Universidade de Lisboa
Faculdade de Ciências
Departamento de Matemática

Instituto Universitário de Lisboa
ISCTE Business School
Departamento de Finanças

Option Pricing Through Fourier Transforms

Tiago Miguel de Andrade Martins

Dissertação orientada pelo
Professor Doutor João Pedro Vidal Nunes

Mestrado em Matemática Financeira

2015

Universidade de Lisboa
Faculdade de Ciências
Departamento de Matemática

Instituto Universitário de Lisboa
ISCTE Business School
Departamento de Finanças

Option Pricing Through Fourier Transforms

Tiago Miguel de Andrade Martins

Dissertação orientada pelo
Professor Doutor João Pedro Vidal Nunes

Mestrado em Matemática Financeira

2015

Acknowledgements

First of all I have to say that it was a great pleasure working with the Professor João Pedro Vidal Nunes, so the first person to who I need to say thank is him. Thank you for the support, and for the great help while I was in London.

I would also like to say thank you to my Master colleagues, especially to Pedro Alves and Carla Amaral, because they always worked really hard during our work groups, and together we had some good marks.

Now I would like to leave a word to my family, especially to my grandmothers, as they have always supported me a lot. My grandmother Adélia Martins is not between us anymore so she can not see my professional achievements, but I hope, if she was here, I was making her proud.

Finally and the most important, as I think this will be the last Master thesis I do, I want to dedicate this work to my parents. They have supported me since I was born, and they are the ones that I know will never let me down, so this is for you.

Abstract

Financial markets have been evolving so much in the last few years that the whole system had to be adapted to the changes. With the evolution of the computers, and with the entry of mathematicians and physicists into the financial industry, everything has changed so much that any Financial Institution can not afford to be slower than some competitor.

This master thesis focus on the pricing of European-style call and put options using Fourier analytic methods, which can be much faster than other calculation methods for the same accuracy level.

We analyze and prove Lee [2004] [1] pricing equations for the specific case of European vanilla options, and we also implement the Matlab code which will allow us to see the application of those equations into practice.

Keywords

Option pricing

Fourier transform

Sampling error

Truncation error

Index

List of figures

List of tables

Main variables

G_1 – Payoff from a European-style call option.

G_1^* – Payoff represented by the minimum between the spot value of an underlying, and the strike price defined in the option contract.

G_1^{**} – Payoff from a European-style put option.

C_{G_1} – Time-0 price of an European-style option with payoff G_1.

c_{α,G_1} – Damped option price from an European-style option with payoff G_1.

$\hat{c}_{\alpha,G_1}$ – Fourier transform from c_{α,G_1}.

$f(\xi)$ – Discounted characteristic function of X.

1. Introduction

1.1 Motivation and Objectives

Option pricing has changed a lot during the last decades, and more importantly after the development of Black-Scholes-Merton model. But this change is not over, and we will keep watching the many changes that will happen in the next decades.

The competition in the financial markets is really high, so the financial institutions even need to struggle to be closer to the Stock Exchange, as each microsecond counts, and means the difference between the big winners and the losers.

It has not always been like this, but since the mathematicians, physicists and engineers got into the financial markets, and started applying their quantitative knowledge into the financial markets, everything started to be automated, and each price of each product is calculated within microseconds, in some cases.

With this huge change, as well as with algorithm trading, each second started to be seen as a minute, and each minute started to be seen as an hour. Each financial institution, specially hedge funds, can not afford to be slower than their competitors, as the opportunities are taken instantly.

This challenge is one of the main reasons which led us to develop this Master thesis, and also kept us really motivated, as the topic is really actual.

Our work consists in a deep analysis and extension of Lee (2004) [1] work, as we follow his assumptions.

In our specific case, we focus in the case of European-style call and put options from which we know the characteristic function of their underlying state variable, and we will price those options using Fourier-analytic methods.

Given the characteristic functions of the state variables, the aim of this work is to compute efficiently and accurately the prices of European-style call and put options on those underlying state variables.

Fourier-analytic solutions to various forms of this problem have appeared in the finance literature. They express option prices in terms of the Fourier-inversion integrals, which are in practice, evaluated numerically.

Our work, in a general overview, bounds the error in the numerical evaluation of these integrals as N-point sums, which may be computed as a discrete Fourier transform (DFT) by schemes including the fast Fourier transform (FFT). Finally, we show how these bounds lead to algorithms that make efficient choices of quadrature parameters and compute prices with numerical accuracy.

1.2 Organization of the text

This Master thesis is composed by 5 chapters, and in those chapters we present all the necessary proofs and information required to justify the equations and the assumptions we make.

The first chapter presents our motivation and objectives, as well as the organization of the text.

The second chapter is divided into different subsections, which will explain our procedures, starting by the main principles, where we define our first variables and present in some detail our pricing problem. Then we will explain how we get the bounds which are going to be used during the our work, and subsequently we will show how we are going to use the Fourier transforms. Finally we will prove how we get our pricing equation, and define both the sampling and truncation errors, which will vary with the value of the dampening parameter alpha.

In the third chapter we apply the theoretical work from the previous chapters, and by the utilization of the Heston model, we will determine the price of some European-style options.

2. Option Pricing

2.1 *Main Principles*

Financial options are contracts which give the buyer the right, but not the obligation, to buy (call option) or sell (put option) an underlying asset or instrument at a specified strike price on or before a specified date.

This Master thesis deals only with European-style options, which give its owner the right to buy or sell the underlying asset at a specified date, called maturity.

In the case of a call option, its owner makes money only if the price of the underlying asset is greater than the strike price (which is defined in the settlement of the contract). On the other way, the put options behave symmetrically.

First of all, we need to define the option pricing problem, which will consist in the numerical calculation of the price C_0 of a European-style call at instant $t_0 = 0$, with maturity occurring at time T (>0).

To make this calculation, we will consider a filtered probability space, represented by $(\Omega, \mathbb{P}, \mathcal{F}_X)$, and the major variables will be represented by

- r_t – Interest rate process, which can be considered to be stochastic;
- $M_t := exp\left(\int_0^t r_s \, ds\right)$ – The value of a money market account at instant t;
- B_t – The value, at instant t, of a pure discount bond maturing at T.

As it is normal in the valulation of every financial instrument, we will assume that all the assets under consideration do not admit arbitrage. This assumption implies the existence of a risk-neutral and equivalent probability measure $\mathbb{Q}$, under which asset prices discounted by M, the numeraire, are martingales.

The option price and the zero-coupon bond price, represented in terms of the expectation with respect to $\mathbb{Q}$, and conditional on the sigma-algebra $\mathcal{F}_0$ can be defined as

$$C_0 = \mathbb{E}_{\mathbb{Q}}[M_T^{-1} C_T], \tag{2.1}$$

and

$$B_0 = \mathbb{E}_{\mathbb{Q}}[M_T^{-1}] \ ,$$

(2.2)

respectively.

For any numeraire N there exists a probability measure, $\mathbb{Q}_N$, said to be risk-neutral with respect to N, meaning that the N_t-discounted price of any financial asset is a $\mathbb{Q}_N$-martingale. Following El Karoui, Geman, and Rochet (1995), the change of measure from $\mathbb{Q}$ to $\mathbb{Q}_N$ is given by the following Radon-Nikodym derivative:

$$\left.\frac{d\mathbb{Q}_N}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \frac{N_T/N_0}{M_T/M_0}$$

(2.3)

When the numeraire is chosen to be the price, B_t, of a T-maturity zero-coupon bond, the risk-neutral measure, $\mathbb{Q}_B$, is known as the T-forward measure.

In terms of the notation for the expectation with respect to $\mathbb{Q}_B$, we are going to simply define it by $\mathbb{E}$.

Even though this work comprises the valuation of both European-style calls and puts, we are going to devote the main focus to the proof of the pricing equations for the call options, due to the fact that the put prices can be found trough the put-call parity. Nevertheless and although we focus more in the calls, all the steps for the determination of the price of the put options are also going to be documented.

First of all, we start by defining the payoff of the European-style call options as

$$G_1(x,\kappa) \coloneqq (\exp(x) - \exp(\kappa))^+, \tag{2.4}$$

where $x, \kappa \in \mathbb{R}$. As we want to price options with future payoffs, we will have the state variable X as an $\mathcal{F}_T$-measurable random variable with values in $\mathbb{R}$.

For the payoff function $G_1\colon \mathbb{R}^n \times \to \mathbb{R}$, we define

$$C_{G_1}(\kappa) \coloneqq B_0 \mathbb{E}\big(G_1(X,\kappa)\big), \tag{2.5}$$

which is the time-0 price of an option on $\exp(X)$, paying the payoff $G_1(X,\kappa)$ at time T. The trigger κ is a contract variable, which in this case will represent the logarithm of a strike price.

With the payoff G_1 defined above, if one chooses X to be a bond yield, or the logarithm of a stock price or of a FX rate, then one obtains a call on a bond, stock, or currency, respectively.

2.2 Upper Bounds at Extreme Strikes

Since we are going to work with numerical methods, we will need to define the types of errors that we should limit and minimize, and, therefore we also need to clearly define the upper bounds for extreme strikes.

For practical use in bounding the sampling errors that we will define and prove in subsection 2.5., it is important that C_{G_1} be dominated by an expression that is easily evaluated in terms of the characteristic function of X.

For the payoff G_1 defined in the previous subsection, we are going to give two bounds, both valid for all κ: The first is intended for use with large positive κ, whereas the second is intended for use with large negative κ.

Broadie, Cvitanic and Soner (1998) [9] determine the minimum cost of super-replicating a nonnegative contingent claim when there are convex constraints on portfolio weights. They define the payoff of a standard call option as $b(S) = (\mathrm{S} - \mathrm{K})^+$, for $\mathrm{S}, \mathrm{K} \in \mathbb{R}^+$, and according to their constraints they get the following expression for the cost of the calls, dominating claim:

$$\hat{b}(S) = \begin{cases} \mathrm{S} - \mathrm{K} & , \quad \text{if } \mathrm{S} \geq \dfrac{Ku}{u-1} \\ \dfrac{K}{u-1}\left(\dfrac{(u-1)S}{Ku}\right)^{u} & , \quad \text{if } \mathrm{S} < \dfrac{Ku}{u-1} \end{cases} \tag{2.6}$$

Lee (2004) [1] uses equation (2.6) to determine the bounds for large positive and negative κ. In this Master thesis, we are going to prove the result from Lee (2004), using as starting point the initial equation (2.6) of Broadie, Cvitanic and Soner (1998) [9].

Proposition 1

For any $p > 0$,

$$C_{G_1}(\kappa) \leq \frac{B_0 \mathbb{E}\exp\left((p+1)X\right)}{(p+1)exp(p\kappa)}\left(\frac{p}{p+1}\right)^{p}, \tag{2.7}$$

and

$$C_{G_1}(\kappa) \leq B_0 \mathbb{E}\exp(X).$$

(2.8)

Proof: From equation (2.6) and for $S < \frac{Ku}{u-1}$,

$$\hat{b}(S) = \frac{K}{u-1}\left(\frac{(u-1)S}{Ku}\right)^u,$$

(2.9)

Considering the following change of variables,

$$\begin{cases} K = e^{\kappa} \\ u = p + 1 \end{cases},$$

(2.10)

we then have:

$$\begin{aligned} \hat{b}(S) &= \frac{e^{\kappa}}{p}\left(\frac{p \times S}{e^{\kappa}(p+1)}\right)^{p+1} \\ &= \frac{e^{\kappa}S^{p+1}}{pe^{\kappa(p+1)}}\left(\frac{p}{p+1}\right)^{p+1} \\ &= \frac{S^{p+1}}{pe^{\kappa p}}\left(\frac{p}{p+1}\right)^{p+1} \\ &= \frac{S^{p+1}}{e^{\kappa p}(p+1)}\left(\frac{p}{p+1}\right)^{p}. \end{aligned}$$

(2.11)

If we consider, as in Lee (2004) [1], $S = \frac{(p+1)e^{\kappa}}{p}$ and apply it to equation (2.11), we get

$$\hat{b}(S) = \left(\frac{(p+1)e^{\kappa}}{p}\right)^{p} \frac{(p+1)e^{\kappa}}{p} \frac{1}{(p+1)e^{\kappa p}} \left(\frac{p}{p+1}\right)^{p}$$
$$= \frac{e^{\kappa}}{p} \ .$$

(2.12)

On the other way, if we take $S - e^{\kappa}$, we are going to get the same result as the one above:

$$S - e^{\kappa} = \frac{(p+1)e^{\kappa}}{p} - e^{\kappa}$$
$$= \frac{(p+1)e^{\kappa} - pe^{\kappa}}{p}$$
$$= \frac{e^{\kappa}}{p} \ .$$

(2.13)

Remark: Lee (2004) [1] mentions that $\forall S \leq 0$ the following inequality can be applied:

$$S - e^{\kappa} \leq \frac{S^{p+1}}{e^{\kappa p}(p+1)} \left(\frac{p}{p+1}\right)^{p} \ .$$

(2.14)

We disagree, and reformulate it indicating that the inequality above applies $\forall S$.

We define the following function of S:

$$f(S) = S - e^{\kappa} - \frac{S^{p+1}}{e^{\kappa p}(p+1)} \left(\frac{p}{p+1}\right)^{p} \ .$$

(2.15)

It is really simple to find its critical point, by computing the first derivative and equalizing it to zero (0):

$$
\begin{aligned}
&f'(S) = 0\\
&\Leftrightarrow 1-(p+1)\frac{S^p}{e^{\kappa p}(p+1)}\left(\frac{p}{p+1}\right)^p = 0\\
&\Leftrightarrow \frac{S^p}{e^{\kappa p}}\left(\frac{p}{p+1}\right)^p = 1\\
&\Leftrightarrow S^p = e^{\kappa p}\left(\frac{p+1}{p}\right)^p\\
&\Leftrightarrow S = e^{\kappa}\left(\frac{p+1}{p}\right).
\end{aligned}
\tag{2.16}
$$

We prove then the value of S that we have already used before.

Now that we have already proved equation (2.14), it is simple to see that both the left-hand and right-hand sides have equal values and first derivatives at S. The right-hand side has a positive second derivative $\forall S$.

If we analyze in terms of the function $f(S)$, we see that $f''(S) < 0$ which let us conclude that $f(S)$ has a maximum at point $S = e^{\kappa}\left(\frac{p+1}{p}\right)$. Substituting the previous value of S into f, we get that $f\left(e^{\kappa}\left(\frac{p+1}{p}\right)\right) = 0$. We can then conclude that $\forall S\colon f(S) \leq 0$.

The left-hand side of equation (2.14) can thus be expressed as $(S - e^{\kappa})^{+}$. If we substitute S by e^X, using the expected value and multiplying by B_0 we get the first bound, equation (2.7). The second bound, equation (2.8), arises analogously.

■

2.3 Fourier Transforms

We will produce formulas for the price of options with payoff G_1, and for that purpose we will use Fourier Transforms which will express the option price in terms of a discounted characteristic function f. Finally we will need to invert those Fourier Transforms.

Our starting point will be the discounted characteristic function f of the state variable X. The function f will include a discount factor inside the expression, which will allow the option pricing under stochastic interest rates.

Before achieving the final equation for the function f, we need to define all its parameters.

Considering X as being an $\mathbb{R}_+$ - valued random variable, we define A_X, which will denote the interior of the set

$$A_X := \{\nu \in \mathbb{R}_+ : \mathbb{E}e^{\nu X} < \infty\}. \tag{2.17}$$

The negated imaginary parts of the complex vectors which are in A_X form the following strip/tube:

$$\Lambda_X := \{\xi \in \mathbb{C}_+ : -Im(\xi) \in A_X\} \tag{2.18}$$

We can now define the discounted characteristic function f, with respect to a discount factor $exp\left(-\int_0^T r_t\, dt\right)$. In terms of the terminology, we will adopt the one suggested in Bakshi and Madan (2000), i.e.

$$f(\xi) := \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_0^T r_t\, dt} e^{i\xi X}\right). \tag{2.19}$$

It is important to mention that even though the expected value (2.19) is defined with respect to $\mathbb{Q}$, the discounted characteristic function is also related to measure $\mathbb{Q}_B$, since

$$\frac{f(\xi)}{f(0)} = \mathbb{E}e^{i\xi X}.$$

(2.20)

Now that we have defined the discounted characteristic function f, we can determine the Fourier Transform of the damped option price.

And why do we need the Fourier Transform of the damped option price, instead of the usual Fourier Transform of C_{G_1}? Because the usual one does not exist, as $C_{G_1}(\kappa)$ does not decay as $\kappa \to \infty$.

Using the same topology as Carr and Madon (1999), we can then, for each damping constant $\alpha > 0$, define the damped option price function $\mathrm{c}_{\alpha,G_1}: \mathbb{R} \times \mathbb{R}$ by

$$\mathrm{c}_{\alpha,G_1}(\kappa) \coloneqq exp(\alpha\kappa)C_{G_1}(\kappa).$$

(2.21)

The next step will consist in showing that, provided an α is chosen appropriately, the damped option price c_{α,G_1} does have a Fourier transform $\hat{\mathrm{c}}_{\alpha,G_1}: \mathbb{R} \to \mathbb{C}$, defined by

$$\hat{\mathrm{c}}_{\alpha,G_1}(u) \coloneqq \int_{-\infty}^{+\infty} e^{iu\kappa}\mathrm{c}_{\alpha,G_1}(\kappa)d\kappa.$$

(2.22)

The upper bounds on option prices at extreme strikes that were defined in subsection 2.2. imply that $c_{G_1}(\kappa)$ decays exponentially for $|\kappa| \to \infty$. Also, $c_{G_1}(\kappa)$ is bounded. Therefore $c_{G_1}(\kappa)$ is L^1 and has a Fourier transform; moreover, the use of Fubini in the following computation of $\hat{c}$ is justified:

$$
\begin{aligned}
\hat{c}_{\alpha,G_1}(u) &:= \int_{-\infty}^{+\infty} e^{iu\kappa} c_{\alpha,G_1}(\kappa) d\kappa \\
&= \int_{-\infty}^{+\infty} e^{iu\kappa} e^{\alpha\kappa} B_0 \mathbb{E}\big(G_1(X,\kappa)\big) d\kappa \\
&= f(0)\mathbb{E} \int_{-\infty}^{+\infty} e^{(iu+\alpha)\kappa} G_1(X,\kappa) d\kappa
\end{aligned}
\tag{2.23}
$$

Proposition 2:

There exists one $\alpha > 0$ with $\alpha + 1 \in A_X$, such that the Fourier transform $\hat{c}_{\alpha,G_1}$ of c_{α,G_1} exists and

$$
\hat{c}_{\alpha,G_1}(u) = \frac{f(u - (\alpha+1)i)}{\alpha^2 + \alpha - u^2 + u(2\alpha+1)i} \ . \tag{2.24}
$$

Proof:

As referred previously, we will present the proofs not just for the European-style call options, but also for the European-style put options.

The value of α has to be chosen appropriately, depending on the type of contract each equation should refer to. We will then present the proofs for all the three cases where the value of α will have an impact.

Call Option: $[\alpha > 0]$

Using equation (2.23), and the payoff $G_1(x,\kappa) := (\exp(x) - \exp(\kappa))^+$, then

$$
\begin{aligned}
\hat{c}_{\alpha,G_1}(u) &= f(0)\mathbb{E} \int_{-\infty}^{+\infty} (e^x - e^\kappa)^+ e^{(iu+\alpha)\kappa} d\kappa \\
&= f(0)\mathbb{E} \int_{-\infty}^{x} (e^x - e^\kappa) e^{(iu+\alpha)\kappa} d\kappa
\end{aligned}
$$

$$= f(0)\mathbb{E}\left(\int_{-\infty}^{x} e^{x+(iu+\alpha)\kappa} - e^{(1+iu+\alpha)\kappa} d\kappa\right)$$

$$= f(0)\mathbb{E}\left(\left[\frac{e^{x+(iu+\alpha)\kappa}}{iu+\alpha}\right]_{-\infty}^{x} - \left[\frac{e^{(1+iu+\alpha)\kappa}}{1+iu+\alpha}\right]_{-\infty}^{x}\right)$$

$$= f(0)\mathbb{E}\left(\frac{e^{x+(iu+\alpha)x}}{iu+\alpha} - \frac{e^{(1+iu+\alpha)x}}{1+iu+\alpha}\right)$$

$$= f(0)\mathbb{E}\left(\frac{(1+iu+\alpha)e^{x+(iu+\alpha)x} - (iu+\alpha)e^{(1+iu+\alpha)x}}{\alpha^2+\alpha+\alpha iu+iu+\alpha iu-u^2}\right)$$

$$= \frac{f(0)\mathbb{E}e^{x+(iu+\alpha)x}}{\alpha^2+\alpha-u^2+i(2\alpha+1)u} \quad . \tag{2.25}$$

Put Option: $[\alpha < -1]$

Using equation (2.23), and the payoff $G_1(x,\kappa) := (\exp(\kappa) - \exp(x))^+$, then

$$\hat{c}_{\alpha,G_1}(u) := f(0)\mathbb{E}\int_{-\infty}^{+\infty} (e^{\kappa} - e^{x})^+ e^{(iu+\alpha)\kappa} d\kappa$$

$$= f(0)\mathbb{E}\int_{x}^{+\infty} (e^{\kappa} - e^{x}) e^{(iu+\alpha)\kappa} d\kappa$$

$$= f(0)\mathbb{E}\left(\int_{x}^{+\infty} e^{(1+iu+\alpha)\kappa} - e^{x+(iu+\alpha)\kappa} d\kappa\right)$$

$$= f(0)\mathbb{E}\left(\left[\frac{e^{(1+iu+\alpha)\kappa}}{1+iu+\alpha}\right]_{x}^{+\infty} - \left[\frac{e^{x+(iu+\alpha)\kappa}}{iu+\alpha}\right]_{x}^{+\infty}\right)$$

$$= f(0)\mathbb{E}\left(\frac{e^{x+(iu+\alpha)x}}{iu+\alpha} - \frac{e^{(1+iu+\alpha)x}}{1+iu+\alpha}\right)$$

$$= f(0)\mathbb{E}\left(\frac{(1+iu+\alpha)e^{x+(iu+\alpha)x} - (iu+\alpha)e^{(1+iu+\alpha)x}}{\alpha^2+\alpha+\alpha iu+iu+\alpha iu-u^2}\right)$$

$$= \frac{f(0)\mathbb{E}e^{x+(iu+\alpha)x}}{\alpha^2+\alpha-u^2+i(2\alpha+1)u} \quad .$$

■

Remark: For: $-1 < \alpha < 0$, we can not use neither the payoff from the call option nor from the put option. Hence, Lee (2004) [1] defined the following equation for the payoff when $-1 < \alpha < 0$:

$$G_1^*(x,\kappa) := min(\exp(x), \exp(\kappa))$$

(2.26)

Combining equations (2.23) and (2.26),

$$\begin{aligned}
\hat{c}_{\alpha,G_1^*}(u) &= f(0)\mathbb{E}\int_{-\infty}^{+\infty} min(\exp(x), \exp(\kappa))e^{(iu+\alpha)\kappa}d\kappa \\
&= f(0)\mathbb{E}\int_{-\infty}^{x} e^{\kappa}e^{(iu+\alpha)\kappa}d\kappa \\
&= f(0)\mathbb{E}\left(\left[\frac{e^{(1+iu+\alpha)\kappa}}{1+iu+\alpha}\right]_{-\infty}^{x}\right) \\
&= \frac{f(0)\mathbb{E}e^{(1+iu+\alpha)x}}{1+iu+\alpha}
\end{aligned}$$

(2.27)

We can see then, that we get a final result different from equation (2.24).

Finally, we just need to recover the option prices using Fourier inversion. Before defining the equation that will give us the option prices, it is important to notice that the damped option price $c_{\alpha,G_1}(\kappa)$ is L^1, and is continuous by the dominated convergence theorem. Its transform is also L^1, and therefore the usual Fourier inversion recovers $c_{G_1}(\kappa)$.

The option price is then given by

$$\begin{aligned}
C_{G_1}(\kappa) &= \frac{e^{-\alpha\kappa}}{2\pi}\int_{-\infty}^{\infty} \hat{c}_{\alpha,G_1}(u)e^{-i\kappa u}du \\
&= \frac{e^{-\alpha\kappa}}{\pi}\int_0^{\infty} \Re\left[\hat{c}_{\alpha,G_1}(u)e^{-i\kappa u}\right]du,
\end{aligned}$$

(2.28)

where $\Re(z)$ denotes the real part of $z \in \mathbb{C}$.

2.4 Pricing Equation

Lewis (2001) uses a technique to shift the contour of an integral in the complex plane across a pole of the integrand. This shift changes the value of the integral, and will also be used by us for the representation of the Fourier transform of option prices, which can be viewed as contour integrals in the complex plane.

Our approach will consist in deriving equations for the transforms of option prices with respect to the variable κ. Alternatively Lewis (2001) derived his equations with respect to the spot variable X_0.

The equations that we will be using are not subject to the same type of restrictions as Lewis (2001) equations are. He has to define certain type of assumptions, which require the option to be written on the exponential of a variable X_T where the distribution of $X_T - X_0$ is not permitted to depend on X_0. Our equations apply to a wider class of underlying state variables X, including variables with mean-reversion.

To allow the direct application of FFT to calibrate parameters to the prices of options at multiple strikes, we need to transform our equation into a transform-in-strike equation.

According to Proposition 2 from Section 2.3, for positive α with $\alpha + 1 \in A_X$, and for simplicity, we can define

$$\hat{C}_{G_1}(z) = \frac{f(z-i)}{iz - z^2} \ . \tag{2.29}$$

We can easily notice that if we apply, in equation (2.29), a change of variables from z to $u - \alpha i$ we will get equation (2.24):

$$\begin{aligned} \hat{c}_{\alpha,G_1} &= \hat{C}_{G_1}(u - \alpha i) \\ &= \frac{f(u - \alpha i - i)}{i(u - \alpha i) - (u - \alpha i)^2} \end{aligned}$$

$$= \frac{f(u - (\alpha + 1)i)}{\alpha^2 + \alpha - u^2 + u(2\alpha + 1)i}$$

We can also define the complex Fourier transform, $\hat{C}_{G_1}(z)$, of the unmodified option price, $C_{G_1}(\kappa)$, as:

$$\hat{C}_{G_1}(z) = \int_{-\infty}^{\infty} C_{G_1}(\kappa) e^{i\kappa z} d\kappa$$

(2.30)

According to equation (2.28) and assuming a positive α, we can see that $C_{G_1}(\kappa)$ may be inverted by integrating along the contour $Im(z) = -\alpha$.

$$C_{G_1}(\kappa) = \frac{1}{2\pi} \int_{-\infty-\alpha i}^{\infty-\alpha i} \hat{C}_{G_1}(z) e^{-i\kappa z} dz$$
$$= \frac{1}{\pi} \int_{0-\alpha i}^{\infty-\alpha i} \Re\left[\hat{C}_{G_1}(z) e^{-i\kappa z}\right] dz$$

(2.31)

<u>Proposition 3</u>: Assume α to be any real number such that $\alpha + 1 \in A_X$. We can define $C_{G_1}(\kappa)$ as

$$C_{G_1}(\kappa) = R_{\alpha,G_1} + \tfrac{1}{\pi} \int_{0-\alpha i}^{\infty-\alpha i} \Re\left[\hat{C}_{G_1}(z) e^{-i\kappa z}\right] dz \ ,$$

(2.32)

where

$$R_{\alpha,G_1} := \begin{cases} f(-i) - e^k f(0), \quad \alpha < -1 \\ f(-i) - e^k f(0)/2 \,, \quad \alpha = -1 \\ f(-i), \qquad -1 < \alpha < 0 \\ f(-i)/2 \,, \ \alpha = 0 \\ 0, \ \alpha > 0 \end{cases}$$

Proof:

For the European call option, and substituting equation (2.24) into (2.28):

$$C_{G_1}(\kappa) = R_{\alpha,G_1} + \frac{e^{-\alpha\kappa}}{\pi}\int_0^\infty \Re\left[e^{-i\kappa u}\frac{f\big(u - i(1+\alpha)\big)}{\alpha^2 + \alpha + 2\alpha i u + iu - u^2}\right]du \ . \tag{2.33}$$

We then apply the following change of variables,

$$\begin{gathered} z - i = u - i - i\alpha \\ \Leftrightarrow z = u - i\alpha \\ \Leftrightarrow u = z + i\alpha \end{gathered} \tag{2.34}$$

to the integral part of equation (2.33), which results in the following:

$$\begin{aligned} &\frac{e^{-\alpha\kappa}}{\pi}\int_{0-i\alpha}^{\infty-i\alpha} \Re\left[e^{-i(z+i\alpha)\kappa}\frac{f(z-i)}{\alpha^2 + \alpha + (2\alpha+1)i(z+i\alpha) - (z+i\alpha)^2}\right]dz \\ &= \frac{1}{\pi}\int_{0-i\alpha}^{\infty-i\alpha} \Re\left[e^{-iz\kappa}\frac{f(z-i)}{\alpha^2 + \alpha + iz - 2\alpha^2 - \alpha - z^2 + \alpha^2}\right]dz \\ &= \frac{1}{\pi}\int_{0-i\alpha}^{\infty-i\alpha} \Re\left[e^{-iz\kappa}\frac{f(z-i)}{z(i-z)}\right]dz \end{aligned} \tag{2.35}$$

We can now determine the residues of the integral (2.35), so that we can compute its value by applying the residues theorem. It is easy to see that there are two singularities, one for $z = 0$ and the other one for $z = i$. The two residues will then be

$$
\begin{aligned}
&Res\left(e^{-iz\kappa}\frac{f(z-i)}{z(i-z)},i\right)\\
&=2\pi i\lim_{z\to i}e^{-iz\kappa}\frac{f(z-i)}{-z(i-z)2\pi}\\
&=2\pi ie^{\kappa}\frac{f(0)}{-2\pi i}\\
&=-e^{\kappa}f(0)\,,
\end{aligned}
\tag{2.36}
$$

and

$$
\begin{aligned}
&Res\left(e^{-iz\kappa}\frac{f(z-i)}{z(i-z)},0\right)\\
&=2\pi i\lim_{z\to 0}ze^{-iz\kappa}\frac{f(z-i)}{z(i-z)2\pi}\\
&=f(-i)\,.
\end{aligned}
\tag{2.37}
$$

We will explain the residues theorem, in more detail in Appendix B, so that it is easier to understand the logic behind equations (2.36) and (2.37), but we will focus on its application to the determination of improper integrals involving exponential functions.

For now, and as it is easy to notice, for $\alpha < 0$, the transform $\hat{c}_{\alpha,G}$ does not exist. Nonetheless, equation (2.31) is integrable for $\alpha < 0$, but it does not recover C_{G_1}, because the integration path has shifted across the pole $z = 0$. However, it recovers C_{G_1} less the contribution of the residue of $\hat{C}_{G_1}$ at $z = 0$.

The two poles will generate two additional pricing equations. For $\alpha = 0$ and $\alpha = -1$, the integral is again integrable, however the integration contour passes through a pole, which will result in a contribution of the residue cut in half. This fact generates two additional pricing equations.

■

2.5 Truncation and Sampling Errors

In the previous subsection we have been discussing a way of getting the pricing equation for the determination of the prices of European-style options. That pricing equation is going to be implemented using a numerical method, and numeric methods generate errors which should be well defined so that they can be minimized.

In this section we want to clearly identify the two types of errors that we are going to face. It is thus clear that equation (2.32) can be numerically calculated using an N-point sum with a grid spacing Δ. This process has two types of errors: The truncation error, because instead of having an integration from 0 to infinite, we will have a finite limit; and a sampling error, because in a numeric method we do not integrate in continuous time (instead small squares in a grid are used).

The resultant total error is defined as the absolute difference between the true value given by equation (2.32), and the discrete approximation given by the N-point sum

$$\sum\nolimits^{N}(\kappa) := \sum\nolimits_{\alpha,G_1}^{N,\Delta}(\kappa) := R_{\alpha,G_1} + e^{-\alpha\kappa}\frac{\Delta}{\pi}\Re\left[\sum_{n=0}^{N-1}\hat{c}_{\alpha,G_1}\left(\left(n+\tfrac{1}{2}\right)\Delta\right)e^{-i(n+1/2)\kappa\Delta}\right]. \tag{2.38}$$

The total error will then be bounded by the sum of the sampling error and the truncation error,

$$|C - \textstyle\sum^{N}| \leq |C - \sum^{\infty}| + |\sum^{\infty} - \sum^{N}| \,, \tag{2.39}$$

where $\sum^{\infty}$ is defined as $\sum^{N}$, except with an infinite upper limit for the summation.

Finally, it is important to notice that as it is going to be shown in the next two subsections, the truncation error does not depend on the sign of α. However, for the determination of the sampling error we will need to separate the analysis for the different signs of α.

2.5.1 Truncation Error

Proposition 4: Assuming the hypotheses of Proposition 3, and considering that $\hat{c}_{\alpha,G_1}$ has a power decay, i.e. $\left|\hat{c}_{\alpha,G_1}(u)\right| \leq \Phi(u)/u^{1+\gamma}$ for all $u > u_0$ where $\gamma > 0$ and $\Phi(u)$ is decreasing in u, then the truncation error can be defined as

$$|\Sigma^{\infty}(\kappa) - \Sigma^{N}(\kappa)| \leq \frac{\Phi(N\Delta)}{\pi e^{\alpha\kappa}\gamma\big((N+1/2)\Delta\big)^{\gamma}},$$

(2.40)

provided that $N\Delta > u_0$.

Proof:

In Lee (2004) [1] paper, he defined equation (2.40) without the constant $1/2$ in its denominator, but we are going to prove that the way we define it, is the correct one.

Knowing that

$$|\Sigma^{\infty}(\kappa) - \Sigma^{N}(\kappa)| \leq e^{-\alpha\kappa}\frac{\Delta}{\pi}\sum_{n=N}^{\infty}\left|\hat{c}_{\alpha,G_1}\left(\left(n+\tfrac{1}{2}\right)\Delta\right)\right|$$

(2.41)

and as we are assuming $\left|\hat{c}_{\alpha,G_1}(u)\right| \leq \Phi(u)/u^{1+\gamma}$, we can apply it to equation (2.41) which will lead us to

$$e^{-\alpha\kappa}\frac{\Delta}{\pi}\sum_{n=N}^{\infty}\hat{c}_{\alpha,G_1}\left(\left(n+\tfrac{1}{2}\right)\Delta\right) \leq e^{-\alpha\kappa}\frac{\Delta}{\pi}\sum_{n=N}^{\infty}\frac{\Phi\left(\left(n+\frac{1}{2}\right)\Delta\right)}{[(n+1/2)\Delta]^{\gamma+1}}$$

$$
\begin{aligned}
&\leq \frac{e^{-\alpha\kappa}}{\pi}\frac{\Phi(N\Delta)}{\Delta^{\gamma}}\int_{N}^{\infty}\frac{dx}{(x+1/2)^{\gamma+1}}\\
&=\frac{e^{-\alpha\kappa}}{\pi}\frac{\Phi(N\Delta)}{\Delta^{\gamma}}\left[\frac{(x+1/2)^{-\gamma}}{-\gamma}\right]_{N}^{\infty}\\
&=\frac{e^{-\alpha\kappa}}{\pi}\frac{\Phi(N\Delta)}{\Delta^{\gamma}}\frac{(N+1/2)^{-\gamma}}{\gamma}\\
&=\frac{\Phi(N\Delta)}{\pi e^{\alpha\kappa}\gamma\big((N+1/2)\Delta\big)^{\gamma}}
\end{aligned}
\tag{2.42}
$$

■

In the first step of the proof of equation (2.42) we have replaced the summation by an integral, as we want to define a bound for that series, so we just need an estimation which, in this specific case, maximizes its value. For this specific case and as our series is represented by a function that is continuous and decreasing, we applied a principle which can be seen in further detail in Appendix A.

In terms of the expression for the function $\Phi(u)$ it can be seen, in Appendix D, how this function can be determined.

Finally, it is important to refer that, for practical purposes, it is desirable to improve the power γ, by factoring in the contribution from the large-u decay of $|f(-(\alpha+1)i)|$, as we do in our code.

2.5.2 Sampling Error

Before starting to discuss the major topic of this subsection, “sampling error”, it is important to define a principle which will impact with all the equations and explanations that will follow.

This principle is “aliasing”, which according to signal theory is an effect that causes different signals to become indistinguishable, when sampled. It also refers to the distortion or artifact that results when the signal reconstructed from samples is different from the original continuous signal.

In our specific case, we have to deal with the aliasing effect, because we are going to create samples of $\hat{c}_{\alpha,G_1}$ at regular discrete intervals, which imply that we do not fully recover c_{α,G_1}, but instead a periodic function equal to a combination of c_{α,G_1} and infinitely many shifted copies of itself.

As it has been already said in section 2.5., the sampling error depends on the values of α, so our analysis will focus separately on each of the different intervals of α that have different behaviors and, consequently, different sampling error equations.

<u>Proposition 5</u>: The sampling error equation for $\alpha > 0$, is given by

$$\left|C_{G_1} - \Sigma^{\infty}_{\alpha,G_1}\right| \leq \underset{p > \alpha : p+1 \in A_X}{inf} \left[\frac{e^{-2\pi\alpha/\Delta} f(-i)}{1 - e^{-4\pi\alpha/\Delta}} + \frac{e^{2\pi(\alpha-p)/\Delta} f\big(-i(p+1)\big)}{(p+1)e^{p\kappa}\left(1 - e^{4\pi(\alpha-p)/\Delta}\right)} \left(\frac{p}{p+1}\right)^{p}\right].$$

(2.43)

Proof:

Considering $\Delta > 0$ and a positive integer j, we obtain the following relation from the definition of the Fourier transform:

$$c_{\alpha,G_1}(\kappa - 2\pi j/\Delta) + c_{\alpha,G_1}(\kappa + 2\pi j/\Delta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \hat{c}_{\alpha,G_1}(u)\cos(2\pi j u/\Delta)\, e^{-\iota\kappa u} du$$

$$= 2\int_0^{\Delta} F(u)\cos(2\pi ju/\Delta)\,du \ ,$$

(2.44)

where

$$F(u) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \hat{c}_{\alpha,G_1}(\mathrm{u} + \mathrm{n}\Delta)e^{-i(u+n\Delta)\kappa} \ .$$

(2.45)

Since F is Lipschitz, the Fourier cosine series may be summed:

$$c_{\alpha,G_1}(\kappa) + \sum_{n=-\infty}^{\infty}\left[c_{\alpha,G_1}(\kappa - 2\pi j/\Delta) + c_{\alpha,G_1}(\kappa + 2\pi j/\Delta)\right]\cos(2\pi ju/\Delta) = F(u)\Delta \ .$$

(2.46)

However equation (2.46) is much simpler to understand if we prove it by applying the Trigonometric Fourier Series:

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n\cos(\omega_0 nt) + b_n\sin(\omega_0 nt)) \ ,$$

(2.47)

where $a_0 = \frac{1}{T}\int_0^T f(t)\,dt$, $a_n = \frac{2}{T}\int_0^T f(t)\cos(\omega_0 nt)\,dt$, and $b_n = 0$ as $f(t)$ is an even function, which means that $f(t) = f(-t)$.

In our case, we are going to define the variables in the following way:$F(u) \equiv f(t)$, $\omega_0 \equiv 2\pi/\Delta$, $n \equiv j$, $T \equiv \Delta$ and $t \equiv u$. The values of a_n and a_0 will then be given by

$$\begin{aligned}
&a_n = \frac{2}{\Delta}\int_0^{\Delta} F(u)\cos(2\pi j u/\Delta)\,du \\
&\Leftrightarrow a_n = \Delta\left[c_{\alpha,G_1}(\kappa - 2\pi j/\Delta) + c_{\alpha,G_1}(\kappa + 2\pi j/\Delta)\right] \ ,
\end{aligned}$$

(2.48)

and

$$\begin{aligned}
&a_0 = \frac{1}{\Delta}\int_0^{\Delta} F(u)\,du \\
&\Leftrightarrow a_0 = \frac{1}{\Delta}\int_0^{\Delta} \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \hat{c}_{\alpha,G_1}(u + n\Delta)e^{-i(u+n\Delta)\kappa}\,du \\
&\Leftrightarrow a_0 = \frac{1}{2\pi\Delta}\sum_{n=-\infty}^{\infty}\int_{n\Delta}^{(n+1)\Delta} \hat{c}_{\alpha,G_1}(z)e^{-iz\kappa}\,dz \\
&\Leftrightarrow a_0 = \frac{1}{2\pi\Delta}\int_{-\infty}^{\infty} \hat{c}_{\alpha,G_1}(z)e^{-iz\kappa}\,dz \\
&\Leftrightarrow \Delta a_0 = c_{\alpha,G_1}(\kappa)
\end{aligned}$$

(2.49)

Replacing a_n and a_0 by equations (2.48) and (2.49) in the Trigonometric Fourier Series (2.47), we get

$$\begin{aligned}
F(u)\Delta &= a_0 + \sum_{n=1}^{\infty}(a_n\cos(\omega_0 n u)) \\
\Leftrightarrow F(u)\Delta &= c(\kappa) + \sum_{j=1}^{\infty}\left[c_{\alpha,G_1}(\kappa - 2\pi j/\Delta) + c_{\alpha,G_1}(\kappa + 2\pi j/\Delta)\right]\cos(2\pi j u/\Delta) \ ,
\end{aligned}$$

(2.50)

and we recover equation (2.46).

In particular, taking $u = \Delta/2$, equation (2.48) yields

$$\left|c_{\alpha,G_1}(\kappa) - F(\Delta/2)\Delta\right| = \left|\sum_{j=1}^{\infty}(-1)^j\left[c_{\alpha,G_1}(\kappa - 2\pi j/\Delta) + c_{\alpha,G_1}(\kappa + 2\pi j/\Delta)\right]\right| .$$

(2.51)

Notice that we have replaced $\cos(2\pi ju/\Delta)$ by $(-1)^j$, which gives us exaclty the same result.

Then, if we recover equation (2.21), by multiplying equation (2.51) by $exp(-\alpha\kappa)$ to undamp the call prices, we get

$$\left|C_{G_1}(\kappa) - \Sigma^{\infty}(\kappa)\right| = \left|\sum_{j=1}^{\infty}(-1)^j\left[e^{-2\pi j\alpha/\Delta}C_{G_1}(\kappa - 2\pi j/\Delta) + e^{2\pi j\alpha/\Delta}C_{G_1}(\kappa + 2\pi j/\Delta)\right]\right| .$$

(2.52)

To start, it can be noticed that we used to have $\left|c_{\alpha,G_1}(\kappa) - F(\Delta/2)\Delta\right|$ in equation (2.51), and now we have $\left|C_{G_1}(\kappa) - \Sigma^{\infty}(\kappa)\right|$ in equation (2.52). As the first term is really simple to understand, the second one is a little bit trickier but we are going to use the integral version of the equations as an analogy to make it simple.

As we already know from section 2.3,

$$\begin{aligned}
c_{\alpha,G_1}(\kappa) &= \int_{-\infty}^{\infty}\hat{c}_{\alpha,G_1}(u)e^{-i\kappa u}du \\
&= \int_{-\infty}^{0}\hat{c}_{\alpha,G_1}(u)e^{-i\kappa u}du + \int_{0}^{\infty}\hat{c}_{\alpha,G_1}(u)e^{-i\kappa u}du \\
&= \int_{0}^{\infty}\hat{c}_{\alpha,G_1}(u)e^{-i\kappa u}du - \int_{-\infty}^{0}\hat{c}_{\alpha,G_1}(u)e^{-i\kappa v}dv \\
&= \int_{0}^{\infty}\left[\hat{c}_{\alpha,G_1}(u)e^{-i\kappa u} + \hat{c}_{\alpha,G_1}(-u)e^{i\kappa u}\right]du \\
&= \int_{0}^{\infty}2\Re\left[\hat{c}_{\alpha,G_1}(u)e^{-i\kappa u}\right]du
\end{aligned}$$

(2.53)

And what we want to prove is that $\sum^{\infty}(\kappa) = e^{-\alpha\kappa}F(\Delta/2)\Delta$, so now it is easier to see that if we compare equation (2.53) with equation (2.38), that they both represent the same, as we know that the real part of a complex number is equal to half the sum of its value with its conjugate.

Then, for the right and side of equation (2.52), as we have already seen in section 2.2., we have an upper bound for extreme positive strikes given by equation (2.7), and an upper bound for extreme negative strikes given by (2.8).

If we apply $C_{G_1}(\kappa - 2\pi j/\Delta)$ to equation (2.7) and $C_{G_1}(\kappa + 2\pi j/\Delta)$ to equation (2.8), we will have

$$\begin{aligned} C_{G_1}(\kappa + 2\pi j/\Delta) &\leq \frac{B_0 \mathbb{E}\exp\big((p+1)X\big)}{(p+1)exp\big(p(\kappa + 2\pi j/\Delta)\big)}\left(\frac{p}{p+1}\right)^p \\ &\leq \frac{B_0 \mathbb{E}\exp\big((p+1)X\big)}{(p+1)exp(p\kappa)exp(2\pi jp/\Delta)}\left(\frac{p}{p+1}\right)^p \\ &\leq \frac{B_0 \mathbb{E}\exp\big((p+1)X\big)exp(-2\pi jp/\Delta)}{(p+1)exp(p\kappa)}\left(\frac{p}{p+1}\right)^p \ . \end{aligned} \tag{2.54}$$

Multiplying inequality (2.54) by $exp(2\pi j\alpha/\Delta)$, we have

$$\begin{aligned} C_{G_1}(\kappa + 2\pi j/\Delta) &\leq \frac{B_0 \mathbb{E}\exp\big((p+1)X\big)exp(-2\pi jp/\Delta)exp(2\pi j\alpha/\Delta)}{(p+1)exp(p\kappa)}\left(\frac{p}{p+1}\right)^p \\ &\leq \frac{B_0 \mathbb{E}\exp\big((p+1)X\big)exp(2\pi j(\alpha - p)/\Delta)}{(p+1)exp(p\kappa)}\left(\frac{p}{p+1}\right)^p \ . \end{aligned} \tag{2.55}$$

For $C_{G_1}(\kappa + 2\pi j/\Delta)$ we can see from equation (2.8) that it does not depend from κ, so its value is exactly equation (2.8). We just need to multiply that result by $exp(-2\pi j\alpha/\Delta)$, as it can be seen from equation (2.52).

Finally it is easy to see that by applying Proposition 1, we get the following result:

$$\left|C_{G_1}(\kappa) - \Sigma^{\infty}(\kappa)\right| \leq B_0 \sum_{\substack{j=1 \\ j\,odd}}^{\infty} \left[e^{-2\pi j\alpha/\Delta} \mathbb{E} e^{X} + \frac{e^{2\pi j(\alpha-p)/\Delta} \mathbb{E} e^{(p+1)X}}{(p+1)\mathbb{E} e^{\kappa p}} \left(\frac{p}{p+1}\right)^{p} \right] .$$

(2.56)

It can also be noticed that we got rid of $(-1)^j$, because we added a new condition for the summation above by setting j as an odd number.

Now we are not far from reaching the result from equation (2.43), but for that purpose we need to get the final result of the summation as well as the expected values in equation (2.56).

Starting with the summations, we know that the sum of a geometric progression is given by:

$$S_{\infty} = \frac{a_1}{1-q} \ ,$$

(2.57)

where a_1 is the first term of the series, and $a_n = a_1 q^{n-1}$.

So, the geometric progression from $e^{-2\pi j\alpha/\Delta}$ with j odd will then be

$$a_1 = e^{-2\pi\alpha/\Delta}, \qquad a_3 = e^{-6\pi\alpha/\Delta}, \qquad a_5 = e^{-10\pi\alpha/\Delta} \ ,$$

$$q = \frac{a_3}{a_1} = \frac{e^{-6\pi\alpha/\Delta}}{e^{-2\pi\alpha/\Delta}} = e^{-4\pi\alpha/\Delta} \ ,$$

$$S_{\infty} = \frac{e^{-2\pi\alpha/\Delta}}{1 - e^{-4\pi\alpha/\Delta}} \ .$$

(2.58)

And the geometric progression from $e^{2\pi j(\alpha-p)/\Delta}$ with j odd will then be

$$a_1 = e^{2\pi(\alpha-p)/\Delta}, a_3 = e^{6\pi(\alpha-p)/\Delta} \ ,$$

$$q = \frac{a_3}{a_1} = \frac{e^{6\pi(\alpha-p)/\Delta}}{e^{2\pi(\alpha-p)/\Delta}} = e^{4\pi(\alpha-p)/\Delta} \ ,$$

$$S_\infty = \frac{e^{2\pi(\alpha-p)/\Delta}}{1 - e^{4\pi(\alpha-p)/\Delta}} \ . \tag{2.59}$$

Note that Lee (2004) [1] has a typo, because in his final equation he indicates that the summation above is given by

$$S_\infty = \frac{e^{2\pi(\alpha-p)/\Delta}}{1 - e^{-4\pi(\alpha-p)/\Delta}} \ . \tag{2.60}$$

We clearly know that it is not true, and we will use the correct result given by equation (2.59).

Finally, we just have to determine the expected values contained in equation (2.56), which will be given by:

$$\begin{aligned} f(u) &= \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_0^T r_t dt} e^{iux_T}\right) \\ &= P(0,T)\mathbb{E}_{\mathbb{Q}}(e^{iux_T}) \ , \end{aligned} \tag{2.61}$$

$$B_0 = P(0,T) \ , \tag{2.62}$$

$$f\big(-i(p+1)\big) = B_0\mathbb{E}_{\mathbb{Q}}\big(e^{i(-i)(p+1)x_T}\big)$$
$$\Leftrightarrow \mathbb{E}_{\mathbb{Q}}\big(e^{(p+1)X}\big) = \frac{f\big(-i(p+1)\big)}{B_0} \ ,$$

(2.63)

and

$$f(-i) = B_0\mathbb{E}_{\mathbb{Q}}(e^X) \qquad \text{so} \qquad \mathbb{E}_{\mathbb{Q}}(e^X) = \frac{f(-i)}{B_0} \ .$$

(2.64)

It is now really simple to see that if we replace equations (2.58), (2.59), (2.63) and (2.64) into equation (2.56), we get the desired result given by equation (2.43).

■

Now that we have bounded the sampling error for positive α, we will focus in the case of $\alpha < 0$. For negative α there will be two sampling error equations.

<u>Proposition 6</u>:

For $-1 < \alpha < 0$,

$$\left|C_{G_1}(\kappa) - \Sigma_{\alpha,G_1}^{\infty}\right| \leq \frac{e^{\kappa-2\pi(\alpha+1)/\Delta} f(0)}{1 - e^{-4\pi(\alpha+1)/\Delta}} + \frac{e^{2\pi\alpha/\Delta} f(-i)}{1 - e^{4\pi\alpha/\Delta}} \ , \tag{2.65}$$

and for $\alpha < -1$,

$$\left|C_{G_1} - \Sigma_{\alpha,G_1}^{\infty}\right| \leq \underset{\substack{q>-(\alpha+1): \\ -q \in A_X}}{inf} \left[\frac{e^{\kappa+2\pi(\alpha+1)/\Delta} f(0)}{1 - e^{4\pi(\alpha+1)/\Delta}} + \frac{e^{(q+1)\kappa} e^{-2\pi(1+\alpha+q)/\Delta} f(-iq)}{(q+1)e^{p\kappa}(1 - e^{-4\pi(1+\alpha+q)/\Delta})} \left(\frac{q}{q+1}\right)^{q}\right] . \tag{2.66}$$

Proof:

For negative α, the payoff G_1 no longer applies, because it is not integrable. Although this might seem an issue, we have already seen that we can use the put/call parity. Nevertheless, we will have a specific case for $-1 < \alpha < 0$ where we will have to define a new payoff function that is not the payoff for a call neither for a put, but which will guarantee the integrability of the function, and as a consequence the determination of the sampling error.

We will then define one complementary payoff G^{*}, and a second complementary payoff G^{**} by:

$$G_1^{*}(x,\kappa) \coloneqq min(\exp(x), \exp(\kappa)) \ , \tag{2.67}$$

and

$$G_1^{**}(x,\kappa) \coloneqq (\exp(\kappa) - \exp(x))^{+}$$

(2.68)

Considering G_1^* with $-1 < \alpha < 0$, we just need to apply equation (2.31) to get $C_{G_1^*}(\kappa)$.

With $\alpha < -1$, this holds after replacing the G_1^* with G_1^{**}. And for that specific case we have,

$$G_1^{**}(x, \kappa) = P_0$$

(2.69)

where p_0 represents the time-0 value of a standard European-style put option.

By using the discounted characteristic function f and applying the results from Proposition 3, we will now show the application of the put/call parity in this specific situation.

Given the discounted characteristic function, we have that:

$$\begin{aligned} f(u) &= \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_0^T r_t dt} e^{iux_T}\right) \\ &= P(0,T)\mathbb{E}(e^{iux_T}) \ , \end{aligned}$$

(2.70)

and we know that the discounted factor can be expressed by

$$B_0 = P(0,T) \ ,$$

(2.71)

If we recover the results from Proposition 3, for $\alpha < -1$, we have

$$R_{\alpha, G_1^{**}} = f(-i) - e^k f(0) \ .$$

(2.72)

According to equation (2.70),

$$
\begin{aligned}
f(-i) &= B_0\mathbb{E}(e^{x_T}) \\
&= e^{-rT}\mathbb{E}(S_T \backslash \mathcal{F}_0) \\
&= e^{-rT}S_0e^{(r-q)T} \\
&= S_0e^{-qT} \ ,
\end{aligned}
$$

(2.73)

$$f(0) = P(0,T) = e^{-rT} \ ,$$

(2.74)

It is really simple to see that if we replace equations (2.73) and (2.74) into equation (2.72), we get:

$$
\begin{aligned}
R_{\alpha,G_1^{**}} &= S_0e^{-qT} - e^{k}e^{-rT} \\
&= C_0 - P_0 \ .
\end{aligned}
$$

(2.75)

Now that we have already shown how the different payoffs correlate between each other, we need to prove how we will get the final equations for the sampling errors of both payoffs G_1^{*} and G_1^{**}.

Following the same logic we applied to the proof of the sampling error for positive α, we already know what we will need to do. First of all we have to recall equation (2.52), and second of all we need to use once again the upper bounds from subsection 2.2.

For both payoffs G_1^{*} and G_1^{**}, we just have to follow the same reasoning applied in Proposition 1, and the upper bounds will then be given by

$$C_{G_1^{*}}(\kappa) \leq B_0e^{\kappa}, \text{ and } \qquad C_{G_1^{*}}(\kappa) \leq B_0\mathbb{E}e^{X} \ .$$

(2.76)

And for payoff G_1^{**},

$$C_{G_1^{**}}(\kappa) \leq \frac{B_0 \mathbb{E}e^{-qX}}{1+q}\left(\frac{q}{q+1}\right)^q e^{(q+1)\kappa}, \text{ and } \qquad C_{G_1^{**}}(\kappa) \leq B_0 e^{\kappa} \ .$$

(2.77)

Recalling equation (2.52) for the case of $-1 < \alpha < 0$, with $C_{G_1^*}$ instead of C_{G_1}, where

$$C_{G_1^*}(\kappa - 2\pi j/\Delta) \leq B_0 \mathbb{E}e^X \qquad , \qquad C_{G_1^*}(\kappa + 2\pi j/\Delta) \leq B_0 \mathbb{E}e^X$$

(2.78)

$$C_{G_1^*}(\kappa - 2\pi j/\Delta) \leq B_0 e^{\kappa - 2\pi j/\Delta} \qquad , \qquad C_{G_1^*}(\kappa + 2\pi j/\Delta) \leq B_0 e^{\kappa + 2\pi j/\Delta} \ .$$

(2.79)

If we replace those values into equation (2.52), then

$$\left|C_{G_1^*}(\kappa) - \Sigma^{\infty}(\kappa)\right| = \left|\sum_{j=1}^{\infty} (-1)^j \left[e^{-2\pi j\alpha/\Delta} B_0 e^{\kappa - 2\pi j/\Delta} + e^{2\pi j\alpha/\Delta} B_0 \mathbb{E}e^X\right]\right|$$
$$\leq B_0 \sum_{\substack{j=1 \\ j\, odd}}^{\infty} \left[e^{\kappa - 2\pi j(\alpha+1)/\Delta} + e^{2\pi j\alpha/\Delta} \mathbb{E}e^X\right] \ .$$

(2.80)

Once again we just need to find the result of the sum from equations (2.80), which corresponds to a geometric series, because we have already seen that $\mathbb{E}e^X = \frac{f(-i)}{B_0}$ and $B_0 = f(0)$.

For $e^{\kappa - 2\pi j(\alpha+1)/\Delta}$:

$$a_1 = e^{\kappa - 2\pi(\alpha+1)/\Delta}, \qquad a_2 = e^{\kappa - 4\pi(\alpha+1)/\Delta}, \qquad a_3 = e^{\kappa - 6\pi(\alpha+1)/\Delta}$$

$$q = \frac{a_3}{a_1} = \frac{e^{\kappa - 6\pi(\alpha+1)/\Delta}}{e^{\kappa - 2\pi(\alpha+1)/\Delta}} = e^{-4\pi(\alpha+1)/\Delta} \ ,$$

(2.81)

$$S_\infty = \frac{e^{\kappa - 2\pi(\alpha+1)/\Delta}}{1 - e^{-4\pi(\alpha+1)/\Delta}} \ .$$

(2.82)

And for $e^{2\pi j\alpha/\Delta}$:

$$a_1 = e^{2\pi\alpha/\Delta}, \qquad a_2 = e^{4\pi\alpha/\Delta}, \qquad a_3 = e^{6\pi\alpha/\Delta}$$

$$q = \frac{a_3}{a_1} = \frac{e^{6\pi\alpha/\Delta}}{e^{2\pi\alpha/\Delta}} = e^{4\pi\alpha/\Delta} \ ,$$

(2.83)

$$S_\infty = \frac{e^{2\pi\alpha/\Delta}}{1 - e^{4\pi\alpha/\Delta}} \ .$$

(2.84)

Finally, it is really simple to see that if we replace equations (2.82) and (2.84) into equation (2.80), we have the result that we wanted to prove from equation (2.65), which is the equation of the sampling error for payoff $C_{G_1^*}$.

For the payoff $C_{G_1^{**}}$, the process is the same, but this time we will be analyzing the situation where $\alpha < -1$.

The upper bounds are now given by

$$C_{G_1^{**}}(\kappa - 2\pi j/\Delta) \leq \frac{B_0 \mathbb{E} e^{-qX}}{1+q} \left(\frac{q}{q+1}\right)^q e^{(q+1)(\kappa - 2\pi j/\Delta)},$$

(2.85)

and

$$C_{G_1^{**}}(\kappa + 2\pi j/\Delta) \leq B_0 e^{\kappa + 2\pi j/\Delta}$$

(2.86)

And if we replace both equations (2.85) and (2.86) into equation (2.52), we get

$$\begin{aligned} \left|C_{G_1^{**}}(\kappa) - \Sigma^{\infty}(\kappa)\right| &= \left|\sum_{j=1}^{\infty} (-1)^j \left[e^{-2\pi j\alpha/\Delta} \frac{B_0 \mathbb{E} e^{-qX}}{1+q} \left(\frac{q}{q+1}\right)^q e^{(q+1)(\kappa - 2\pi j/\Delta)} + e^{2\pi j\alpha/\Delta} B_0 e^{\kappa + 2\pi j/\Delta}\right]\right| \\ &\leq B_0 \sum_{\substack{j=1 \\ j\,odd}}^{\infty} \left[e^{\kappa + 2\pi j(\alpha+1)/\Delta} + e^{(q+1)\kappa} e^{-2\pi j(1+q+\alpha)/\Delta} \frac{\mathbb{E} e^{-qX}}{1+q} \left(\frac{q}{q+1}\right)^q\right]. \end{aligned}$$

(2.87)

Now we just have to determine the geometric series of both $e^{\kappa + 2\pi j(\alpha+1)/\Delta}$ and $e^{-2\pi j(\alpha+q+1)/\Delta}$ to find the final result.

For $e^{\kappa + 2\pi j(\alpha+1)/\Delta}$:

$$a_1 = e^{\kappa + 2\pi(\alpha+1)/\Delta}, \qquad a_2 = e^{\kappa + 4\pi(\alpha+1)/\Delta}, \qquad a_3 = e^{\kappa + 6\pi(\alpha+1)/\Delta}$$

$$q = \frac{a_3}{a_1} = \frac{e^{\kappa+6\pi(\alpha+1)/\Delta}}{e^{\kappa+2\pi(\alpha+1)/\Delta}} = e^{4\pi(\alpha+1)/\Delta} \ ,$$

(2.88)

$$S_\infty = \frac{e^{\kappa+2\pi(\alpha+1)/\Delta}}{1 - e^{4\pi(\alpha+1)/\Delta}} \ .$$

(2.89)

And for $e^{-2\pi j(\alpha+q+1)/\Delta}$:

$a_1 = e^{-2\pi(\alpha+q+1)/\Delta}$, $a_2 = e^{-4\pi(\alpha+q+1)/\Delta}$, $a_3 = e^{-6\pi(\alpha+q+1)/\Delta}$

$$q = \frac{a_3}{a_1} = \frac{e^{-6\pi(\alpha+q+1)/\Delta}}{e^{-2\pi(\alpha+q+1)/\Delta}} = e^{-4\pi(\alpha+q+1)/\Delta} \ ,$$

(2.90)

$$S_\infty = \frac{e^{-2\pi(\alpha+q+1)/\Delta}}{1 - e^{-4\pi(\alpha+q+1)/\Delta}} \ .$$

(2.91)

The sampling error equation is now obvious, as by applying equations (2.89) and (2.91) in (2.87), we get the result we wanted to prove from equation (2.66).

■

We have proved the sampling error equations for positive α, and for negative α, but we have not talked yet about the sampling error equations when α is zero.

In this specific case ($\alpha = 0, \alpha = -1$) we are going to bound the sampling error along integration contours that intercept a pole.

Proposition 7:

For $\alpha = -1$ and $\alpha = 0$, sampling errors are bounded by

$$\left|C_{G_1} - \textstyle\sum_{-1,G_1}^{\infty}\right| \leq max\left[\underset{\substack{q>0:\\ -q\in A_X}}{inf}\ \frac{e^{(q+1)\kappa}f(iq)}{(q+1)e^{2\pi q/\Delta}}\left(\frac{q}{q+1}\right)^q, e^{-2\pi/\Delta}f(-i)\right], \tag{2.92}$$

and

$$\left|C_{G_1} - \textstyle\sum_{0,G_1}^{\infty}\right| \leq max\left[e^{\kappa-2\pi/\Delta}f(0), \underset{\substack{p>0:\\ p+1\in A_X}}{inf}\ \frac{f\big(-i(p+1)\big)}{(p+1)e^{p(\kappa+2\pi/\Delta)}}\left(\frac{p}{p+1}\right)^p\right]. \tag{2.93}$$

Proof:

After normalization, the option price function can be interpreted as a cumulative distribution function, which allow us to directly apply bounds from probability literature.

First of all, what we need to show is that for $\alpha = -1$, there will be a probability space $(\Omega_1, P_1, \mathcal{F})$ where there exists a real valued random variable, Y, with density function $\varphi(y) \coloneqq e^{-y}\left[e^X \mathrm{I}_{(x<y)}\right]$. Note that Lee (2004) [1] states that $\varphi(y) \coloneqq e^{-y}\left[e^X \mathrm{I}_{(x>y)}\right]$, which we do not agree and will now show why.

Starting by the payoff function, we have:

$$\begin{aligned}
&C_{G^{**}}(\kappa) = B_0\mathbb{E}\big(G^{**}(X,\kappa)\big) \\
&= B_0\mathbb{E}((e^{\kappa} - e^{x})^{+}) \\
&= B_0\int (e^{\kappa} - e^{x})^{+}\, pdf(x)dx \ ,
\end{aligned} \tag{2.94}$$

where pdf(x) denotes the probability density function of x.

We can see that we will need to have $\mathrm{I}_{(x<y)}$ in the definition of the density function, and not the other way around, because

$$\begin{aligned}
&C_{G^{**}}(\kappa) = f(0)e^{\kappa}P_1(Y < \kappa) \\
&= B_0e^{\kappa}\mathbb{E}\left[\int_{-\infty}^{\kappa} e^{-y}e^{x}\mathrm{I}_{(y>x)}\, dy\right] \\
&= B_0e^{\kappa}\mathbb{E}\left[\int_{x}^{\kappa} e^{-y}e^{x}\, dy\right] \\
&= B_0e^{\kappa}\mathbb{E}\big(e^{x}\mathrm{I}_{(x<\kappa)}[-e^{-y}]_{x}^{\kappa}\big) \\
&= -B_0e^{\kappa}\mathbb{E}\big(e^{x}\mathrm{I}_{(x<\kappa)}[e^{-\kappa} - e^{-x}]\big) \\
&= -B_0\mathbb{E}\big(\mathrm{I}_{(x<\kappa)}[e^{x} - e^{\kappa}]\big) \\
&= B_0\mathbb{E}[(e^{\kappa} - e^{x})^{+}] \ .
\end{aligned} \tag{2.95}$$

As we are talking about a probability density function, we know that its integration along the whole domain needs to be 1, and that is just possible with $\mathrm{I}_{(x<y)}$:

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{-y}e^{x}\,\mathrm{I}_{(y>x)}dy \\
&= e^{x}\int_{x}^{\infty} e^{-y}\, dy \\
&= e^{x}[-e^{-y}]_{x}^{\infty}
\end{aligned}$$

$$= e^{x}[-0 + e^{-x}] = 1 \ .$$

(2.96)

Continuing with the proof of the sampling error equation for $\alpha = -1$, equation (2.95) shows that

$$C_{G_1^{**}}(\kappa) = f(0)e^{\kappa}P_1(Y < \kappa) \ .$$

(2.97)

Moreover Y has P_1-characteristic function $f(u)/[f(0)(1 - iu)]$, because

$$\begin{aligned} \mathbb{E}(e^{iuy}) &= \mathbb{E}\left(\int_{-\infty}^{\infty} e^{iuy} e^{-y} e^{x}\, \mathrm{I}_{(y>x)} dy\right) \\ &= \mathbb{E}\left(\int_{x}^{\infty} e^{(iu-1)y} e^{x}\, dy\right) \\ &= \mathbb{E}\left(e^{x}\left[\frac{e^{(iu-1)y}}{iu-1}\right]_{x}^{\infty}\right) \\ &= \mathbb{E}\left(e^{x}\,\frac{-e^{(iu-1)x}}{iu-1}\right) \\ &= \mathbb{E}\left(\frac{e^{iux}}{1-iu}\right) , \end{aligned}$$

(2.98)

and

$$f(u) = B_0\mathbb{E}_{\mathbb{Q}}(e^{iux_T}) \ .$$

(2.99)

Combining equations (2.98) and (2.99),

$$\begin{aligned}\mathbb{E}(e^{iuy}) &= \frac{\mathbb{E}(e^{iux})}{1-iu} \\ &= \frac{f(u)/B_0}{1-iu} \\ &= \frac{f(u)}{B_0(1-iu)}\ .\end{aligned}$$

(2.100)

To get the sampling error equation, we just need to apply the Gil-Pelaez (1951) [25] formula which will lead us to

$$\begin{aligned}C_{G_1^{**}}(\kappa) &= f(0)e^{\kappa}\left(\frac{1}{2}-\frac{1}{\pi}\int_0^{\infty}\Re\left[\frac{f(u)/f(0)}{iu(1-iu)}e^{-i\kappa u}\right]du\right) \\ &= \frac{f(0)e^{\kappa}}{2}+\frac{1}{\pi}\int_{0+i}^{\infty+i}\Re\left[\frac{f(z-i)}{iz(1+iz)}e^{-i\kappa z}\right]dz\ .\end{aligned}$$

(2.101)

By the application of the bounds used by Davies (1973) [18] to set a maximum error, we can get our sampling error bound:

$$\left|C_{G_1}-\Sigma_{-1,G_1}^{\infty}\right| \leq f(0)e^{\kappa}max\big(P_1(Y<\kappa-2\pi/\Delta),P_1(Y>\kappa+2\pi/\Delta)\big)\ .$$

(2.102)

If we consider equation (2.97), for an argument of $\kappa-2\pi/\Delta$ we have

$$\begin{aligned}&C_{G_1^{**}}(\kappa-2\pi/\Delta)=f(0)e^{\kappa-2\pi/\Delta}P_1(Y<\kappa-2\pi/\Delta) \\ &\Leftrightarrow P_1(Y<\kappa-2\pi/\Delta)=\frac{C_{G_1^{**}}(\kappa-2\pi/\Delta)}{f(0)e^{\kappa-2\pi/\Delta}}\ .\end{aligned}$$

(2.103)

Now we can apply the bound for the price equation $C_{G_1^{**}}$, which is given by the first bound from equation (2.77). Replacing it into equation (2.103) for $\kappa - 2\pi/\Delta$, we get

$$P_1(Y < \kappa - 2\pi/\Delta) = \frac{\dfrac{B_0 \mathbb{E} e^{-qX}}{1+q}\left(\dfrac{q}{q+1}\right)^q e^{(q+1)(\kappa - 2\pi/\Delta)}}{f(0)e^{\kappa - 2\pi/\Delta}}$$

$$= \frac{\dfrac{B_0 \mathbb{E} e^{-qX}}{1+q}\left(\dfrac{q}{q+1}\right)^q e^{(q+1)\kappa} e^{-(q+1)2\pi/\Delta}}{f(0)e^{\kappa - 2\pi/\Delta}} \ .$$

(2.104)

Multiplying equation (2.104) by $f(0)e^{\kappa}$,

$$f(0)e^{\kappa} P_1(Y < \kappa - 2\pi/\Delta) = \frac{\dfrac{B_0 \mathbb{E} e^{-qX}}{1+q}\left(\dfrac{q}{q+1}\right)^q e^{(q+1)\kappa} e^{-(q+1)2\pi/\Delta}}{f(0)e^{\kappa - 2\pi/\Delta}}$$

$$= \frac{B_0 \mathbb{E} e^{-qX}}{1+q}\left(\frac{q}{q+1}\right)^q e^{(q+1)\kappa} e^{-(q+1)2\pi/\Delta} e^{2\pi/\Delta}$$

$$= \frac{B_0 \mathbb{E} e^{-qX}}{1+q}\left(\frac{q}{q+1}\right)^q e^{(q+1)\kappa} e^{-2\pi q/\Delta} \ .$$

(2.105)

So, for any $q > 0$,

$$\left| C_{G_1} - \Sigma_{-1,G_1}^{\infty} \right| \leq max\left(\frac{B_0 \mathbb{E} e^{(q+1)\kappa} e^{-qX}}{(q+1)e^{2\pi q/\Delta}} \left(\frac{q}{q+1}\right)^q , e^{-2\pi/\Delta} B_0 e^{X} \right) ,$$

(2.106)

as claimed.

For $\alpha = 0$ the process will be the same. We know that there is a probability space $(\Omega_1, P_1, \mathcal{F})$ where there is a real valued random variable, Y, with density function $\varphi(y) \coloneqq e^y P_{B(x>y)} f(0)/f(-i)$.

Following the same reasoning as we did for $\alpha = -1$, it is then easy to verify that

$$C_{G_1}(\kappa) = f(-i)P_1(Y > \kappa) \ , \tag{2.107}$$

and that Y has P_1-characteristic function $f(u-i)/[f(-i)(1+iu)]$. Again by the application of Gil-Pelaez (1951) formula, we get

$$C_{G_1}(\kappa) = f(-i)\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left[\frac{f(z-i)/f(-i)}{iz(1+iz)} e^{-i\kappa z}\right] dz\right) \ , \tag{2.108}$$

and according to Davies (1973) [18],

$$\left|C_{G_1} - \textstyle\sum_{0,G_1}^{\infty}\right| \leq f(-i) max[P_1(Y < \kappa - 2\pi/\Delta), P_1(Y > \kappa + 2\pi/\Delta)] \ . \tag{2.109}$$

If we consider equation (2.107), for an argument of $\kappa + 2\pi/\Delta$ we have

$$\begin{gathered} C_{G_1}(\kappa + 2\pi/\Delta) = f(-i)P_1(Y > \kappa + 2\pi/\Delta) \\ \Leftrightarrow P_1(Y > \kappa + 2\pi/\Delta) = \frac{C_{G_1}(\kappa + 2\pi/\Delta)}{f(-i)} \ . \end{gathered} \tag{2.110}$$

Now we can apply the bound for payoff C_{G_1} , which is given by equation (2.7), and by replacing it into equation (2.110) for $\kappa + 2\pi/\Delta$, we get

$$P_1(Y > \kappa + 2\pi/\Delta) = \frac{\frac{B_0 \mathbb{E}\exp\left((p+1)X\right)}{(p+1)exp\left(p(\kappa + 2\pi/\Delta)\right)}\left(\frac{p}{p+1}\right)^p}{f(-i)} \ .$$

(2.111)

Multiplying equation (2.111) by $f(-i)$,

$$f(-i)P_1(Y > \kappa + 2\pi/\Delta) = \frac{\frac{B_0 \mathbb{E}\exp\left((p+1)X\right)}{(p+1)exp\left(p(\kappa + 2\pi/\Delta)\right)}\left(\frac{p}{p+1}\right)^p}{f(-i)}$$
$$= \frac{B_0 \mathbb{E}\exp\left((p+1)X\right)}{(p+1)exp\left(p(\kappa + 2\pi/\Delta)\right)}\left(\frac{p}{p+1}\right)^p \ .$$

(2.112)

So for any $p > 0$,

$$\left|C_{G_1} - \Sigma_{0,G_1}^{\infty}\right| \leq max\left(f(0)e^{\kappa - 2\pi/\Delta}, \frac{B_0 \mathbb{E}e^{(p+1)X}}{(p+1)e^{p(\kappa + 2\pi/\Delta)}}\left(\frac{p}{p+1}\right)^p\right) ,$$

(2.113)

as claimed.

3. Applications under the Heston (1993) Model

3.1 The Heston (1993) model

The Heston model offers a closed-form solution for pricing European-style options that seeks to overcome the shortcomings in the Black-Scholes option pricing model related to return skewness and strike-price bias.

The Heston model (1993) [27] assumes that the underlying price process (S_t) is governed by the SDE, given by equation (3.1) under the Risk-Neutral measure:

$$\frac{dS_t}{S_t} = (r-q)dt + \sqrt{v_t}dW_1^{\mathbb{Q}}(t) \ , \tag{3.1}$$

where $dW_1^{\mathbb{Q}}(t)$ is a standard Wiener increment.

The risk-neutral dynamics of the instantaneous variance is given by the square-root process

$$dv_t = k(\theta - v_t)dt + \sigma\sqrt{v_t}dW_2^{\mathbb{Q}}(t) \ , \tag{3.2}$$

Where $dW_2^{\mathbb{Q}}(t)$ is another standard Brownian motion such that

$$d\langle W_1^{\mathbb{Q}}, W_2^{\mathbb{Q}}\rangle_t = \rho dt \ , \tag{3.3}$$

for $\rho \in [-1, 1]$.

To price European-style options under the Heston (1993) model and through Proposition 3 and Heston characteristic function, it is necessary to define the characteristic function of log-spot price. Following Lee (2004),

$$f(\xi) = exp[-rT + i\xi(\log S_0 + rT) + C(\xi) + D(\xi)V_0] \ , \tag{3.4}$$

with,

$$C(\xi) := \frac{\kappa\theta}{\sigma^2}\left[(\kappa - \rho\sigma\xi i + d)T - 2\log\left(\frac{1 - ge^{dT}}{1 - g}\right)\right] \ , \tag{3.5}$$

$$D(\xi) := \frac{\kappa - \rho\sigma\xi i + d}{\sigma^2}\left(\frac{1 - e^{dT}}{1 - ge^{dT}}\right) \ , \tag{3.6}$$

$$g \coloneqq g(\xi) \coloneqq \frac{\kappa - \rho\sigma\xi i + d}{\kappa - \rho\sigma\xi i - d} \ , \tag{3.7}$$

and,

$$d \coloneqq d(\xi) \coloneqq \sqrt{(\rho\sigma\xi i - \kappa)^2 + \sigma^2(\xi i + \xi^2)} \ . \tag{3.8}$$

3.2 Matlab Simulation

Now that we have already presented all the theoretical concepts needed for the calculation of European-style call and put options, we will apply the equations from the previous sections by using our Matlab code, and defining some call options with different maturities.

The code used to obtain the following results can be seen in Appendix C.

T = 2					
Strike:	80	90	100	110	120
Call Price:	28.2845	20.6761	13.9701	8.4982	4.5485
Error:	2.1630e-63	2.4334e-63	2.7038e-63	2.9742e-63	3.2445e-63
Optimal α:	-1	-1	-1	-1	-1

T = 4					
Strike:	80	90	100	110	120
Call Price:	34.9839	28.2138	22.0755	16.6812	12.1222
Error:	1.4712e-68	1.1277e-68	8.8901e-69	7.1692e-69	5.8907e-69
Optimal α:	1.2574	1.2574	1.2574	1.2574	1.2574

T = 6					
Strike:	80	90	100	110	120
Call Price:	40.7098	34.5654	28.9059	23.7816	19.2298
Error:	2.0499e-68	1.5895e-68	1.2531e-68	1.0105e-68	8.3031e-69
Optimal α:	0	1.2574	1.2574	1.2574	1.2574

T = 8					
Strike:	80	90	100	110	120
Call Price:	45.7495	40.1240	34.8877	30.0656	25.0749
Error:	1.8923e-68	2.1289e-68	1.7662e-68	1.4243e-68	1.1703e-68
Optimal α:	0	0	1.2574	1.2574	1.2574

Table 3.1 – Simulation results for different European-style call options

For this simulation we have used an underlying with $S_0 = 100$, and we have set the Heston parameters in the following way: $\kappa = 1.15$, $\theta = 0.034782609$, $\sigma = 0.39$ and $\rho = -0.64$.

Optimal α was obtained by the minimization of the sampling and truncation errors, and by the application of Proposition 3 and Heston characteristic function.

4. Conclusions

This Master thesis allowed us to analyze and understand all the possibilities of the implemented methodology for option pricing.

We were able to see that the Fourier transform method for option pricing applies on different types of options, and not just European-style options, such as calls and puts. The use of this method within different types of option classes just requires the correct definition of the parameters and the implementation of the subsequent changes in the pricing equations.

In terms of the computational results that we obtained by the implementation of this method using MATLAB, they were very good. And as we can see from the results of Table 3.1, we were able to see that all the results were exactly the same as if calculated using another option pricing method.

We have also noticed that the optimal values for the α parameter, the ones that minimize the truncation and sampling errors, have not changed that much for different strikes and for different times to maturity. In the specific cases that we presented in subsection 3.3, we just got two different values for the alpha parameter (0 and 1.2574).

The speed of the calculation was high, but it could be even higher depending on the size of the grid used in the calculation, while guaranteeing a strong precision in the values obtained.

Finally, we would suggest the possibility of a future work, with the implementation of the same method, but using other types of options, such as American-style options or Asian options.

5. Bibliography

[1] Lee, R. (2004)Option pricing by transform methods: extensions, unification and error control.

[2] Bakshi, G., Cao, C., and Chen Z. (1997). Empirical performance of alternative option pricing models. Journal of Finance 52, 2003–49.

[3] Bakshi, G., and Madan, D. (2000). Spanning and derivative-security valuation. Journal of Financial Economics 55, 205–38.

[4] Barndorff-Nielsen, O. (1998). Processes of normal inverse Gaussian type. Finance and Stochastics 2, 41–68.

[5] Barndorff-Nielsen, O., Nicolato, E., and Shephard, N. (2002). Some recent developments in stochastic volatility modelling. Quantitative Finance 2, 11–23.

[6] Bates, D. (1996). Jumps and stochastic volatility: Exchange rate processes implicit in Deutsche mark options. Review of Financial Studies 9(1), 69–107.

[7] Bates, D. (2000). Post-’87 crash fears in the S&P500 futures option market. Journal of Econometrics 94, 181–238.

[8] Boyarchenko, S. I., and Levendorski, S. Z. (2002). Non-Gaussian Merton–Black–Scholes Theory. World Scientific.

[9] Broadie, M., Cvitanic, J., and Soner, H. M. (1998). Optimal replication of contingent claims under portfolio constraints. Review of Financial Studies 11(1), 59–79.

[10] Carr, P., Geman, H., Madan, D., and Yor, M. (2002). The fine structure of asset returns: An empirical investigation. Journal of Business 75(2), 305–32.

[11] Carr, P., Geman, H., Madan, D., and Yor, M. (2003). Stochastic volatility for Lévy processes. Mathematical Finance 13(3), 345–82.

[12] Carr, P., and Madan, D. (1999). Option valuation using the fast Fourier transform. Journal of Computational Finance 3, 463–520.

[13] Carr, P., and Schröder, M. (2003). Bessel processes, the integral of geometric Brownian motion, and Asian options. Theory of Probability and Its Applications.

[14] Carr, P., and Wu, L. (2002). Time-changed Lévy processes and option pricing. Journal of Financial Economics.

[15] Carr, P., and Wu, L. (2003). The finite moment log stable process and option pricing. Journal of Finance 58(2), 753–77.

[16] Chacko, G., and Das, S. (2002). Pricing interest rate derivatives: A general approach. Review of Financial Studies 15(1), 195–241.

[17] Champeney, D. (1987). A Handbook of Fourier Theorems. Cambridge University Press.

[18] Davies, R. (1973). Numerical inversion of a characteristic function. Biometrika 60(2), 415–7.

[19] Delbaen, F. and W. Schachermayer (1994). A general version of the fundamental theorem of asset pricing. Mathematische Annalen 2(4), 61–73.

[20] Duffie, D., Pan, J., and Singleton, K. (2000). Transform analysis and option pricing for affine jump-diffusions. Econometrica 68(6), 1343–76.

[21] Eberlein, E., and Prause, K. (2002). The generalized hyperbolic model: Financial derivatives and risk measures. In H. Geman, D. Madan, S. Pliska, and T. Vorst (eds), Mathematical Finance – Bachelier Congress 2000, pp. 245–67. Springer.

[22] El Karoui, N., Geman, H., and Rochet, J.-C. (1995). Changes of numeraire, changes of probability measure, and option pricing. Journal of Applied Probability 32, 443–58.

[23] Fu, M., Madan, D., and Wang T. (1999). Pricing continuous Asian options: A comparison of Monte Carlo and Laplace transform inversion methods. Journal of Computational Finance 2(2), 49–74.

[24] Geman, H. and Yor, M. (1996). Pricing and hedging double-barrier options: A probabilistic approach. Mathematical Finance 6(4), 365–78.

[25] Gil-Pelaez, J. (1951). Note on the inversion theorem. Biometrika 38(3/4), 481–2.

[26] Harrison, J. M., and Kreps, D. (1979). Martingales and arbitrage in multiperiod securities markets. Journal of Economic Theory 20, 381–408.

[27] Heston, S. L. (1993). A closed-form solution for options with stochastic volatility and applications to bond and currency options. Review of Financial Studies 6(2), 327–43.

[28] Lee, R. (2003). The moment formula for implied volatility at extreme strikes. Mathematical Finance.

[29] Lewis, A. (2000). Option Valuation under Stochastic Volatility. Finance Press.

[30] Lewis, A. (2001). A simple option formula for general jump-diffusion and other exponential Lévy processes. Envision Financial Systems and OptionCity.net.

[31] Madan, D., Carr, P., and Chang, E. (1998). The variance gamma process and option pricing. European Finance Review 2, 79–105.

[32] Schöbel, R., and Zhu, J. (1999). Stochastic volatility with an Ornstein–Uhlenbeck process: An extension. European Finance Review 3, 23–46.

[33] Zemanian, A. (1966). The distributional Laplace and Mellin transformations. SIAM Journal on Applied Mathematics 14(1), 41–59.

[34] Zhu, J. (2000). Modular Pricing of Options. Springer.

Appendix A

Estimating the Sum of a Series

When we know that a series is convergent, we can find the approximation to the sum S of the series. For that purpose, we first need to estimate the size of the remainder.

The remainder R_n is going to be represented by the difference between the real value of the sum of a series, and an approximation to S, which is going to be given by S_n.

The reminder is then given by:

$$\begin{aligned} R_n &= \lim_{n\to\infty} S_n - S_n \\ &= S - S_n \\ &= a_{n+1} + a_{n+2} + a_{n+3} + \cdots \end{aligned} \tag{A.1}$$

To understand how we can get the bounds that will approximate the value of the remainder, we need to think about the principle behind the *Integral Test*, which states that:

> If we have a positive, continuous, and decreasing function f on the interval [1, ∞[, let $a_n = f(n)$. The series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. This leads us to:
>
> - If $\int_1^{\infty} f(x)\,dx$ is convergent, $\sum_{n=1}^{\infty} a_n$ is convergent;
> - If $\int_1^{\infty} f(x)\,dx$ is divergent, $\sum_{n=1}^{\infty} a_n$ is divergent.

In the specific case of the value of the remainder that we want to estimate, we use the same idea from the *Integral Test*, but this time the interval considered will be [n, ∞[.

If we now try to determine the area below a positive, continuous, and decreasing function $f(x)$, by using rectangles we are able to see that:

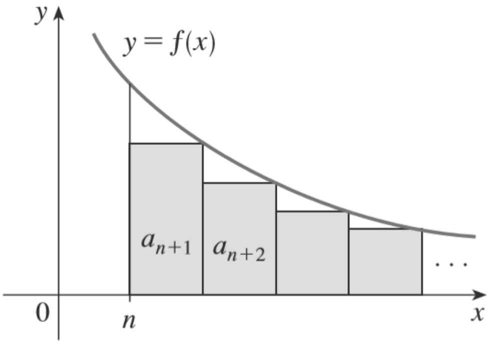

Figure A.1

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$
$$\leq \int_n^\infty f(x)\,dx$$

(A.2)

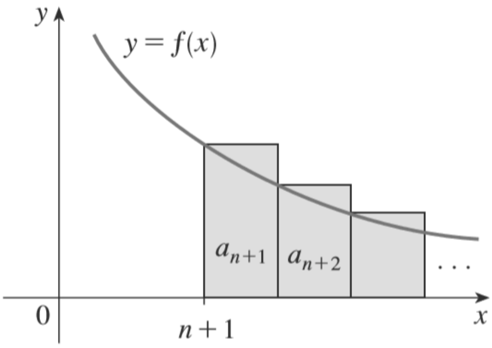

Figure A.2

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$
$$\geq \int_{n+1}^\infty f(x)\,dx$$

(A.3)

Finally we are able to conclude that if we have a function $f(n) = a_n$, which is positive, continuous, decreasing for $x \geq n$ and $\sum_{n=1}^{\infty} a_n$ is convergent, we will have the reminder limited by:

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

(A.4)

Appendix B

Improper integrals

Let a and b be arbitrary real numbers, and consider the following integrals

$$\int_{-\infty}^{b} f(x)\,dx \qquad \text{and} \qquad \int_{a}^{\infty} f(x)\,dx$$

(B.1)

where in each case f is continuous in the interval of integration and at its finite endpoint. These are called improper integrals, because the interval of integration is infinite.

Integral $\int_a^\infty f(x)\,dx$ is convergent if $\lim_{b\to\infty}\int_a^b f(x)\,dx$ exists as a finite number. Similarly, $\int_{-\infty}^b f(x)\,dx$ is convergent if $\lim_{a\to-\infty}\int_a^b f(x)\,dx$ exists as a finite number.

Now let $f(x)$ be continuous on the real line. The integral $\int_{-\infty}^\infty f(x)\,dx$ is also improper since the interval of integration is infinite, but here it is infinite in both the positive and negative direction. Such an integral is said to be convergent if both $\int_0^\infty f(x)\,dx$ and $\int_{-\infty}^0 f(x)\,dx$ are convergent.

If this applies, we define the Cauchy principal value of the integral $\int_{-\infty}^\infty f(x)\,dx$ to be

$$P.V. = \int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to\infty}\int_{-a}^{a} f(x)\,dx$$

(B.2)

The Cauchy principal value of an integral may exist even though the integral itself is not convergent. However, whenever $\int_{-\infty}^\infty f(x)\,dx$ is convergent, $P.V. = \int_{-\infty}^\infty f(x)\,dx$ exists, and the two integrals will be the same. This is because both $\lim_{a\to\infty}\int_{-a}^0 f(x)\,dx$ and $\lim_{a\to\infty}\int_0^a f(x)\,dx$ exist, so

$$
\begin{aligned}
P.V. &= \lim_{a\to\infty}\int_{-a}^{a} f(x)\,dx \\
&= \lim_{a\to\infty}\left(\int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx\right) \\
&= \lim_{a\to\infty}\int_{-a}^{0} f(x)\,dx \;+\; \lim_{a\to\infty}\int_{0}^{a} f(x)\,dx \\
&= \int_{-\infty}^{\infty} f(x)\,dx
\end{aligned}
\tag{B.3}
$$

Because of this fact, we can compute a convergent integral over the real line by computing its principal value, which can often be obtained by use of complex methods and the residue theorem.

Contours

Consider the contour integral of a complex rational function $f(z)$ taken around the special contour illustrated in the Figure B.1. Assume that the rational function $f(z)$ has poles at the points $z1$, $z2$,..., zn lying in the upper half plane. The contour of integration consists of two parts. The first part is the upper semicircle C_R centered at the origin and having a radius R large enough to enclose all the poles of $f(z)$ in the upper half plane. The second part of the contour is the straight line segment from $(-R, 0)$ to $(R, 0)$. The integration of $f(z)$ around this special contour C is performed in the positive sense. The semicircular part of the path of integration is given the notation C_R to emphasize that this portion of the path depends upon the radius R.

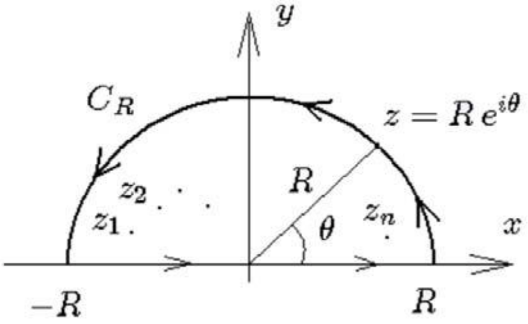

Figure B.1 – Contour path C consisting of semicircle C_R and straight line path from $-R$ to R on the x-axis. It is assumed that $f(z)$ has poles at points $z1$, $z2$,..., zn in the upper half-plane which lie inside the contour C.

Consider the contour integral of $f(z)$ around the path C of Figure B.1 in the limit as the radius R increases without bound. This type of integral can be evaluated using the residue theorem by writing

$$\lim_{R\to\infty} \oint f(z)dz = \lim_{R\to\infty} \left(\int_{-R}^{R} f(x)\, dx + \int_{C_R} f\left(Re^{i\theta}\right) Re^{i\theta} i\, d\theta \right) = 2\pi i \sum_{j=1}^{n} Res\left[f, z_j\right]$$

(B.4)

where $z = x$ is the value of z on the line segment and $z = Re^{i\theta}$ is the value of z on the semicircular path C_R. If we can show that the line integral along C_R approaches zero as R increases without bound, then one can write

$$\lim_{R\to\infty} \int_{C_R} f(z)\, dz = \lim_{R\to\infty} \int_{0}^{\pi} f\left(Re^{i\theta}\right) Re^{i\theta} i\, d\theta = 0$$

(B.5)

and consequently the integral in equation (B.4) reduces to the Cauchy principal value of an improper integral

$$\lim_{R\to\infty} \int_{-R}^{R} f(z)\, dz = \int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum_{j=1}^{n} Res\left[f, z_j\right]$$

(B.6)

If the integrand $f(z)$, for $z = Re^{i\theta}$ on the semicircle C_R, satisfies $|f(z)| \leq \frac{M}{R^k}$ where M and k are constants with $k > 1$, then integrals of the type given by equations (B.5) and (B.6) can be shown to be valid. If $f(z)$ satisfies the above boundedness property, then one can employ the magnitude of a line integral property, to obtain the result

$$\left|\int_{C_R} f(z)\,dz\right| \leq \frac{M}{R^k}\pi R = \frac{\pi M}{R^{k-1}} \ , \qquad k > 1$$

(B.7)

where πR is the lenght of the semicircle arc C_R. It then follows that $\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0$, like we have seen in equation (B.5).

Whenever $f(z) = \frac{P(z)}{Q(z)}$ is a rational function with $P(z)$ a polynomial of degree m with real coefficients $a_0, a_1, \dots, a_m$ and $Q(z)$ is a polynomial of degree $n \geq m + 2$ with real coefficients $b_0, b_1, \dots, b_m$, then we can write

$$\frac{P(z)}{Q(z)} = \frac{z^m p(z)}{z^n q(z)}$$

where,

$$p(z) = a_0, a_1 z^{-1}, \dots, a_m z^{-m} \qquad \text{and} \qquad q(z) = b_0, b_1 z^{-1}, \dots, b_n z^{-n}$$

In the special case $z = Re^{i\theta}$ lies on the path C_R in figure B.1 we find that

$$|zf(z)| = \left|\frac{zP(z)}{Q(z)}\right| \leq \left|\frac{R^{m+1}}{R^n}\right| \frac{|p(z)|}{|q(z)|}$$

and in the limit as R increases without bound we can write

$$\lim_{R\to\infty}\left|\frac{R^{m+1}}{R^n}\right|\frac{|p(z)|}{|q(z)|} \leq \lim_{R\to\infty}\frac{R^{m+1}}{R^n}\frac{a_0}{b_0} = 0$$

because $n \ \geq \ m \ + \ 2$. Therefore, one can select a radius R so large that $\left|\frac{zP(z)}{Q(z)}\right| \leq \epsilon$ which implies the inequality $\left|\frac{P(z)}{Q(z)}\right| < \frac{\epsilon}{R}$ whenever $z = Re^{i\theta}$ is on the semicircle C_R. It then follows from the magnitude of an integral property, that

$$\left|\int_{C_R}\frac{P(z)}{Q(z)}dz\right| \le \left|\int_{C_R}\frac{\epsilon}{R}dz\right| = \frac{\epsilon}{R}\pi R = \pi\epsilon$$

(B.8)

In the limit, as R increases without bound, the integral on the left-hand side of equation (B.8) is some constant which is less than $\pi\epsilon$ which can be made arbitrarily small by making ϵ small. This implies that the constant on the left-hand side must be zero. Hence, for $P(z)$ and $Q(z)$ polynomials of degree m and n respectively, with $n \ge m + 2$, one can write

$$\lim_{R\to\infty}\int_{C_R}\frac{P(z)}{Q(z)}dz = 0$$

Appendix C

Matlab code

```
function [OtimeSpent,MMHestonEcall] =
ErfwscallHestonDFT2(VecContractNr,VecSpot,VecPerStrike,Vecr,Vecq,VecV,
Veckv,Vecthetav,Vecsigmav,Vecrho,Vectau,Vectstar,VecNN,Vecalpha)


%European standard calls under Heston model

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%
%clc
%clear

%Main program
tic

InputMatrix =
[VecContractNr,VecSpot,VecPerStrike,Vecr,Vecq,VecV,Veckv,Vecthetav,Vec
sigmav,Vecrho,Vectau,Vectstar,VecNN,Vecalpha];


OERFWScall = zeros(size(InputMatrix,1),1);

Oexitflag = zeros(size(InputMatrix,1),1);

OAlphas = zeros(size(InputMatrix,1),1);

MinfAlpha = zeros(size(InputMatrix,1),1);

Odeltas = zeros(size(InputMatrix,1),1);


for jcount = 1:1:size(InputMatrix,1)

    OneContractNr = InputMatrix(jcount,1);    %Contract Number
    OneSpot = InputMatrix(jcount,2);          % Inicial Asset Price.
    OnePerStrike = InputMatrix(jcount,3);           % Percentage
Strike Price.
    Oner = InputMatrix(jcount,4);         %Initial short-term interest
rate.
    Oneq = InputMatrix(jcount,5);         % Dividend Yield
    OneV = InputMatrix(jcount,6);         % Variance of asset returns
- 2nd state variable
    Onekv = InputMatrix(jcount,7);         % velocity of mean
reversion kv for dvt
    Onethetav = InputMatrix(jcount,8);         % long-term mean thetav
of vt
    Onesigmav = InputMatrix(jcount,9);         % Annualized volatility
sigmav of vt
    Onerho = InputMatrix(jcount,10);         % linear correlation
coefficient between St and vt
```

```
    Onetau = InputMatrix(jcount,11);           % Time to Maturity of
the option.
    OneNN = InputMatrix(jcount,13);    %NN is number of log-stikes and
u-steps; must be an integer power of 2.
    OneAlpha = InputMatrix(jcount,14);   %one alpha.

    %Optimization of alpha; no restrictions on alpha: alpha can be any
real
    %number.

    %Computation of qbarr and pbarr via Lee 2004
    [OnealphaMin, OnealphaMax] =
AlphaRange(Onekv,Onesigmav,Onerho,Onetau,OneAlpha);

    OAlphaMin(jcount,1) = OnealphaMin;

    OAlphaMax(jcount,1) = OnealphaMax;


    [Optimalalpha, Optimaldelta,fval,exitflag] =
OptimalAlphaDelta(log(OneSpot),OnePerStrike,Oner,Oneq,Onekv,Onethetav,
Onesigmav,Onerho,Onetau,OneV,OneNN,OnealphaMin, OnealphaMax,OneAlpha);

    OAlphas(jcount,:) = Optimalalpha;

    Odeltas(jcount,:) = Optimaldelta;

    MinfAlpha(jcount,1) = fval;

    Oexitflag(jcount,1) = exitflag;


    OERFWScall(jcount,1) = Scallfft(log(OneSpot),OnePerStrike, Onekv,
Onethetav, Onesigmav, Onerho, Oneq, Oner,
Onetau,OneV,OneNN,OAlphas(jcount,:),Odeltas(jcount,:));


end                                                   %end jcount for


OContractNr = InputMatrix(:,1);

MMHestonEcall = [OContractNr OAlphaMin OAlphaMax MinfAlpha Oexitflag
OAlphas Odeltas OERFWScall];

toc

OtimeSpent = toc;

Val = OERFWScall

Al = OAlphas

end


%Subfunctions
```

```
%Computation of standard call

function [Cfunction, Dfunction] =
ChFunctionMarginalxQ(Intrate,divyield,kv,thetav,sigmav,rho,tau,phi)
      %ChFunctionMarginalx Computes complex-valued functions C and D
from
        %marginal characteristic function of xt under measure Q
        %Uses alternative and stable specification of Heston ch
function

        rhosigmaiphi = rho.*sigmav.*phi.*complex(0,1);

        dminus = -sqrt( ((rhosigmaiphi-kv).^2)
+(sigmav.^2).*phi.*(complex(0,1)+phi) );

        kminusplus = kv -rhosigmaiphi +dminus;

        qfunction = kminusplus ./ (kv -rhosigmaiphi -dminus);

        expdtau = exp(dminus.*tau);

        Oneminusqexpd = 1-qfunction.*expdtau;

        Dfunction = ( kminusplus .* (1-expdtau) ) ./ ( (sigmav.^2) .*
Oneminusqexpd  );

        Cfunction = (Intrate-divyield).*phi.*tau.*complex(0,1) +
(kv.*thetav./(sigmav.^2)) *...
            ( kminusplus.*tau -2*log( Oneminusqexpd./(1-qfunction) )
);
end

function y =
Chfunctionx(lnS0,Intrate,divyield,kv,thetav,sigmav,rho,TT,phix,vt)
      %Chfunctionx Computes the characteristic function of x=ln(ST)
under measure Q
        %i.e. EQ(i*phiz*x)
        %returns a complex number

        [OneCfunctionS, OneDfunctionS] =
ChFunctionMarginalxQ(Intrate,divyield,kv,thetav,sigmav,rho,TT,phix);

        y = exp(-
Intrate*TT+complex(0,1)*phix*lnS0+OneCfunctionS+OneDfunctionS*vt);

end

function y = QSI(
lnS0,Intrate,divyield,kv,thetav,sigmav,rho,TT,vt,alpha,u)



            y =
Chfunctionx(lnS0,Intrate,divyield,kv,thetav,sigmav,rho,TT, u-
complex(0,1).*(alpha+1),vt)./( alpha.*(alpha+1)-
(u.^2)+complex(0,1).*(2.*alpha+1).*u );

end
```

```
%For FFT

function y = CalcV2base( NN, logStrike,
lnS0,Intrate,divyield,kv,thetav,sigmav,rho,TT, vt, alpha,delta)

%NN is number of log-stikes and u-steps; must be an integer power of 2


        Sum = sum( exp(-complex(0,1)*delta*((1:NN)-0.5)*logStrike)
.*...

QSI(lnS0,Intrate,divyield,kv,thetav,sigmav,rho,TT,vt,alpha,((1:NN)-
0.5)*delta) );


        gammakmstar = delta * real( Sum );


        %compute Vj
        y = ( exp(-alpha*logStrike) *gammakmstar ) / pi;



end

function y = Scallfft(lnS0,Strike, kv, thetav, sigmav, rho, divyield,
Intrate, TT, vt,NN,alpha,delta)
     % Scallfft Computes European standard call under the HESTON model
     %and using FFT


        %residue


            if (alpha==0.0)
                Residue =
0.5*Chfunctionx(lnS0,Intrate,divyield,kv,thetav,sigmav,rho,TT,-
complex(0,1),vt);
            elseif ( (alpha<0.0) && (alpha>-1.0) )
                Residue =
Chfunctionx(lnS0,Intrate,divyield,kv,thetav,sigmav,rho,TT,-
complex(0,1),vt);
            elseif (alpha==-1.0)
                Residue =
Chfunctionx(lnS0,Intrate,divyield,kv,thetav,sigmav,rho,TT,-
complex(0,1),vt) -(0.5*Strike)*
Chfunctionx(lnS0,Intrate,divyield,kv,thetav,sigmav,rho,TT,0,vt);
            elseif (alpha<-1.0)
                Residue =
Chfunctionx(lnS0,Intrate,divyield,kv,thetav,sigmav,rho,TT,-
complex(0,1),vt) -(1.0*Strike)*
Chfunctionx(lnS0,Intrate,divyield,kv,thetav,sigmav,rho,TT, 0,vt);
            else
                Residue = 0.0;
            end
```

```
        OneIntegral = CalcV2base( NN, log(Strike),
lnS0,Intrate,divyield,kv,thetav,sigmav,rho, TT, vt, alpha,delta);


        y = Residue + OneIntegral;

end


%Optimal alpha

 function y = Aux22 (kv,sigmav,rho,TT,Onezeta)

    beta = kv -complex(0,1)*sigmav*rho*(-Onezeta*complex(0,1));

    Delta = sqrt( (beta^2) +(sigmav^2)*(-Onezeta*complex(0,1))*(-
Onezeta*complex(0,1)+complex(0,1)) );

    c = 0.5*abs(Delta);

    y = -TT +(log( (beta-c)/(beta+c) ) /c);

 end

 function y = Aux23 (kv,sigmav,rho,TT,Onezeta)

    beta = kv -complex(0,1)*sigmav*rho*(-Onezeta*complex(0,1));

    Delta = sqrt( (beta^2) +(sigmav^2)*(-Onezeta*complex(0,1))*(-
Onezeta*complex(0,1)+complex(0,1)) );

    c = 0.5*abs(Delta);

    if (beta>0.0)
        Ifpi = pi;
    else
        Ifpi = 0.0;
    end

    y = -TT +( 2.0* (Ifpi +atan(-c/beta) ) ) /c;

 end

 function [alphaMin, alphaMax] =
AlphaRange(kv,sigmav,rho,TT,TrialAlpha)

 %Computation of alphaMin and alphaMax via Lord-Kahl 2006


    optionszeta = optimset('Display','off','TolFun',1e-12,'TolX',1e-
12);

    %Compute initial values zetaDminus and zetaDplus

    undersqrt = ((sigmav-2*kv*rho)^2) +4*(1-(rho^2))*(kv^2);
```

```
    zetaDminus = ( sigmav-2*kv*rho -sqrt( undersqrt ) ) /
(2*sigmav*(1-(rho^2)));

    zetaDplus = ( sigmav-2*kv*rho +sqrt( undersqrt ) ) / (2*sigmav*(1-
(rho^2)));


    if (undersqrt<0.0)
        disp('zetaDminus and zetaDplus are complex...');
    end


    %zetaminus and alphaMin

        beta = kv -complex(0,1)*sigmav*rho*(-zetaDminus*complex(0,1));

        Deltasqr = (beta^2) +(sigmav^2)*(-zetaDminus*complex(0,1))*(-
zetaDminus*complex(0,1)+complex(0,1));


        if ( (Deltasqr>=0.0) && (beta>=0.0) )
            zetaminus = zetaDminus;
        end

        if ( (Deltasqr>=0.0) && (beta<0.0) )
                zetaminus = fsolve(@(x) Aux22 (kv,sigmav,rho,TT,x),
zetaDminus, optionszeta);
        end

        if (Deltasqr<0.0)
                zetaminus = fsolve(@(x) Aux23 (kv,sigmav,rho,TT,x),
zetaDminus, optionszeta);
        end



        if not(imag(zetaminus)==0.0)
            disp('zetaminus is complex...');
            zetaminus = TrialAlpha -1.0;
        end


        alphaMin = zetaminus -1.0;




    %zetaplus and alphaMax

        beta = kv -complex(0,1)*sigmav*rho*(-zetaDplus*complex(0,1));

        Deltasqr = (beta^2) +(sigmav^2)*(-zetaDplus*complex(0,1))*(-
zetaDplus*complex(0,1)+complex(0,1));
```

```
        if ( (Deltasqr>=0.0) && (beta>=0.0) )
            zetaplus = zetaDplus;
        end

        if ( (Deltasqr>=0.0) && (beta<0.0) )
                zetaplus = fsolve(@(x) Aux22 (kv,sigmav,rho,TT,x),
zetaDplus, optionszeta);
        end

        if (Deltasqr<0.0)
                zetaplus = fsolve(@(x) Aux23 (kv,sigmav,rho,TT,x),
zetaDplus, optionszeta);
        end



        if not(imag(zetaplus)==0.0)
            disp('zetaplus is complex...');
            zetaplus = TrialAlpha +1.0;
        end


        alphaMax = zetaplus -1.0;






 end

 function y =
TruncationError(PerStrike,Intrate,divyield,kv,thetav,sigmav,rho,TT,lnS
0,vt,alpha,NN,delta)
      %TruncationError Computes truncation error using power decay
      %Lee (2004, Eq 6.2 and Remark 6.1)

      %Should be improved
      gamma = 1.0;

      y =
abs(Chfunctionx(lnS0,Intrate,divyield,kv,thetav,sigmav,rho,TT,NN*delta
-
complex(0,1)*(alpha+1),vt))/(pi*exp(alpha*log(PerStrike))*gamma*((NN*d
elta)^gamma));


 end

 function y =
TotalErrorAlphaPositive(lnS0,PerStrike,Intrate,divyield,kv,thetav,sigm
av,rho,TT,vt,Vecalphadeltap,NN)
      %TotalErrorAlphaPositive Computes, for alpha>0, truncation error
+
      %sampling error,
      %The later from Lee (2004, Theorem 6.2)

      alpha = Vecalphadeltap(1);
      delta = Vecalphadeltap(2);
```

```
      p = Vecalphadeltap(3);

      if (alpha<=0)
          disp('alpha out of admissible range...');
      end

      %Sampling error
      SE = (exp(-
2*pi*alpha/delta)*Chfunctionx(lnS0,Intrate,divyield,kv,thetav,sigmav,r
ho,TT,-complex(0,1),vt)/(1-exp(-4*pi*alpha/delta)))...
          +( exp(2*pi*(alpha-
p)/delta)*Chfunctionx(lnS0,Intrate,divyield,kv,thetav,sigmav,rho,TT,-
complex(0,1)*(p+1),vt)...
          /( (p+1)*exp(p*log(PerStrike))*(1-exp(4*pi*(alpha-p)/delta))
) ) * ((p/(p+1))^p);

      y =
TruncationError(PerStrike,Intrate,divyield,kv,thetav,sigmav,rho,TT,lnS
0,vt,alpha,NN,delta) + SE;

 end

 function y =
TotalErrorAlphaMinusOneZero(lnS0,PerStrike,Intrate,divyield,kv,thetav,
sigmav,rho,TT,vt,Vecalphadelta,NN)
      %TotalErrorAlphaMinusOneZero Computes, for -1<alpha<0,
truncation error + sampling error,
      %The later from Lee (2004, Theorem 6.5 eq1)

      alpha = Vecalphadelta(1);
      delta = Vecalphadelta(2);

      if ( (alpha<=-1) || (alpha>=0) )
          disp('alpha out of admissible range...');
      end

      %Sampling error
      SE = (exp(log(PerStrike)-
2*pi*(alpha+1)/delta)*Chfunctionx(lnS0,Intrate,divyield,kv,thetav,sigm
av,rho,TT,0.0,vt)...
          /(1-exp(-4*pi*(alpha+1)/delta)))...
          +(
exp(2*pi*alpha/delta)*Chfunctionx(lnS0,Intrate,divyield,kv,thetav,sigm
av,rho,TT,-complex(0,1),vt)...
          /(1-exp(4*pi*alpha/delta)) );

      y =
TruncationError(PerStrike,Intrate,divyield,kv,thetav,sigmav,rho,TT,lnS
0,vt,alpha,NN,delta) + SE;

 end

 function y =
TotalErrorAlphaLowerMinusOne(lnS0,PerStrike,Intrate,divyield,kv,thetav
,sigmav,rho,TT,vt,Vecalphadeltaq,NN)
      %TotalErrorAlphaLowerMinusOne Computes, for alpha<-1, truncation
error + sampling error,
      %The later from Lee (2004, Theorem 6.5 eq2)
```

```
      alpha = Vecalphadeltaq(1);
      delta = Vecalphadeltaq(2);
      q = Vecalphadeltaq(3);

      if (alpha>=-1)
          disp('alpha out of admissible range...');
      end

      %Sampling error
      SE =
(exp(log(PerStrike)+2*pi*(1+alpha)/delta)*Chfunctionx(lnS0,Intrate,div
yield,kv,thetav,sigmav,rho,TT,0.0,vt)...
          /(1-exp(4*pi*(1+alpha)/delta)))...
          +( exp((1+q)*log(PerStrike)-
2*pi*(alpha+1+q)/delta)*Chfunctionx(lnS0,Intrate,divyield,kv,thetav,si
gmav,rho,TT,complex(0,1)*q,vt)...
          /( (q+1)*(1-exp(-4*pi*(alpha+1+q)/delta)) ) ) *
((q/(q+1))^q);

      y =
TruncationError(PerStrike,Intrate,divyield,kv,thetav,sigmav,rho,TT,lnS
0,vt,alpha,NN,delta) + SE;

 end

 function y =
TotalErrorAlphaZero(lnS0,PerStrike,Intrate,divyield,kv,thetav,sigmav,r
ho,TT,vt,NN,Vecdeltap)
      %TotalErrorAlphaPositive Computes, for alpha>0, truncation error
+ sampling error,
      %The later from Lee (2004, Theorem 6.6b)

      delta = Vecdeltap(1);
      p = Vecdeltap(2);

      alpha=0.0;

      %Sampling error
      SE = max(
Chfunctionx(lnS0,Intrate,divyield,kv,thetav,sigmav,rho,TT,0.0,vt)*exp(
log(PerStrike)-2*pi/delta),...
          (
Chfunctionx(lnS0,Intrate,divyield,kv,thetav,sigmav,rho,TT,-
complex(0,1)*(p+1),vt)/...
          ((p+1)*exp(p*(log(PerStrike)+2*pi/delta)) )) * ((p/(p+1))^p)
);

      y =
TruncationError(PerStrike,Intrate,divyield,kv,thetav,sigmav,rho,TT,lnS
0,vt,alpha,NN,delta) + SE;

 end

 function y =
TotalErrorAlphaMinusOne(lnS0,PerStrike,Intrate,divyield,kv,thetav,sigm
av,rho,TT,vt,NN,Vecdeltaq)
      %TotalErrorAlphaMinusOne Computes, for alpha=-1, truncation
error + sampling error,
      %The later from Lee (2004, Theorem 6.6a)
```

```
      delta = Vecdeltaq(1);
      q = Vecdeltaq(2);

      alpha=-1.0;

      %Sampling error
      SE = max( (
exp((1+q)*log(PerStrike))*Chfunctionx(lnS0,Intrate,divyield,kv,thetav,
sigmav,rho,TT,complex(0,1)*q,vt)...
          /( (q+1)*exp(2*pi*q/delta) ) ) * ((q/(q+1))^q),...
          Chfunctionx(lnS0,Intrate,divyield,kv,thetav,sigmav,rho,TT,-
complex(0,1),vt)*exp(-2*pi/delta) );


      y =
TruncationError(PerStrike,Intrate,divyield,kv,thetav,sigmav,rho,TT,lnS
0,vt,alpha,NN,delta) + SE;

 end

 function [Optimalalpha Optimaldelta,Minimumfval,Optimalexitflag] =
OptimalAlphaDelta(lnS0,PerStrike,Intrate,divyield,kv,thetav,sigmav,rho
,TT,vt,NN,alphaMin,alphaMax,InitialAlpha)
      %OptimalAlphaDelta Computes optimal alpha and delta
      %through minimization of truncation error + sampling error,
      %following Lee (2004)

      Initialdelta = (2.0*pi)/((NN)*0.01);

      optionsalpha = optimset('Display','none','Algorithm','interior-
point','TolFun',1e-12,'TolX',1e-12);


      %alpha>0
      %Vecx = Vecalphadeltap
      A = [1 0 -1];
      b = 0;
      [Vecx,fval,exitflag] =
fmincon(@(Vecx)TotalErrorAlphaPositive(lnS0,PerStrike,Intrate,divyield
,kv,thetav,sigmav,rho,TT,vt,Vecx,NN),...
        [InitialAlpha Initialdelta InitialAlpha+1],A,b,[],[],[1.0e-10
1.0e-10 1.0e-10],[alphaMax Inf alphaMax],[],optionsalpha);
      %funciona com alphatxt
      Optimalalpha = Vecx(1);
      Optimaldelta = Vecx(2);
      Minimumfval = fval;
      Optimalexitflag = exitflag;


      %-1<alpha<0
      %Vecx = Vecalphadelta
      [Vecx,fval,exitflag] =
fmincon(@(Vecx)TotalErrorAlphaMinusOneZero(lnS0,PerStrike,Intrate,divy
ield,kv,thetav,sigmav,rho,TT,vt,Vecx,NN),...
        [-0.5 Initialdelta],[],[],[],[],[-1 1.0e-10],[0.0
Inf],[],optionsalpha);
      if (fval<Minimumfval)
          Minimumfval = fval;
          Optimalalpha = Vecx(1);
```

```
            Optimaldelta = Vecx(2);
            Optimalexitflag = exitflag;
        end


        %alpha<-1
        [Vecx,fval,exitflag] =
fmincon(@(Vecx)TotalErrorAlphaLowerMinusOne(lnS0,PerStrike,Intrate,div
yield,kv,thetav,sigmav,rho,TT,vt,Vecx,NN),...
          [-1.5 Initialdelta 1.5],[],[],[],[],[alphaMin 1.0e-10 -
alphaMax-1],[-1.0 Inf alphaMin],[],optionsalpha);
        if (fval<Minimumfval)
            Minimumfval = fval;
            Optimalalpha = Vecx(1);
            Optimaldelta = Vecx(2);
            Optimalexitflag = exitflag;
        end


        %alpha=0
        [Vecx,fval,exitflag] =
fmincon(@(Vecx)TotalErrorAlphaZero(lnS0,PerStrike,Intrate,divyield,kv,
thetav,sigmav,rho,TT,vt,NN,Vecx),...
          [Initialdelta 1.0],[],[],[],[],[1.0e-10 1.0e-10],[Inf
alphaMax],[],optionsalpha);
        if (fval<Minimumfval)
            Minimumfval = fval;
            Optimalalpha = 0.0;
            Optimaldelta = Vecx(1);
            Optimalexitflag = exitflag;
        end



        %alpha=-1
        [Vecx,fval,exitflag] =
fmincon(@(Vecx)TotalErrorAlphaMinusOne(lnS0,PerStrike,Intrate,divyield
,kv,thetav,sigmav,rho,TT,vt,NN,Vecx),...
          [Initialdelta 1.0],[],[],[],[],[1.0e-10 1.0e-10],[Inf
alphaMin+1],[],optionsalpha);
        if (fval<Minimumfval)
            Minimumfval = fval;
            Optimalalpha = -1.0;
            Optimaldelta = Vecx(1);
            Optimalexitflag = exitflag;
        end




  end
```

Appendix D

Function $\boldsymbol{\phi(u,\omega)}$ from the Truncation Error of subsection 2.5.1

According to the characteristic function, equation (3.4) from subsection 3.1, to define $f(\omega i)$ for real ω, the correct choice of $\log z$ is the principal branch $\log|z| + \arg(z)$, where $-\pi < \arg(z) < \pi$. However, as pointed out by Schodel and Zhu (1999), to define $f(\xi)$ for general ξ, the correct choice of log is not necessarily the principal branch. Instead, the value of log when $\xi = u + \omega i$ is determined by the analyticity of f, which implies that log mus vary continuously as ξ varies from $0 + \omega i$ to $u + \omega i$.

The domain of f is the strip Λ_X induced by $A_X = (a_-, a_+)$, where $a_- < 0$ and $a_+ > 1$ solve

$$g(-ia)\exp(d(-ia)T) = 1 \tag{D.1}$$

Specifically, a_- is the largest (closest to 0) solution in $(-\infty, y_-)$, and a_+ is the smallest solution in (y_+, ∞), where

$$y_\pm := \frac{\sigma - 2\kappa\rho \pm \sqrt{\sigma^2 - 4\kappa\rho\sigma + 4\kappa^2}}{2\sigma(1-\rho^2)} \tag{D.2}$$

For $\xi = u + \omega i$ we bound the large-u decay of f, as follows. Define

$$H_{R_1}(u) := u^2\sigma^2(1-\rho^2) \tag{D.3}$$

$$H_{R_2}(\omega) := \omega^2\sigma^2(1-\rho^2) - \omega(2\kappa\rho\sigma - \sigma^2) - \kappa^2 \tag{D.4}$$

$$H_R(u,\omega) := Re(d^2) = H_{R_1}(u) - H_{R_2}(\omega)$$

(D.5)

$$H_I(u,\omega) := Im(d^2) = \sigma u(2\omega\sigma(1-\rho^2) + \sigma - 2\kappa\rho)$$

(D.6)

$$h(u,\omega) := \sqrt{H_R(u,\omega)}$$

(D.7)

and define,

$$g^*(u,\omega) := \frac{\kappa}{\sigma\sqrt{u^2+\omega^2}} + \frac{|\sigma - 2\kappa\rho| + \kappa^2/\left(\sigma\sqrt{u^2+\omega^2}\right)}{h(u,\omega) + \sqrt{(u^2-\omega^2)\sigma^2(1-\rho^2)}}$$

(D.8)

$$\underline{g}(u,\omega) := \frac{1-g^*(u,\omega)}{1+g^*(u,\omega)}$$

(D.9)

$$J(u,\omega) := \left(1 + \frac{1}{\underline{g}(u,\omega)}\right)\left(1 + \frac{1}{\underline{g}(u,\omega)exp\left(T.h(u,\omega)\right) - 1}\right)$$

(D.10)

Let $u_0 > |\omega|$ satisfy $1 > g^*(u_0,\omega)$ and $h(u_0,\omega) > \left({}^{1}\!/_{T}\right)max\left(\log\left({}^{1}\!/_{\underline{g}(u_0,\omega)}\right), 1\right)$ and $H_{R_1}(u_0) > \left|H_{R_2}(\omega)\right|$. Then for all $u > u_0$, we have

$$|f(u+\omega i)| \le \phi(u,\omega)exp\left(-\sqrt{1-\rho^2}\frac{V_0+\kappa\theta T}{\sigma}u\right)$$

(D.11)

Where (suppressing the arguments (u, ω) for convenience) we let

$$\begin{aligned}\phi(u,\omega) &:= J^{2\kappa\theta/\sigma^2} exp\left[-rT - (\log S_0 + rT)\omega + \frac{V_0 + \kappa\theta T}{\sigma^2}\left(\kappa + \rho\sigma\omega + \sqrt{max\left(0, H_{R_2}\right)}\right)\right] \\ &\times exp\left[\frac{V_0}{\sigma^2}\frac{J}{exp(T.h)}\left(\kappa + |\rho\sigma u| max\left(1, \sqrt{H_R + H_{R_1}}\right) + |\rho\sigma\omega| + \sqrt{H_R + H_I}\right)\right]\end{aligned} \tag{D.12}$$

Hence the square-root stochastic volatility model's discounted characteristic function satisfies the exponential decay condition.