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On a Schrödinger system arizing in nonlinear optics

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Abstract
We study the nonlinear Schrödinger system

$$\begin{cases} iu_t + \Delta u - u + \left(\frac{1}{9}|u|^2 + 2|w|^2\right)u + \frac{1}{3}\overline{u}^2 w = 0, \\ i\sigma w_t + \Delta w - \mu w + (9|w|^2 + 2|u|^2)w + \frac{1}{9}u^3 = 0, \end{cases}$$

for $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, $1 \leq n \leq 3$ and $\sigma, \mu > 0$. This system models the interaction between an optical beam and its third harmonic in a material with Kerr-type nonlinear response. We prove the existence of ground state solutions, analyse its stability, and establish local and global well-posedness results as well as several criteria for blow-up.

Keywords Nonlinear Schrödinger systems · Blow-up · Ground states · Orbital stability

Mathematics Subject Classification 35Q60 · 35Q41 · 35Q51 · 35C07

1 Introduction

In recent years, cascading nonlinear processes have attracted an increasing interest. It is now well understood that this phenomena leads to effective higher-order nonlinearities in materials with $\chi^{(2)}$ and $\chi^{(3)}$ susceptibilities, in particular in the framework of second and third-order generation (see for instance [5,13,14,18,19] and references therein). In [20], Sammut et al. introduced a new model for the resonant interaction between

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a monochromatic beam with frequency ω propagating in a Kerr-type medium and its third harmonic (with frequency 3ω). The third-harmonic generation leads to features typical of non-Kerr $\chi^{(2)}$ media. We begin by briefly detailing its derivation. For a more thorough explanation of the computations and approximations involved we refer the reader to [4] and [20]. Let $(\vec{E}, \vec{B})$ the electromagnetic field, μ_0 and ϵ_0, respectively, the vacuum permeability and permittivity, c the speed of light in the vacuum and $vvecD$ the electric displacement vector. From the Maxwell–Faraday's equation

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}$$

and Ampère's Law (for nonmagnetic materials and in the absence of free currents)

$$\vec{\nabla} \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t},$$

we obtain

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} + \mu_0 \frac{\partial^2 \vec{D}}{\partial t^2} = 0.$$

Using the constitutive law $\vec{D} = \mathrm{n}^2 \epsilon_0 \vec{E} + 4\pi \epsilon_0 \vec{P}_{NL}$, where $\vec{P}_{NL}$ is the nonlinear part of the polarization vector and n the linear refractive index, the identity $\mu_0 \epsilon_0 c^2 = 1$ and noticing that $\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\Delta \vec{E} + \vec{\nabla}(\vec{\nabla} \cdot \vec{E})$, we get, after neglecting the last term in this identity, the vectorial wave equation

$$\Delta \vec{E} - \frac{\mathrm{n}^2}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \frac{4\pi}{c^2} \frac{\partial^2 \vec{P}_{NL}}{\partial t^2}. \tag{1.1}$$

Assuming that the beams propagate in a slab waveguide, in the direction of the (Oz) axis, we decompose one of the transverse directions of $\vec{E}$ in two frequency components as

$$E = \Re e\Big(E_1 e^{i(k_1 z - \omega t)} + E_3 e^{i(k_3 z - 3\omega t)}\Big),$$

where $\Re e(Z)$ stands for the real part of the complex number Z. Each one of these frequency components satisfy equation (1.1) for suitable values of the polarization, namely, $P_{NL}(\omega)e^{-i\omega t}$ and $P_{NL}(3\omega)e^{-3i\omega t}$, where the nonlinear polarization can be written in terms of the $\chi^{(3)}$ susceptibility as

$$P_{NL} = \chi^{(3)} E^3 = \chi^{(3)} \sum_{\omega_j} P_{NL}(\omega_j) e^{-\omega_j t}, \quad \omega_j = j\omega, |j| \leq 9.$$

A simple computation yields

$$P_{NL}(\omega)e^{-i\omega t} = \frac{1}{8}\chi^{(3)}(3|E_1|^2 E_1 + 6|E_3|^2 E_1 + 3E_3 \overline{E}_1^2 e^{-i(3k_1 - k_3)z}) e^{i(k_1 z - \omega t)}$$

and

$$P_{NL}(3\omega)e^{-3i\omega t} = \frac{1}{8}\chi^{(3)}(6|E_1|^2E_3 + 3|E_3|^2E_3 + E_1^3 e^{i(3k_1-k_3)z})e^{i(k_3z-3\omega t)}.$$

By plugging into (1.1) the quantities $E_1 e^{i(k_1z-\omega t)}$ and $E_3 e^{i(k_3z-3\omega t)}$, and under the slowly-varying amplitude approximation, we obtain the system

$$\begin{cases} \Delta_\perp E_1 + 2ik_1\frac{\partial E_1}{\partial z} + \Big(\frac{(\mathrm{n}(\omega))^2\omega^2}{c^2} - k_1^2\Big)E_1 + \chi(|E_1|^2E_1 + 2|E_3|^2E_1 + E_3\overline{E}_1^2 e^{-i(3k_1-k_3)z}) = 0 \\ \Delta_\perp E_3 + 2ik_3\frac{\partial E_3}{\partial z} + \Big(\frac{9(\mathrm{n}(3\omega))^2\omega^2}{c^2} - k_3^2\Big)E_3 + 9\chi(2|E_1|^2E_3 + |E_3|^2E_3 + \frac{1}{3}E_1^3 e^{i(3k_1-k_3)z}) = 0, \end{cases}$$

where $\chi = \dfrac{3\pi\omega^2\chi^{(3)}}{2c^2}$.

Using the dispersion relations $k_1^2 = \frac{(\mathrm{n}(\omega))^2\omega^2}{c^2}$, $k_3^2 = \frac{9(\mathrm{n}(3\omega))^2\omega^2}{c^2}$ and introducing the dimensionless variables $t = z_d z$, $(x_1, x_2) = x_0(x, y)$ for a given beam width x_0 with associated diffraction length $z_d = 2x_0^2 k_1$, this system can be reduced to

$$\begin{cases} iU_t + \Delta U + \left(\frac{1}{9}|U|^2 + 2|W|^2\right)U + \frac{1}{3}\overline{U}^2 W = 0, \\ i\sigma W_t + \Delta W - \alpha\sigma W + \left(9|w|^2 + 2|u|^2\right)W + \frac{1}{9}U^3 = 0, \end{cases} \tag{1.2}$$

where $U = 3(k_1x_0\chi)^{\frac{1}{2}}E_1$, $W = 3(k_1x_0\chi)^{\frac{1}{2}}E_3e^{-i(3k_1-k_3)z})$, $\sigma = k_3/k_1$ and $\alpha = 2k_1(3k_1-k_3)x_0^2$.

Finally, considering the nonlinearity-induced propagation constant β, and introducing u and w trough the relations

$$U(x,t) = \sqrt{\beta}e^{i\omega t}u(\sqrt{\beta}x, \sqrt{\beta}t), \quad W(x,t) = \sqrt{\beta}e^{i3\omega t}w(\sqrt{\beta}x, \sqrt{\beta}t),$$

we get the nonlinear Schrödinger system

$$\begin{cases} iu_t + \Delta u - u + \left(\frac{1}{9}|u|^2 + 2|w|^2\right)u + \frac{1}{3}\overline{u}^2 w = 0, \\ i\sigma w_t + \Delta w - \mu w + \left(9|w|^2 + 2|u|^2\right)w + \frac{1}{9}u^3 = 0, \end{cases} \tag{1.3}$$

where $\mu = (3 + \frac{\alpha}{\beta})\sigma$. Note that at resonance ($k_3 = 3k_1$), $\sigma = 3$ and $\mu = 3\sigma$. This equality will play a major role in several results presented in this paper.

From a mathematical point of view, the system (1.3) has been studied in [1] and [20] in one space dimension. In [1], the authors established local and global well-posedness results for the associated Initial Value Problem with periodic initial data. Furthermore, they showed the existence of smooth curves of periodic standing-wave solutions (dnoidal waves) and proved several results concerning their linear and nonlinear stability. In [20], the linear stability of localized stationary solutions was adressed and some numerical simulations presented.

In the present paper we are concerned with the study of (1.3) in Euclidean space $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, $1 \leq n \leq 3$. Our main goal is to study the Cauchy problem associated with (1.3) in the L^2-based Sobolev space of order one, $H^1(\mathbb{R}^n)$, the so-called *energy space*. This terminology comes from the fact that such a system conserves the energy functional

$$E(u, w) = \frac{1}{2}\int \left(|\nabla u|^2 + |\nabla w|^2 + |u|^2 + \mu|w|^2\right) - \int \left(\frac{1}{36}|u|^4 + \frac{9}{4}|w|^4 + |u|^2|w|^2 + \frac{1}{9}\Re e(\overline{u}^3 w)\right) \tag{1.4}$$

and the mass

$$M(u, w) = \int \left(|u|^2 + 3\sigma|w|^2\right). \tag{1.5}$$

It is well-known that for Schrödinger-type equations with cubic nonlinearities, the space dimension $n = 2$ is critical in the sense that global existence in the energy space is guaranteed provided that the initial data has L^2 norm below the one of the ground state (see for instance [24]). Hence, since we are interested in addressing this type of issue for (1.3), the associated stationary problem must also be studied. Recall that standing waves are special solutions of (1.3) of the form

$$u(x, t) = e^{i\omega t}P(x), \qquad w(x, t) = e^{3i\omega t}Q(x), \tag{1.6}$$

where P and Q are real functions with a suitable decay at infinity. By replacing (1.6) into (1.3) we see that (P, Q) must satisfy

$$\begin{cases} \Delta P - (\omega + 1)P + \left(\frac{1}{9}P^2 + 2Q^2\right)P + \frac{1}{3}P^2 Q = 0, \\ \Delta Q - (\mu + 3\sigma\omega)Q + \left(9Q^2 + 2P^2\right)Q + \frac{1}{9}P^3 = 0. \end{cases} \tag{1.7}$$

The rest of this paper is organized as follows: in Sect. 2 we will show the existence of solutions for (1.7) and study their properties. By a *solution* of (1.7) we mean a pair of functions $(P, Q) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ such that

$$\int (\nabla P \cdot \nabla f + (\omega + 1)Pf = \int \left(\frac{1}{9}P^3 + 2Q^2 P + \frac{1}{3}P^2 Q\right) f$$

and

$$\int (\nabla Q \cdot \nabla g + (\mu + 3\sigma\omega)Qg = \int \left(9Q^3 + 2P^2 Q + \frac{1}{9}P^3\right) g,$$

for any pair $(f, g) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$. So, a solution is *a priori* understood in the weak sense. However, as it is standard from the elliptic regularity theory, such a weak solution is indeed a strong solution in the usual sense (see, for instance, [9]). It is easy to check that solutions of (1.7), also called *bound states*, are the critical points of the action functional defined by

$$S(P, Q) := E(P, Q) + \frac{\omega}{2} M(P, Q), \tag{1.8}$$

that is, denoting by $\mathcal{B} = \mathcal{B}(\omega, \mu, \sigma)$ the set of all solutions of (1.7), we have

$$\mathcal{B}(\omega, \mu, \sigma) := \{(P, Q) \in H^1 \times H^1 \ : \ S'(P, Q) = 0\}.$$

Among all bound states, we will single out the *ground states*, i.e., the bound states which minimize the action S among all other bound states. We will prove that such a set of solutions is indeed nonempty (Theorem 2.1). The method we use to prove this result is a variational one, by minimizing S in the so-called *Nehari manifold*. In addition, we also study when a ground state has both components nontrivial.

In Sect. 3 we study the Cauchy problem associated to (1.3) for initial data in the energy space $(u_0, w_0) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$. After establishing local-well posedness and a blow-up alternative (Theorem 3.1) we show that the Cauchy problem is globally well-posed in dimension $n = 1$ (Corollary 3.2). In what concerns dimensions $n = 2$ and $n = 3$, we will give sufficient conditions for global well-posedness in terms of the size of the initial data with respect to the size of ground states at resonance $\mu = 3\sigma$ (Theorems 3.8 and 3.10).

In Sect. 4 we study the blow-up of solutions to (1.3). We will begin by showing in Theorem 4.2 that, at resonance, Theorem 3.8 is sharp. In dimension $n = 3$, we also show that Theorem 3.10 is sharp at resonance provided that the initial data (u_0, w_0) lies in $\mathbb{H} = H^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^n, |x|^2 dx)$ (Theorem 4.6). Moreover, we exhibit several conditions implying that the solution blows up either forward or backward in time (Theorems 4.7 and 4.8).

Finally, in Sect. 5, we deal with the stability/instability of the ground states (P, Q). We will show that the ground states are orbitally stable in dimension one provided $\omega + 1 = \mu + 3\sigma\omega$ (Theorem 5.4). On the other hand, we prove that ground states are unstable if either $n = 3$ and $\mu > 0$ or $n = 2$ and $\mu \neq 3\sigma$.

Throughout the paper we will use standard notation in PDEs. Unless otherwise stated, the domain of the different integrals is $\mathbb{R}^n$, hence, for convenience, we will denote $\int_{\mathbb{R}^n} f dx$ simply by $\int f$. Also, C will represent a generic constant which may vary from inequality to inequality.

2 Existence of ground states

The main goal of this section is to prove the existence of ground states. More precisely, we will establish the following result:

Theorem 2.1 *Let $1 \le n \le 3$, $\sigma, \mu > 0$ and $\omega > \max\{-1, -\mu/3\sigma\}$. Then the set of ground states, denoted by $\mathcal{G}(\omega, \mu, \sigma)$, is nonempty, that is,*

$$\mathcal{G}(\omega, \mu, \sigma) := \Big\{(P_0, Q_0) \in \mathcal{B} \setminus \{(0,0)\} \,:\, S(P_0, Q_0) \le S(P, Q), \forall (P, Q) \in \mathcal{B}, \Big\} \neq \emptyset.$$

In addition, there exists at least one ground state, say, (P_0, Q_0), which is radially symmetric, Q_0 is positive and P_0 is either positive or identically zero.

Before proceeding, let us establish some Pohojaev-type identities for the solutions of (1.7), which will be useful later.

Lemma 2.2 *Assume that (1.7) has a solution $(P, Q) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$. Then the following identities hold:*

$$\int \left(-|\nabla P|^2 - (\omega + 1) P^2 + \frac{1}{9} P^4 + 2P^2 Q^2 + \frac{1}{3} P^3 Q \right) = 0, \tag{2.1}$$

$$\int \left(-|\nabla Q|^2 - (\mu + 3\sigma\omega) Q^2 + 9Q^4 + 2P^2 Q^2 + \frac{1}{9} P^3 Q \right) = 0, \tag{2.2}$$

and

$$(n-4) \int \left(|\nabla P|^2 + |\nabla Q|^2 \right) + n(\omega + 1) \int P^2 + n(\mu + 3\sigma\omega) \int Q^2 = 0. \tag{2.3}$$

Proof By multiplying the first equation in (1.7) by P, the second one by Q, integrating over $\mathbb{R}^n$ and using integration by parts, we obtain (2.1) and (2.2).
On the other hand, by the same procedure but multiplying this time the two equations by $x \cdot \nabla P$ and $x \cdot \nabla Q$ respectively, we deduce

$$\int \left(\frac{(n-2)}{2} |\nabla P|^2 + \frac{n(\omega + 1)}{2} P^2 - \frac{n}{36} P^4 + 2Q^2 P x \cdot \nabla P + \frac{1}{3} P^2 Q x \cdot \nabla P \right) = 0 \tag{2.4}$$

and

$$\int \left(\frac{(n-2)}{2} |\nabla Q|^2 + \frac{n(\mu + 3\sigma\omega)}{2} Q^2 - \frac{9n}{4} Q^4 + 2P^2 Q x \cdot \nabla Q + \frac{1}{9} P^3 x \cdot \nabla Q \right) = 0. \tag{2.5}$$

Now, integration by parts yields

$$\int \left(2P^2 Q x \cdot \nabla Q + \frac{1}{9} P^3 x \cdot \nabla Q \right) = -\int \left(2Q^2 P x \cdot \nabla P + \frac{1}{3} P^2 Q x \cdot \nabla P + n P^2 Q^2 + \frac{n}{9} P^3 Q \right).$$

By replacing this last identity into (2.5) and summing the resulting equation with (2.4),

$$\frac{(n-2)}{2}\int(|\nabla P|^2+|\nabla Q|^2)+\frac{n(\omega+1)}{2}\int P^2+\frac{n(\mu+3\sigma\omega)}{2}\int Q^2 - \frac{n}{4}\int\left(\frac{1}{9}P^4+9Q^4+4P^2Q^2+\frac{4}{9}P^3Q\right)=0. \tag{2.6}$$

Also, summing equations (2.1) and (2.2), we obtain

$$\int\left(\frac{1}{9}P^4+9Q^4+4P^2Q^2+\frac{4}{9}P^3Q\right)=\int(|\nabla P|^2+|\nabla Q|^2)+\int\left((\omega+1)P^2+(\mu+3\sigma\omega)Q^2\right). \tag{2.7}$$

Identity (2.3) then follows by combining (2.7) and (2.6). □

Remark 2.3 As an immediate consequence of Lemma 2.2 we see that, under the assumption $\omega > \max\{-1, -\mu/3\sigma\}$, ground state solutions in $H^1(\mathbb{R}^n)\cap L^4(\mathbb{R}^n)$ do not exist if $n \geq 4$.

In order to prove Theorem 2.1, we will study a minimization problem in the Nehari manifold.

Lemma 2.4 *Let*

$$\mathcal{N} := \{(u, w) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \,:\, (u, w) \neq (0, 0),\, S'(u, w) \perp (u, w)\}$$

be the Nehari manifold associated to the action S. Then any solution of the minimization problem

$$\inf\{S(u, w) \,:\, (u, w) \in \mathcal{N}\}, \tag{2.8}$$

is a ground state.

Proof Since $\mathcal{B} \subset \mathcal{N}$, it is enough to prove that all critical points of (2.8) are indeed bound states.
We begin by noticing that $(u, w) \in \mathcal{N}$ if and only if $(u, w) \neq (0, 0)$ and

$$\tau(u, w) :=\int |\nabla u|^2+|\nabla w|^2+(1+\omega)u^2+(\mu+3\sigma\omega)w^2 -\frac{1}{9}u^4-4u^2w^2-9w^4-\frac{4}{9}u^3w=0. \tag{2.9}$$

Furthermore,

$$\langle\tau'(u, w), (u, w)\rangle_{L^2}=2\left(\int |\nabla u|^2+|\nabla w|^2+(1+\omega)u^2+(\mu+3\sigma\omega)w^2 -\frac{2}{9}u^4-8u^2w^2-18w^4-\frac{8}{9}u^3w\right),$$

and, if $(u, w) \in \mathcal{N}$,

$$\langle \tau'(u, w), (u, w)\rangle_{L^2} = -2\Big(\int |\nabla u|^2 + |\nabla w|^2 + (1+\omega)u^2 + (\mu + 3\sigma\omega)w^2\Big) \neq 0, \tag{2.10}$$

which shows that $\mathcal{N}$ is locally smooth.

In addition, it is easy to check that $[h_1, h_2]\text{Hess}\,\tau_{(0,0)}\,{}^t[h_1, h_2] > 0$ for all $(h_1, h_2) \neq (0, 0)$, which means that $(0, 0)$ is a strict minimizer of τ, hence an isolated point of the set $\{\tau(u, w) = 0\}$, implying that $\mathcal{N}$ is a complete manifold. Finally, any critical point of S constrained to $\mathcal{N}$ is a (unconstrained) critical point of S. Indeed, let us consider $(u_0, w_0) \in \mathcal{N}$ a critical point of S constrained to $\mathcal{N}$. There exists a Lagrange multiplier λ such that $S'(u_0, w_0) = \lambda\tau'(u_0, w_0)$. By taking the L^2 scalar product with (u_0, w_0),

$$\langle S'(u_0, w_0), (u_0, w_0)\rangle_{L^2} = \lambda\langle \tau'(u_0, w_0), (u_0, w_0)\rangle_{L^2},$$

that is, in view of (2.10), $0 = -2\lambda\Big(\int |\nabla u_0|^2 + |\nabla w_0|^2 + (1+\omega)u_0^2 + (\mu + 3\sigma\omega)w_0^2\Big)$. Hence $\lambda = 0$ and $S'(u_0, w_0) = 0$, which establishes the claim. □

As a consequence of Lemma 2.4, in order to show Theorem 2.1 we will prove the existence of a minimizer to problem (2.8).

Proof (Proof of Theorem 2.1)

Notice that for $(u, w) \in H^1 \times H^1$, $(u, w) \neq (0, 0)$, with $\tau(u, w) \leq 0$, there exists $t \in]0, 1]$ such that $(tu, tw) \in \mathcal{N}$. Indeed, if $\tau(u, w) = 0$, one chooses $t = 1$. If $\tau(u, w) < 0$ we simply observe that

$$\begin{aligned}\tau(tu, tw) = t^2 \Bigg\{\int \Big[&|\nabla u|^2 + |\nabla w|^2 + (1+\omega)u^2 + (\mu + 3\sigma\omega)w^2 \\ &-t^2\left(\frac{1}{9}u^4 + 4u^2w^2 + 9w^4 + \frac{4}{9}u^3 w\right)\Big]\Bigg\} := t^2 T_{u,w}(t),\end{aligned}$$

with $T_{u,w}(0) > 0$ and $T_{u,w}(1) < 0$. The Intermediate Value Theorem allows us to conclude.

We now take a minimizing sequence $(u_j, w_j) \in \mathcal{N}$ for the problem

$$m = \inf\{S(u, w) : (u, w) \in \mathcal{N}\}.$$

Since $(u_j, w_j) \in \mathcal{N}$,

$$S(u_j, w_j) = \frac{1}{4}\left(\int |\nabla u_j|^2 + |\nabla w_j|^2 + (1+\omega)u_j^2 + (\mu + 3\sigma\omega)w_j^2\right),$$

hence it is clear that $m \geq 0$ and that (u_j, w_j) is bounded in $H^1 \times H^1$.

In order to get compactness we will replace (u_j, w_j) by a suitable symmetric rearrangement (see, for instance, [22]). Indeed, let u_j^* and w_j^* be the decreasing radial rearrangements of $|u_j|$ and $|w_j|$, respectively. It is well-known that this rearrangement preserves the L^p norm $(1 \leq p \leq +\infty)$. Furthermore, the Pólya-Szegö inequality,

$$\|\nabla f^*\|_{L^2} \leq \|\nabla |f|\|_{L^2},$$

in addition with the inequality $\|\nabla |f|\|_{L^2} \leq \|\nabla f\|_{L^2}$ (see [16]) shows that

$$S(u_j^*, v_j^*) \leq S(u_j, v_j).$$

On the other hand, the Hardy-Littlewood inequality,

$$\int |uw| \leq \int u^* w^*,$$

combined with the monotonicity of the map $\lambda \mapsto \lambda^4$ (see for instance [12] for details) yields

$$\int u^2 w^2 \leq \int (u^*)^2 (w^*)^2 \quad \text{and} \quad \int |u^3 w| \leq \int (u^*)^3 w^*.$$

A combination of these inequalities give

$$\tau(u_j^*, w_j^*) \leq \tau(|u_j|, |w_j|) \leq \tau(u_j, w_j) = 0.$$

Next, let $t_j \in]0, 1]$ be such that $(t_j u_j^*, t_j w_j^*) \in \mathcal{N}$. We have

$$S(t_j u_j^*, t_j w_j^*) = t_j^2 S(u_j^*, w_j^*) \leq S(u_j^*, w_j^*)$$

and hence, we obtained a minimizing sequence $(t_j u_j^*, t_j v_j^*)$ of radially decreasing functions, denoted again, in what follows, by (u_j, v_j). Since this sequence is bounded in $H^1 \times H^1$, up to a subsequence, $(u_j, v_j) \rightharpoonup (u_*, v_*)$ weakly in $H^1 \times H^1$.

To obtain a convergence in a strong topology, it is often necessary to treat the unidimensional $n = 1$ separately due to the lack of compactness of the injection $H_d^1(\mathbb{R}) \hookrightarrow L^4(\mathbb{R})$, where $H_d^1(\mathbb{R})$ denotes the space of the radially symmetric functions of $H^1(\mathbb{R})$. This lack of compactness is, in a sense, a consequence of the inequality

$$|u(x)| \leq C|x|^{\frac{1-n}{2}} \|u\|_{H^1(\mathbb{R}^n)} \tag{2.11}$$

for $u \in H_d^1(\mathbb{R}^n)$, which provides no decay in the case $n = 1$. However, if u is also radially decreasing, it is easy to establish that

$$|u(x)| \leq C|x|^{-\frac{n}{2}} \|u\|_{L^2(\mathbb{R}^n)},$$

which provides decay in all space dimensions, hence compactness by applying the classical Strauss' compactness lemma ([21]). Therefore, putting

$$H^1_{rd}(\mathbb{R}^n) = \{u \in H^1_d(\mathbb{R}^n) \,:\, u \text{ is radially decreasing}\},$$

we get the compactness of the injection $H^1_{rd}(\mathbb{R}^n) \hookrightarrow L^4(\mathbb{R}^n)$ for all $n \geq 1$ (see the Appendix of [3] or Section 1.7 in [6] for more details). Consequently, up to a subsequence, $(u_j, v_j) \to (u_*, v_*)$ strongly in L^4 and almost everywhere. In particular this shows that (u_*, v_*) is radially symmetric and nonnegative.

Next, since

$$\int \frac{1}{36}u_j^4 + \frac{9}{4}w_j^4 + u_j^2 w_j^2 + \frac{1}{9}u_j^3 w_j \to \int \frac{1}{36}u^4 + \frac{9}{4}w^4 + u^2w^2 + \frac{1}{9}u^3 w,$$

we deduce that

$$\tau(u_*, w_*) \leq \liminf \tau(u_j, w_j) = 0.$$

Once again, let $t \in]0, 1]$ such that $(tu_*, tv_*) \in \mathcal{N}$. Thus,

$$m \leq S(tu_*, tw_*) = t^2 S(u_*, w_*) \leq \liminf S(u_j, v_j) = m.$$

This implies that (tu_*, tw_*) is a minimizer. In particular, all inequalities above are in fact equalities, which means that $t = 1$, $(u_*, w_*) \in \mathcal{N}$ and $(u_j, w_j) \to (u_*, w_*)$ strongly in H^1.

Finally, it is easy to see that $(P_0, Q_0) = (u_*, w_*)$ is a ground state accordingly to the conclusions of the theorem. Indeed, by elliptic regularity (P_0, Q_0) is a C^2 solution and satisfies

$$\begin{cases} \Delta P_0 - (\omega + 1)P_0 = -(\frac{1}{9}P_0^2 + 2Q_0^2)P_0 - \frac{1}{3}P_0^2 Q_0 \leq 0, \\ \Delta Q_0 - (\mu + 3\sigma\omega)Q_0 = -(9Q_0^2 + 2P_0^2)Q_0 - \frac{1}{9}P_0^3 \leq 0. \end{cases}$$

Therefore, from the maximum principle (see, for example, Theorem 3.5 in [9]) both P_0 and Q_0 are either positive or identically zero. Note that Q_0 is not identically zero; otherwise so is P_0. This completes the proof of Theorem 2.1. □

Next we will pay particular attention to the question of when both components of a ground state are non-trivial. First of all, recall that a ground state of the scalar equation

$$\Delta w - (\mu + 3\sigma\omega)w + 9w^3 = 0, \tag{2.12}$$

is a solution (in the weak sense) that minimizes the action $S_0(w) := S(0, w)$ among all solutions of (2.12). As is well known (see, for instance, [3] or [6]), for $\mu + 3\sigma\omega > 0$, (2.12) has a unique (up to translation) ground state which is positive, radially symmetric and decays exponentially at infinity.

It is easily seen that if $(0, Q)$ is a ground state of (1.7) then Q is a ground state of (2.12). Thus, a natural question is if the reciprocal is also true, that is, if Q is a ground state of (2.12), is it true that $(0, Q)$ is a ground state of (1.7)? As we will see below, depending on the parameters μ and σ, the answer to this question may be negative or positive:

Proposition 2.5 *In addition to the assumptions of Theorem 2.1, assume $\mu = 3\sigma$ and $\mu \geq 9^{\frac{4}{4-n}}$. Then there exists a pair (P^*, Q^*) in the Nehari manifold $\mathcal{N}$ such that*

$$S(P^*, Q^*) < S(0, Q),$$

where Q is the ground state of (2.12). In particular $(0, Q)$ is not a ground state of (1.7).

Proof In what follows, for real functions $u, w \in H^1$, we introduce the functional

$$N(u, w) := \int \left(\frac{1}{36}u^4 + \frac{9}{4}w^4 + u^2w^2 + \frac{1}{9}u^3w \right). \tag{2.13}$$

and

$$K(u, w) = \|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2. \tag{2.14}$$

According to Lemma 2.4 it suffices to prove the existence of $\theta, t \in \mathbb{R}$ and $W \in H^1$ such that $(t\theta W, tQ) \in \mathcal{N}$ and $S(t\theta W, tQ) < S(0, Q)$. But from the proof of Lemma 2.4 we have $(t\theta W, tQ) \in \mathcal{N}$ if and only if $\tau(t\theta W, tQ) = 0$, where τ is defined in (2.9). Since

$$\tau(t\theta W, tQ) = K(t\theta W, tQ) + (1 + \omega)M(t\theta W, tQ) - 4N(t\theta W, tQ),$$

by taking $t \in \mathbb{R}$ satisfying

$$t^2 = \frac{K(\theta W, Q) + (1 + \omega)M(\theta W, Q)}{4N(\theta W, Q)} \tag{2.15}$$

we see that $\tau(t\theta W, tQ) = 0$ (we will choose $\theta > 0$ and $W > 0$, so that $N(\theta W, Q) > 0$). Consequently, from this point on, we take t as in (2.15).

Now, in view of the identity,

$$K(t\theta W, tQ) + (1 + \omega)M(t\theta W, tQ) = 4N(t\theta W, tQ)$$

and (2.15), we deduce

$$\begin{aligned} S(t\theta W, tQ) &= \frac{1}{2}\Big(K(t\theta W, tQ) + (1+\omega)M(t\theta W, tQ)\Big) - N(t\theta W, tQ)) \\ &= \frac{1}{4}\Big(K(t\theta W, tQ) + (1+\omega)M(t\theta W, tQ)\Big) \\ &= \frac{t^2}{4}\Big(K(\theta W, Q) + (1+\omega)M(\theta W, Q)\Big) \\ &= \frac{\Big(K(\theta W, Q) + (1+\omega)M(\theta W, Q)\Big)^2}{16N(\theta W, Q)}. \end{aligned}$$

Thus $S(t\theta W, tQ) < S(0, Q)$ if and only if

$$\Big(K(\theta W, Q)+(1+\omega)M(\theta W, Q)\Big)^2 < 4N(\theta W, Q)\Big(K(0, Q)+(\omega+1)M(0, Q)\Big), \tag{2.16}$$

where we used that $S(0, Q) = (K(0, Q) + (\omega+1)M(0, Q))/4$. Both sides of (2.16) are polynomials of degree four in θ. The leading coefficient of the polynomial in the left-hand side is $(K(W, 0) + (\omega+1)M(W, 0))^2$ whereas the leading coefficient of the polynomial in the right-hand side is

$$\frac{1}{9}\left(\int W^4\right)\Big(K(0, Q) + (\omega+1)M(0, Q)\Big).$$

Therefore, (2.16) holds, for θ sufficient large, provided that

$$(K(W, 0) + (\omega+1)M(W, 0))^2 < \frac{1}{9}\left(\int W^4\right)\Big(K(0, Q) + (\omega+1)M(0, Q)\Big). \tag{2.17}$$

So, we are left to show that (2.17) holds for some $W \in H^1$. For that, assume $W(x) = Q(\lambda x)$ for some $\lambda \in \mathbb{R}$ to be determined. With this definition, (2.17) is equivalent to

$$\lambda^2 \int |\nabla Q|^2 + (\omega+1)\int Q^2 < \frac{\lambda^{n/2}}{3}\Big(K(0, Q)+(\omega+1)M(0, Q)\Big)^{1/2}\left(\int Q^4\right)^{1/2}.$$

In view of (2.3),

$$K(0, Q) = \int |\nabla Q|^2 = \frac{n\mu(\omega+1)}{4-n}\int Q^2, \tag{2.18}$$

Also, by using (2.2) and (2.18), we deduce

$$\int Q^4 = \frac{4}{9}\frac{\mu(\omega+1)}{4-n}\int Q^2. \tag{2.19}$$

By replacing (2.18) and (2.19) into (2.17), we then obtain that (2.17) is equivalent to

$$\frac{n\mu}{4-n}\lambda^2 + 1 - \frac{4\mu}{9(4-n)}\lambda^{n/2} < 0. \tag{2.20}$$

Let $f(\lambda)$ denotes the left-hand side of (2.20). It is easy to see that such a function has a global minimum at the point $\lambda_0 = 9^{-2/(4-n)}$. In addition, $f(\lambda_0) = 1 - \mu\lambda_0^2$. Finally, under the assumption $f(\lambda_0) < 0$, which means to say $\mu \geq 9^{4/(4-n)}$, we then see that (2.20) holds for $\lambda = \lambda_0$ and the proof of the proposition is complete. □

Next, we shall show that under the condition $\omega + 1 = \mu + 3\sigma\omega$, the ground states of (1.7) are precisely of the form $(0, Q)$, where Q is a ground state of (2.12). We will closely follow the strategy in [8]. Define the functionals

$$I(u, w) = \int (|\nabla u|^2 + |\nabla w|^2) + \int \left((\omega + 1)u^2 + (\mu + 3\sigma\omega)w^2 \right), \tag{2.21}$$

$$\widetilde{N}(u, w) = \frac{1}{4}N(u, w) = \int \left(\frac{1}{9}u^4 + 9w^4 + 4u^2w^2 + \frac{4}{9}u^3w \right) \tag{2.22}$$

and, for $\lambda > 0$, consider the minimization problem

$$I_\lambda = \inf\{I(f, g) :\ (f, g) \in H^1 \times H^1 \text{ with } \widetilde{N}(f, g) = \lambda\}. \tag{2.23}$$

Our goal will be to prove that for a certain specific λ such an infimum is attained by the ground states of (1.7). Initially, note that, from the homogeneity of I and $\widetilde{N}$, if follows that

$$I_\lambda = \lambda^{1/2} I_1. \tag{2.24}$$

Also, from Young and Gagliardo-Nirenberg's inequality,

$$\widetilde{N}(u, w) \leq C(\|u\|_{L^4}^4 + \|w\|_{L^4}^4) \leq CK(u, w)^{n/2} M(u, w)^{2-n/2} \leq CI(u, w)^2,$$

which implies that $I_\lambda > 0$, for any $\lambda > 0$. To motivate which λ would be the correct one, we recall that if $(u, w) \in \mathcal{G}(\omega, \mu, \sigma)$ then, by (2.7), $\widetilde{N}(u, w) = I(u, w)$. Hence, we must choose λ such that $I_\lambda = \lambda$. In view of (2.24), we must choose $\lambda = \lambda_1$, where

$$\lambda_1 := (I_1)^2. \tag{2.25}$$

Lemma 2.6 *Let assumptions of Theorem 2.1 hold and let* $m = \inf\{S(u, w) :\ (u, w) \in \mathcal{N}\}$. *Then*

$$\lambda_1 = 4m.$$

Proof From the proof of Theorem 2.1 we already know the minimization problem (2.8) has a solution (a ground state). So, we may fix $(u,w) \in H^1 \times H^1$ satisfying $m = S(u,w)$. Since $(u,w) \in \mathcal{G}(\omega,\mu,\sigma)$, we have $\widetilde{N}(u,w) = I(u,w)$ and

$$m = S(u,w) = E(u,w) + \frac{\omega}{2}M(u,w) = \frac{1}{2}I(u,w) - \frac{1}{4}\widetilde{N}(u,w) = \frac{1}{4}I(u,w). \tag{2.26}$$

Hence, $I(u,w) = 4m$. Next, define $(U,W) = (1/4m)^{1/4}(u,w)$. Then, $\widetilde{N}(U,W) = 1$ and

$$I(U,W) = \left(\frac{1}{4m}\right)^{1/2} I(u,w) = (4m)^{1/2}.$$

This identity implies that $I_1 \le (4m)^{1/2}$, which yields $\lambda_1 \le 4m$.
We shall have established the lemma if we prove that $\lambda_1 \ge 4m$, that is, $I_1 \ge (4m)^{1/2}$. Take any $(z,v) \in H^1 \times H^1$ with $\widetilde{N}(z,v) = 1$. It then suffices to prove that $4m \le I(z,v)^2$. To prove this, first observe that, for any $(f,g) \in H^1 \times H^1$,

$$\langle S'(f,g),(f,g)\rangle = I(f,g) - \widetilde{N}(f,g).$$

In particular, for any $\ell > 0$ we have $\langle S'(\ell z,\ell v),(\ell z,\ell v)\rangle = \ell h'(\ell)$, where $h(\ell) = S(\ell z,\ell v)$. But since

$$h'(\ell) = \ell I(z,v) - \ell^3 \widetilde{N}(z,v) = \ell\left(I(z,v) - \ell^2\right),$$

by choosing $\ell_0 > 0$ such that $\ell_0^2 = I(z,v)$ we deduce that $h'(\ell_0) = 0$ and $(Z,V) = (\ell_0 z,\ell_0 v) \in \mathcal{N}$. Consequently,

$$m \le S(Z,V) = \frac{\ell_0^2}{2}I(z,v) - \frac{\ell_0^4}{4}\widetilde{N}(z,v) = \frac{\ell_0^4}{4} = \frac{I(z,v)^2}{4},$$

which give the desired assertion. □

Next, we show the following:

Proposition 2.7 *Under the assumptions of Theorem 2.1, $(u,w) \in \mathcal{G}(\omega,\mu,\sigma)$ if and only if $I(u,w) = I_{\lambda_1}$ and $\widetilde{N}(u,w) = \lambda_1$.*
In particular, the set of solutions of the minimization problem (2.23) with $\lambda = \lambda_1$ is nonempty.

Proof Let us first take $(u,w) \in \mathcal{G}(\omega,\mu,\sigma)$. By reasoning as in (2.26) and using Lemma 2.6, we get

$$I(u,w) = 4m = \lambda_1 = I_{\lambda_1} \quad \text{and} \quad \widetilde{N}(u,w) = I(u,w) = 4m = \lambda_1,$$

which shows one of the assertions.
Let us now assume that (u, w) satisfies $I(u, w) = I_{\lambda_1}$ and $\widetilde{N}(u, w) = \lambda_1$. By the Lagrange multiplier theorem, there exists $\eta \in \mathbb{R}$ such that, for any $(f, g) \in H^1 \times H^1$,

$$\int (\nabla u \cdot \nabla f + (\omega + 1)uf = 2\eta \int \left(\frac{1}{9}u^3 + 2w^2u + \frac{1}{3}u^2w\right) f,$$
$$\int (\nabla w \cdot \nabla g + (\mu + 3\sigma\omega)wg = 2\eta \int \left(9w^3 + 2u^2w + \frac{1}{9}u^3\right) g.$$

By taking $(f, g) = (u, w)$, and adding the last two identities, we deduce that $I(u, w) = 2\eta\widetilde{N}(u, w)$. But, from

$$\lambda_1^{1/2} I_1 = I_{\lambda_1} = I(u, w) = 2\eta\widetilde{N}(u, w) = 2\eta\lambda_1,$$

we obtain $I_1 = 2\eta\lambda_1^{1/2}$, which compared to (2.25) gives $2\eta = 1$. Consequently, $(u, w) \in \mathcal{B}(\omega, \mu, \sigma)$ and $I(u, w) = \widetilde{N}(u, w)$.

It remains to show that (u, w) is indeed a ground state. To do so, take any (z, v) in $\mathcal{B}(\omega, \mu, \sigma)$ and let $\kappa := \widetilde{N}(z, v) > 0$. Recalling (2.7), we then have $I(z, v) = \widetilde{N}(z, v) = \kappa$ and,

$$S(z, v) = \frac{1}{2}I(z, v) - \frac{1}{4}\widetilde{N}(z, v) = \frac{1}{4}I(z, v) = \frac{\kappa}{4}.$$

Define $(\widetilde{z}, \widetilde{v}) = (\lambda_1/\kappa)^{1/4}(z, v)$. Then,

$$\widetilde{N}(\widetilde{z}, \widetilde{v}) = \frac{\lambda_1}{\kappa}\widetilde{N}(z, v) = \lambda_1$$

and

$$\lambda_1^{1/2} I_1 = I(u, w) \le I(\widetilde{z}, \widetilde{v}) = \left(\frac{\lambda_1}{\kappa}\right)^{1/2} I(z, v) = \left(\frac{\lambda_1}{\kappa}\right)^{1/2} \kappa = \lambda_1^{1/2}\kappa^{1/2}.$$

This last inequality implies that $\kappa \ge (I_1)^2 = \lambda_1$. Thus,

$$S(z, v) = \frac{\kappa}{4} \ge \frac{\lambda_1}{4} = S(u, w), \tag{2.27}$$

which proves that $(u, w) \in \mathcal{G}(\omega, \mu, \sigma)$. □

Finally, we prove the previously announced result:

Proposition 2.8 *In addition to the assumptions of Theorem 2.1, suppose that* $\omega + 1 = \mu + 3\sigma\omega$. *If* $(u, w) \in \mathcal{G}(\omega, \mu, \sigma)$ *then* $u \equiv 0$ *and* w *is a ground state of* (2.12). *In particular, up to translation, ground states are unique.*

Proof Take (u, w) in $\mathcal{G}(\omega, \mu, \sigma)$. From Proposition 2.7 we have $I(u, w) = I_{\lambda_1}$ and $\widetilde{N}(u, w) = \lambda_1$. Let us introduce the function $F : \mathbb{R}^2 \to \mathbb{R}$ by $F(x, y) = \frac{1}{9}x^4 + 9y^4 + 4x^2y^2 + \frac{4}{9}x^3y$. It is easily seen that, restricted to the unit circle $\mathbb{S}^1$, F has two maximum points, namely, $(0, 1)$ and $(0, -1)$. In addition, its maximum value is $F(0, \pm 1) = 9$.

Now, define $U(x) := |(u(x), w(x))| = \sqrt{u(x)^2 + w(x)^2} > 0$. Thus,

$$\begin{aligned}\widetilde{N}(u, w) &= \int F(u(x), w(x)) = \int F\left(\frac{1}{U(x)}(u(x), w(x))\right) U(x)^4 \leq \int F(0, 1)U(x)^4 \\ &= \int F(0, U(x)) = \widetilde{N}(0, U).\end{aligned} \tag{2.28}$$

Also, because $|\nabla U|^2 \leq |\nabla u|^2 + |\nabla w|^2$,

$$\begin{aligned}I(0, U) &= \int |\nabla U(x)|^2 + (\mu + 3\sigma\omega)|U(x)|^2 \\ &\leq \int |\nabla u(x)|^2 + |\nabla w(x)|^2 + (\omega + 1)u(x)^2 + (\mu + 3\sigma\omega)w(x)^2 = I(u, w),\end{aligned}$$

where we used that $\omega + 1 = \mu + 3\sigma\omega$. In view of (2.28) and the homogeneity of $\widetilde{N}$, there exists $0 < t \leq 1$ such that $\widetilde{N}(0, tU) = \widetilde{N}(u, w) = \lambda_1$. Hence,

$$I(0, tU) = t^2 I(0, U) \leq I(0, U) \leq I(u, w).$$

By recalling that (u, w) is a minimum of I restricted to $\widetilde{N} = \lambda_1$, it must be the case that $t = 1$. Thus,

$$\widetilde{N}(0, U) = \lambda_1 \quad \text{and} \quad I(0, U) = I(u, w) = I_{\lambda_1}.$$

Another application of Proposition 2.7 yields that $(0, U) \in \mathcal{G}(\omega, \mu, \sigma)$. Consequently, U must be a ground state of (2.12).

By defining $(z, v) = U^{-1}(u, w)$, we see that we can write $(u(x), w(x)) = U(x)(z(x), v(x))$, with $(z(x), v(x)) \in \mathbb{S}^1$, for any $x \in \mathbb{R}^n$. From

$$\begin{aligned}\int 9U(x)^4 &= \int F(0, U(x)) = \widetilde{N}(0, U) \\ &= \widetilde{N}(u, w) = \int F\Big(U(x)(z(x), v(x))\Big) = \int F(z(x), w(x))U(x)^4\end{aligned}$$

it follows that

$$\int U(x)^4(9 - F(z(x), w(x))) = 0,$$

Therefore, $F(z(x), v(x)) = 9$ for a.e. $x \in \mathbb{R}^n$, which implies that either $(z(x), v(x)) = (0, 1)$ or $(z(x), w(x)) = (0, -1)$ for a.e. $x \in R^n$. Consequently, $(u(x), w(x)) = (0, U(x))$ or $(u(x), w(x)) = (0, -U(x))$, which is the desired conclusion. □

Remark 2.9 In the case $\omega + 1 = \mu + 3\sigma\omega$, besides the solutions of the form $(0, w)$ (with w a solution of (2.12)), (1.7) has another interesting solution. Indeed, assume that $Q = bP$, where b is the (negative) real solution of the equation

$$2b^2 + \frac{1}{3}b + \frac{1}{9} = \frac{1}{b}\left(9b^3 + 2b + \frac{1}{9}\right).$$

Then, equations in (1.7) reduce to the same one, namely,

$$\Delta P - (\mu + 3\sigma\omega)P + \left(2b^2 + \frac{1}{3}b + \frac{1}{9}\right)P^3 = 0. \tag{2.29}$$

Hence, if P_b is a solution of (2.29) it follows that (P_b, bP_b) is a solution (1.7). Note that, according to Proposition 2.8, even if P_b is a ground state of (2.29) (which clearly exist), (P_b, bP_b) is not a ground state of (1.7).

In the case $n = 1$, the unique ground state of (2.12) is explicitly given by

$$w(x) = \frac{1}{3}\sqrt{2(\mu + 3\sigma\omega)}\,\mathrm{sech}(\sqrt{(\mu + 3\sigma\omega)}x). \tag{2.30}$$

So, according to Proposition 2.8, the unique ground sate of (1.7) is $(0, w)$, with w given in (2.30).

3 Global well-posedness

In this section we are interested in the study of the Cauchy problem associated with (1.3) in the energy space; so, we couple (1.3) with an initial data (u_0, w_0) in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ and consider the problem

$$\begin{cases} iu_t + \Delta u - u + \left(\frac{1}{9}|u|^2 + 2|w|^2\right)u + \frac{1}{3}\overline{u}^2 w = 0, \\ i\sigma w_t + \Delta w - \mu w + \left(9|w|^2 + 2|u|^2\right)w + \frac{1}{9}u^3 = 0, \\ u(x, 0) = u_0(x), \quad w(x, 0) = w_0(x). \end{cases} \tag{3.1}$$

By using the contraction mapping principle combined with the well-known Strichartz estimates, one can easily show the local well-posedness of (3.1) (see [6] or [15] for details). More precisely, one may establish the following result:

Theorem 3.1 *Assume $1 \leq n \leq 3$ and $u_0, w_0 \in H^1(\mathbb{R}^n)$. Then, the Cauchy problem (3.1) admits a unique solution,*

$$(u, w) \in C((-T_*, T^*); H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n))$$

defined in the maximal interval of existence $(-T_*, T^*)$, where $T_*, T^* > 0$.

In addition, the following blow-up alternative holds: if $T^* < \infty$ then

$$\lim_{t\to T^*}\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right)=+\infty.$$

A similar statement holds with T_* instead of T^*.

Since the quantity M defined in (1.5) is conserved and is equivalent to the standard norm in $L^2\times L^2$, in order to prove the global well-posedness of (3.1) in $H^1\times H^1$, one only needs to get an *a priori* bound on the L^2-norm of the gradients of u and w. With this in mind, let us recall the functional

$$K(u,w)=\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2. \tag{3.2}$$

To obtain an upper bound for K, we may use the conservation of the energy and Hölder's inequality combined with the Gagliardo-Nirenberg inequality

$$\|f\|_{L^4}^4\le C\|\nabla f\|_{L^2}^n\|f\|_{L^2}^{4-n}\;:$$

$$\begin{aligned}K(u,w)&\le K(u,w)+\|u\|_{L^2}^2+\mu\|w\|_{L^2}^2\\&=2E(u_0,v_0)+2\int\left(\frac{1}{36}|u|^4+\frac{9}{4}|w|^4+|u|^2|w|^2+\frac{1}{9}\Re e(\overline{u}^3w)\right)\\&\le 2E(u_0,v_0)+2\int\left(\frac{1}{36}|u|^4+\frac{9}{4}|w|^4+|u|^2|w|^2+\frac{1}{9}|u|^3|w|\right)\\&\le 2E(u_0,v_0)+2C\left(\|u\|_{L^4}^4+\|w\|_{L^4}^4\right)\\&\le 2E(u_0,v_0)+2C\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right)^{n/2}\left(\|u\|_{L^2}^2+3\sigma\|w\|_{L^2}^2\right)^{2-n/2}\\&=2E(u_0,v_0)+2CK(u,w)^{n/2}M(u_0,w_0)^{2-n/2},\end{aligned} \tag{3.3}$$

where C is a positive universal constant. An immediate consequence of (3.3) is that if $n=1$ then $K(u(t),w(t))$ is bounded. Indeed, for all $\epsilon>0$,

$$K(u,w)\le 2E(u_0,w_0)+C\Big(\epsilon K(u,w))+\frac{1}{\epsilon}M(u_0,w_0)^3\Big),$$

and, choosing $\epsilon=1/2C$,

$$K(u,w)\le 4E(u_0,w_0)+4C^2M(u_0,w_0)^3.$$

In view of the blow-up alternative stated in Theorem 3.1, this yields the following corollary:

Corollary 3.2 *Assume $n = 1$ and $u_0, w_0 \in H^1(\mathbb{R})$. Then, the Cauchy problem (3.1) is globally well-posed.*

Now, if $n = 2$, (3.3) does not give an immediate *a priori* bound. However, in this case, we can rewrite it as

$$(1 - 2CM(u_0, w_0))K(u, w) \leq 2E(u_0, w_0).$$

Hence, if $M(u_0, w_0) < 1/2C$ then the last inequality provides a bound for $K(u(t), w(t))$ and we deduce:

Corollary 3.3 *Assume $n = 2$ and $u_0, w_0 \in H^1(\mathbb{R}^2)$. Then, the Cauchy problem (3.1) is globally well-posed, provided that the initial mass $M(u_0, w_0)$ is sufficiently small.*

Next we focus on the question of how small $M(u_0, w_0)$ must be for the conclusion of Corollary 3.3 to hold. As we observed above, the constant C appearing in (3.3) plays a crucial role in this question. So, in some sense, the problem is related with the best constant we can place in the inequality

$$\int \left(\frac{1}{36}|u|^4 + \frac{9}{4}|w|^4 + |u|^2|w|^2 + \frac{1}{9}|u|^3|w| \right) \leq CK(u, w)^{n/2} M(u, w)^{2-n/2}. \tag{3.4}$$

Recall that for $u, w \in H^1$,

$$N(u, w) := \int \left(\frac{1}{36}u^4 + \frac{9}{4}w^4 + u^2w^2 + \frac{1}{9}u^3w \right).$$

Also, define

$$J(u, w) := \frac{K(u, w)^{n/2} M(u, w)^{2-n/2}}{N(u, w)}. \tag{3.5}$$

It is easily seen that (3.4) is equivalent to

$$\frac{1}{C} \leq J(u, w)$$

for functions (u, w) in the set

$$\mathbf{N} := \{(u, w) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n);\ N(u, w) > 0\}.$$

In particular, the infimum of J on $\mathbf{N}$ is clearly the reciprocal of the best constant in (3.4). In the sequel we will show that this infimum is indeed attained on $\mathbf{N}$. We start with the following preliminary result:

Lemma 3.4 *Assume $1 \leq n \leq 3$. Let (P, Q) be any solution of (1.7) with $\omega = 0$ and $\mu = 3\sigma$. Then,*

$$N(P, Q) = S(P, Q), \tag{3.6}$$
$$K(P, Q) = nS(P, Q), \tag{3.7}$$
$$K(P, Q) = \frac{n}{4-n} M(P, Q). \tag{3.8}$$

In particular,

$$J(P, Q) = n^{n/2}(4-n)^{2-n/2} S(P, Q). \tag{3.9}$$

Proof By summing (2.1) and (2.2) we promptly deduce that

$$K(P, Q) + M(P, Q) = 4N(P, Q). \tag{3.10}$$

Thus, since for $\omega = 0$, $S = E$, we obtain

$$\begin{aligned} S(P, Q) = E(P, Q) &= \frac{1}{2}\Big(K(P, Q) + M(P, Q)\Big) - N(P, Q) \\ &= 2N(P, Q) - N(P, Q) = N(P, Q), \end{aligned}$$

which proves (3.6). Also, the identity (3.8) follows directly from (2.3).

Furthermore, from (3.8), $M(P, Q) + K(P, Q) = \frac{4}{n} K(P, Q)$, and, from

$$N(P, Q) = E(P, Q) = \frac{1}{2}(K(P, Q) + M(P, Q)) - N(P, Q),$$

one obtains (3.7).
Finally, (3.9) is a consequence of (3.6)–(3.8). The proof of the lemma is thus completed. □

Lemma 3.5 *Suppose $1 \leq n \leq 3$. The infimum of J is attained on $\mathbf{N}$ at a pair of real functions (P, Q), that is,*

$$\inf_{\mathbf{N}} J(u, w) = J(P, Q),$$

if and only if, up to scaling, (P, Q) is a ground state solution of (1.7) with $\omega = 0$ and $\mu = 3\sigma$.

Proof Assume (P, Q) is a minimum of J on $\mathbf{N}$. Since (P, Q) is a critical point we have $J'(P, Q) = 0$, which implies that

$$\begin{cases} -\frac{n}{K(P,Q)}\Delta P + \frac{4-n}{M(P,Q)}P = \frac{1}{N(P,Q)}\left(\frac{1}{9}P^3 + 2Q^2P + \frac{1}{3}P^2Q\right), \\ -\frac{n}{K(P,Q)}\Delta Q + \frac{(4-n)3\sigma}{M(P,Q)}Q = \frac{1}{N(P,Q)}\left(9Q^3 + 2P^2Q + \frac{1}{9}P^3\right). \end{cases} \tag{3.11}$$

Now take $\lambda, \nu > 0$ such that

$$\lambda^2 = \frac{nM(P,Q)}{(4-n)K(P,Q)} \quad \text{and} \quad \nu^2 = \frac{M(P,Q)}{(4-n)N(P,Q)}$$

and define

$$\widetilde{P}(x) = \nu P(\lambda x), \qquad \widetilde{Q}(x) = \nu Q(\lambda x).$$

A straightforward calculation reveals that $(\widetilde{P}, \widetilde{Q})$ satisfies

$$\begin{cases} -\Delta\widetilde{P} + \widetilde{P} = \left(\frac{1}{9}\widetilde{P}^3 + 2\widetilde{Q}^2\widetilde{P} + \frac{1}{3}\widetilde{P}^2\widetilde{Q})\right), \\ -\Delta\widetilde{Q} + 3\sigma\widetilde{Q} = \left(9\widetilde{Q}^3 + 2\widetilde{P}^2\widetilde{Q} + \frac{1}{9}\widetilde{P}^3\right), \end{cases} \tag{3.12}$$

which is exactly system (1.7) with $\omega = 0$ and $\mu = 3\sigma$. In addition, it is not difficult to see that $J(\widetilde{P}, \widetilde{Q}) = J(P, Q)$ and $N(\widetilde{P}, \widetilde{Q}) = \nu^4\lambda^{-n}N(P, Q) > 0$, which means that $(\widetilde{P}, \widetilde{Q})$ is also a minimizer of J on $\mathbf{N}$. Relation (3.9) then yields that $(\widetilde{P}, \widetilde{Q})$ is a minimizer of S on $\mathbf{N}$. In view of (2.7), it is easy to conclude that any bound state belongs to $\mathbf{N}$ and we deduce that $(\widetilde{P}, \widetilde{Q})$ is a ground state.

Conversely, if (P, Q) is a ground state of (1.7) with $\omega = 0$ and $\mu = 3\sigma$, we have $(P, Q) \in \mathbf{N}$ and (P, Q) is a minimum of S. The identity (3.9) again implies that (P, Q) is also a minimum of J. □

The above results allow us to obtain the best constant in the Gagliardo-Nirenberg inequality (3.4). More precisely, we have:

Corollary 3.6 *Assume* $1 \le n \le 3$. *Then the inequality*

$$N(u, w) \le C_{GN}K(u, w)^{n/2}M(u, w)^{2-n/2}$$

holds, for any $(u, v) \in \mathbf{N}$, *with*

$$C_{GN} = \frac{(4-n)^{n/2-1}}{n^{n/2}} \frac{1}{M(P, Q)} = \frac{(4-n)^{n/2-2}}{n^{n/2}} \frac{1}{S(P, Q)},$$

where (P, Q) *is any ground state of* (1.7) *with* $\omega = 0$ *and* $\mu = 3\sigma$.

Proof It suffices to recall that

$$\frac{1}{C_{GN}} = \inf_{\mathbf{N}} J(u, w)$$

and use Lemmas 3.4 and 3.5. □

Remark 3.7 Note that the constant C_{GN} does not depend on the choice of the ground state (P, Q) since all ground states have the same mass M (and the same action S). Hence, the question of uniqueness of ground states is not an issue here.

With Corollary 3.6 in hand we can to prove the following Theorem:

Theorem 3.8 *Assume* $n = 2$ *and* $u_0, w_0 \in H^1(\mathbb{R}^2)$. *Then the Cauchy problem* (3.1) *is globally well-posed provided that*

$$M(u_0, w_0) < M(P, Q),$$

where (P, Q) *is any ground state of* (1.7) *with* $\omega = 0$ *and* $\mu = 3\sigma$.

Proof Indeed, It suffices to use (3.3) with the constant C replaced by C_{GN} given in Corollary 3.6. □

Next we turn attention to the global well-posedness for $n = 3$. We begin by stating the following Lemma, whose proof can be found in [2] and [17]:

Lemma 3.9 *Let* I *be an open interval with* $0 \in I$. *Let* $a \in \mathbb{R}$, $b > 0$ *and* $q > 1$. *Define* $\gamma = (bq)^{-\frac{1}{q-1}}$ *and* $f(r) = a - r + br^q$, *for* $r \geq 0$. *Let* $G(t)$ *be a nonnegative continuous function such that* $f \circ G \geq 0$ *on* I. *Assume that* $a < \left(1 - \frac{1}{q}\right)\gamma$.

(i) *If* $G(0) < \gamma$, *then* $G(t) < \gamma$, $\forall t \in I$.
(ii) *If* $G(0) > \gamma$, *then* $G(t) > \gamma$, $\forall t \in I$.

In addition if $a < (1 - \delta_1)\left(1 - \frac{1}{q}\right)\gamma$ *and* $G(0) > \gamma$, *for some* $\delta_1 > 0$, *then there exists* δ_2, *depending only on* δ_1 *such that* $G(t) > (1 + \delta_2)\gamma$, $\forall t \in I$.

Our main theorem here reads as follows.

Theorem 3.10 *Assume $n = 3$ and $u_0, w_0 \in H^1(\mathbb{R}^3)$. Suppose that*

$$E(u_0, w_0)M(u_0, w_0) < \frac{1}{2}E(P, Q)M(P, Q) \tag{3.13}$$

and

$$K(u_0, w_0)M(u_0, w_0) < K(P, Q)M(P, Q), \tag{3.14}$$

where (P, Q) is any ground state of (1.7) with $\omega = 0$ and $\mu = 3\sigma$. Then, as long as the local solution given in Theorem 3.1 exists, there holds

$$K(u(t), w(t))M(u(t), w(t)) < K(P, Q)M(P, Q). \tag{3.15}$$

In particular, this implies that the Cauchy problem (3.1) is globally well-posed under conditions (3.13) and (3.14).

Proof Let $a = 2E(u_0, w_0)$, $b = 2C_{GN}M(u_0, w_0)^{1/2}$, and $q = 3/2$. If $G(t) = K(u(t), w(t))$, from (3.3), with C_{GN} instead of C, we obtain $f \circ G \geq 0$, where $f(r) = a - r + br^{3/2}$. Also, by using Lemma 3.4 we see that

$$\gamma = \frac{3M(P, Q)^2}{M(u_0, w_0)}.$$

In addition, a simple calculation using Lemma 3.4 also reveals that

$$a < \left(1 - \frac{1}{q}\right)\gamma \quad \Leftrightarrow \quad E(u_0, w_0)M(u_0, w_0) < \frac{1}{2}E(P, Q)M(P, Q)$$

and

$$G(0) < \gamma \quad \Leftrightarrow \quad K(u_0, w_0)M(u_0, w_0) < K(P, Q)M(P, Q).$$

Hence, Lemma 3.9 implies that (3.15) holds. This completes the proof of the theorem. □

4 Blow up

In this section we will show some blow up results.

Definition 4.1 We say that the solution of (3.1), given in Theorem 3.1, blows up forward in time if $T^* < \infty$ and backward in time if $T_* < \infty$. We say that the solution blows up if it blows up forward and backward in time.

Our results of this Section will show that the condition in Theorem 3.8 is sharp, at least for some parameters σ and μ. Actually, in the case $n = 2$ we can construct an explicit solution that blows up, say, forward in time.

Theorem 4.2 *Assume $n = 2$, $\sigma = 3$, and $\mu = 9$. Let (P, Q) be any ground state of (1.7) with $\omega = 0$ (and $\mu = 3\sigma$). Then, there exists $u_0, w_0 \in H^1$ satisfying $M(u_0, w_0) = M(P, Q)$ such that the corresponding solution of the Cauchy problem (3.1) blows up forward in time.*

Proof First we note that (u, w) is a solution of (3.1) if and only if

$$\widetilde{u}(x,t) = e^{it}u(x,t), \qquad \widetilde{w}(x,t) = e^{3it}w(x,t)$$

is a solution of

$$\begin{cases} i\widetilde{u}_t + \Delta\widetilde{u} + (\frac{1}{9}|\widetilde{u}|^2 + 2|\widetilde{w}|^2)\widetilde{u} + \frac{1}{3}\overline{\widetilde{u}}^2\widetilde{w} = 0, \\ i\sigma\widetilde{w}_t + \Delta\widetilde{w} + (9|\widetilde{w}|^2 + 2|\widetilde{u}|^2)\widetilde{w} + \frac{1}{9}\widetilde{u}^3 = 0, \\ \widetilde{u}(x,0) = u_0(x), \quad \widetilde{w}(x,0) = w_0(x). \end{cases} \tag{4.1}$$

Actually, this equivalence is true only under the condition $\mu = 3\sigma$. So the problem is reduced to showing that (4.1) has a solution with $M(u_0, w_0) = M(P, Q)$ that blows up forward in time.

Next, a tedious but straightforward calculation gives that if $(\widetilde{u}, \widetilde{w})$ is a solution of the differential equations in (4.1) so is the pair $(\widehat{u}, \widehat{w})$ defined by

$$\widehat{u}(x,t) = \frac{1}{1-t}e^{-\frac{i|x|^2}{4(1-t)}}\widetilde{u}\left(\frac{x}{1-t}, \frac{t}{1-t}\right), \quad \widehat{w}(x,t) = \frac{1}{1-t}e^{-\frac{3i|x|^2}{4(1-t)}}\widetilde{w}\left(\frac{x}{1-t}, \frac{t}{1-t}\right).$$

In addition,

$$\widehat{u}(x,0) = e^{-\frac{i|x|^2}{4}}u_0(x), \qquad \widehat{u}(x,0) = e^{-\frac{3i|x|^2}{4}}w_0(x).$$

Finally, by taking

$$\widetilde{u}(x,t) = e^{it}P(x), \qquad \widetilde{w}(x,t) = e^{3it}Q(x),$$

it is easily seen that $(\widetilde{u}, \widetilde{w})$ is a solution of the equations in (4.1). Consequently,

$$\widehat{u}(x,t) = \frac{1}{1-t}e^{-\frac{i|x|^2}{4(1-t)}}e^{\frac{it}{1-t}}P\left(\frac{x}{1-t}\right), \quad \widehat{w}(x,t) = \frac{1}{1-t}e^{-\frac{3i|x|^2}{4(1-t)}}e^{\frac{3it}{1-t}}Q\left(\frac{x}{1-t}\right)$$

is a solution of (4.1) that blows up at time $t = 1$ and satisfies $M(\widehat{u}(0), \widehat{w}(0)) = M(P, Q)$. □

Remark 4.3 By using the same ideas as in the proof of Theorem 4.2 one can construct a blowing up solution at any time $T \neq 0$. In particular, we can also construct a solution that blows up backward in time.

The Theorem 4.2 holds only in dimension $n = 2$, the critical dimension. Next we will obtain some virial identities to system (1.3). First observe that (1.3) can be written in the pseudo-Hamiltonian form

$$\frac{d}{dt}X(t) = \Lambda E'(X(t)), \tag{4.2}$$

where $X(t) = (u(t), w(t))$, E' stands for the Fréchet derivative of E, and Λ is the skew-adjoint operator given by

$$\Lambda = \begin{pmatrix} -i & 0 \\ 0 & -i/\sigma \end{pmatrix}. \tag{4.3}$$

Proposition 4.4 *Assume*

$$u_0, w_0 \in H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, |x|^2 dx) =: \mathbb{H}$$

and define

$$V(t) = \int |x|^2(|u(t)|^2 + 3\sigma|w(t)|^2),$$

where $(u(t), w(t))$ *is the maximal solution of* (3.1)*, with initial data* (u_0, w_0)*, and defined in the maximal time interval* $[0, T^*)$*. Then* $V \in C^2([0, T^*))$*. In addition,*

$$V'(t) = 4Im \int (\overline{u}(t)x \cdot \nabla u(t) + 3\overline{w}(t)x \cdot \nabla w(t)) \tag{4.4}$$

and

$$\begin{aligned} V''(t) = &\int \left(8|\nabla u|^2 + 8|\nabla w|^2 - \frac{2n}{9}|u|^4 - \frac{54n}{\sigma}|w|^4 - 8n|u|^2|w|^2\right) \\ &+ 2\left(\frac{24}{\sigma} - 8\right)\Re e \int \overline{u}|w|^2 x \cdot \nabla u + \frac{1}{9}\left(\frac{12}{\sigma} - 12\right) n \Re e \int \overline{u}^3 w \\ &+ \frac{1}{9}\left(\frac{24}{\sigma} - 8\right)\Re e \int 3\overline{u}^2 w x \cdot \nabla \overline{u}. \end{aligned} \tag{4.5}$$

Proof We proceed formally. Introduce the functional

$$\mathcal{V}(u, w) = \int |x|^2(|u|^2 + 3\sigma|w|^2)$$

and note that $V(t) = \mathcal{V}(u(t), w(t)) \equiv \mathcal{V}(X(t))$. Thus,

$$V'(t) = \frac{d}{dt}\mathcal{V}(X(t)) = \langle \mathcal{V}'(X(t)), \frac{d}{dt}X(t)\rangle = \langle \mathcal{V}'(X(t)), \Lambda E'(X(t))\rangle =: P(X(t)). \tag{4.6}$$

Thus, in order to determine $V'(t)$, it suffices to determine the functional P. To do so, we use a dual Hamiltonian system. Indeed, given $Y_0 = (\widetilde{u}_0, \widetilde{w}_0) \in \mathbb{H}$, assume the initial-value problem

$$\frac{d}{dt}Y(t) = \Lambda\mathcal{V}'(Y(t)), \qquad Y(0) = Y_0 \tag{4.7}$$

is (at least) locally well-posed. Then

$$\begin{aligned} &\frac{d}{dt}E(Y(t)) = \langle E'(Y(t)), \\ &\frac{d}{dt}Y(t)\rangle = \langle E'(Y(t)), \Lambda\mathcal{V}'(Y(t))\rangle = -\langle \mathcal{V}'(Y(t)), \Lambda E'(Y(t))\rangle = -P(Y(t)). \end{aligned} \tag{4.8}$$

Evaluating at $t = 0$, we deduce

$$P(Y_0) = -\frac{d}{dt}E(Y(t))\Big|_{t=0}.$$

In conclusion, in order to determine the first derivative of $V(t)$, it suffices to solve (4.7) and then take the derivative of the energy at this solution evaluated at $t = 0$.

Next we solve (4.7). Indeed, if $Y(t) = (\widetilde{u}(t), \widetilde{w}(t))$, it easy to see that (4.7) is equivalent to

$$\begin{cases} \dfrac{d}{dt}(\widetilde{u}(t), \widetilde{w}(t)) = (-2i|x|^2\widetilde{u}, -6i|x|^2\widetilde{w}) \\ \widetilde{u}(0) = \widetilde{u}_0, \quad \widetilde{w}(0) = \widetilde{w}_0, \end{cases}$$

whose solution is

$$Y(t) = (\widetilde{u}(t), \widetilde{w}(t)) = (e^{-2i|x|^2t}\widetilde{u}_0, e^{-6i|x|^2t}\widetilde{w}_0).$$

Hence,

$$\begin{aligned} P(Y_0) &= -\frac{d}{dt}E(Y(t))\Big|_{t=0} = -\frac{1}{2}\frac{d}{dt}\left(\int |\nabla\widetilde{u}(t)|^2 + |\nabla\widetilde{w}(t)|^2\right)\Big|_{t=0} \\ &= 4Im\int \left(\overline{\widetilde{u}}_0 x\cdot\nabla\widetilde{u}_0 + 3\overline{\widetilde{w}}_0 x\cdot\nabla\widetilde{w}_0\right). \end{aligned}$$

This establishes (4.4).

To compute $V''(t)$ we use the above argument replacing $V(t)$ by $V'(t)$ and $\mathcal{V}(u, w)$ by

$$G(u, w) = 4Im\int (\overline{u}x\cdot\nabla u + 3\overline{w}x\cdot\nabla w).$$

Since

$$G'(u,w)=-4i\,(2x\cdot\nabla u+nu,6x\cdot\nabla w+3nw)\,,$$

we see that

$$\frac{d}{dt}Y(t)=JG'(Y(t)),\qquad Y(0)=Y_0 \tag{4.9}$$

is equivalent to

$$\begin{cases}\dfrac{d}{dt}(\widetilde{u}(t),\widetilde{w}(t))=(-8x\cdot\nabla\widetilde{u}-4n\widetilde{u},-\frac{24}{\sigma}x\cdot\nabla\widetilde{w}-\frac{12n}{\sigma}\widetilde{w})\\ \widetilde{u}(0)=\widetilde{u}_0,\quad \widetilde{w}(0)=\widetilde{w}_0,\end{cases}$$

It is not difficult to check that the solution of the above initial-value problem is

$$Y(t)=(\widetilde{u}(t),\widetilde{w}(t))=(e^{-4nt}\widetilde{u}_0(e^{-8t}x),e^{-\frac{12}{\sigma}t}\widetilde{w}_0(e^{-\frac{24}{\sigma}t}x)).$$

Hence,

$$\begin{aligned}E(Y(t))=\int\Big(&\frac{1}{2}e^{-16t}|\nabla\widetilde{u}_0|^2+\frac{1}{2}e^{-\frac{48}{\sigma}t}|\nabla\widetilde{w}_0|^2-\frac{1}{36}e^{-8nt}|\widetilde{u}_0|^4-\frac{9}{4}e^{-\frac{24}{\sigma}nt}|\widetilde{w}_0|^4\\ &+\frac{1}{2}|\widetilde{u}_0|^2+\frac{\mu}{2}|\widetilde{w}_0|^2\Big)-e^{-8nt}\int|\widetilde{u}_0(e^{\left(\frac{24}{\sigma}-8\right)t}x)|^2|\widetilde{w}_0(x)|^2dx\\ &-\frac{1}{9}e^{\left(\frac{12}{\sigma}-12\right)nt}Re\int\overline{\widetilde{u}}_0^3(e^{\left(\frac{24}{\sigma}-8\right)t}x)\widetilde{w}_0(x)dx\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}E(Y(t))\Big|_{t=0}=&\int\Big(-\frac{16}{2}|\nabla\widetilde{u}_0|^2-\frac{48}{2\sigma}|\nabla\widetilde{w}_0|^2+\frac{8n}{36}|\widetilde{u}_0|^4+\frac{9}{4}\frac{24n}{\sigma}|\widetilde{w}_0|^4+8n|\widetilde{u}_0|^2|\widetilde{w}_0|^2\Big)\\ &-2\left(\frac{24}{\sigma}-8\right)Re\int\overline{\widetilde{u}}_0|\widetilde{w}_0|^2x\cdot\nabla\widetilde{u}_0-\frac{1}{9}\left(\frac{12}{\sigma}-12\right)nRe\int\overline{\widetilde{u}}_0^3\widetilde{w}_0\\ &-\frac{1}{9}\left(\frac{24}{\sigma}-8\right)Re\int3\overline{\widetilde{u}}_0^2\widetilde{w}_0x\cdot\nabla\overline{\widetilde{u}}_0.\end{aligned}$$

Consequently, by recalling that $V''(t)$ must be the above expression (with the opposite sign) when we replace $(\widetilde{u}_0,\widetilde{w}_0)$ by $(u(t),w(t))$, (4.5) follows. The proof of the proposition is thus completed. □

Corollary 4.5 *Under the assumptions of Proposition 4.4, if $\sigma=3$ then*

$$V''(t)=8nE(u_0,w_0)+4(2-n)\int(|\nabla u|^2+|\nabla w|^2)-4n\int(|u|^2+\mu|w|^2)$$

Proof It follows easily from Proposition 4.4. Indeed, a simple computation yields

$$V''(t) = 16E(u_0, w_0) + 8(2-n)\int \left(\frac{1}{36}|u|^4 + \frac{9}{4}|w|^4 + |uw|^2 + \frac{1}{9}Re\overline{u}^3 w\right) \\ - 8\int (|u|^2 + \mu|w|^2),$$

and, by the definition of the energy functional,

$$V''(t) = 16E(u_0, w_0) + 8(2-n)\left[\frac{1}{2}\int \left(|\nabla u|^2 + |\nabla w|^2 + |u|^2 + \mu|w|^2\right) - E(u_0, v_0)\right] \\ - 8\int (|u|^2 + \mu|w|^2) \\ = 8nE(u_0, w_0) + 4(2-n)\int (|\nabla u|^2 + |\nabla w|^2) - 4n\int (|u|^2 + \mu|w|^2),$$

as claimed. □

With Corollary 4.5 in hand we can also show that, under the assumption (3.13), the condition (3.14) is sharp (at least in the case $\sigma = 3$ and $\mu = 9$) to obtain the global well posedness of (3.1). More precisely, we have

Theorem 4.6 *Assume $n = 3$, $\sigma = 3$, $\mu = 9$. Suppose that*

$$E(u_0, w_0)M(u_0, w_0) < \frac{1}{2}E(P, Q)M(P, Q) \tag{4.10}$$

and

$$K(u_0, w_0)M(u_0, w_0) > K(P, Q)M(P, Q), \tag{4.11}$$

where (P, Q) is any ground state of (1.7) with $\omega = 0$ (and $\mu = 3\sigma$). Then, as long as the local solution given in Theorem 3.1 exist there holds

$$K(u(t), w(t))M(u(t), w(t)) > K(P, Q)M(P, Q). \tag{4.12}$$

In particular, if $u_0, w_0 \in \mathbb{H}$ then the solution blows up in finite time.

Proof In view of (ii) in Lemma 3.9, the proof of the first part is similar to the one of the Theorem 3.10; so we omit the details.

Assume now $u_0, w_0 \in \mathbb{H}$. From assumption (4.10), we can find a sufficiently small $\delta_1 > 0$ satisfying

$$E(u_0, w_0)M(u_0, w_0) < \frac{1}{2}(1-\delta_1)E(P, Q)M(P, Q).$$

Consequently, using Lemma 3.9, there exists $\delta_2 > 0$ (depending only on δ_1) such that

$$K(u(t), w(t))M(u_0, w_0) > (1 + \delta_2)K(P, Q)M(P, Q).$$

Thus, from Corollary 4.5, we deduce that

$$\begin{aligned} M(u_0, w_0)V''(t) &< 24E(u_0, w_0)M(u_0, w_0) - 4K(u(t), w(t))M(u_0, w_0) \\ &< 12(1 - \delta_1)E(P, Q)M(P, Q) - 4(1 + \delta_2)K(P, Q)M(P, Q) \\ &= 4(1 - \delta_1)K(P, Q)M(P, Q) - 4(1 + \delta_2)K(P, Q)M(P, Q) \\ &= -4(\delta_1 + \delta_2)K(P, Q)M(P, Q), \end{aligned}$$

where we have used that $K(P, Q) = 3E(P, Q)$. Since the right-hand side of this last inequality is negative, a standard convexity argument allows us to conclude. □

Next, we state some sufficient conditions which imply that the solution blows up either forward or backward in time.

Theorem 4.7 *Assume* $2 \le n \le 3$, $\sigma = 3$ *and* $\mu > 0$. *Suppose* $u_0, w_0 \in \mathbb{H}$ *and let*

$$(u, v) \in C((-T_*, T^*); \mathbb{H} \times \mathbb{H})$$

be the maximal solution of (3.1) *given in Theorem* 3.1. *The following statements hold:*

(i) *If* $E(u_0, w_0) < 0$ *then* $T_* < \infty$ *and* $T^* < \infty$.

(ii) *If* $E(u_0, w_0) = 0$ *and*

$$Im \int (\overline{u}_0 x \cdot \nabla u_0 + 3\overline{w}_0 x \cdot \nabla w_0) < 0,$$

then $T^* < \infty$.

(iii) *If* $E(u_0, w_0) = 0$ *and*

$$Im \int (\overline{u}_0 x \cdot \nabla u_0 + 3\overline{w}_0 x \cdot \nabla w_0) > 0,$$

then $T_* < \infty$.

(iv) *If* $E(u_0, w_0) > 0$ *and*

$$Im \int (\overline{u}_0 x \cdot \nabla u_0 + 3\overline{w}_0 x \cdot \nabla w_0) < -\sqrt{nE(u_0, w_0)M(xu_0, xw_0)}$$

then $T^* < \infty$.

(v) *If* $E(u_0, w_0) > 0$ *and*

$$Im \int (\overline{u}_0 x \cdot \nabla u_0 + 3\overline{w}_0 x \cdot \nabla w_0) > \sqrt{nE(u_0, w_0)M(xu_0, xw_0)}$$

then $T_* < \infty$.

Proof It is clear from Corollary 4.5 that $V''(t) \leq 8nE(u_0, w_0)$. So, the proof follows the standard convexity method and we shall omit the calculations. The interested reader will find the details for the classical Schrödinger equation in [6, Sect. 6.5]. □

In the particular case $\mu = 9$, the above result can be improved in the following sense:

Theorem 4.8 *Assume* $2 \leq n \leq 3$, $\sigma = 3$ *and* $\mu = 9$. *Suppose* $u_0, w_0 \in \mathbb{H}$ *and let*

$$(u, v) \in C((-T_*, T^*); \mathbb{H} \times \mathbb{H})$$

be the maximal solution of (3.1) *given in Theorem* 3.1. *Then,*

(i) If $2E(u_0, w_0) < M(u_0, w_0)$ *then* $T_* < \infty$ *and* $T^* < \infty$.

(ii) If $2E(u_0, w_0) = M(u_0, w_0)$ *and*

$$Im \int (\overline{u}_0 x \cdot \nabla u_0 + 3\overline{w}_0 x \cdot \nabla w_0) < 0,$$

then $T^* < \infty$.

(iii) If $2E(u_0, w_0) = M(u_0, w_0)$ *and*

$$Im \int (\overline{u}_0 x \cdot \nabla u_0 + 3\overline{w}_0 x \cdot \nabla w_0) > 0,$$

then $T_* < \infty$.

(iv) If $2E(u_0, w_0) > M(u_0, w_0)$ *and*

$$\sqrt{2} Im \int (\overline{u}_0 x \cdot \nabla u_0 + 3\overline{w}_0 x \cdot \nabla w_0) < -\sqrt{n(2E(u_0, w_0) - M(u_0, w_0))M(xu_0, xw_0)}$$

then $T^* < \infty$.

(v) If $2E(u_0, w_0) > M(u_0, w_0)$ *and*

$$\sqrt{2} Im \int (\overline{u}_0 x \cdot \nabla u_0 + 3\overline{w}_0 x \cdot \nabla w_0) > \sqrt{n(2E(u_0, w_0) - M(u_0, w_0))M(xu_0, xw_0)}$$

then $T_* < \infty$.

Proof In this case, the last integral in 4.5 becomes $M(u_0, w_0)$. Hence, $V''(t) \leq 4n(2E(u_0, w_0) - M(u_0, w_0))$. □

Remark 4.9 Under the assumption $2E(u_0, w_0) < M(u_0, w_0)$ (and $\sigma = 3$, $\mu = 9$) a simple calculation using the definition of the energy and Lemma 3.6 shows that

$$K(u_0, w_0)^{n-2} M(u_0, w_0)^{4-n} > \frac{n^n}{4(4-n)^{n-2}} M(P, Q)^2.$$

Hence, for $n = 2$, we obtain $M(u_0, w_0) > M(P, Q)$, which does not contradict Theorem 4.2. On the other hand, for $n = 3$, using that $K(P, Q) = 3M(P, Q)$,

$$K(u_0, w_0)M(u_0, w_0) > \frac{9}{4}K(P, Q)M(P, Q),$$

which implies that (4.11) holds.

5 Stability/instability of ground states $(u, v) = (e^{i\omega t}P(x), e^{i\omega t}Q(x))$

This section is devoted to study the (orbital) stability/instability of the standing waves (1.6) in some particular cases. Let (P, Q) be a real ground state of (1.7). In particular $Q \neq 0$ and (P, Q) must satisfy

$$\begin{cases} \Delta P - (\omega + 1)P + (\frac{1}{9}P^2 + 2Q^2)P + \frac{1}{3}P^2 Q = 0, \\ \Delta Q - (\mu + 3\sigma\omega)Q + (9Q^2 + 2P^2)Q + \frac{1}{9}P^3 = 0. \end{cases} \tag{5.1}$$

To start with, let us make clear our notion of stability and instability. Recall that (1.3) is invariant by translations and rotations, that is, if (u, w) is a solution of (1.3) so are $(u(\cdot + y), w(\cdot + y))$ and $(e^{i\theta}u, e^{3i\theta}w)$, for any $\theta \in \mathbb{R}$ and $y \in \mathbb{R}^n$. Thus, the orbit generated by (P, Q) is defined by

$$\Omega = \{(e^{i\theta}u(\cdot + y), e^{3i\theta}u(\cdot + y)) : \ \theta \in \mathbb{R}, y \in \mathbb{R}^n\}.$$

Definition 5.1 (Orbital stability) We say that a standing wave $(e^{i\omega t}P, e^{3i\omega t}Q)$ is orbitally stable by the flow of (1.3) if for any $\epsilon > 0$ there exists a $\delta > 0$ with the following property: if $(u_0, w_0) \in H^1 \times H^1$ satisfies $\|(u_0, w_0) - (P, Q)\|_{H^1 \times H^1} < \delta$ then the solution of (1.3), with initial data (u_0, w_0) is global and satisfies

$$\sup_{t\in\mathbb{R}} \inf_{(\theta, y)\in\mathbb{R}\times\mathbb{R}^n} \|(u(t), w(t)) - (e^{i\theta}u(\cdot + y), e^{3i\theta}u(\cdot + y))\|_{H^1\times H^1} < \epsilon.$$

Otherwise, we say that $(e^{i\omega t}P, e^{3i\omega t}Q)$ is orbitally unstable by the flow of (1.3).

Roughly speaking, this means that there exists an ϵ-neighborhood of Ω such that any solution of (1.3) starting in this neighborhood remains close to the orbit generated by (P, Q). As usual in the current literature we say that (P, Q) is orbitally stable (unstable) instead of saying that $(e^{i\omega t}P, e^{3i\omega t}Q)$ is orbitally stable (unstable).

5.1 Instability

In order to establish our main theorem concerning instability let us introduce

$$\Sigma := \left\{(u, w) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) : \ M(u, w) = M(P, Q)\right\}. \tag{5.2}$$

Recall the following criterion for instability.

Theorem 5.2 (Instability Criterion for ground states) *Assume there exists* $\Psi \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ *satisfying*

(i) Ψ *belongs to the tangent space* $T_{(P,Q)}\Sigma$*;*
(ii) $\Lambda^{-1}\Psi$ *is* L^2*-orthogonal to* $i(P, 3Q)$ *and* $\partial_{x_j}(P, Q)$*,* $j = 1, \dots, n$*;*
(iii) $i(P, 3Q)$ *and* $\partial_{x_j}(P, Q)$*,* $j = 1, \dots, n$ *are linearly independent;*
(iv) $\langle S''(P, Q)\Psi, \Psi\rangle < 0$*, where* $S = E + \frac{\omega}{2}M$*.*

Then, (P, Q) *is orbitally unstable by the flow of* (1.3).

Proof See [7,10]. □

We are now in position of proving the following result:

Theorem 5.3 *Assume either* $n = 3$ *and* $\mu > 0$ *or* $n = 2$ *and* $\mu \neq 3\sigma$*. Let* (P, Q) *be a ground state. Then, the standing wave* $(e^{i\omega t}P, e^{3i\omega t}Q)$ *is orbitally unstable by the flow of* (1.3).

Proof We will check the assumptions in Theorem 5.2. To do so, let us introduce the smooth curve

$$\Gamma(t) = \left(\gamma(t)\lambda^{\frac{n}{2}}(t)P(\lambda(t)\cdot), \alpha(t)\lambda^{\frac{n}{2}}(t)Q(\lambda(t)\cdot)\right),$$

where α, γ, and λ are smooth functions to be chosen later satisfying,

$$\alpha(0) = \gamma(0) = \lambda(0) = 1. \tag{5.3}$$

In particular we have $\Gamma(0) = (P, Q)$. Define the real number k by

$$k := \frac{\int P^2}{3\sigma \int Q^2}.$$

The assumption that $\Gamma(t) \subset \Sigma$ is equivalent to

$$\gamma^2 k + \alpha^2 = k + 1. \tag{5.4}$$

So, from now on we will assume that (5.4) holds; so that once we choose the function α, γ is completely determined. By defining

$$\Psi = \Gamma'(0) \tag{5.5}$$

we promptly see that $\Psi \in T_{(P,Q)}\Sigma$; and condition (i) in Theorem 5.2 holds.

Next we recall that

$$\Lambda^{-1} = \begin{pmatrix} i & 0 \\ 0 & \sigma i \end{pmatrix}.$$

Hence $\Lambda^{-1}\Psi$ has purely imaginary components. This immediately implies that $\Lambda^{-1}\Psi$ is orthogonal to $\partial_{x_j}(P, Q)$, $j = 1, \dots, n$. On the other hand, if $\Psi = (\Psi_1, \Psi_2)$, we have

$$\Lambda^{-1}\Psi \perp i(P, 3Q) \Leftrightarrow (\Psi_1, \sigma\Psi_2) \perp (P, 3Q) \Leftrightarrow (\Psi_1, \Psi_2) \perp (P, 3\sigma Q) \Leftrightarrow \Psi \perp \nabla M(P, Q).$$

Since $M(\Gamma(t)) = M(P, Q)$, by taking the derivative with respect to t and evaluating at $t = 0$, it is clear that $\Psi \perp \nabla M(P, Q)$ and assumption (ii) in Theorem 5.2 is checked.

Note that $i(P, 3Q)$ and $\partial_{x_j}(P, Q)$ are orthogonal in $L^2 \times L^2$ which yields (iii). So it remains to check (iv). To do so, first recall that $S(\Gamma(t)) = E(\Gamma(t)) + \frac{\omega}{2}M(P, Q)$, because $\Gamma(t) \subset \Sigma$. Thus,

$$\frac{d^2}{dt^2}E(\Gamma(t)) = \frac{d^2}{dt^2}S(\Gamma(t)) = \langle S''(\Gamma(t))\Gamma'(t), \Gamma'(t)\rangle + \langle S'(\Gamma(t)), \Gamma''(t)\rangle.$$

Evaluating at $t = 0$ and using that $S'(P, Q) = 0$, we see that (iv) is equivalent to

$$\frac{d^2}{dt^2}E(\Gamma(t))\Big|_{t=0} < 0. \tag{5.6}$$

Hence our task is to prove that we can choose α and λ such that (5.6) holds. But, by using (5.4), a simple calculation reveals that

$$\frac{d}{dt}E(\Gamma(t)) = \alpha'(t)A(t) + \lambda'(t)B(t)$$

where

$$\begin{aligned} A(t) = &\int \left(-\frac{\alpha\lambda^2}{k}|\nabla P|^2 + \alpha\lambda^2|\nabla Q|^2 + \frac{1}{9k^2}(k+1-\alpha^2)\alpha\lambda^n P^4 - 9\alpha^3\lambda^n Q^4\right) \\ &+ \int \left(\frac{2}{k}\alpha^3\lambda^n P^2 Q^2 - \frac{2}{k}(k+1-\alpha^2)\alpha\lambda^n P^2 Q^2 - \frac{\alpha}{k}P^2 + \mu\alpha Q^2\right) \\ &+ \int \left(\frac{1}{3k^{3/2}}(k+1-\alpha^2)^{1/2}\alpha^2\lambda^n P^3 Q - \frac{1}{9k^{3/2}}(k+1-\alpha^2)^{3/2}\lambda^n P^3 Q\right) \end{aligned}$$

and

$$\begin{aligned} B(t) = &\int \Bigg(\frac{1}{k}(k+1-\alpha^2)\lambda|\nabla P|^2 + \alpha^2\lambda|\nabla Q|^2 - \frac{n}{36k^2}(k+1-\alpha^2)^2\lambda^{n-1}P^4 \\ &- \frac{9n}{4}\alpha^4\lambda^{n-1}Q^4\Bigg) \\ &+ \int \left(-\frac{n}{k}(k+1-\alpha^2)\alpha^2\lambda^{n-1}P^2Q^2 - \frac{n}{9k^{3/2}}(k+1-\alpha^2)^{3/2}\alpha\lambda^{n-1}P^3 Q\right) \end{aligned}$$

In view of (2.7) and (2.3), we have

$$
\begin{aligned}
B(0) &= \int (|\nabla P|^2 + |\nabla Q|^2) - \frac{n}{4}\int \left(\frac{1}{9}P^4 + 9Q^4 + 4P^2Q^2 + \frac{4}{9}P^3Q\right)\\
&= -\frac{1}{4}\left((n-4)\int \left(|\nabla P|^2 + |\nabla Q|^2\right) + n(\omega+1)\int P^2 + n(\mu+3\sigma\omega)\int Q^2\right)\\
&= 0.
\end{aligned}
$$

Also, in view of (2.2) and (2.1),

$$
\begin{aligned}
A(0) &= \int \left(-\frac{1}{k}|\nabla P|^2 + |\nabla Q|^2 + \frac{1}{9k}P^4 - 9Q^4 + \left(\frac{2}{k}-2\right)P^2Q^2 - \frac{1}{k}P^2 + \mu Q^2\right)\\
&\quad + \frac{1}{3k}\int P^3Q - \frac{1}{9}\int P^3Q\\
&= \int \left(-\frac{1}{k}|\nabla P|^2 + |\nabla Q|^2 + \frac{1}{9k}P^4 - 9Q^4 + \left(\frac{2}{k}-2\right)P^2Q^2 - \frac{1}{k}P^2 + \mu Q^2\right)\\
&\quad + \frac{1}{3k}\int P^3Q - \left(\int (|\nabla Q|^2 + (\mu+3\sigma\omega)Q^2 - 9Q^4 - 2P^2Q^2)\right)\\
&= \frac{1}{k}\left(\int \left(-|\nabla P|^2 - (\omega+1)P^2 + \frac{1}{9}P^4 + 2P^2Q^2 + \frac{1}{3}P^3Q\right)\right)\\
&= 0.
\end{aligned}
$$

Therefore, by denoting $\alpha_0 = \alpha'(0)$ and $\lambda_0 = \lambda'(0)$, we deduce, after some calculations using Lemma 2.2,

$$
\begin{aligned}
&\frac{d^2}{dt^2}E(\Gamma(t))\Big|_{t=0} = \alpha_0 A'(0) + \lambda_0 B'(0)\\
&= \alpha_0^2\left[\int \left(-\frac{2}{9k^2}P^4 + \frac{8}{k}P^2Q^2 - 18Q^4 + \left(\frac{2}{3k}+\frac{1}{9}-\frac{1}{3k^2}\right)P^3Q\right)\right]\\
&\quad + 2\alpha_0\lambda_0\Bigg[2(3\sigma-\mu)\int Q^2 + (n-2)\int \left(\frac{1}{9k}P^4 - 9Q^4 + \left(\frac{2}{k}-2\right)P^2Q^2\right.\\
&\quad \left. + \left(\frac{1}{3k}-\frac{1}{9}\right)P^3Q\right)\Bigg] + \lambda_0^2\frac{n(2-n)}{4}\int \left(\frac{1}{9}P^4 + 9Q^4 + 4P^2Q^2 + \frac{4}{9}P^3Q\right)\\
&\equiv A_0\alpha_0^2 + 2B_0\alpha_0\lambda_0 + C_0\lambda_0^2.
\end{aligned}
$$

In particular, the second derivative of $E(\Gamma(t))$ at $t = 0$ can be identified as a quadratic form associated with a symmetric matrix. Hence, it suffices to show that this quadratic form assumes negative values.

Assume first $n = 2$. Then, it suffices to show that the discriminant

$$
D = A_0C_0 - B_0^2 = -\left(2(3\sigma-\mu)\int Q^2\right)^2
$$

is negative. But this statement is true provided $\mu \neq 3\sigma$.

Assume now $n = 3$. By taking $(\alpha_0, \lambda_0) = (0, 1)$ and using (2.7) and (2.3), we obtain

$$\frac{d^2}{dt^2} E(\Gamma(t))\Big|_{t=0} = -\frac{3}{4}\int \left(\frac{1}{9}P^4 + 9Q^4 + 4P^2Q^2 + \frac{4}{9}P^3Q\right)$$
$$= -\int (|\nabla P|^2 + |\nabla Q|^2), \tag{5.7}$$

from which we deduce (5.6). The proof of Theorem 5.3 is thus completed. □

5.2 Stability

In this last section we study the orbital stability of the ground state given in Proposition 2.8. First of all, we shall rewrite (1.3) as a real pseudo-Hamiltonian system in the form

$$\frac{\partial X}{\partial t}(t) = \Lambda E'(X(t)),$$

where we have written $u = u_1 + iu_2$, $w = w_1 + iw_2$, $X = (u_1, w_1, u_2, w_2)$, Λ is the skew-symmetric linear operator defined by

$$\Lambda = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/\sigma \\ -1 & 0 & 0 & 0 \\ 0 & -1/\sigma & 0 & 0 \end{pmatrix} \tag{5.8}$$

and E is the energy function now given as

$$E(u_1, w_1, u_2, w_2) = \frac{1}{2}\int \Big\{ |\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla w_1|^2 + |\nabla w_2|^2 + u_1^2 + u_2^2$$
$$+ \mu(w_1^2 + w_2^2) - \frac{1}{18}(u_1^4 + 2u_1^2u_2^2 + u_2^4) - \frac{9}{2}(w_1^4 + 2w_1^2w_2^2 + w_2^4)$$
$$- 2(u_1^2 + u_2^2)(w_1^2 + w_2^2) - \frac{2}{9}(u_1^3w_1 + 3u_1^2u_2w_2$$
$$- 3u_1u_2^2w_1 - u_2^3w_2 \Big\} dx. \tag{5.9}$$

Our main theorem here reads as follows.

Theorem 5.4 *Assume $n = 1$ and $\omega + 1 = \mu + 3\sigma\omega$. Let $(0, Q)$ be a ground state of (1.7) according to Proposition 2.8. Then $(0, e^{3i\omega t}Q)$ is orbitally stable by the flow of (1.3).*

Here, if necessary, we will use Q_ω instead of Q to emphasize that Q depends on ω. In addition, throughout the section, we assume $\omega + 1 = \mu + 3\sigma\omega$. In order to prove Theorem 5.4 we will use the well-known Grillakis, Shatah and Strauss' theory [11]. To simplify the notations, let $\Phi = (0, Q, 0, 0)$ and

$$\mathcal{L}_{\rightarrow} = S''(\Phi) = \begin{pmatrix} \mathcal{L}_R & 0 \\ 0 & \mathcal{L}_I \end{pmatrix}, \tag{5.10}$$

where $\mathcal{L}_R$ and $\mathcal{L}_I$ are the 2×2 matrix diagonal operators defined by

$$\mathcal{L}_R = \begin{pmatrix} -\Delta + (\omega + 1) - 2Q^2 & 0 \\ 0 & -\Delta + (\mu + 3\sigma\omega) - 27Q^2 \end{pmatrix} \tag{5.11}$$

and

$$\mathcal{L}_I = \begin{pmatrix} -\Delta + (\omega + 1) - 2Q^2 & 0 \\ 0 & -\Delta + (\mu + 3\sigma\omega) - 9Q^2 \end{pmatrix}. \tag{5.12}$$

In order to describe the spectrum of $\mathcal{L}_\omega$, we first study the spectral properties of the following operators:

$$\mathcal{L}_1 = -\Delta + (\mu + 3\sigma\omega) - 27Q^2, \qquad \mathcal{L}_2 = -\Delta + (\mu + 3\sigma\omega) - 9Q^2 \tag{5.13}$$

and

$$\mathcal{L}_3 = -\Delta + (\omega + 1) - 2Q^2 \tag{5.14}$$

More precisely, we have:

Theorem 5.5 *Let* $(0, Q)$ *be as in Proposition 2.8. Then:*

(i) The operator $\mathcal{L}_1$ *in (5.13) defined in* $L^2(\mathbb{R}^n)$ *has only one negative eigenvalue. Its kernel is given by* $Ker(\mathcal{L}_1) = span\{Q_{x_i};\ i = 1, \dots, n\}$ *and the remainder of the spectrum is bounded away from zero.*

(ii) The operator $\mathcal{L}_2$ *in (5.13) defined in* $L^2(\mathbb{R}^n)$ *has no negative eigenvalues. Zero is a simple eigenvalue with associated eigenfunction* Q*. Moreover, the remainder of the spectrum is bounded away from zero.*

(iii) The operator $\mathcal{L}_3$ *in (5.14) defined in* $L^2(\mathbb{R}^n)$ *is a positive operator. Moreover, the remainder of the spectrum is bounded away from zero.*

Proof These are well-known results, see for instance [23,25]. Note that (iii) is a consequence of (ii). □

As an immediate consequence, we have.

Corollary 5.6 *Let* $(0, Q)$ *be as in Proposition 2.8. Then the operator* $\mathcal{L}_{\rightarrow}$ *has exactly one negative eigenvalue,* $Ker(\mathcal{L}_{\rightarrow})$ *is* $(n + 1)$*-dimensional and spanned by the set* $\{(0, 0, 0, Q), (0, Q_{x_i}, 0, 0);\ i = 1, \dots, n\}$*. Moreover, the remainder of the spectrum is bounded away from zero.*

Now we proof Theorem 5.4.

Proof *(Proof of Theorem 5.4)* In view of Corollary 5.6 and the theory in [11] it suffices to prove that the second derivative of the function $d(\omega) = S(0, Q_\omega)$ is positive. But since $(0, Q_\omega)$ is a critical point of S we have

$$d'(\omega) = \frac{1}{2}M(0, Q_\omega) = \frac{3\sigma}{2}\int Q_\omega^2.$$

Note that if Q_0 is the ground state of the equation

$$-\Delta Q + (\omega + 1)Q - 9Q^3 = 0, \tag{5.15}$$

with $\omega = 0$, then (by uniqueness)

$$Q_\omega(x) = (\omega + 1)^{1/2} Q_0\left((\omega + 1)^{1/2}x\right)$$

is the ground state of (5.15) with $\omega > -1$. Thus,

$$\int Q_\omega^2 = \frac{1}{(\omega + 1)^{n/2-1}}\int Q_0^2$$

and

$$d''(\omega) = \left(1 - \frac{n}{2}\right)\frac{3\sigma}{2(\omega + 1)^{n/2}}\int Q_0^2,$$

from which we deduce $d''(\omega) > 0$ for $n = 1$. This completes the proof of the theorem. □

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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