(rB)}{r}e^{-\frac{W(rB)}{r}h}x_1(h) + e^{-\frac{W(rB)}{r}h}x_1'(h)\right]$$
$$= -\frac{W(rB)}{r}\phi(0) + B\phi(-r).$$

At the interior points of $A_1 = ]0, r]$, we have

$$\begin{aligned} M'(t) &= -\frac{W(rB)}{r}e^{-\frac{W(rB)}{r}t}x_1(t) + e^{-\frac{W(rB)}{r}t}x_1'(t) \\ &= -\frac{W(rB)}{r}M(t) + Be^{-\frac{W(rB)}{r}t}\phi(t-r), \end{aligned}$$

by (2).

At $t = r$

$$\begin{aligned} M'(r^-) &= \lim_{h\to 0^-}\frac{M(r+h)-M(r)}{h} = \lim_{h\to 0^-}\frac{e^{-\frac{W(rB)}{r}(r+h)}x_1(r+h) - e^{-W(rB)}x_1(r)}{h} \\ &= \lim_{h\to 0^-}\left(-x_1(r+h)e^{-\frac{W(rB)}{r}(r+h)}\frac{W(rB)}{r} + e^{-\frac{W(rB)}{r}(r+h)}x_1'(r+h)\right) \\ &= -\frac{W(rB)}{r}e^{-W(rB)}x_1(r) + e^{-W(rB)}x_1'(r) \\ &= B\phi(0)e^{-W(rB)} - \frac{W(rB)}{r}M(r), \end{aligned}$$

and

$$\begin{aligned} M'(r^+) &= \lim_{h\to 0^+}\frac{M(r+h)-M(r)}{h} = \lim_{h\to 0^+}\frac{e^{-\frac{W(rB)}{r}(r+h)}x_2(r+h) - e^{-W(rB)}x_1(r)}{h} \\ &= \lim_{h\to 0^+}\left(-x_2(r+h)e^{-\frac{W(rB)}{r}(r+h)}\frac{W(rB)}{r} + e^{-\frac{W(rB)}{r}(r+h)}x_2'(r+h)\right) \\ &= \lim_{h\to 0^+}\left(-\frac{W(rB)}{r}M(r+h) + Be^{-W(rB)-\frac{W(rB)}{r}h}x_1(h)\right) \\ &= B\phi(0)e^{-W(rB)} - \frac{W(rB)}{r}M(r), \end{aligned}$$

which ends the proof for the first branch of Equation (6).

For $t > r$

$$\begin{aligned} M'(t) &= -\frac{W(rB)}{r}e^{-\frac{W(rB)}{r}t}x_n(t) + e^{-\frac{W(rB)}{r}t}x_n'(t) \\ &= -\frac{W(rB)}{r}M(t) + Be^{-\frac{W(rB)}{r}t}x_{n-1}(t-r) \end{aligned}$$

$$= -\frac{W(rB)}{r}M(t) + Be^{-W(rB)}M(t-r)$$
$$= -Be^{-W(rB)}[M(t) - M(t-r)],$$

which ends the proof for the second branch of Equation (6). □

**Corollary 5.** *The solution $X(t)$ of Problem (1), presented in Equation (7) of Theorem 3, verifies*

$$X(t)|_{t\in A_n} = x_n(t) \quad \textit{for } n \geq 1.$$

*Proof.* For every $n \geq 1$

$$X(t)|_{t\in A_n} = M(t)e^{\frac{W(rB)}{r}t}|_{t\in A_n}$$
$$= e^{-\frac{W(rB)}{r}t}x_n(t)e^{\frac{W(rB)}{r}t} = x_n(t),$$

by Proposition 4. □

## 4. CONCLUSION

In this article, we have defined a new class of integrals in the delay. This procedure follows from the application of the method of step algorithm.

The significant advantage of this method, as regards the traditional method, consists in allowing for an explicit formula for all $x_n(t)$, defined on each interval $A_n = ](n-1)r, nr]$, with $n \geq 1$.

In the future, we propose to extend this method to a much wider class of DDEs, such as $x'(t) = Ax(t) + Bx(t-r)$.

We also proved how the *Lambert W-function*, that is used in the solution of $x'(t) = Bx(t-r)$, through $x(t) = C\exp(W(rB)/r)t)$, is related to both the polynomials $P_j^n(rB)$ and the integrals $I_j^n[\phi(s-r)]$, which appear in the solutions of the DDE, depending on the initial condition, $\phi(t)$, being a constant or a general continuous function, respectively.

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