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JOSÉ A. AZEVEDO PEREIRA
DAVID P. NEWTON
DEAN A. PAXSON

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DEPARTAMENTO DE GESTÃO

R. Miguel Lupi, 20 – 1249-078 Lisboa – Fax: 21.392.28.08 – Telf. 21.392.28.12/23

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José A. Azevedo Pereira
David P. Newton
Dean A. Paxson

Instituto Superior de Economia e Gestão
R. Miguel Lupi, 20
1249-078 LISBOA
Telf. 21.392.28.12 / 23
Fax: 21.392.28.08

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ABSTRACT

We use a mean-reverting interest rate model and a lognormal house price diffusion model to evaluate British fixed rate mortgage contracts with (embedded) default and prepayment options. The valuation model also provides values for mortgage indemnity guarantees and the corresponding lenders' coinsurance.

Since the partial differential equation incorporating the general features of these mortgage contracts does not have a closed-form solution, an explicit finite difference method was used to solve the problem.

Changes in contractual features in common mortgage products lead to different equilibrium coupon rates and different values for mortgage components. Our numerical results suggest that mortgage modelling include both of these contractual provisions and the embedded options in order to prevent biased and misleading mortgage valuation.

UK Fixed Rate Repayment Mortgage

and Mortgage Indemnity Valuation *

I. Introduction

We value UK mortgage products, including those embedded features which are significantly different from their American counterparts. We note that some British insurers were severely hurt in the last house price recession and have since developed capped products in which the liability for a loss suffered by a lender has an upper limit. Such Mortgage Indemnity Guarantees (MIGs) automatically involve the lenders in the process of loss coverage, making them responsible for part of the costs as so-called coinsurance. We consider the valuation of repayment mortgages and MIGs. The value of a MIG depends on the behaviour of the borrower, which in turn is influenced by the value of the property and the options intrinsic to the mortgage contract. Thus, during the life of a mortgage contract, the borrower may find it financially advantageous to prepay or even to default. It follows that the value of the mortgage is coupled to property value, the term structure of interest rates and time. Simultaneously, since insurance is directly related to default and default depends on the value of the whole mortgage, it is necessary to have valued the mortgage and its embedded options prior to valuing the MIG.

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Our approach to the valuation of mortgages is broadly comparable with the line of reasoning given by McDonald and Siegel (1986) in which the decision to invest in a non-repeatable project excludes the possibility of making the same investment in the future, with consequent loss of a valuable option. More particularly, we follow the work of Kau et al. (1993a), who modelled mortgages in the U.S., though these are significantly different in contractual details and, hence, valuation. The options to terminate the mortgage, by default or by prepayment, have positive value for the borrower. By defaulting, the borrower not only gives away the house but also loses the joint option to terminate the loan by default or prepayment. Conversely, as each monthly payment is made and as the time to expiry of the mortgage lessens, the value of this joint option is reduced.

The next section outlines the valuation framework. The third section identifies the separate components of a mortgage. The fourth section presents the mortgage parameters and valuation results. The final section summarises the valuation approach and limitations.

II. Valuation Framework

We apply a contingent claims framework for pricing residential mortgages as derivative assets, using two state variables: spot interest rate and house price. We use the CIR model of spot interest rate, r, as a mean-reverting square root diffusion process (Cox, Ingersoll and Ross, 1985b). This allows only positive nominal interest rates, through the square root term, and allows for bounded movement of the interest rate, through mean reverting drift.

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dz_r \tag{1}$$

where

κ ≡ speed of adjustment in the mean reverting process,

θ ≡ central location or long term mean of the short-term interest rate r(t) (steady state spot rate),

σ ≡ instantaneous standard deviation of the (interest rate) disturbance,

z_r ≡ standardized Wiener process,

We treat the house price, H, as a lognormal diffusion process (for a discussion, see Merton 1973) shown in equation (2). Since the householder receives benefits from living in the house which would otherwise be paid for as part of the rent for living in comparable property as a tenant, we include the term δ, for the "service flow" provided by the house.

$$\frac{dH}{H} = (\mu - \delta)dt + \nu dz_H \qquad (2)$$

where:

$\mu \equiv$ instantaneous long-term average rate of house price appreciation,

$\delta \equiv$ "dividend-type" per unit service flow provided by the house,

$\nu \equiv$ instantaneous standard deviation of the house price,

$z_H \equiv$ a second standardised Wiener process.

Correlation between these two processes is captured by equation (3).

$$dz_r(t)dz_H(t) = \rho dt \qquad (3)$$

where ρ denotes the instantaneous correlation coefficient between the Wiener processes.

The complex considerations of customer preferences, building type and quality differences and any other practical or market considerations may be said to determine the evolution of the state variables. However, once the house price and term structure have been determined, the value of the mortgage is set through a process of arbitrage inference. All influential factors are condensed into the market price of risk associated with each state variable. In addition, since the house price can be said to derive from a traded asset, no risk adjustment is required beyond the inclusion of the service flow (a dividend-like payment), leaving the spot interest rate as the external factor impounding risk.

The CIR framework provides a general methodology for valuation of contingent claims. Standard arguments in finance allow us to write equation (4), relating the mortgage asset value, F(r,H,t) to the state variables house price, H, and spot interest rate, r (Cox Ingersoll and Ross, 1985a,b; Epperson et al., 1985, Kau et al., 1992, 1993a).

$$\frac{1}{2}H^2 v^2 \frac{\partial^2 F}{\partial H^2} + \rho H \sqrt{r} v \sigma \frac{\partial^2 F}{\partial H \partial r} + \frac{1}{2} r \sigma^2 \frac{\partial^2 F}{\partial r^2} + \kappa(\theta - r)\frac{\partial F}{\partial r} + (r - \delta) H \frac{\partial F}{\partial H} + \frac{\partial F}{\partial t} - rF = 0 \qquad (4)$$

Solution of this equation must include the borrower's two options to terminate before maturity by prepayment or by default, which cannot be valued independently. The prepayment option is American in style, with the inherent free-boundary, and the default option is of compound European style (i.e. it is actually a series of options). No analytic solution is available and so a numerical solution must be sought.

Methods for numerical solution may be categorised into those which work forwards in time, in particular Monte Carlo methods, and those which work backwards, for example finite difference methods. Although successful in many applications, Monte Carlo methods, which use large numbers of simulations of the stochastic processes followed by the underlying variables, cannot readily deal with early exercise features. The reason is that, working forwards in time in a standard simulation in an asset-time mesh, it is unknown at each mesh point whether or not early exercise would be optimal. In other words, an ordinary Monte Carlo simulation cannot determine that there is no arbitrage. It is possible to value the option at each point on the asset-time mesh, using a new simulation each time another option value is required, and working through a process to mark points in the mesh at which the option must be exercised, prior to the Monte Carlo run. However, the computation time needed grows exponentially with the number of points and is, for our purposes, impractical.

Working backwards in time, using a finite difference mesh, the problem of identifying points at which early exercise is optimal disappears and we are left only with the technical intricacies of implementation. Given specific details of the contract, the value of the financial assets (options) embedded in a mortgage are known at expiry. Using appropriately small time steps, equation (4) can be used to work backwards from the final mortgage payment, calculating the asset values sequentially to the previous mortgage payment, then using that new set of terminal conditions to work back to a still earlier payment until eventually the origination of the contract is reached.

III. Components of the Mortgage Contract

The value of a mortgage to the borrower is composed not only of the (negative) present value of promised future monthly payments to the lender but also of the (positive) options to prepay or to default. These options are valuable to the borrower and reduce the absolute value (to borrower or lender) of the outstanding mortgage. At some point in time, t, the options are represented by equation (5), where C represents the value of the call option to prepay, D the default option and J the joint option.

$$J(H,r,t) = C(H,r,t) + D(H,r,t) \quad (5)$$

Representing the value of remaining payments by A(r,t), the value of the mortgage to the borrower is given by equation (6):

$$V_B(H,r,t) = A(r,t) - C(H,r,t) - D(H,r,t) = A(r,t) - J(H,r,t) \quad (6)$$

Notice that if a mortgage has an associated MIG and if the borrower is financially rational and circumstances arise in which he chooses to default, then the MIG only benefits the lender. Consequently, the MIG has no value for the borrower. Although the MIG's value depends on the contract via market conditions and the borrower's rational behaviour, it is not part of the contract and renders the mortgage value for the borrower different from its value for the lender. The value of the contract for the lender, therefore, is the sum of its value to the borrower and the value of the MIG. This is shown in equation (7).

$$V_L(H,r,t) = V_B(H,r,t) + I(H,r,t) \quad (7)$$

where:

$I(H,r,t) \equiv$ the value of the MIG at time t;

It follows that the value of the mortgage to the borrower can be calculated first and so from here onwards "mortgage value" will refer to the value to the borrower.

The value of each monthly payment, MP, is determined in order to allow the principal to be paid in full by the end of the contract:

$$MP = \frac{\left(\frac{c}{12}\right)\left[1+\left(\frac{c}{12}\right)\right]^{n} O(0)}{\left[1+\left(\frac{c}{12}\right)\right]^{n} - 1} \tag{8}$$

where O(0) represents the amount of the debt at the origination of the loan.

The outstanding balance after each payment date, O(i), is given by the following expression:

$$O(i) = \frac{\left[\left(1+\frac{c}{12}\right)^{n} - \left(1+\frac{c}{12}\right)^{i}\right]}{\left(1+\frac{c}{12}\right)^{n} - 1} O(0) \tag{9}$$

Valuation of the promised future payments is relatively straightforward, involving only the term structure of interest rates. The option valuations are more difficult. Default is rational only when payments are immediately due (obviously, since the possibility of defaulting does not arise until payment is contractually required). Since there is a series of monthly payment dates, the default option is actually a compound European option. Prepayment may prove optimal at any time during the life of the contract and so this is an American option, with all the difficulties inherent in the solution of a free-boundary problem.

At payment dates, a distinction will be made between the value of an asset immediately before and immediately after each payment. The notation used is:

$F^{-}(H,r,t)$ = Value of the asset F immediately before a payment is made;

$F^{+}(H,r,t)$ = Value of the asset F immediately after a payment is made.

Assuming monthly mortgage payments by the borrower, we need to be able to distinguish mortgage payment dates. The notation used is:

n = the life of the mortgage in months;

$\eta(i)$ = the time of the i^{th} month (i^{th} payment date);

Valuation begins at the maturity of the mortgage, when the terminal condition must be that the function representing the value of remaining payments must be equal to the final monthly payment due, MP, (equation (10)).

$$A^{-}(r,t) = MP \qquad \text{for } t = \eta(n) \qquad (10)$$

Moving backwards in time, as each monthly payment date is reached, the borrower's debt changes abruptly by the amount MP, as $A^{+}(r,t)$ becomes $A^{-}(r,t)$. This leads to solution of equation (4) by a finite difference method, starting with the terminal condition at maturity, working backwards in time until another monthly payment date is reached. Then a new boundary condition, equation (11), is applied and the backwards process is continued until the next payment date is reached. The process is repeated until valuation has been completed, at the starting time of the mortgage contract.

$$A^-(r,t) = A^+(r,t) + MP \qquad \text{for } t = \eta(1), \ldots, \eta(n-1) \qquad (11)$$

During this process, we apply boundary conditions for the options held by the borrower. The option to default is directly affected by the house price. If the house price equals or exceeds the value of the remaining payments, the financially rational borrower either does nothing or sells; otherwise, he defaults and gives up the house to the lender. The value of the prepayment option depends on the prevailing term structure of interest rates but not directly on the house price. However, there is an indirect relationship, since the exercise of the option to default automatically causes the prepayment option to expire worthless. Thus, the two options interact and cannot be separately valued and added. At expiry of the mortgage the borrower can choose to make the final monthly payment or to default and give up the house, and so the value of the mortgage at that time is given by equation (12)

$$V_B^-(H,r,t) = \min(MP, H) \qquad \text{for } t = \eta(n) \qquad (12)$$

At earlier payment dates we have a series of similar conditions, given by equation (13)

$$V_B^-(H,r,t) = \min\left([V_B^+(H,r,t) + MP], H\right) \qquad \text{for } t = \eta(1), \ldots, \eta(n-1) \qquad (13)$$

In addition to the boundary conditions for the compound European options (the default option), there is also a boundary condition imposed by the prepayment option. This is a time-dependent free-boundary whose position cannot be known a priori but must be determined as part of the solution of the valuation problem. This may be done either by a boundary tracking method (for a synthesis see Crank, 1984) or via a transformation which converts the problem into one with a fixed boundary, which was the method selected for this work (for a review see Wilmott, Dewynne and Howison, 1993).

If the borrower prepays the mortgage, the amount to be paid is calculated from the outstanding balance and the accrued interest since the most recent scheduled monthly payment. In the UK there is likely to be an additional penalty payment required in the terms of the contract. The details of such penalties are not standardised and so here a penalty is modelled as a percentage of the outstanding balance plus accrued interest at the time of early termination. This is represented in equation (14).

$$TD(t) = \{(1+\pi)\{1+c[t-\eta(i)]\}O(i) \quad \text{for} \quad \eta(i) \leq t \leq \eta(i+1) \qquad (14)$$

where $TD(t)$ is the total debt, π represents the early termination penalty imposed on the borrower and c is the fixed mortgage payment, equivalent to a coupon on a bond.

The default decision is assumed not to be simply triggered if the present value of the remaining payments exceeds the current market value of the house, H, but rather if V_B, the value of the mortgage to the borrower including options, exceeds the house value. This termination condition is shown by equation (15).

$$A(r,t) > H + [C(H,r,t)+D(H,r,t)] \quad (15)$$

At maturity of the mortgage, when the decision on whether or not to make the final mortgage payment is made, the default option will be worthless if the house is worth more than the final payment and otherwise equal to the difference between the two:

$$D^{-}(H,r,t) = \max(0,[MP - H]) \qquad \text{for } t = \eta(n) \quad (16)$$

On other monthly payment dates the default option value is unchanged by the payment under conditions of no default and is adjusted to the difference between the value of the remaining payments and the house price when there is default. The conditions for default are set as follows for default and no default:

$$\begin{aligned} D^{-}(H,r,t) &= D^{+}(H,r,t) && \text{if } V_B^{-}(H,r,t) = V_B^{+}(H,r,t) + MP && \text{(no default)} \\ &= A^{-}(r,t) - H && \text{if } V_B^{-}(H,r,t) \leq H && \text{(default)} \\ & && \text{for } t = \eta(1), \ldots, \eta(n-1) && (17) \end{aligned}$$

It remains only to consider the value of the prepayment component of the joint option to terminate. The terminal condition for the prepayment option at maturity is trivial, since prepayment cannot then have any value for the borrower. At other payment dates, prepayment can only have value in the absence of default and so the conditions must be as in equation 18

$$C^-(H,r,t) = C^+(H,r,t) \quad \text{if } V_B^-(H,r,t) = V_B^+(H,r,t) + MP \quad \text{(no default)}$$

$$= 0 \quad \text{if } V_B^-(H,r,t) = H \quad \text{(default)}$$

$$\text{for } t = \eta(1), \ldots, \eta(n-1) \quad (18)$$

Alternatively, C can be calculated as in (19), using earlier results:

$$C = A - V_B - D \quad (19)$$

Table 1 shows the boundary conditions for the mortgage and its components.

Table 1.

Boundary Conditions for the Value of the Components of the Mortgage Contract

Components of the Mortgage Contract	Value in Case of Continuation (no default)	Value in Case of Default
A^-	$A^+ + MP$	$A^+ + MP$
C^-	C^+	0
D^-	D^+	$A^- - H$
V_B^-	$V_B^+ + MP$	H
$C^- = A^- - V_B^- - D^-$	$(A^+ + MP) - (V_B^+ + MP) - D^+ = A^+ - V_B^+ - D^+ = C^+$	$(A^+ + MP) - (A^- - H) - H = 0$

We next develop a framework for valuing British MIGs and the corresponding coinsurance assumed by lenders. The MIG is contractually separate from the package of the mortgage, the house and the embedded options but its value is dependent on the expected performance of this package. The MIG is a contract in which an insurer agrees to pay a fraction of the total loss suffered by a mortgage lender on each loan included in a specific pool of mortgages. The precise characteristics of British MIGs vary from case to case, between different insurers and according to the economic environment when the contract is signed. Here we will consider common features of recent MIGs.

In the case of default by the borrower, the loss suffered by the lender is considered to be the difference between the value of the borrower's total debt and the value of the house, TD(t) - H. The insurer agrees, in a "guarantee", to pay a fraction, γ, of the lender's total loss but only up to a maximum indemnity (i.e. a cap), Γ. Define x as the initial ratio of loans to value for mortgage or, in practice, for a group or "pool" of mortgages, (so that x = 0.95 if the loan is 95% of the house value). A second ratio value, y, is then set so that the insurer's maximum exposure is given by $\Gamma = (x-y)H(0)$. Normally, a lender might expect the difference (x-y) to be around 0.2. We summarise the values of the guarantee in equation (20)

$$\Gamma\,[\eta(i)] = \gamma\{TD^{-}[\eta\,(i)] - H\} \qquad \text{if} \qquad \left\{\left\{TD[\eta(i)]\right\} - H\right\} \le \frac{1}{\gamma}\left\{[xH(0)] - [yH(0)]\right\}$$

$$\Gamma\,[\eta(i)] = \left\{[xH(0)] - [yH(0)]\right\} \qquad \text{if} \qquad \left\{\left\{TD[\eta(i)]\right\} - H\right\} > \frac{1}{\gamma}\left\{[xH(0)] - [yH(0)]\right\}$$

$$\text{for all } \eta(i) \qquad (20)$$

The terminal condition for the MIG must be that either the MIG has no value and the value of the mortgage to the borrower just before the moment of final payment is equal to that final payment, or that the MIG has some value because it is worthwhile for the borrower to default. These conditions are shown in (21)

$$I^{-}(H,r,t) = 0 \qquad \text{if} \quad V_B^{-}(H,r,t) = MP \quad \text{(no default)}$$

$$= \min\{[\gamma\,(MP - H)], \Gamma\} \qquad \text{if} \quad V_B^{-}(H,r,t) = H \quad \text{(default)}$$

$$\text{for } t = \eta(n) \qquad (21)$$

At earlier payment dates, these equations need only slight modifications, as in (22)

$$I^{-}(H,r,t) = I^{+}(H,r,t) \qquad \text{if} \quad V_B^{-}(H,r,t) = V_B^{+}(H,r,t) + MP \quad \text{(no default)}$$

$$= \min\{\{\gamma\,[TD^{-}(t) - H]\}, \Gamma\} \quad \text{if} \quad V_B^{-}(H,r,t) = H \quad \text{(default)}$$

$$\text{for } t = \eta(1), \ldots, \eta(n-1) \qquad (22)$$

The portion of the potential loss not covered by the MIG is called the coinsurance, CI. At each payment date, the coinsurance is simply the difference between the values of the potential loss and the insurance coverage provided by the MIG.

$$CI^{-}(H,r,t) = 0 \qquad \text{if} \quad V_B^{-}(H,r,t) = MP \quad \text{(no default)}$$

$$= \max\{[(1-\gamma)(MP - H)], [(MP - H) - \Gamma]\} \quad \text{if} \quad V_B^{-}(H,r,t) = H \quad \text{(default)}$$

for $t = \eta(n)$ (23)

Other Payment Dates:

$CI^{-}(H,r,t) = CI^{+}(H,r,t)$ if $V_B^{-}(H,r,t) = V_B^{+}(H,r,t) + MP$ (no default)

$= \max\{\{(1-\gamma)\{TD^{-}[\eta(i)] - H\}\}, \{\{TD^{-}[\eta(i)] - H\} - \Gamma\}\}$

if $V_B^{-}(H,r,t) = H$ (default)

for $t = \eta(1), ..., \eta(n-1)$ (24)

Mortgage rates and contract provisions vary widely over time. The economic environment changes continuously and contract specifications are also subject to frequent readjustment, not only in order to accommodate those changes but also for marketing reasons. In the equilibrium framework proposed in this paper, a contract can only be acceptable if it represents a fair deal. In order for two economic agents to trade assets freely, it is necessary that neither is able to make any "a priori" profit. In other words, it is necessary to ensure that the borrower is not able to make an instantaneous profit by prepaying the loan at origination and, similarly that the contract is not structured in such a way that allows the lender to make any immediate profit. This is a condition of no arbitrage.

The values assumed by the two state variables considered in the model, r(0) and H(0), are known at the origination of each mortgage. Consequently, the identification of an equilibrium fixed mortgage rate or "coupon" rate is an iterative exercise in which, starting with these initial values for the state variables and the functional form specification for the contract provisions, a search is done in order to find a coupon rate capable of allowing the mortgage to meet the condition of no-arbitrage for each of the parties involved.

The mortgage contract clauses which exert a direct influence upon the possibility of early termination by the borrower and, subsequently upon the value of the mortgage to the lender, are the arrangement fee, ξ, the early termination penalty, π, and the Mortgage Indemnity Guarantee, I.

Both the arrangement fee and the mortgage indemnity guarantee can be treated independently of the other components of the loan (the value of the future payments or the options to prepay or default) because neither of them affects the value of these components. These two provisions have very different effects on equilibrium coupon rates. The inclusion of an arrangement fee only leads to a linear increase in the value of the lender's position in the contract. The influence of the insurance component is not so straightforward. Not only does its value change according to the nature of the underlying contracts but also it changes non-linearly with the coupon rate charged to the borrower.

At origination of the mortgage, the equilibrium condition can now be applied in order to find the coupon rate charged to the borrower which avoids arbitrage, so that the assets exchanged are of equal value to borrower and lender. This is shown in equation (25), where the influences of an arrangement fee, ξ and an early termination fee, π, have been recognised.

$$V_B(H_0, r_0, t_0, c, \pi) + I(H_0, r_0, t_0, c, \pi) = (1-\xi)L \qquad (25)$$

where L = amount of the loan;

Notice that the MIG, despite not being part of the contract, influences the mortgage value via its effect on the equilibrium coupon rate, c. In order to find this rate, a secant iteration technique was used, as described by Gerald and Wheatley (1994) and Press et al. (1992). This technique iteratively searches for the root of the problem (coupon rate) capable of making the equilibrium condition hold. The iteration process was stopped for errors less than 0.0001 of the house value, a margin of error of £10 on a £100,000 house.

Rearranging equation (25), equilibrium holds when the following equation is true:

$$V_B(c, \pi) - (1-\xi)L + I(c,\pi) = 0 \qquad (26)$$

All elements are now in place for solution of equation (4). In the absence of analytical solutions, the equation was solved numerically using an explicit finite difference technique. In the next section we set base parameters appropriate for UK mortgages and show graphically some of the valuation results.

The framework presented can be extended to cope with the variable rate counterparts of both products analysed. In these cases it is necessary to cope with additional problems related to the need to overcome the path-dependency inherent in the evolution of variable rates. The methodologies proposed by Kishimoto (1989), Kau et al. (1993a) and Hilliard et al. (1995) permit those difficulties to be overcome. The only constraint in these cases is related to the computing power necessary to tackle the problem. The need to add an additional auxiliary state variable to track the evolution of the coupon rate adds an extra spatial dimension to the problem. This requires the use of a powerful computing system in order reach a solution in a reasonable period of time.

IV. Mortgage Parameters and Mortgage Values

We present illustrative results for two contract specifications. First, a mortgage with an arrangement fee but no early termination penalty, secondly one with a lower arrangement fee and an early termination penalty. One point which must be emphasised is that American mortgages differ significantly from British mortgages. They differ first in the amount of the arrangement fee ("points" in the US) and secondly in the types of the insurance coverage associated with the products. In the American case the insurance coverage seems to be a simple pre-defined percentage of the value of the debt (see, for instance, Kau et al. 1993a). In the British case, the loss coverage is shared between the insurer and the lender, but the liability of the former is capped to a pre-defined amount which is a function of the difference between the original loan-to-value (LTV) ratio of the loan and a defined "normal" LTV ratio (usually 75%). Both these features affect the equilibrium coupon rates. Consequently, even for two mortgages that coincide in every other detail, the coupon rates could differ, giving different values for the mortgages. In addition, British mortgages have early termination penalties which affect the exercise of options by the borrower and consequently the equilibrium coupon rates.

Table 2 shows the basic set of economic parameters used in this work, in line with common assumptions in the literature (see Buser and Hendershott, 1984; Dunn and McConnell, 1981a,b; Kau et al. 1993b, 1995; Leung and Sirmans, 1990; Stanton, 1995; Stanton and Wallace, 1995).

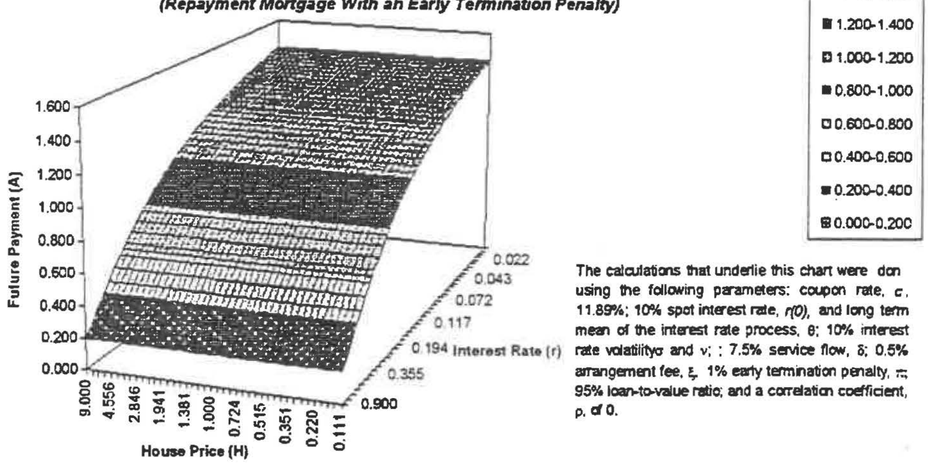
Table 2. Base Values

LIST OF PARAMETERS	CONTRACT	
	Repayment Mortgage With Arrangement Fee and Without Early Termination Penalty	Repayment Mortgage With Arrangement Fee and Early Termination Penalty
ECONOMIC ENVIRONMENT:		
. Spot interest rate, r(0)	10%	10%
. Long term average of interest rate (steady state), θ	10%	10%
. Speed of reversion, κ	25%	25%
. House service flow, δ	7.5%	7.5%
. Correlation coefficient, ρ	0	0
CONTRACT		
. Maturity, η	300 months	300 months
. Value of the house at origination, H	£ 100,000	£ 100,000
. Arrangement fee, ξ	1%	0.5%
. Early termination penalty, π	0	1%

Figures 2 to 7 illustrate values at the origination [$t = \eta(0)$] of contracts with an early termination penalty (the data are available from the authors). See the Appendix for a summary of the finite difference approach and graphical synthesis of the structure of the solution. All assets are valued in relation to a par value of unity for the house. These figures show that the numerical solutions evolved across the state space in a smooth way without any sort of instability. Also, they show that the shapes of the graphs make economic sense.

The value of remaining mortgage payments, A, depends only on the discount rate, r, and so values parallel to the H axis are constant, as shown in Figure 1. As would be expected, A shows an inverse relationship with r. The different equilibrium coupon rates for the two mortgage types cause a difference in the values of the mortgage payments which generates the small divergence in the value of the function across contracts.

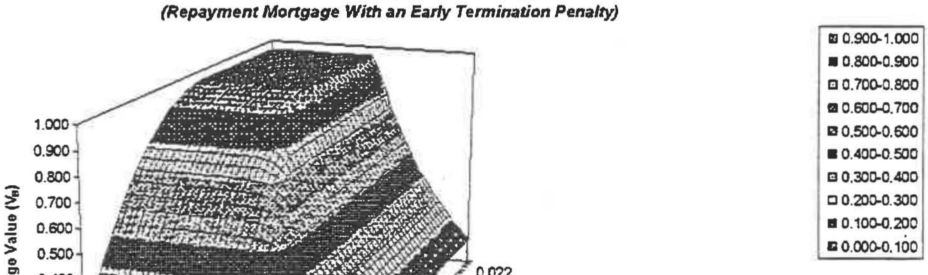
Figure 1 Value of Future Payments, A(r,t=0)
(Repayment Mortgage With an Early Termination Penalty)



The value of the mortgage, V_B, is a complicated function of the value of the remaining mortgage payments, A, the option to prepay, C, and the option to default, D, and is illustrated in Figure 2. Low levels of house prices tend to increase the value of the default option, D, held by the borrower and, consequently, to reduce the value of the mortgage contract. At other house price levels, D is displaced by C as the significant option. Changes in interest rates then impact both A and C inversely but these produce opposite effects on the value of V_B; for example, increases in A increase V_B whereas increases in C diminish V_B. Obviously, C cannot be bigger than A and normally is substantially smaller. Therefore, the relationship between interest rates and the value of the mortgage contract tends to be dominated by the effect of the interest rate on A. An exception occurs when the conjunction of low interest rates and high house prices leads to a situation in which it becomes preferable for the borrower to prepay the loan. This situation corresponds to the top section of the graph presented in Figure 2. In the case of the repayment mortgage with arrangement fee and prepayment penalty, the prepayment region is a plane but now levelled slightly higher than the original value of the loan. When house prices assume very low levels, default is certain to occur at the next payment date and the level of the house price exerts a major influence on the value of the mortgage contract.

Figure 2 Mortgage Value (V_B)

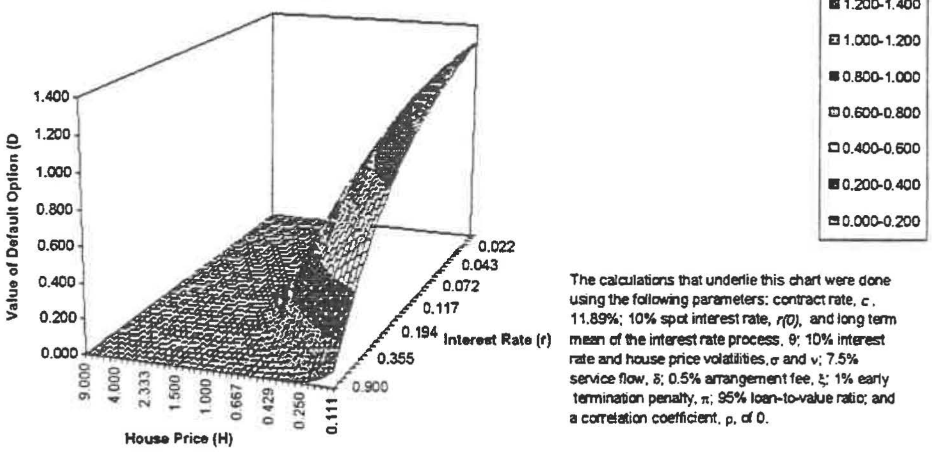
(Repayment Mortgage With an Early Termination Penalty)



The calculations that underlie this chart were done using the following parameters: contract rate, c, 11.89%; 10% spot interest rate, $r(0)$, and long term mean of the interest rate process, θ; 10% interest rate and house price volatilities, σ and v; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 1% early termination penalty, π; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0.

Figure 3 portrays the value of the default option, D. The major influence on the value of this option is the relationship between the level of H and the value of the mortgage contract. The value of D is positive in almost all of the subset of state space in which H < H(0). As the increase in the level of r leads to decreases in the value of A and V_B, the value of the default option, whenever positive, tends to be inversely related to r.

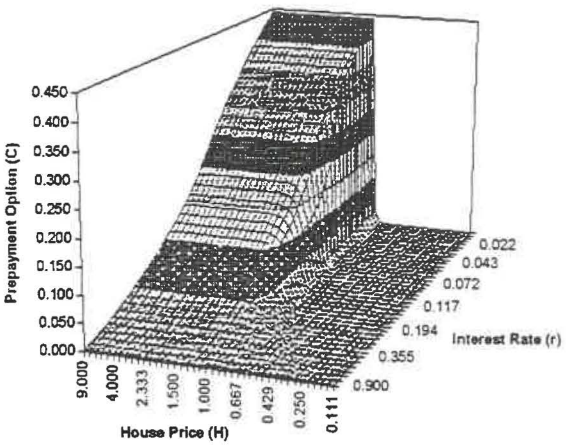
Figure 3 Default Option (D)
(Repayment Mortgage With an Early Termination Penalty)



The calculations that underlie this chart were done using the following parameters: contract rate, *c*, 11.89%; 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 10% interest rate and house price volatilities, σ and v; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 1% early termination penalty, π; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0.

Figure 4 presents the value of the prepayment option, C. The primary determinant of its value is the level of the interest rate. As can be observed, the function only assumes high values for low levels of r coinciding with high levels of house prices. This happens because low house prices tend to generate default and, of course, a defaulted mortgage cannot be prepaid.

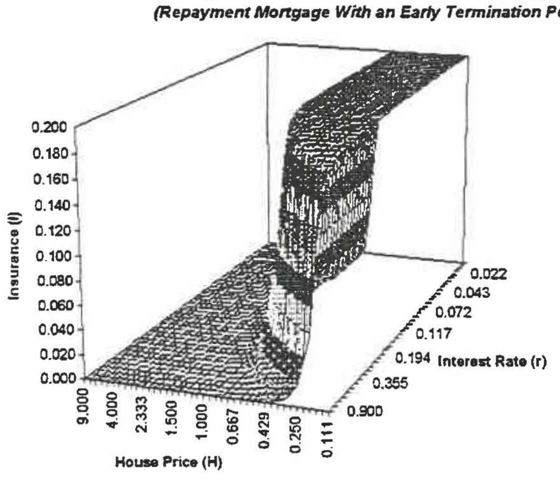
Figure 4 Value of Prepayment Option (C)
(Repayment Mortgage With an Early Termination Penalty)



The calculations that underlie this chart were done using the following parameters: contract rate, *c*, 11.89%; 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 10% interest rate and house price volatilities, σ and ν; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 1% early termination penalty, π; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0.

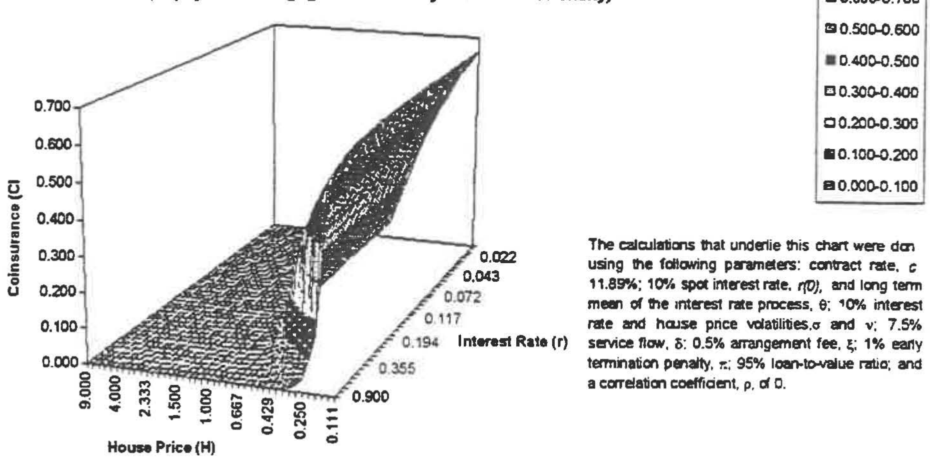
Figures 5 and 6 present the values assumed by the insurance related variables, I and CI along the state space. Both are directly related to the evolution of the default option. At high interest rate levels, the value of the contract is significantly reduced and it is necessary for house prices to fall to very low levels for the borrower to default and the insurance policy to be exercised. The value of the insurance coverage is capped. Consequently, for low levels of H, the function reaches its maximum level quite quickly. As expected, the coinsurance assumes higher values only after that level is surpassed.

Figure 5 Value of Insurance Coverage (I)
(Repayment Mortgage With an Early Termination Penalty)



The calculations that underlie this chart were done using the following parameters: contract rate, *c*, 11.89%; 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 10% interest rate and house price volatilities, σ and ν; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 1% early termination penalty, π; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0.

Figure 6 Value of Coinsurance (CI)
(Repayment Mortgage With an Early Termination Penalty)



The calculations that underlie this chart were don using the following parameters: contract rate, c 11.89%; 10% spot interest rate, $r(0)$, and long term mean of the interest rate process, θ; 10% interest rate and house price volatilities, σ and v; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 1% early termination penalty, π; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0.

Comparative Versions of the Mortgage Contract

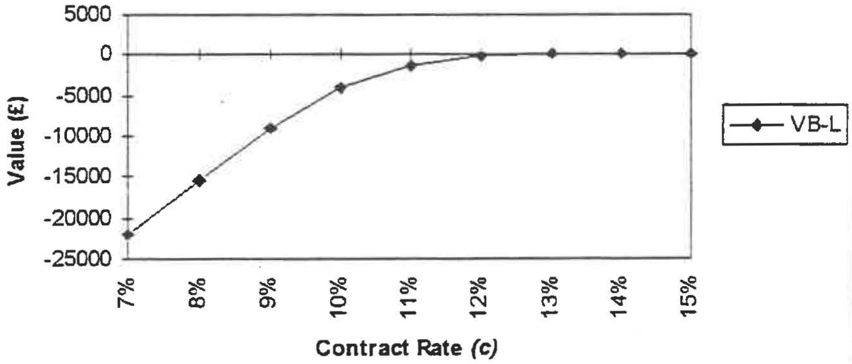
First we analyse a repayment mortgage in which the mortgage indemnity guarantee, the arrangement fee and the early termination penalty do not exist. Under these circumstances, equation (26) may be written as:

$$V_B(c) - L = 0 \tag{27}$$

In order for this contract to be viable, it is necessary that the value of the mortgage to the borrower, V_B, is equal to the amount lent, L. As Kau et al. (1995) correctly point out, for this to happen it is necessary that the prepayment region expands in such a way that (H(0), r(0)) becomes situated within the prepayment boundary (free-boundary), and immediate prepayment constitutes a possible optimal strategy for the borrower. Figure 7.A illustrates this. Here, the borrower faces a situation of indifference between the alternatives of continuation and immediate repayment. Any increase in the coupon rate, c, that corresponds to this initial equilibrium situation generates a peculiar effect.

It results in higher present value for future payments to be made by the borrower, but at the same time it also increases the value of the option of early repayment, C, by a similar amount. As these are compensating effects, the borrower repays the loan immediately after taking it. This is reflected in Figure 7.A as zero mortgage value for all contract (coupon) rates above around 12-13%. Consequently, despite the possibility of finding coupon rates capable of generating fair deals for both borrower and lender, no stable equilibrium exists, because those coupon rates correspond to situations in which the mortgage is immediately terminated.

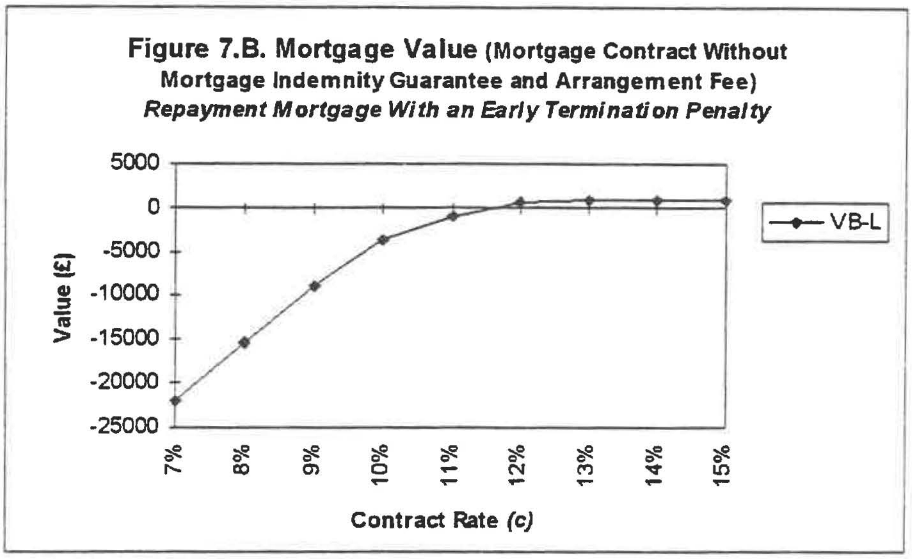
Figure 7.A. Mortgage Value (Mortgage Contract Without Mortgage Indemnity Guarantee and Arrangement Fee)
Repayment Mortgage Without an Early Termination Penalty



Adding a prepayment penalty to the mortgage contract, equation (27) will be rewritten as:

$$V_B(c,\pi) - L = 0 \qquad (28)$$

This particular situation is considered in Figure 7.B. As can be observed, there is only one level of c capable of generating an equilibrium contract.



The introduction of a prepayment penalty allows for the rectification of the "anomalous" situation of artificial equilibrium that exists in the absence of any negative incentive to early termination. As a result of the introduction of this new feature, the borrower faces an additional cost that is translated into an upward move of the line representing the value of the mortgage for the lender in Figure 7.B. The consequence of this move is that there is a single combination of c and π capable of attaining equilibrium (capable of leading the corresponding function to reach a value of zero).

It is also noteworthy that if the loan to value ratio were 1 (100%), attainment of equilibrium combinations would be impossible. In that case L = H, and consequently equation (27) assumes the following form:

$$V_B(c) - H = 0 \tag{29}$$

In this situation, a rational borrower would become indifferent between default and continuation. In spite of that, it is obvious that by defaulting the borrower would lose the service flow of the house from which benefit could be obtained until the first payment was due, if the decision to default were delayed until that moment. Therefore, there can be no coupon rate capable of making (29) hold. The common imposition of LTV ratios below unity constitutes one of the instruments that helps equilibrium mortgage contract combinations to be reached (for an argument along the same lines, see Kau et al., 1995).

Mortgage Contracts with the Inclusion of Arrangement Fees

The inclusion of an arrangement fee modifies equation (26) as follows:

$$V_B(c) - (1-\xi)L = 0 \tag{30}$$

Equation (28) can be modified similarly by including the same arrangement fee:

$$V_B(c,\pi) - (1-\xi)L = 0 \tag{31}$$

In any case, the curves represented in Figures 8.A and 8.B will shift vertically by the amount of the arrangement fee, ξL. The effect caused by the introduction of the arrangement fee is qualitatively similar to that induced by the early termination penalty.

Both constitute costs to the borrower and, consequently, reduce the value the borrower attributes to the mortgage contract. However, the effects generated by the arrangement fee are more linear, since its amount is fixed whereas the amount of the early repayment penalty varies with time. A single equilibrium combination is reached, in the case of a contract that does not include an early termination penalty, as a result of the change in the value of the mortgage deal produced by the inclusion of an arrangement fee (see Figure 8.B).

Figure 8.A. Mortgage Value **(Mortgage Contract Without Mortgage Indemnity Guarantee and Arrangement Fee)**
Repayment Mortgage With an Early Termination Penalty

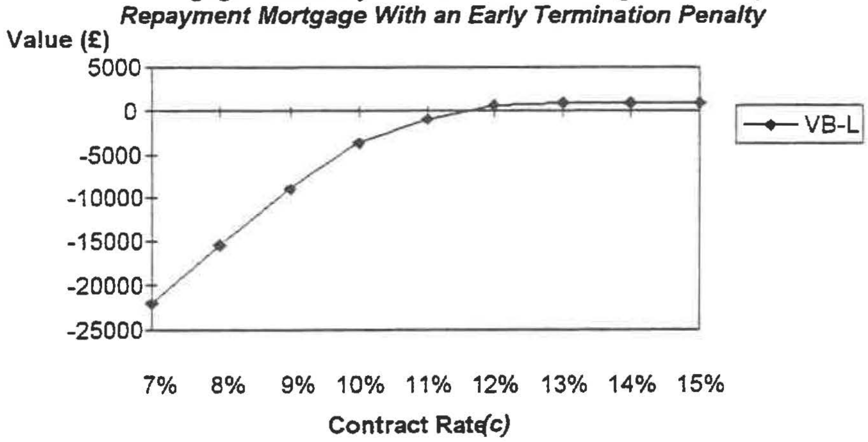
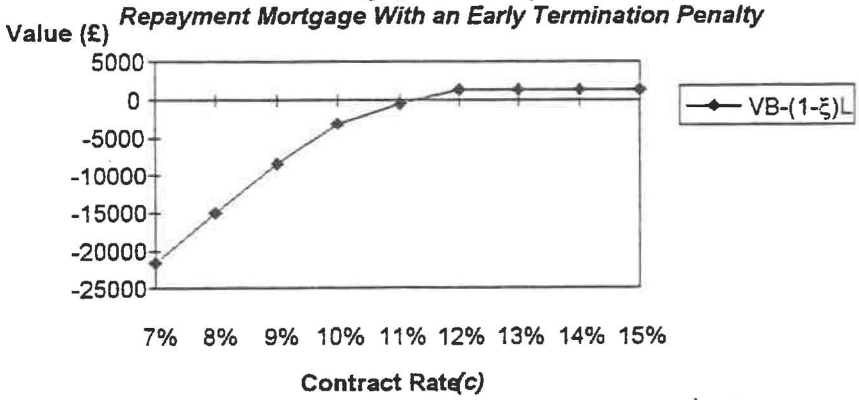


Figure 8.B. Mortgage Value
(Mortgage Contract With Inclusion of Arrangement Fee and Without Mortgage Indemnity Guarantee)
Repayment Mortgage With an Early Termination Penalty



The Full Mortgage Contract, With Arrangement Fee, Early Termination and MIG

The case where all the common mortgage contract features are simultaneously considered is illustrated by Figures 9.A and 9.B. As expected, the simultaneous consideration of arrangement fees, early termination penalties and MIGs always leads to situations of unique equilibrium combinations. All those contractual features generate a net benefit to the lender and, consequently, the equilibrium combinations are now reached at slightly lower levels of the coupon rate.

Figure 9.A. Mortgage Value
(Mortgage Contract With Arrangement Fee and Mortgage Indemnity Guarantee)
Repayment Mortgage Without an Early Termination Penalty

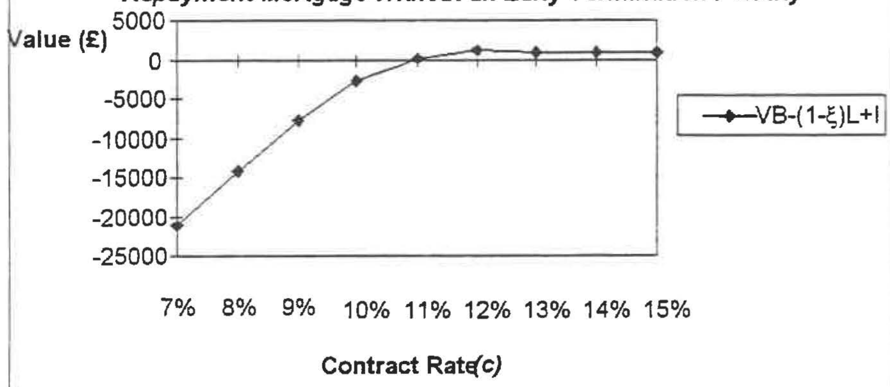
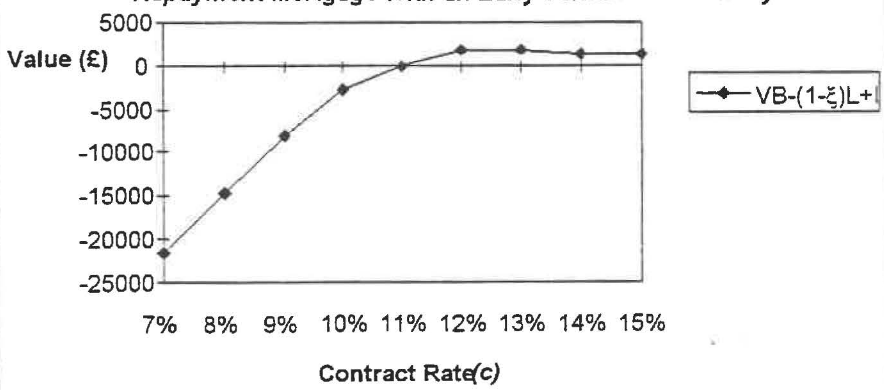


Figure 9.B. Mortgage Value
(Mortgage Contract With Arrangement Fee and Mortgage Indemnity Guarantee)
Repayment Mortgage With an Early Termination Penalty



Effects Induced by Changes in the Economic Environment

As noted earlier, the economic environment is characterised in the present work through the set of parameters given in Table 1. This section presents an analysis of the effects induced by changes in these parameters.

(a) Volatility of the State Variables

In order to examine the effects of the risk created by changes in the state variables, a series of numerical results will be presented in which, for a previously determined equilibrium coupon rate, the risk parameters will be changed. This will allow an examination of the partial effects induced by changes in the volatilities of both state variables. In a second stage, the global effect will be analysed, based on equilibrium values for different combinations of σ and υ.

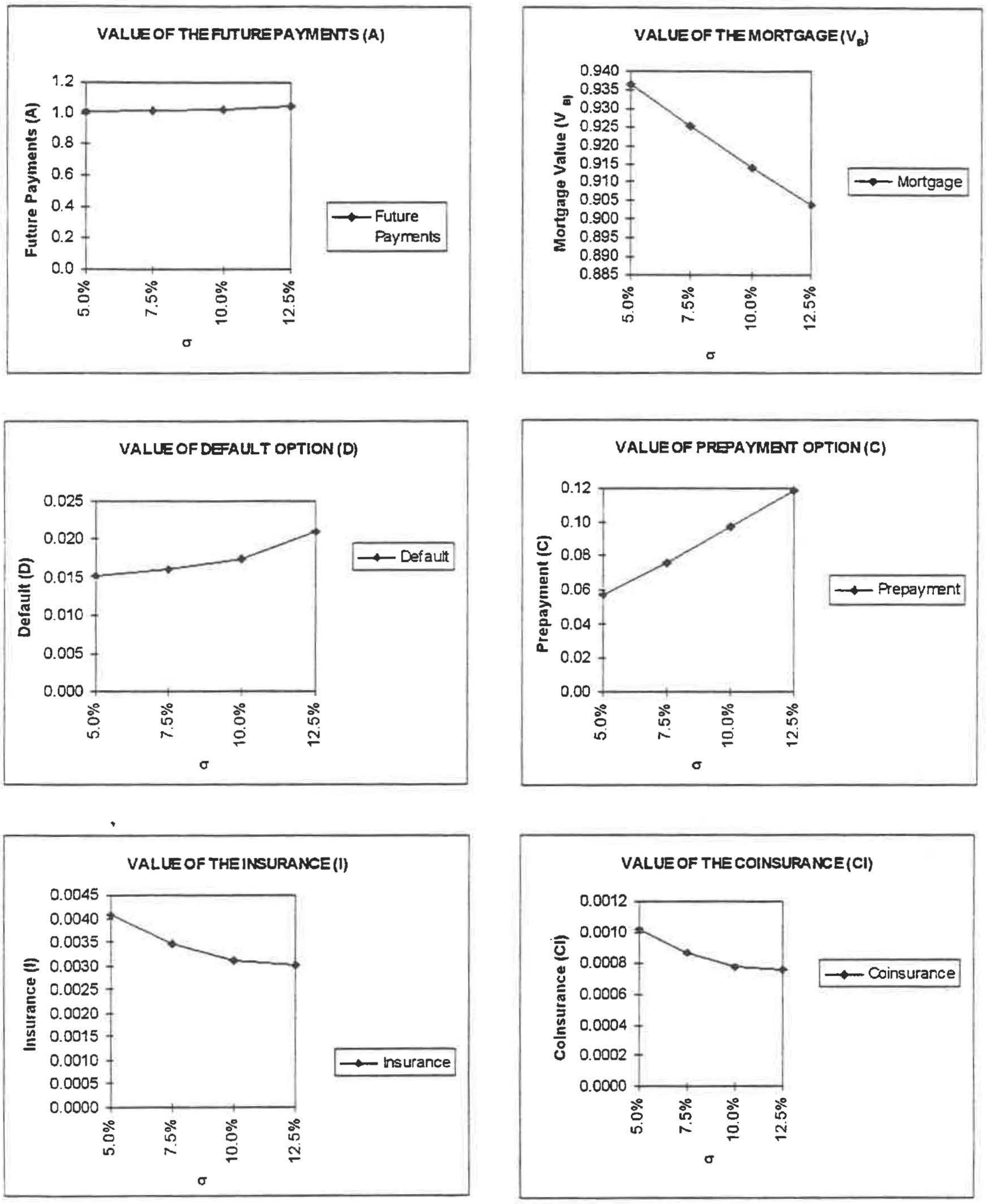
(b) Interest Rate Volatility

Interest rate volatility affects the values of all mortgage-related evaluated in this work. Figure 10 presents some of the simulations that were performed in order to analyse this subject for each of the contracts under study.

Figure 10

Interest Rate Volatility and Mortgage Related Assets:

Repayment Mortgage With An Early Termination Penalty



The calculations that underlie these charts were done using the following parameters: 10% spot interest rate, *r(0)* and long term mean of the interest rate process, θ; 5% house price volatility, ν; 7.5% service flow, δ; 1% arrangement fee, ξ; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0.

In the first place, the value of the future mortgage payments increases with the increase of interest rate volatility. This apparently unanticipated relationship is caused by the nature of the process that underlies the valuation of expected cash-flows. Two changes of the same magnitude (in absolute terms), but opposite signs, in the value of the discount rate that will be used to determine the present value of a future cash-flow, will result in changes of different magnitude in the present value itself. The gains observed in case of a fall in the discount rate will surpass the losses generated by an increase of the same magnitude in the discount rate. In other words, the additional likelihood of the interest rate attaining unexpectedly high and low levels, which emanates from an increment in σ, results in an increase in the expected value of future payments due to Jensen's inequality.

In the case of the early repayment option, C, an increase in σ will tend to increase the likelihood that the contract will reach the prepayment region instead of reaching the continuation and the default regions. The size of each of these regions can also change as a result of the change in the interest rate. Additionally, C is also a function of A. Therefore, as a result of all these influences, the value of the prepayment option tends naturally to move in direct relationship with the changes in σ.

The impact of σ in the relative size of the different regions inside the grid favours a negative relationship between σ and D. However, the value of the default option is directly related to the value of A (see equation 6). This factor leads the impact of the changes in C in terms of D to be overturned. The value of the default option tends to move in direct relationship with σ.

The insurance related mortgage assets I and CI constitute a clear example of the fact that the Jensen's inequality effect is not always capable of dictating the relationship between the present values of expected cash-flows and the evolution of the interest rate volatility. In both cases the values of the assets tend to move in inverse relation to the evolution of the interest rate volatility. For this to happen, something must overturn the effect generated by Jensen's inequality. Increases in interest rate volatility tend to be translated into higher levels of prepayment and consequently a smaller number of default paths. Additionally, in contrast with D and C, I and CI do not benefit from the evolution in the value of A, since the values of the loss and the corresponding indemnity depend on the outstanding balance. This is almost entirely determined by the value of the unpaid principal and not by the value of the future payments, A. Consequently, both variables, I and CI, tend to have a negative relationship with σ in the contract specifications under study.

As a result of all those influences, the value of the lender's position also evolves in inverse relationship with σ. There are two main reasons for this: in the first place, the positive effect on the value of A is more than offset by the evolution of the joint option to terminate the loan (C+D) - the value of V_B, which is the overall result of the progress of these variables, evolves in a negative relationship with σ. In the second place, the value of the MIG, I, also presents a negative relationship with σ, providing an additional incentive to the final nature of the overall result.

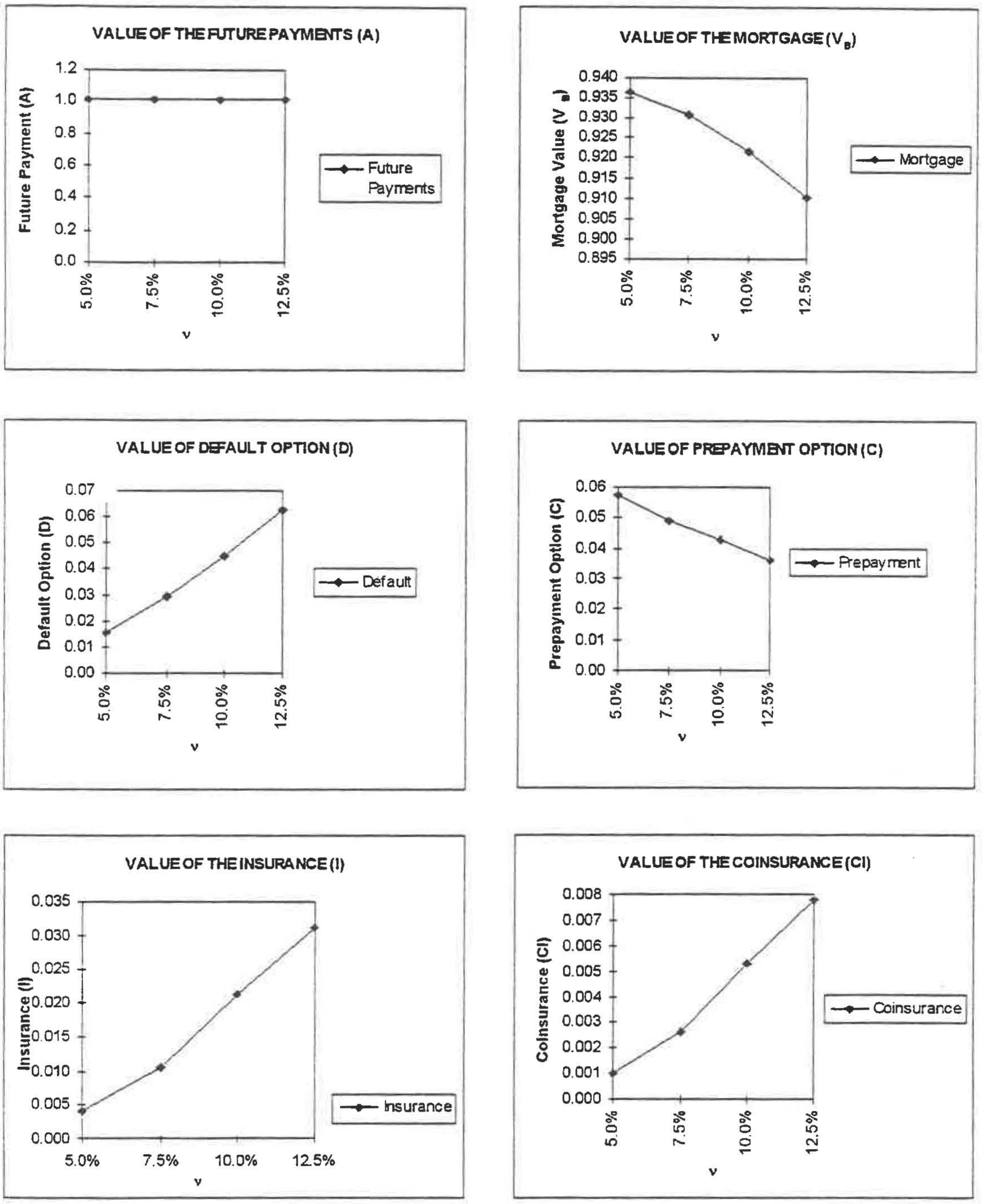
(c) House Price Volatility

The partial effects induced by changes in house price volatility are significantly different from those generated by interest rate variation. The first reason for this is related to the valuation of the future payments, A. The valuation of A obeys a degenerated version of equation (4) in which the only state variable is r. This means that the valuation of A is completely independent of H and consequently its evolution is not constrained in any way by the corresponding volatility, ν.

Figure 11

House Price Volatility and The Value Of Mortgages and Mortgage-Related Assets:

Repayment Mortgage With An Early Termination Penalty



The calculations that underlie these charts were done using the following parameters: 10% spot interest rate, *r(0)* and long term mean of the interest rate process, θ; 5% interest rate volatility, σ; 7.5% service flow, δ; 1% arrangement fee, ξ; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0

Under these circumstances, the relationship between ν and V_B is dictated by the way in which the joint option to terminate the mortgage, (C+D), relates to the house price variation. In the case of default, D, the numerical results point towards a strong direct relationship between this variable and ν (see Figure 11). This is explained by the fact that increases in ν tend to create a relatively higher likelihood for the contract to reach the default region instead of the prepayment and the continuation regions. A direct result of this phenomenon is the tendency for C to decline with increases in ν. Therefore, changes in ν tend to produce opposite effects in the components of the joint option to terminate the loan. According to our numerical results, V_B presents an inverse relationship with ν. Thus, the effects of the evolution of D tend to dominate the effects of the evolution of C at this level.

In contrast with the previous case, the insurance related variables, I and CI, are directly related to ν. The house price volatility impacts default much more directly than prepayment and the values of the insurance-related assets are a direct function of the probability of default and the expected amount of the corresponding losses. Consequently, the evolution of I and CI is very much in line with the evolution of D.

(d) Global Effects Induced by the Volatilities of the State Variables

The total effects induced by changes in the volatility parameters are portrayed in Table 3. An analysis of these tables, within each level of LTV (loan-to-value) ratio, gives an insight on the fundamental aspects at this level.

Table 6

The Combined Effects Of Changes In LTV Ratios and House Price and Interest Rate Volatilities

Repayment Mortgage With Arrangement Fee and Early Termination Penalty

LTV	Int. Rate Volatility	Eq. Coupon rate (c)		Future Payments (A)		Default (D)		Prepayment (C)		Insurance (I)		Coinsurance (CI)	
		v = 5%	v=10%	v = 5%	v=10%	v = 5%	v=10%	v = 5%	v=10%	v = 5%	v=10%	v = 5%	v=10%
80%	σ = 5%	10.86%	10.85%	84598	84514	26	693	4976	4483	8	260	3	175
	σ = 10%	12.03%	12.02%	93829	93764	27	682	14204	13609	3	135	1	95
85%	σ = 5%	10.85%	10.79%	89819	89405	104	1431	5181	4092	36	692	9	193
	σ = 10%	12.02%	11.92%	99671	98947	160	1737	14952	13087	17	456	4	129
90%	σ = 5%	10.83%	10.77%	94961	94541	389	2627	5192	3682	161	1316	40	331
	σ = 10%	12.00%	11.86%	105396	104340	422	4042	15517	11742	93	992	23	250
95%	σ= 5%	10.82%	10.77%	100131	99776	1672	4394	4500	3230	569	2373	142	593
	σ= 10%	11.97%	11.85%	110973	110069	2764	7626	14127	9838	451	1917	113	480
100%	σ= 5%	10.86%	10.86%	105724	105763	5166	7588	2632	2509	1567	3826	392	957
	σ= 10%	12.09%	11.90%	117788	116238	14708	12616	5058	7723	1477	3605	369	901

The calculations that underlie this table were done using the following parameters: 10% spot interest rate, $r(0)$, and long term mean of the interest rate process, θ; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 1% early termination penalty, π; and a correlation coefficient, ρ, of 0.

The effects induced by the interest rate volatility are more or less straightforward. Increases in σ lead to the growth of A and simultaneously tend to generate increases in D and C. However, I tends to move in a direction opposite to the movement in σ. Consequently, the overall result in terms of the position of the lender depends on the global effect induced in terms of the evolution of V_B (dictated by the relationship between the increases registered by A and those observed by D and C) and the evolution in I. Both tend to decrease with increases in σ. As a result, in these circumstances, in order to reach an equilibrium it is necessary to increase the coupon rate and consequently the value of A to compensate for those declines in I and increases in C+D.

Changes in v involve complex relationships and make the analysis more intricate. Increases in house price volatility, v, tend to correspond to increases in D and decreases in C. The joint effect of these changes tends to be translated into a reduction of V_B. Simultaneously, I tends to increase with v. The magnitude of impact tends to be similar at both levels, with the change in I slightly surpassing that in V_B. In order to compensate for this small increase in the value of the lender's position, it is necessary to adjust the coupon rate. Therefore, increases in v tend to be accompanied by slight reductions in the coupon rate.

Another relevant aspect, at this level, is related to the value of both options to terminate the loan. According to the numerical results, the value of the prepayment option tends to exceed the value of the default option, with the exception of the cases in which the LTV ratio reaches very high levels. This is in line with the results reported in Kau et al. (1995) for the American case.

(e) Changes in the Relationship Between the Spot Rate and the Long Term Average of the Interest Rate Process

Table 4 shows the effects induced by different types of yield curves in terms of the value of the mortgage-related assets.

Table 4

The Effects Of The Relationship Between The Spot Rate, r(0),

and The Long-Term Average of the Interest Rate, θ

Repayment Mortgage With Arrangement Fee and Early Termination Penalty

r(0)	Int Rate Volatility (σ)	Eq. Coupon rate (c)		Future Payments (A)		Default (D)		Prepayment (C)		Insurance (I)		Coinsurance (CI)	
		v = 5%	v =10%	v = 5%	v =10%	v = 5%	v=10%	v = 5%	v=10%	v = 5%	v=10%	v= 5%	v=10%
8%	5%	9.70%	9.68%	97057	96905	1328	4030	1923	1260	715	2911	179	728
10%		10.82%	10.77%	100131	99776	1672	4394	4500	3230	569	2373	142	593
12%		12.22%	12.21%	104819	104762	2322	5480	8499	6618	531	1856	133	464
8%	10%	10.52%	10.37%	105389	104137	2625	6738	8563	5624	626	2751	157	689
10%		11.97%	11.85%	110973	110069	2764	7626	14127	9838	451	1917	113	480
12%		13.53%	13.48%	116950	116510	2867	8540	19944	14945	384	1504	96	376

The calculations that underlie this table were done using the following parameters: 8%, 10% and 12% spot interest rates, *r(0)*; 10% long term mean of the interest rate process, θ ; 7.5% service flow, δ ; 0.5% arrangement fee, ξ ; 1% early termination penalty, π ; 95% LTV ratio; and a correlation coefficient, ρ , of 0.

The approach used in this work to capture different yield curve shapes consisted of changing the initial level assumed by the spot interest rate, r(0), while holding constant the long-run mean spot interest rate, θ, for all the runs that underlie the construction of the tables.

There are three main levels at which the effects induced by different types of yield curves can be analysed. In the first place, it is important to look at the impact produced by changes in the yield curve in cases in which all the other parameters are held constant. According to our numerical solutions, higher levels of r(0) tend to be related to decreases in V_B and in I. Both these effects lead to the reduction of $V_B - (1-\xi)L + I$. As a consequence, in order to reach equilibrium, it is necessary for the coupon rate to attain higher levels. Therefore, there is a direct relationship between the level of the initial spot rate and the coupon rate for a fixed θ. This relationship informs our understanding of the evolution of the default option, D. This is negatively influenced by the increase in the size of the prepayment region which occurs as a natural consequence of higher spot interest rates. However, simultaneously, there is an increase in the coupon rate, leading to the rise of A. As previously mentioned, D has a direct relationship with A. This effect overshadows the influence induced by the evolution of the prepayment region in terms of the value of D, and provides the rationale for the direct relationship between D and the initial level of the spot rate.

A further important aspect is the effect of increases in σ, given different slopes of the yield curve. The increase in the interest rate volatility is directly related to the evolution of D and, especially, of C. Both of these contribute to the reduction in the value of the mortgage to the borrower, V_B. Besides this, it is also necessary to take into account the inverse relationship between I and σ. The overall effect induced by increases in σ is a significant reduction in the value of $V_B - (1-\xi)L + I$, for all types of yield curves and contracts under study. In order to reach equilibrium combinations, it is necessary to compensate for all these adverse effects in the value of the lender's position by increasing the coupon rate. This means that, for higher levels of σ, the coupon rate tends to increase and normally to reach levels that are higher than θ. Even in cases where the original level of the spot interest rate was clearly below 8%, the long term average of the interest rate, the equilibrium coupon rate is above θ, when $\sigma = 10\%$, for all contracts under study.

Finally, there is the joint effect produced by increases in ν and changes in the slope of the yield curve. As previously noted, the effects induced by increases in ν in terms of the evolution of D and C are of opposite nature and tend partially to compensate each other. Consequently, the influence of house price volatility upon the equilibrium coupon rates is moderate. Since the positive effect on value, generated by increases in υ, upon D tends to be of a higher magnitude than the combined effects produced upon C and I, the overall result tends to be translated into a slight reduction of the coupon rate as a result of the increase of ν, for all the yield curve environments under study.

(f) Changes in the Correlation Coefficient Between State Variables

Table 5 presents the effects induced by a change in the correlation coefficient, ρ, between the state variables. Increases in ρ tend to be translated into increases in the value of default. In the first place, this phenomenon implies a direct relationship between the evolution of r and H. This is equivalent to saying that when r is high default tends not to happen because of the low level of V_B. However, when r is low, the reduced levels of H will make default more likely. Besides this, when D is high the likelihood of prepayment tends to be lower. Therefore, both factors generate a tendency not only for D to move in direct relationship with ρ, but also for C to move in an opposite direction.

Table 5

Effects Induced By Changes In The Correlation Coefficient, ρ

Repayment Mortgage With Arrangement Fee And Early Termination Penalty

(ρ)	Coupon rate (c)	Value of Payments (A)	Default (D)	Prepayment (C)	Insurance (I)	Coinsurance (CI)
-0.20	11.81%	109748	6429	10575	1788	448
-0.15	11.82%	109825	6751	10368	1815	454
-0.10	11.83%	109903	7048	10185	1849	463
-0.05	11.84%	109981	7344	10001	1883	471
0.00	11.85%	110069	7626	9838	1917	480
0.05	11.86%	110137	7904	9663	1954	489
0.10	11.87%	110215	8181	9494	1991	498
0.15	11.88%	110269	8444	9320	2024	506
0.20	11.89%	110331	8711	9147	2054	514

The calculations that underlie this table were done using the following parameters: 10% spot interest rate, $r(0)$, and long term mean of the interest rate process, θ ; 10% interest rate volatility, σ ; 10% house price volatility, ν ; 7.5% service flow, δ ; 0.5% arrangement fee, ξ ; 1% early termination penalty, π ; 95% LTV ratio.

Under these circumstances, it is normal for the values of the MIG and the coinsurance to follow the trend registered by D and move in direct relationship with ρ. The magnitude of the movement in D is greater than that registered by C. Consequently, V_B tends to decrease. As the movements in I are minimal, this is equivalent to saying that, with an increase of ρ, $V_B-(1-\xi)L+I$ is reduced. In these circumstances, the coupon rate needs to be increased in order for equilibrium to be reached.

In summary, the borrower and the insurer both have options that they can exercise at the lender's expense, and which it is important to value. Our numerical results suggest that mortgage modelling include contractual provisions in order to prevent misleading conclusions regarding mortgage valuation.

V Conclusion

This paper presents a theoretical framework for valuing UK fixed-rate mortgages and Mortgage Indemnity Guarantees (MIGs). The terminal conditions imposed take into account the specific nature of the early repayment penalties included in most UK fixed rate mortgages. At the insurance valuation level, the terminal conditions account for the features included in some recent UK policies, such as the MIG cap and the definition of the guarantee as a proportion of the loss. Simultaneously, the framework allows for the valuation of coinsurance, the potential loss not covered by the MIG but covered by the lender.

Numerical results were determined for different contract specifications. In the first place, an analysis of the equilibrium combinations for the contract specifications under study was undertaken and the likely nature and determinants of equilibrium coupon rates were defined. Numerical solutions were also found to explore different contract specifications, in order to analyse the partial and global effects induced by changes in the main parameters associated with the evolution of the economic environment. In spite of the complexity of the product and the intricate relationships that exist between the variables, it was possible to find a rationale for the results based on underlying economic knowledge. The implications induced by changes in the contract specifications were likewise considered. The corresponding numerical results suggest that model specifications that exclude important mortgage contractual features might produce misleading results and consequently should be avoided or, at least, used cautiously.

APPENDIX

A Graphical Synthesis of the Structure of the Solution

The method of solution of partial differential equations by finite difference methods is well established and the reader will find textbooks on the subject from a mathematician's point of view. There are also several excellent descriptions in finance texts (Wilmott et al. 1993, Wilmott 1998). Essentially, the continuum of underlying asset prices, on which option values depend, versus time is approximated by a grid in which only small but finite changes in each are considered. Partial derivatives are then approximated by linear slopes across the grid. When there are several underlying variables, this grid becomes a multi-dimensional and discrete representation of the "state space". Knowing boundary conditions (such as when it is financially optimal to default on payment of a mortgage) it is possible to work backwards in time, valuing the options at each point on the grid, until initial values are obtained at the start of the grid, time zero.

A general discussion of the evolution of the grid used to represent the state space should facilitate an understanding of the way in which numerical results are produced. Figures 15 to 18 provide a graphical synthesis of the numerical approach used in this work. The state space, (H x r), is compacted into a square, and each figure represents a certain moment in time. Since the solution moves backwards in time, this series of figures proceeds in that order.

At the final moment in the life of a mortgage it only makes sense for the borrower to make the last monthly payment if the value of the house, H, is higher than the amount to be paid, MP. Otherwise, the borrower will default. Figure 15 illustrates this situation. The shadowed column in the left of the picture represents the default region. If default is not in the borrower's best interests, the only alternative is to pay. At termination it is not possible for the borrower to prepay, since prepayment has no meaning at the final last moment of the mortgage. Therefore, there are only two regions inside the state space: the default region, which leads the borrower to exchange the value of the last payment for the house, and the continuation region which implies that the last payment is made.

Figure 15

Evolution of the Options to Terminate the Loan Across the State Space in Different Moments During the Life of the Loan

i) Termination of the Loan (final payment made)

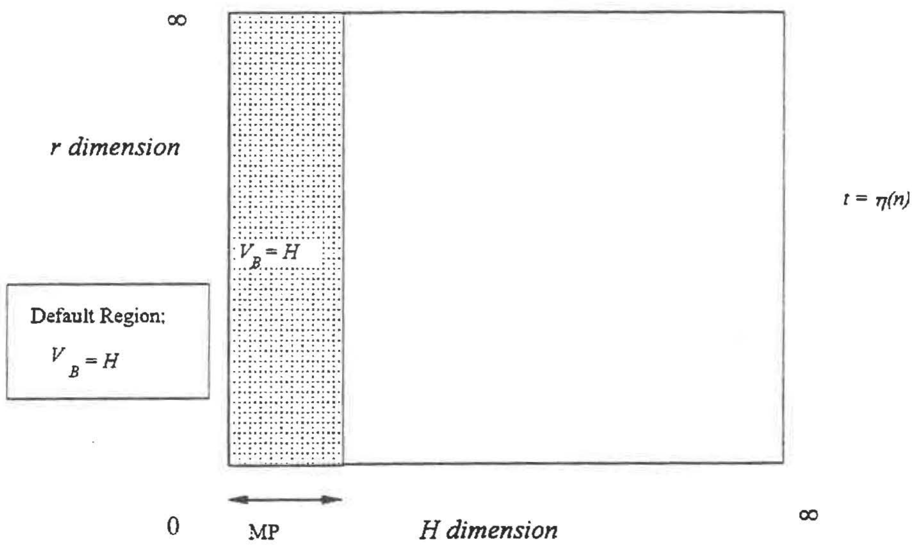
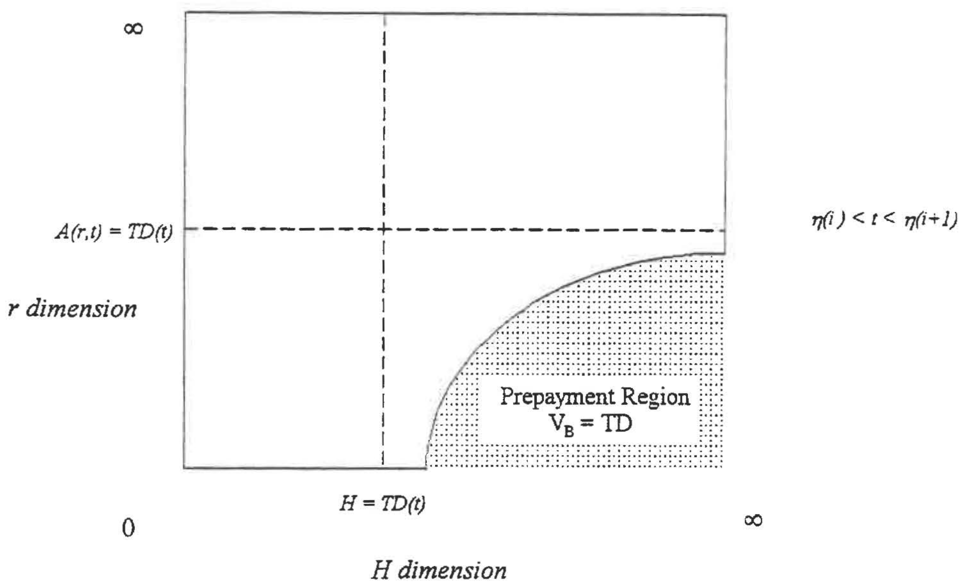


Figure 16 represents a moment located between payment dates. In this case, it is the default region that does not exist since default only makes sense at a payment date. Otherwise, the borrower loses the use of the house ("service flow") for the period of time up to the next payment date during which it could be enjoyed without having to declare default.

Figure 16

Evolution of the Options to Terminate the Loan Across the State Space in Different Moments During the Life of the Loan
ii) Moments Other than Monthly Payment Dates



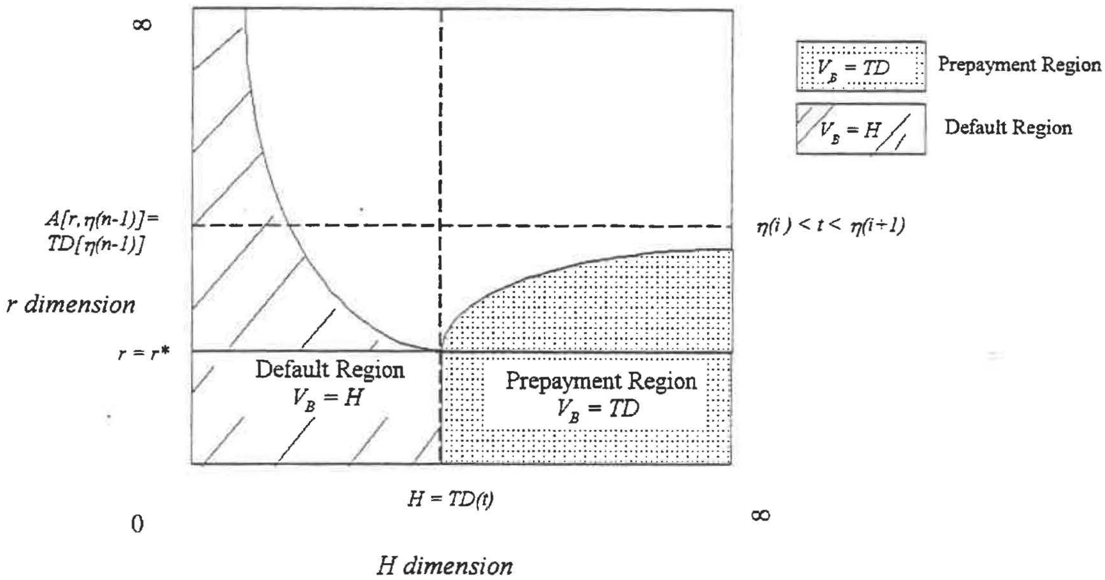
The shadowed area in Figure 16 corresponds to the prepayment region. Prepayment only becomes a reasonable option if the borrower faces simultaneously high house prices and low market interest rates. At high house price levels, the default option becomes less valuable and, consequently, the relative attraction of the alternative early termination option (the prepayment option) increases for higher interest rate levels. This is equivalent to saying that the prepayment boundary (free boundary) is positively sloped, given the borrower's predisposition to prepay at higher interest rates when house prices reach high levels. The prepayment region is bounded in the r dimension by points corresponding to A(r,t) = TD(t), when the value of remaining mortgage payments, at time t, is equal to the borrower's total debt in the case of early termination (including the early termination penalty).

If the interest rate falls below this level, prepayment is "in the money" (Kau et al., 1992). However, it is important to note that prepayment does not necessarily need to take place immediately. The exercise of the call option to prepay the loan renders the put option to default valueless and a rational borrower takes this fact into consideration. The prepayment region is also bounded in the H dimension by points at which H = TD(t), since this is a "border" line corresponding to points at which the appeal of default is equivalent to the appeal of prepayment.

Figure 17 illustrates a situation corresponding to an intermediate payment date in the life of the loan. Default makes economic sense at payment dates and so the figure includes both prepayment and default regions. The dividing line between them corresponds to points at which H = TD(t), where prepayment is as attractive as default, in financial terms. The value r* corresponds to the highest level of r observed along this boundary. This means that for values of r less than r*, the loan is automatically terminated either through default (when H < TD) or through prepayment (when H > TD).

Figure 17

Evolution of the Options to Terminate the Loan Across the State Space in Different Moments During the Life of the Loan
iii) Monthly Payment Dates

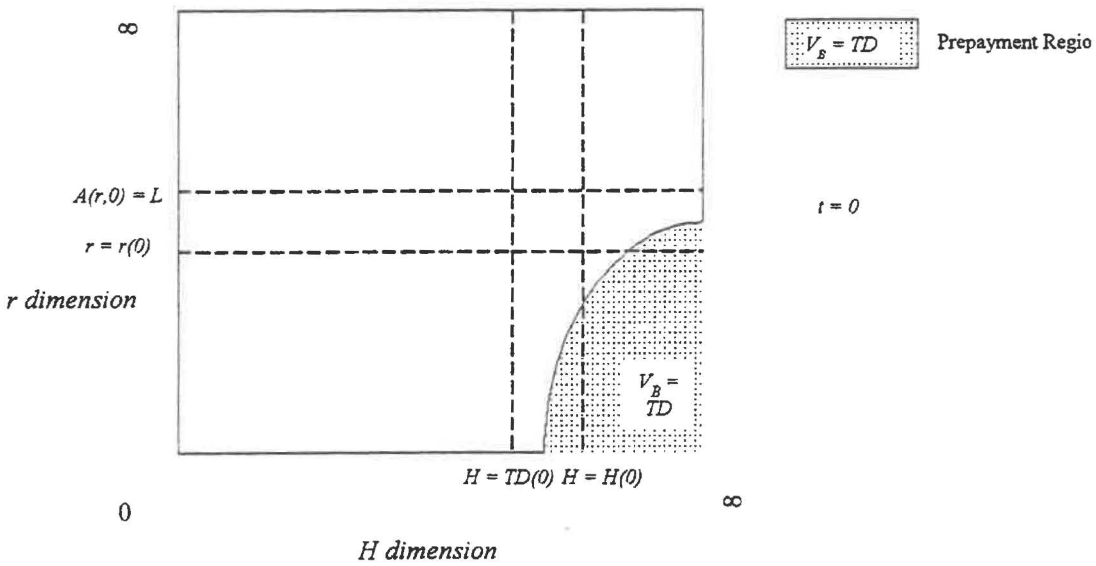


For default to be "in the money", it is necessary that the value of the house, H, is less than the present value of the future mortgage payments, A. However, this is not sufficient for default to take place. As happens with prepayment, exercise of the default option implies the immediate loss of the call option to prepay and, consequently, default tends not to happen immediately. Another important point to note at this level is that higher external interest rates effectively reduce the cost of maintaining the loan (with its constant "coupon") and lead to default being advantageous only at low levels of H. Consequently, the boundary between the default and the continuation regions is negatively sloped.

Finally, Figure 18 portrays the situation at the origination of the loan. As in Figure 16, there is no default region because the moment does not correspond to a payment date. The original combination of state variables corresponds to [H(0), r(0)]. If a sudden rise in H takes place immediately after the contract is signed, the value of the default option (in future payment dates) also suffers an abrupt decline, eventually leading to a situation in which the coupon rate is too high and in which prepayment might become the best alternative to the borrower. An immediate drop in interest rates might likewise lead the borrower to prepay and the mortgage value to decline to the value of the outstanding debt, TD.

Figure 18

Evolution of the Options to Terminate the Loan Across the State Space in Different Moments During the Life of the Loan
iv) Origination of the Loan



REFERENCES

Buser, Stephen A. and Hendershott, Patric H., 1984, "Pricing Default-Free Fixed Rate Mortgages", *Housing Finance Review*, Vol. 3, No. 4, pp. 405-429.

Cox, John C., Ingersoll, Jonathan E., Jr., and Ross, Stephen A., 1985a, "An Intertemporal General Equilibrium Model of Asset Prices", *Econometrica*, Vol. 53, No. 2, pp. 363-384.

Cox, John C., Ingersoll, Jonathan E., Jr., and Ross, Stephen A., 1985b, "A Theory of the Term Structure of Interest Rates", *Econometrica*, Vol. 53, No. 2, pp. 385-407.

Crank, John, 1984, "Free and Moving Boundary Problems", Oxford University Press, Oxford.

Dunn, Kenneth B. and McConnell, John J., 1981a, "A Comparison of Alternative Models for Pricing GNMA Mortgage-Backed Securities", *Journal of Finance*, Vol. 36, No. 2, pp. 471-484.

Dunn, Kenneth B. and McConnell, John J., 1981b, "Valuation of GNMA Mortgage-Backed Securities", *Journal of Finance*, Vol. 36, No. 3, pp. 599-617.

Epperson, James F., Kau, James B., Keenan, Donald C. and Muller III, Walter J., 1985, "Pricing Default Risk in Mortgages", *Journal of the American Real Estate and Urban Economics Association*, Vol. 13, No. 3, pp.152-167.

Gerald, Curtis F., Wheatley, Patrick O., 1994, "Applied Numerical Analysis", 5th ed., Addison-Wesley, Reading, Massachusetts.

Hilliard, Jimmy E., Kau, James B., Keenan, Donald C. and Muller III, Walter J., 1995, "Pricing a Class of American and European Path Dependent Securities", *Management Science*, Vol. 41, No. 2, pp. 1892-1899.

Kau, James B., Keenan, Donald C., Muller, III, Walter J., and Epperson, James F., 1992, "A Generalized Valuation Model for Fixed-Rate Residential Mortgages", *Journal of Money Credit and Banking*, Vol. 24, No. 3, pp. 279-299.

Kau, James B., Keenan, Donald C., Muller, III, Walter J., and Epperson, James F., 1993a, "Option Theory and Floating-Rate Securities with a Comparison of Adjustable- and Fixed-Rate Securities", *Journal of Business*, Vol. 66, No. 4, pp. 595-618.

Kau, James B., Keenan, Donald C. and Muller, III, Walter J., 1993b, "An Option Based Pricing Model of Private Mortgage Insurance", *Journal of Risk and Insurance*, Vol. 60, No. 2, pp. 288-299.

Kau, James B., Keenan, Donald C., Muller, III, Walter J., and Epperson, James F., 1995, "The Valuation at Origination of Fixed-Rate Mortgages with Default and Prepayment", *Journal of Real Estate Finance and Economics*, Vol. 11, No. 1, pp. 5-36.

Kishimoto, N., 1989, "A Simplified Approach to Pricing Path-Dependent Securities", *Working Paper*, Duke University.

Leung, Wai K. and Sirmans, C. F., 1990, " A Lattice Approach to Pricing Fixed-Rate Mortgages with Default and Prepayment Options", *Journal of the American Real Estate and Urban Economics Association*, Vol. 18, No. 1, pp. 91-104.

McDonald, Robert and Siegel, Daniel, 1986, "The Value of Waiting to Invest", *Quarterly Journal of Economics*, Vol. 101, pp. 707-727.

Merton, Robert C., 1973, "The Theory of Rational Option Pricing", *Bell Journal of Economics and Management Science*, Vol. 4, No. 1, pp. 141-183.

Press, William W., Teukolsky, Saul A., Vetterling, William T. and Flannery, Brian P., 1992, "Numerical Recipes in Fortran. The Art of Scientific Computing", 2nd ed., Cambridge University Press, Cambridge.

Stanton, Richard and Wallace, Nancy, 1995b, "ARM Wrestling: Valuing Adjustable Rate Mortgages Indexed to the Eleventh District Cost of Funds", *Real Estate Economics*, Vol. 23, No. 3, pp. 311-345.

Stanton, Richard, 1995, "Rational Prepayment and the Valuation of Mortgage-Backed Securities", *Review of Financial Studies*, Vol. 8, No. 3, pp. 677-708.

Wilmott, Paul, Dewynne, Jeff and Howison, Sam, 1993, "Option Pricing: Mathematical Models and Computation", Oxford Financial Press, Oxford.

Wilmott, Paul, 1998, "Derivatives; The Theory and Practice of Financial Engineering", Wiley.