UNIVERSIDADE DE LISBOA
FACULDADE DE CIÊNCIAS
DEPARTAMENTO DE MATEMÁTICA

$\mathscr{D}[[\hbar]]$-MODULES: FUNCTORIAL PROPERTIES AND ELLIPTIC PAIRS

David Simão Cóias Raimundo

DOUTORAMENTO EM MATEMÁTICA
GEOMETRIA E TOPOLOGIA

2012

UNIVERSIDADE DE LISBOA
FACULDADE DE CIÊNCIAS
DEPARTAMENTO DE MATEMÁTICA

$\mathscr{D}[[\hbar]]$-MODULES: FUNCTORIAL PROPERTIES AND ELLIPTIC PAIRS

David Simão Cóias Raimundo

Tese orientada pela Prof.ª Doutora Maria Teresa de Lemos Monteiro Fernandes, especialmente elaborada para a obtenção do grau de doutor em Matemática na especialidade de Geometria e Topologia

2012

Abstract

We study the functorial properties of modules over the ring $\mathscr{D}_X[[\hbar]]$ of holomorphic differential operators on a complex manifold X with a formal parameter $\hbar$, which is an example of an algebra of formal deformation. In Chapter 2 we consider the following data: X and Y are two complex manifolds and $\mathscr{S}$ and $\mathscr{S}'$ denote stacks of modules over algebras of formal deformation $\mathscr{A}$ on X and $\mathscr{A}'$ on Y, respectively; Φ is a functor from the category of open subsets of X to the category of open subsets of Y; F is a morphism of prestacks from $\mathscr{S}$ to $\mathscr{S}' \circ \Phi$. We study the category $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$ consisting of coherent $\mathscr{A}$-modules such that the cohomologies associated to the action of the formal parameter $\hbar$ belong to $\mathscr{S}$ and we give conditions to extend F in a canonical way to a functor $F^{\hbar}$ from $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$ to $\mathrm{Mod}(\mathscr{A}')$. We give an explicit construction of $F^{\hbar}$ and we prove that $F^{\hbar}$ is (right) exact when F is (right) exact.

In Chapter 3 we apply the results of Chapter 2 to the case $\mathscr{A} = \mathscr{D}_X[[\hbar]]$. This procedure allow us to define the functors of inverse image, Fourier transform, specialization and microlocalization, nearby and vanishing cycles in the framework of $\mathscr{D}[[\hbar]]$-modules. We are also able to define the functor of direct image once we get a suitable transfer module. We study these functors and we prove coherence criteria for direct image and inverse image, a Cauchy-Kowalewskaia-Kashiwara theorem and comparison theorems for regular holonomic $\mathscr{D}[[\hbar]]$-modules.

In Chapter 4 we extend the results of Schapira and Schneiders [34] on relative regularity and finiteness of elliptic pairs to the framework of $\mathscr{D}[[\hbar]]$-modules and $\mathbb{R}$-constructible sheaves of $\mathbb{C}[[\hbar]]$-modules. In the smooth case, we also prove a relative duality theorem for elliptic pairs satisfying the finiteness criteria.

Key-words: $\mathscr{D}$-modules. Algebras of formal deformation. Functorial properties. Elliptic pairs.

Resumo

Consideremos o anel $\mathscr{D}_X$ dos operadores diferenciais holomorfos sobre uma variedade complexa X. Sabe-se que a um sistema de equações diferenciais lineares holomorfas corresponde naturalmente um módulo coerente sobre $\mathscr{D}_X$ e, reciprocamente, a cada $\mathscr{D}_X$-módulo coerente corresponde localmente um tal sistema de EDP's. Esta identificação entre $\mathscr{D}$-módulos coerentes e sistemas de EDP's constituiu a motivação primária para a sistematização da teoria de $\mathscr{D}$-módulos. A teoria de $\mathscr{D}$-módulos tem sido alvo de investigação desde os anos 1970, tendo sido também aplicada a diversos ramos da matemática como a teoria de representação ou a teoria de deformação-quantização.

Recentemente, no âmbito do estudo de módulos de deformação-quantização sobre variedades complexas, M. Kashiwara e P. Schapira [19] introduziram o anel $\mathscr{D}_X[[\hbar]]$ dos operadores diferenciais com um parâmetro formal $\hbar$ sobre uma variedade complexa X. No mesmo artigo, os autores providenciam uma vasta gama de resultados abstratos relativos ao comportamento dos módulos de deformação-quantização. Uma vez que $\mathscr{D}_X[[\hbar]]$ constitui um exemplo de uma álgebra de deformação formal, tais resultados são válidos quando aplicados ao estudo dos $\mathscr{D}[[\hbar]]$-módulos.

O estudo dos $\mathscr{D}[[\hbar]]$-módulos foi feito em primeiro lugar por A. D'Agnolo, S. Guillermou and P. Schapira [3], artigo em que são também fornecidas ferramentas úteis para desenvolver a teoria.

Neste trabalho aplicamos os resultados e métodos de [19] e de [3] para desenvolver o estudo dos $\mathscr{D}[[\hbar]]$-módulos. Mais especificamente, o nosso objetivo é estudar as propriedades functoriais dos $\mathscr{D}[[\hbar]]$-módulos em analogia com as propriedades functoriais clássicas dos $\mathscr{D}$-módulos. Os aspetos básicos da teoria de $\mathscr{D}$-módulos e as ferramentas necessárias sobre álgebras de deformação formal são revistos no Capítulo 1.

O nosso alvo passa por definir e estudar as chamadas seis operações de Grothendieck para a categoria dos $\mathscr{D}[[\hbar]]$-módulos, bem como a especialização de $\mathscr{D}[[\hbar]]$-módulos ao longo de uma subvariedade e outros functores relacionados: ciclos próximos e ciclos evanescentes ao longo de uma hipersuperfície, transformação de Fourier e microlocalização ao longo de uma subvariedade. Estamos também interessados em estender para o contexto dos $\mathscr{D}[[\hbar]]$-módulos os resultados de P. Schapira e J.P. Schneiders [34] relativos aos pares elípticos, isto é, pares formados por um $\mathscr{D}$-módulo coerente $\mathscr{M}$ e um feixe $\mathbb{R}$-construtível F tais que a variedade característica de $\mathscr{M}$ e o micro-suporte de F verificam uma condição de transversalidade micro-local. As propriedades functoriais dos pares elípticos são estudadas em detalhe em [34], onde os autores provam teoremas de regularidade, finitude e dualidade para estes objectos. Tais teoremas generalizam vários resultados clássicos da teoria de $\mathscr{D}$-módulos, geometria complexa analítica e teoria dos sistemas elípticos.

Tendo em vista os nossos objetivos, um dos principais problemas consistia em estender para a categoria dos $\mathscr{D}_X[[\hbar]]$-módulos functores exatos (ou exatos à direita) que, à partida, estavam bem definidos na categoria dos $\mathscr{D}_X$-módulos. Este problema pode ser enunciado no contexto mais geral das álgebras de deformação formal do seguinte modo: dadas duas variedades complexas X e Y, dada uma álgebra de deformação formal $\mathscr{A}$

em X e uma álgebra de deformação formal $\mathscr{A}'$ em Y, dada uma subcategoria $\mathscr{S}$ plena e Serre em $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$ e um functor F exato à direita (respetivamente exato) de $\mathscr{S}$ para uma subcategoria plena $\mathscr{S}'$ de $\mathrm{Mod}(\mathscr{A}')$, queremos encontrar uma subcategoria conveniente de $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$ contendo $\mathscr{S}$ para a qual F se estenda de forma canónica como functor exato à direita (respetivamente exato). Note-se que em rigor, por imperativos técnicos, precisamos de fazer uso da linguagem de stacks uma vez que pretendemos aplicar a nossa construção a categorias Serre definidas por propriedades locais. Assim, $\mathscr{S}$ e $\mathscr{S}'$ serão stacks de subcategorias Serre de $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$ e $\mathrm{Mod}(\mathscr{A}')$, respetivamente. O functor F será definido como um functor de pré-stacks entre $\mathscr{S}$ e um pré-stack $\mathscr{S}' \circ \Phi$, sendo Φ um functor que estabelece uma correspondência conveniente entre a categoria dos abertos de X e a categoria dos abertos de Y. Observe-se que ao longo deste trabalho lidamos apenas com o exemplo mais simples de stacks, os feixes de categorias.

A solução para estender F consiste em considerar a subcategoria de $\mathrm{Mod}(\mathscr{A})$ formada pelos módulos tais que a cohomologia associada à ação do parâmetro formal $\hbar$ pertence a $\mathscr{S}$. No Capítulo 2 estudamos esta categoria, para a qual adotamos a notação $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$, e estabelecemos condições em $\mathscr{S}$ e $\mathscr{S}'$ para a existência de uma extensão canónica de F. Ou seja, sob condições apropriadas, construímos um functor $F^{\hbar} : \mathrm{Mod}_{\mathscr{S}}(\mathscr{A}) \to \mathrm{Mod}(\mathscr{A}')$ que é exato à direita (respetivamente exato) quando $F(X)$ é exato à direita (respetivamente $F(U)$ é exato para cada aberto $U \subset X$). A construção de $F^{\hbar}$ recorre essencialmente ao limite projetivo de feixes, à condição de Mittag-Leffler e ao estudo das álgebras de deformação formal e das suas propriedades. Para além disso, no caso em que F toma valores em $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}')$ então a imagem de um $\mathscr{A}$-módulo por meio de $F^{\hbar}$ também é um $\mathscr{A}'$-módulo coerente e a extensão $F^{\hbar}$ constitui o único functor, a menos de isomorfismo, que satisfaz propriedades naturais de compatibilidade com o functor F. Os resultados deste capítulo são sintetizados no Teorema 2.5.4.

No Capítulo 3 aplicamos a construção geral levada a cabo no Capítulo 2 ao caso $\mathscr{A} = \mathscr{D}_X[[\hbar]]$. Obtemos um quadro natural para a extensão do functor imagem inversa e do functor imagem direta por um mergulho fechado dos $\mathscr{D}$-módulos para os $\mathscr{D}[[\hbar]]$-módulos. No caso da extensão da imagem inversa por meio de um morfismo f, provamos uma versão formal do teorema de Cauchy-Kowalewskaia-Kashiwara quando o functor é restringido ao substack definido pelas subcategorias Serre dos módulos não-característicos para f. Provamos critérios de coerência para imagem direta e imagem inversa de $\mathscr{D}[[\hbar]]$-módulos à semelhança dos critérios existentes para $\mathscr{D}$-módulos. Generalizamos também o functor imagem inversa extraordinária para $\mathscr{D}[[\hbar]]$-módulos. Assim, como consequência da nossa construção geral e em conjunto com os resultados de [3], as seis operações de Grothendieck ficam definidas para $\mathscr{D}[[\hbar]]$-módulos.

O teorema abstrato do Capítulo 2 também se aplica para efetuar a extensão para o quadro dos $\mathscr{D}[[\hbar]]$-módulos do functor especialização ao longo de uma subvariedade $Y \subset X$. Este functor não pode ser construído de modo análogo ao caso dos $\mathscr{D}$-módulos, uma vez que o anel $\mathscr{D}_X[[\hbar]]$ não está naturalmente munido de uma V-filtração de Kashiwara-Malgrange, ao contrário do que acontece com o anel $\mathscr{D}_X$. No entanto, podemos ultrapassar este obstáculo recorrendo à nossa construção geral do capítulo 2 que faz uso de limites projetivos. Mais precisamente, sendo $\mathscr{M}$ um $\mathscr{D}_X[[\hbar]]$-módulo especializável ao longo de Y virá, por definição, que, para todo o $n \geq 0$, o cokernel da ação de $\hbar^{n+1}$ em $\mathscr{M}$ é um $\mathscr{D}_X$-módulo especializável no sentido habitual. Para cada $n \geq 0$, usamos a notação $\mathscr{M}_n$ para o cokernel da ação de $\hbar^{n+1}$ em $\mathscr{M}$ e notamos ν_Y o functor especialização no quadro dos $\mathscr{D}$-módulos. Então o especializado de $\mathscr{M}$ ao longo de Y é o $\mathscr{D}_{T_Y X}[[\hbar]]$-módulo coerente definido pelo limite projetivo dos especializados dos $\mathscr{M}_n$'s enquanto $\mathscr{D}$-módulos:

$$\nu_Y^{\hbar}(\mathscr{M}) := \varprojlim_{n \geq 0} \nu_Y(\mathscr{M}_n).$$

Usando esta construção para a especialização no quadro formal, demonstramos teoremas de comparação para $\mathscr{D}[[\hbar]]$-módulos holónomos regulares, relacionando o functor de especialização para $\mathscr{D}[[\hbar]]$-módulos com o functor de especialização de Sato da teoria dos feixes (Teorema 3.6.5).

Adotando um procedimento análogo ao caso da especialização, definimos os functores de ciclos próximos e ciclos evanescentes ao longo de uma hipersuperfície, transformação de Fourier e microlocalização ao longo de uma subvariedade. Enunciamos também teoremas de comparação para estes functores.

O Capítulo 4 é dedicado ao estudo dos pares elípticos. Introduzimos a noção de par elíptico associado a um morfismo de variedades complexas $f : X \to Y$ no caso formal e estudamos as propriedades destes objectos. Mais especificamente, provamos propriedades de regularidade para pares elípticos sobre o anel $\mathbb{C}[[\hbar]]$ das séries formais com coeficientes complexos. Estes teoremas de regularidade generalizam uma propriedade clássica dos sistemas elípticos no quadro real analítico: o complexo das soluções reais analíticas de um sistema elíptico coincide com o complexo das soluções pertencentes ao feixe das hiperfunções.

Usando o functor imagem direta própria definido no Capítulo 3 e usando o teorema de regularidade, provamos o teorema de finitude para pares elípticos sobre $\mathbb{C}[[\hbar]]$, cujo enunciado é o seguinte: dado um par elíptico $(\mathscr{M}, F)$ tal que o morfismo f é próprio quando restrito à interseção dos suportes de $\mathscr{M}$ e F e tal que $\mathscr{M}$ é um $\mathscr{D}_X$-módulo *good*, então as cohomologias da imagem direta própria do produto $\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X[[\hbar]]} F$ são coerentes sobre $\mathscr{D}_Y[[\hbar]]$.

No caso em que f é um morfismo suave provamos ainda um teorema de dualidade que relaciona a imagem direta de pares elípticos com o functor de dualidade no quadro dos $\mathscr{D}_X[[\hbar]]$-módulos. Mais precisamente, provamos que se f é suave então é possível construir um morfismo de dualidade para $\mathscr{D}_X[[\hbar]]$-módulos e que este morfismo é um isomorfismo quando aplicado a pares elípticos que satisfazem o critério de finitude. A restrição ao caso suave é necessária mas permite-nos, ainda assim, tratar o caso importante de uma aplicação constante.

Palavras chave: $\mathscr{D}$-módulos. Álgebras de deformação formal. Propriedades functoriais. Pares elípticos.

Acknowledgments

Let me start by expressing my deep gratitude to Professor Teresa Monteiro Fernandes, who supervised my thesis and helped me with suggestions and constant incentive through infinitely many mathematical issues during these last five years. This work is only possible due to her patience and support.

Part of the results that I present in this thesis were obtained in a joint work with Professor Teresa Monteiro Fernandes and Professor Ana Rita Martins and I am grateful for the oportunity of working with them. I also thank Professor Ana Rita Martins for her kind words of encouragement throughout the Phd process.

I am grateful to Professor Pierre Schapira for suggesting the theory of $\mathscr{D}[[\hbar]]$-modules as a possible subject for my thesis and for the helpful suggestions that he provided when I had the first contact with the theory, in the beginning of 2009, when I spent a couple of months working in Paris at the Institut des Mathématiques de Jussieu.

I am grateful to Professor Stéphane Guillermou, who kindly corrected many imprecisions, gave many suggestions and answered my questions on $\mathscr{D}[[\hbar]]$-modules. His suggestions allowed me to get a better understanding of $\mathscr{D}[[\hbar]]$-modules and contributed to shape several constructions and results of the thesis.

I express my gratitude to Faculdade de Ciências da Universidade de Lisboa (FCUL), Fundação para a Ciência e Tecnologia (FCT), Fundação Calouste Gulbenkian, Centro de Álgebra da Universidade de Lisboa (CAUL) and Centro de Matemática e Aplicações Fundamentais (CMAF) for providing finantial and institutional support.

On a personal level, I could name tens of people whose friendship was relevant during the preparation of this thesis, but I hope I can personally thank each of them soon. I want only to express my gratitude to my fellows at Grupo Bíblico Universitário and to my colleagues at FCUL, specially those with whom I shared the undergraduate course and those with whom I shared the Phd office during the last five years.

I must emphasize how much I am grateful to Débora Silva, who in the recent years has become my closest friend, confidant and also my beautiful bride.

Finally, I am grateful to my parents Maria Elisa Cóias and Simão Raimundo for providing a safe and stable environment that allowed me to dedicate to school since I was a little kid. My academic achievements are a fruit of their care and devotion.

David Raimundo
Lisbon, October 19th, 2012

Contents

Introduction

$\mathscr{D}$**-modules.** On a complex manifold X we consider the ring $\mathscr{D}_X$ of holomorphic differential operators. It is well-known that any system of linear partial differential equations on X can be regarded as a coherent $\mathscr{D}_X$-module $\mathscr{M}$ and the sheaf of holomorphic solutions of the system can be identified with $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$. Conversely, any coherent $\mathscr{D}_X$-module $\mathscr{M}$ is locally the cokernel of a $\mathscr{D}_X$-linear morphism between free $\mathscr{D}_X$-modules of finite rank, so locally we can think of $\mathscr{M}$ as a system of PDE's. This identification between coherent $\mathscr{D}$-modules and systems of PDE's constituted the first motivation to the study of $\mathscr{D}$-modules as an independent branch of the so-called Algebraic Analysis: the study of problems that belong to the scope of Analysis using algebraic tools.

The foundations of the theory of $\mathscr{D}$-modules were established in classical works by Masaki Kashiwara and Joseph Bernstein in the 1970's and, subsequently, during the last decades of the 20th century, several mathematicians contributed to create a deep theory which knows applications not only to the field of systems of PDE's but also to representation theory, b-functions, deformation-quantization theory, among other areas of Mathematics.

The research on $\mathscr{D}$-modules made clear that the functorial nature of the theory is better described with the use of derived categories and derived functors. For example, for a given system of PDE's and the corresponding coherent $\mathscr{D}$-module $\mathscr{M}$, one identifies the solutions of the system not with the sheaf $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ but rather with the object $\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ that belongs to the derived category of sheaves of $\mathbb{C}$-vector spaces. We assume that the reader is acquainted with derived categories and sheaf theory, thus we recall only a few needed notions of sheaf theory in Section 1.1.

In Section 1.2 we recall the basic results of the theory of $\mathscr{D}$-modules that we need. Namely, we recall how one constructs functors of direct image, inverse image, specialization and microlocalization along a submanifold, vanishing cycles and nearby-cycles along an hypersurface, and we also review the main theorems concerning these functors.

$\mathscr{D}[[\hbar]]$**-modules.** Consider the ring $\mathscr{D}_X[[\hbar]]$ of formal power series of differential operators. On a subset $U \subset X$ the sections $P \in \Gamma(U; \mathscr{D}_X[[\hbar]])$ are operators $\sum_{n\geq 0} P_n \hbar^n$, where each P_n belongs to $\Gamma(U; \mathscr{D}_X)$, and the multiplication of such operators is given by

$$\left(\sum_{n\geq 0} P_n \hbar^n\right)\left(\sum_{i\geq 0} Q_i \hbar^i\right) = \sum_{n\geq 0}\sum_{i\geq 0} P_n Q_i \hbar^{n+i}.$$

This ring was introduced in [19] by Masaki Kashiwara and Pierre Schapira as a local model of an algreboid $\mathscr{D}_X^{\mathscr{A}}$, constructed by the authors in their study of modules of deformation-quantization on complex manifolds. The ring $\mathscr{D}_X[[\hbar]]$ then plays a role in the study of local properties of deformation-quantization modules in loc. cit.

The study of modules over $\mathscr{D}_X[[\hbar]]$ was developed by Andrea D'Agnolo, Stephane Guillermou and Pierre Schapira in [3]. Section 8 of [3] also provides a link between the theory of $\mathscr{D}_X[[\hbar]]$-modules and the study of deformation quantization algebras on complex symplectic manifolds.

Roughly speaking, in analogy with the $\mathscr{D}$-modules case, the study of $\mathscr{D}[[\hbar]]$-modules is the study of systems of a countable number of interdependent PDE's with a formal parameter $\hbar$. Our approach to the theory of $\mathscr{D}[[\hbar]]$-modules is an algebraic approach that mainly exhibits the functorial aspects of the theory and we expect that such approach constitutes a step to a better understanding of deformation-quantization algebras.

Our main tools. Besides the classical works on $\mathscr{D}$-modules, our tools come mainly from the study of algebras of formal deformation of [19] and the study of formal extensions of [3].

Let $\mathbb{K}$ be a commutative unital ring and consider a $\mathbb{K}_X$-algebra $\mathscr{A}$. One says that $\mathscr{A}$ is an algebra of formal deformation if it is endowed with a central formal parameter $\hbar$ such that: $\mathscr{A}$ is $\hbar$-complete and has no $\hbar$-torsion and there exists either a basis of open subsets of X or a basis of compact subsets of X that is acyclic for coherent modules over $\mathscr{A}_0 := \mathscr{A}/\hbar\mathscr{A}$ (more precisely, $\mathscr{A}$ and $\mathscr{A}_0$ satisfy either Assumption 1.3.5 or Assumption 1.3.6).

Following [19], an object $\mathscr{M}$ belonging to the derived category $\mathsf{D}^{\mathrm{b}}(\mathscr{A})$ is said to be cohomologically complete if the complex $\mathrm{R}\mathscr{H}om_{\mathscr{A}}(\mathscr{A} \otimes_{\mathbb{Z}[\hbar]} \mathbb{Z}[\hbar, \hbar^{-1}], \mathscr{M})$ vanishes. A comprehensive study of cohomologically complete objects is made in loc. cit.

Also in [19], the authors introduce the functor

$$\mathrm{gr}_\hbar : \mathsf{D}^{\mathrm{b}}(\mathscr{A}) \to \mathsf{D}^{\mathrm{b}}(\mathscr{A}_0), \quad \mathscr{M} \mapsto \mathrm{gr}_\hbar(\mathscr{M}) := \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} \mathscr{A}_0,$$

and prove that $\mathrm{gr}_\hbar$ is a conservative functor when restricted to the subcategory of $\mathsf{D}^{\mathrm{b}}(\mathscr{A})$ consisting of cohomologically complete objects, that is: if $\mathscr{M}$ and $\mathscr{N}$ are cohomologically complete objects in $\mathsf{D}^{\mathrm{b}}(\mathscr{A})$, a morphism $\varphi : \mathscr{M} \to \mathscr{N}$ is an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathscr{A})$ if and only if $\mathrm{gr}_\hbar(\varphi) : \mathrm{gr}_\hbar(\mathscr{M}) \to \mathrm{gr}_\hbar(\mathscr{N})$ is an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathscr{A}_0)$. The following coherence criteria is also proved in loc. cit.: let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{A})$ be a cohomologically complete object and suppose that the cohomology modules of $\mathrm{gr}_\hbar(\mathscr{M})$ are coherent over $\mathscr{A}_0$; then, the cohomology modules of $\mathscr{M}$ are coherent over $\mathscr{A}$.

Let $\mathscr{R}_0$ be a sheaf of rings on X. Denote by $\mathscr{R}$ (we also use the notation $\mathscr{R}_0^\hbar$) the sheaf of rings $\mathscr{R}_0[[\hbar]] \simeq \Pi_{n \geq 0} \mathscr{R}_0 \hbar^n$. Set $\mathscr{R}_n := \mathscr{R}/\hbar^{n+1}\mathscr{R}$. We shall use the left exact functor below

$$\begin{array}{rcl} (\bullet)^\hbar : \mathrm{Mod}(\mathscr{R}_0) & \to & \mathrm{Mod}(\mathscr{R}) \\ \mathscr{N} & \mapsto & \mathscr{N}^\hbar = \varprojlim\limits_{n \geq 0} (\mathscr{N} \otimes_{\mathscr{R}_0} \mathscr{R}_n), \end{array}$$

as well as its right derived functor $(\bullet)^{\mathrm{R}\hbar} : \mathsf{D}^{\mathrm{b}}(\mathscr{R}_0) \to \mathsf{D}^{\mathrm{b}}(\mathscr{R})$ which one calls the functor of formal extension. Recall that $(\bullet)^{\mathrm{R}\hbar}$ was introduced and studied in [3].

The ring $\mathscr{D}_X^\hbar = \mathscr{D}_X[[\hbar]]$ is the image of the ring $\mathscr{D}_X$ by the functor $(\bullet)^{\mathrm{R}\hbar}$. Moreover, $\mathscr{D}_X^\hbar$ is a $\mathbb{C}_X^\hbar$-algebra and it is an example of an algebra of formal deformation (cf. Example 1.3.1 of [19]). Hence, the results of [19] and [3] apply to the study of $\mathscr{D}_X^\hbar$-modules. For example, denoting by $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ (resp. $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$) the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^\hbar)$ (resp. $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$) consisting on objects with coherent cohomology modules, one has a conservative functor

$$\mathrm{gr}_\hbar : \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^\hbar) \to \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X), \quad \mathscr{M} \mapsto \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar} \mathscr{D}_X.$$

The coherence criteria mentioned above also holds for $\mathscr{D}_X^\hbar$-modules, i.e.: if $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^\hbar)$ is cohomologically complete, then $\mathscr{M}$ is an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ if and only if $\mathrm{gr}_\hbar(\mathscr{M})$ is an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$.

Consider also the following formal extension functor:

$$(\bullet)^{\mathrm{R}\hbar} : \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X) \to \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^\hbar).$$

The composition $\mathrm{gr}_\hbar \circ (\bullet)^{\mathrm{R}\hbar} : \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X) \to \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ coincides with the identity functor on $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$. This shows that the functors $\mathrm{gr}_\hbar$ and $(\bullet)^{\mathrm{R}\hbar}$ form a bridge between the classical theory of $\mathscr{D}$-modules and the theory of $\mathscr{D}[[\hbar]]$-modules. Therefore, the technique that we often apply to obtain results on $\mathscr{D}[[\hbar]]$-modules is to reduce the proofs to the classical results on $\mathscr{D}$-modules using these functors and the machinery provided by [19] and [3].

In sections 1.3 and 1.4 we review the tools we need from [19] and [3] and we also prove some complementary results that we need for later applications.

Our main purpouse. The purpose of our work is to study the functorial properties of $\mathscr{D}[[\hbar]]$-modules. Let us give now a detailed description of our goals.

Recall that the characteristic variety $\mathrm{char}(\mathscr{M})$ of a coherent $\mathscr{D}$-module is a conic involutive subvariety of the cotangent bundle T^*X. In [3], the authors extend the notion of characteristic variety to coherent $\mathscr{D}_X^\hbar$-modules in the following way: the characteristic variety of a coherent $\mathscr{D}_X^\hbar$-module $\mathscr{M}$ is defined as the characteristic variety of the complex $\mathrm{gr}_\hbar(\mathscr{M})$. Let us mention that in loc. cit. the authors study the functors of solutions and De Rham for holonomic $\mathscr{D}_X^\hbar$-modules, i.e., coherent $\mathscr{D}_X^\hbar$-modules whose characteristic variety is lagrangian, and they construct a Riemman-Hilbert correspondence in this framework.

Our goal is to define the so-called Grothendieck six operations for the category $\mathrm{Mod}(\mathscr{D}_X^\hbar)$ so that after applying $\mathrm{gr}_\hbar$ we recover the usual operations for $\mathscr{D}_X$-modules. We also aim to construct in the $\mathscr{D}_X^\hbar$-modules setting the functors of specialization along a submanifold, nearby and vanishing cycles along an hypersurface, Fourier transform and microlocalization along a submanifold.

Our second goal is to extend to $\mathscr{D}^\hbar$-modules the results on elliptic pairs due to Pierre Schapira and Jean-Pierre Schneiders [34]. Recall that in loc. cit. the authors consider a morphism of complex analytic manifolds $f : X \to Y$ and say that a coherent $\mathscr{D}_X$-module $\mathscr{M}$ and an $\mathbb{R}$-constructible sheaf F form an f-elliptic pair if the f-characteristic variety of $\mathscr{M}$ and the micro-support of F do not intersect outside the zero-section of the cotangent bundle T^*X. The f-characteristic variety of a coherent $\mathscr{D}_X$-module is a closed conic analytic subvariety of T^*X, which depends on f. In particular, the f-characteristic variety coincides with the classical characteristic variety $\mathrm{char}(\mathscr{M})$ when f is the constant map $X \to \{\mathrm{pt}\}$.

If X is the complexification of a real analytic manifold M and if $\mathscr{M}$ is an elliptic system on M, then $(\mathscr{M}, \mathbb{C}_M)$ forms an elliptic pair. Therefore, elliptic pairs can be regarded as a generalization of elliptic systems. The functorial properties of f-elliptic pairs are studied in detail in [34], where theorems of regularity, finiteness and duality are proved for these objects. As the authors point out, such theorems generalize several classical results of $\mathscr{D}_X$-module theory, complex analytic geometry and elliptic systems theory. We aim to extend the main results of [34] to the framework of modules over the ring $\mathscr{D}_X^{\hbar}$.

Main achievements. In view of our goals, one main problem was to extend to $\mathrm{Mod}(\mathscr{D}_X^{\hbar})$ (right) exact functors which, *a priori*, were only well defined on $\mathrm{Mod}(\mathscr{D}_X)$. We can instead state this problem in the general setting of algebras of formal deformation: given two complex manifolds X and Y, an algebra of formal deformation $\mathscr{A}$ on X, an algebra of formal deformation $\mathscr{A}'$ on Y, given a right exact (respectively exact) functor F on a Serre full subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$ to a given full subcategory $\mathscr{S}'$ of $\mathrm{Mod}(\mathscr{A}')$, we need to find the natural subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$ containing $\mathscr{S}$ to which F extends canonically as a right exact (respectively exact) functor. We denote the new functor by $F^{\hbar}$. In practice, $\mathscr{S}$ will be a subcategory whose objects are $\hbar$-torsion modules, i.e., each section of such modules is annihilated by a power $\hbar^n$, for some $n \in \mathbb{N}$.

The answer to our problem is to consider the subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$ characterized by the property that, for each n, the kernel and the cokernel of the action of $\hbar^{n+1}$ belong to $\mathscr{S}$ and denoted by $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$.

For $n \geq 0$, denote by $\mathscr{M}_n$ the cokernel $\mathscr{M}/\hbar^{n+1}\mathscr{M}$, and, for $k \geq n \geq 0$, denote by $\rho_{k,n}$ the projection $\mathscr{M}_n \to \mathscr{M}_k$. Assume that $\mathscr{M}_n \in \mathscr{S}$ for each $n \geq 0$. Then, $(F(\mathscr{M}_n), F(\rho_{k,n}))_n$ is a projective system of $\mathscr{A}'$-modules and the corresponding projective limit in $\mathrm{Mod}(\mathscr{A}')$ is the natural candidate for $F^{\hbar}(\mathscr{M})$:

$$F^{\hbar}(\mathscr{M}) = \varprojlim_{n \geq 0} F(\mathscr{M}_n).$$

In fact, $F^{\hbar}$ is a well-defined functor on $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$ and it constitutes the heart of our study in Chapter 2.

To be rigorous, our approach requires more evolved techniques. We must perform our general construction in the framework of stacks and the reason is that in the applications we will be interested in Serre subcategories of $\mathrm{Mod}(\mathscr{D}_X)$ whose objects are defined by local properties. Recall that stacks provide the framework where the notion of sheaves of categories takes a sense. Throughout this work we shall only deal with the easiest example of stacks consisting precisely of sheaves of categories, since they are substacks of modules over a sheaf of $\mathbb{K}$-algebras and the restriction morphisms are nothing more than the usual restriction of sheaves to open subsets. In particular, all these stacks are $\mathbb{K}[[\hbar]]$-linear.

Denote by $\mathrm{Op}(X)$ the category of open subsets of X with the morphisms being defined by the inclusions. Let $\mathfrak{Mod}(\mathscr{A})$ (resp. $\mathfrak{Mod}_{\mathrm{coh}}(\mathscr{A})$) denote the stack $U \mapsto \mathrm{Mod}(\mathscr{A}|_U)$ (resp. the stack $U \mapsto \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}|_U)$), $U \in \mathrm{Op}(X)$. Given an abelian substack $\mathfrak{C}$ of $\mathfrak{Mod}(\mathscr{A})$, a full substack $\mathfrak{C}'$ of $\mathfrak{C}$ is said to be a full Serre substack of $\mathfrak{C}$ if, for each $U \in \mathrm{Op}(X)$, $\mathfrak{C}'(U)$ is a full Serre subcategory of $\mathfrak{C}(U)$.

Let us now restate the above problem in the language of stacks. Assume that we are given a full Serre substack $\mathscr{S}$ of $\mathfrak{Mod}_{\mathrm{coh}}(\mathscr{A})$ and a full Serre substack $\mathscr{S}'$ of an abelian substack of $\mathfrak{Mod}(\mathscr{A}')$. Consider the category $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$ of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$ characterized by the property that, for each n, the kernel and the cokernel of the action of $\hbar^{n+1}$ belong to $\mathscr{S}(X)$. Assume we are given a functor Φ from $\mathrm{Op}(X)$ to $\mathrm{Op}(Y)$ such that $\Phi(X) = Y$ and Φ transforms any open covering of any $\Omega \in \mathrm{Op}(X)$ on an open covering of $\Phi(\Omega)$. Denote by $\Phi^*\mathscr{S}'$ the prestack $U \mapsto \Phi^*\mathscr{S}' = \mathscr{S}'(\Phi(U))$ and assume that we are given a $\mathbb{K}[[\hbar]]$-linear functor of prestacks $F : \mathscr{S} \to \Phi^*\mathscr{S}'$. This means, in particular, that for each pair $V, U \in \mathrm{Op}(X)$ with $V \subset U$, we have the following commutative diagram of functors of categories whose vertical arrows are the restriction functors:

$$\begin{array}{ccc} \mathscr{S}(U) & \xrightarrow{F(U)} & \mathscr{S}'(\Phi(U)) \\ \downarrow & & \downarrow \\ \mathscr{S}(V) & \xrightarrow{F(V)} & \mathscr{S}'(\Phi(V)). \end{array}$$

In these context, under suitable conditions, we want to extend $F(X)$ from $\mathscr{S}(X)$ to a bigger subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$, namely, to $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$.

Let us outline the main results of Chapter 2, which are summarized in Theorem 2.5.4. Assume that $F(X) : \mathscr{S}(X) \to \mathscr{S}'(\Phi(X))$ is right exact; then, under a condition on the vanishing of the cohomology for $\mathscr{S}'(V)$ (with V running on the objects of $\mathrm{Op}(Y)$) which is verified by coherent modules, we obtain a canonical functor

$$F^{\hbar} : \mathrm{Mod}_{\mathscr{S}}(\mathscr{A}) \to \mathrm{Mod}(\mathscr{A}'), \quad \mathscr{M} \mapsto \varprojlim_n F(X)(\mathscr{M}_n)$$

which is still right exact. Moreover, when $\mathscr{S}'$ is a substack of $\mathfrak{Mod}_{\mathrm{coh}}(\mathscr{A}')$, $F^{\hbar}$ takes values in $\mathrm{Mod}_{\mathscr{S}'}(\mathscr{A}')$. If each $F(U)$ is exact, then $F^{\hbar}$ is still exact and it takes values in the subcategory of cohomologically complete objects. This allow us to prove that in this case the extension is, in a certain sense, unique up to isomorphism.

The term canonical means that our construction is indeed functorial in $\mathscr{S}$, $\mathscr{S}'$, Φ and F (cf. Remark 2.5.5).

In Chapter 3, we apply the construction of Chapter 2 to treat the case $\mathscr{A} = \mathscr{D}_X^{\hbar}$. Note that this case is simpler since each $\mathscr{D}_X^{\hbar}/\hbar^{n+1}\mathscr{D}_X^{\hbar}$ is a free module over $\mathscr{A}_0 \simeq \mathscr{D}_X$, so technically we are bound to extend a right exact functor F defined on a Serre substack $\mathscr{S}$ of $\mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ to a suitable subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar})$.

Applying this procedure, we obtain a natural setting to extend to $\mathscr{D}_X^\hbar$-modules the functor of inverse image and the functor of direct image by a closed embedding. Moreover, when we restrict the extended inverse image functor to the Serre substack of non-characteristic modules, we prove a formal version of Cauchy-Kowalewskaia-Kashiwara theorem (Theorem 3.2.10). We also generalize the functor of extraordinary inverse image of $\mathscr{D}$-modules using the concept of duality introduced in [3] and we prove in Proposition 3.2.9 and in Corollary 3.2.12 that the property of holonomicity is stable under inverse image and extraordinary inverse image. So, as a by-product of our general construction together with the results of [3], the usual Grothendieck's six operations are generalized to the formal case.

The construction performed in Chapter 2 also applies to extend the functor of specialization along a submanifold Y to the framework of $\mathscr{D}_X^\hbar$-modules. Note that this functor cannot be constructed in a similar way to the $\mathscr{D}$-modules case, because we don't have in $\mathscr{D}_X^\hbar$ a notion of Kashiwara-Malgrange V-filtration. However, we were able to overcome the lack of V-filtration using our construction based on projective limits. Thus, if $\mathscr{M}$ is a specializable $\mathscr{D}_X^\hbar$-module along a submanifold $Y \subset X$ (which means, by our definition, that $\mathscr{M}_n$ is a specializable $\mathscr{D}_X$-module for any $n \geq 0$), then the specialization of $\mathscr{M}$ along Y is the coherent $\mathscr{D}_{T_Y X}^\hbar$-module defined by the projective limit:

$$\nu_Y^\hbar(\mathscr{M}) := \varprojlim_{n \geq 0} \nu_Y(\mathscr{M}_n),$$

where ν_Y denotes the specialization functor for $\mathscr{D}$-modules. Making use of this definition, we prove a formal version of the comparison theorem of M. Kashiwara (cf. Theorem 3.6.5). Our theorem relates the specialization in the field of $\mathscr{D}^\hbar$-modules with Sato's specialization for sheaves of $\mathbb{C}^\hbar$-modules by means of the De Rham functor.

Similarly to the case of specialization, we are able to define for $\mathscr{D}_X^\hbar$-modules the functors of nearby-cycles and vanishing cycles along an hypersurface, Fourier transform and microlocalization along a submanifold. We also state comparison theorems concerning these functors and the corresponding functors of sheaf theory.

The results of Chapters 2 and 3 were obtained in [25], a joint work with Prof. Ana Rita Martins and Prof. Teresa Monteiro Fernandes.

Chapter 4 is devoted to treat elliptic pairs in the $\hbar$-setting. We introduce the notion of an f-elliptic pair over $\mathbb{C}^\hbar$ in a natural way, such that, if $\mathscr{M}$ is a coherent $\mathscr{D}_X^\hbar$-module and F is an $\mathbb{R}$-constructible sheaf of $\mathbb{C}^\hbar$-modules, then $(\mathscr{M}, F)$ is an f-elliptic pair over $\mathbb{C}^\hbar$ if and only if $(\mathrm{gr}_\hbar(\mathscr{M}), \mathrm{gr}_\hbar(F))$ is an elliptic pair in the sense of [34]. With this definition, we are able to prove theorems of regularity, finiteness and duality for f-elliptic pairs over $\mathbb{C}^\hbar$ by reducing the proofs to the results of loc. cit.

Let us explain with more detail the main results of Chapter 4.

In Theorems 4.2.1 and 4.2.7 we prove regularity properties for f-elliptic pairs over $\mathbb{C}^\hbar$. These regularity theorems generalize a classical regularity property of elliptic systems in the real analytic setting: the complex of real analytic solutions of an elliptic system coincides with the complex of hyperfunctions solutions.

Using the functor of proper direct image defined in Chapter 3 and using also the regularity theorem, we are able to prove the finiteness theorem (Theorem 4.2.9). The statement is the following: given an f-elliptic pair $(\mathscr{M}, F)$, such that f is proper when restricted to $\mathrm{supp}(\mathscr{M}) \cap \mathrm{supp}(F)$ and such that $\mathscr{M}$ is good, then the cohomology modules of the direct image of $\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar} F$ are coherent over $\mathscr{D}_Y^\hbar$.

The duality theorem relates in the $\mathscr{D}_X^\hbar$-modules framework the direct image functor with the duality functor, which we denote by $\underline{\mathrm{D}}_{\hbar,X}$. To obtain such relation we need to restrict ourselves to the smooth case (which enable us to treat the simplest but interesting case of a constant map). More precisely, we show in the first place that if f is smooth, then for $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar\,\mathrm{op}})$ and $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X^\hbar)$ we have a morphism

$$\underline{f}_{!,\hbar}(\mathrm{D}'_{\mathbb{C}_X^\hbar} F \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar} \underline{\mathrm{D}}_{\hbar,X}\mathscr{M}) \to \underline{\mathrm{D}}_{\hbar,Y}\underline{f}_{!,\hbar}(\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar} F). \qquad (**)$$

Here $\underline{f}_{!,\hbar}$ denotes the proper direct image in the $\mathscr{D}_X^\hbar$-modules framework and $\mathrm{D}'_{\mathbb{C}_X^\hbar}$ denotes the *dual prime* functor for $\mathbb{C}_X^\hbar$-modules (cf. (1.1.1)). Secondly, we show that if $(\mathscr{M}, F)$ is an f-elliptic pair over $\mathbb{C}^\hbar$ in the conditions of the finiteness theorem, then (**) is an isomorphism of complexes of $\mathscr{D}_Y^\hbar$-modules with coherent cohomology modules (Theorem 4.2.18).

Finally, we study particular cases of our main results on elliptic pairs. Namely, we obtain finiteness and duality results for the complex of global solutions of a coherent $\mathscr{D}_X^\hbar$-module and also a refinement in the holonomic case (Proposition 4.3.11). We also state finiteness and duality theorems for $\mathscr{O}_X^\hbar$-modules and discuss those theorems in view of our results. Finally, in the case where X is the complexification of a real analytic manifold M, we obtain regularity, finiteness and duality properties concerning the sheaves of formal analytic functions and formal hyperfunctions.

The main results of Chapter 4 are included in the article [29].

Some obstacles and open problems. One of the remaining natural questions is the left derivability of the functor $F^\hbar$ when we start with a right exact functor $F(X) : \mathscr{S}(X) \to \mathscr{S}'(X)$. The difficulty in constructing an $F^\hbar$-projective subcategory comes from the behavior of the functor $\varprojlim$, for which we don't dispose in general of enough injectives, as well as of its lack of good properties with respect to the usual operations in sheaf theory. At our knowledge there is no canonical way of constructing an $F^\hbar$-projective subcategory of $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$, even if there exists an F-projective subcategory of $\mathscr{S}(X)$.

Another natural question is what can one do in the case of a left exact functor $F : \mathscr{S} \to \mathscr{S}'$. The construction that we carry out in Chapter 2 makes clear that the following properties are essential to perform the extension of right exact functors:

- a right exact functor combined with the action of $\hbar^{n+1}$ transforms, for each n, exact sequences of $\mathcal{A}$-modules into right exact sequences of $\mathcal{A}'_n$-modules;
- the exactness of $\Gamma(K, \cdot)$ for K belonging to adequate basis of the topologies of the manifolds;

- the exactness of projective limits on the category of projective systems satisfying Mittag-Lefler's condition.

Although the two last properties could be applied to the left exact case, we cannot say the same about the first one. In fact, if we start with a left exact sequence of $\mathscr{A}$-modules, then the associated sequence of $\mathscr{A}_n$-modules is not necessarily left or right exact and our techniques no longer apply. This indicates that the procedure to treat the case of left exact functors must be rather different than the one we employ here.

In what concerns the functor of direct image, which is defined as the composition of two derived functors, one being left exact, the other being right exact, our method no longer applies (except in particular cases, such as closed embeddings). Even so, we investigate if the general definition of direct image of a $\mathscr{D}_X^\hbar$-module can be translated in terms of projective limits. Such definition makes use of the transfer module in the $\hbar$-setting, which is a left $\mathscr{D}_X^\hbar$-module and right $f^{-1}(\mathscr{D}_Y^\hbar)$-module denoted by $\mathscr{D}_{X\to Y,\hbar}$ and defined by the projective limit below:

$$\mathscr{D}_{X\to Y,\hbar} := \varprojlim_n (\mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}(\mathscr{D}_{Y}^{\hbar}{}_n)).$$

Denote also by $\mathscr{D}_{X\to Y}$ the usual transfer module of $\mathscr{D}$-modules theory. For each right $\mathscr{D}_X^\hbar$-module $\mathscr{M}$, there is the following morphism of right $\mathscr{D}_Y^\hbar$-modules:

$$f_*(\mathscr{M} \otimes_{\mathscr{D}_X^\hbar} \mathscr{D}_{X\to Y,\hbar}) \to \varprojlim_{n\geq 0} f_*(\mathscr{M}_n \otimes_{\mathscr{D}_X} \mathscr{D}_{X\to Y}),$$

We wonder wether or not this is an isomorphism when $\mathscr{M}$ is coherent. Although we were only able to prove that it is an isomorphism under specific conditions, we didn't find a counter-example either.

Finally, in Chapter 4 we are only able to construct a duality morphism in the smooth case and, thus, we obtained a duality theorem only for the smooth case. Indeed, two fundamental properties hold if f is a smooth morphism

- the transfer module $\mathscr{D}_{X\to Y,\hbar}$ is coherent over $\mathscr{D}_X^\hbar$;
- the extension rings $f^{-1}(\mathscr{D}_X^\hbar)$ and $(f^{-1}\mathscr{D}_X)^\hbar$ are isomorphic;

Since these properties are not necessarily verified if f is not smooth, we conclude that our construction of the duality morphism no longer apply in more general situations. The approach to treat the general case should be rather distinct.

Finally, recall that in $\mathscr{D}$-modules theory there is a natural technique to obtain functorial properties for $\mathscr{D}$-modules that consists in reducing those properties to the functorial properties of $\mathscr{O}$-modules. This reduction is possible since one has formulas that make a bridge between the two frameworks. For example, for a $\mathscr{D}_X$-module $\mathscr{M}$ one has the following isomorphism in $\mathrm{Mod}(f^{-1}\mathscr{D}_Y)$

$$\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{D}_{X\to Y} \simeq \mathscr{M} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y.$$

In [34], in order to prove functorial properties for elliptic pairs, the authors often use isomorphisms of this kind. However, such isomorphisms do not have a natural counterpart in the formal case, thus we cannot perform the study of $\mathscr{D}_X^\hbar$-modules by reducing it to the study of $\mathscr{O}_X^\hbar$-modules.

Preliminary notions and conventions

We follow the notations of [15]. Namely, if $\mathscr{R}$ is a sheaf of rings on a topological space, we denote by $\mathrm{Mod}(\mathscr{R})$ the category of left $\mathscr{R}$-modules and by $\mathsf{D}^*(\mathscr{R})$ the derived category $\mathsf{D}^*(\mathrm{Mod}(\mathscr{R}))$ $(* = +, -, \mathrm{b})$. In general and as usual, by a ring (resp. algebra, resp. module) we mean a sheaf of rings (resp. sheaf of algebras, resp. sheaf of modules).

A subcategory $\mathscr{S}$ of an abelian category $\mathscr{C}$ is called thick if it satisfies the following property: let

$$Y \to Y' \to X \to Z \to Z'$$

be an exact sequence in $\mathscr{C}$ such that Y, Y', Z, Z' are objects of $\mathscr{S}$; then X is an object of $\mathscr{S}$. Equivalently, $\mathscr{S}$ is a thick subcategory of $\mathscr{C}$ if and only if it satisfies the following two properties:

(i) $\mathscr{S}$ is a full abelian subcategory of $\mathscr{C}$;

(ii) given a short exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathscr{C}$, if two objects of X', X or X'' are objects of $\mathscr{S}$, then the third also is an object of $\mathscr{S}$.

By a Serre subcategory $\mathscr{S}$ of an abelian category $\mathscr{C}$ we mean that $\mathscr{S}$ is thick and that it contains all subobjects and quotient objects of its objects, i.e., if $0 \to X' \to X \to X'' \to 0$ is a short exact sequence in $\mathscr{C}$ and X is an object of $\mathscr{S}$, then both X' and X'' are objects of $\mathscr{C}$.

Let $\mathscr{R}$ be a sheaf of rings on a topological space X and let $\mathscr{M}$ be an $\mathscr{R}$-module. One says that $\mathscr{M}$ is locally finitely generated if, for each $x \in X$, there exist an open neighborhood U of x, an integer $N \geq 0$, and an epimorphism of $\mathscr{R}|_U$-modules on U, $(\mathscr{R}|_U)^{\oplus N} \to \mathscr{M}|_U \to 0$. One says that $\mathscr{M}$ is locally finitely presented if for each $x \in X$, there exists an open neighborhood U of x in which $\mathscr{M}|_U$ has a finite presentation, that is, there are integers $N_0, N_1 \geq 0$ and an exact sequence of $\mathscr{R}|_U$-modules on U

$$(\mathscr{R}|_U)^{\oplus N_1} \to (\mathscr{R}|_U)^{\oplus N_2} \to \mathscr{M}|_U \to 0.$$

One says that $\mathscr{M}$ is pseudocoherent if given any open set U and any finitely generated $\mathscr{R}|_U$-submodule $\mathscr{N}$ of $\mathscr{M}|_U$, then $\mathscr{N}$ is locally finitely presented. One says that $\mathscr{M}$ is coherent if it is pseudcoeherent and locally finitenely generated as an $\mathscr{R}$-module.

If $\mathscr{R}$ is a coherent ring (i.e. coherent as a module over itself), $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{R})$ denotes the full thick subcategory of $\mathrm{Mod}(\mathscr{R})$ consisting of coherent objects and $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{R})$ denotes the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathscr{R})$ consisting of complexes with coherent cohomology modules.

We denote by $\{\mathrm{pt}\}$ the zero-dimensional manifold consisting of a point and by $a_X : X \to \{\mathrm{pt}\}$ the constant map. If R is a noetherian ring, identified to a sheaf of rings over $\{\mathrm{pt}\}$, we write $\mathsf{D}^{\mathrm{b}}_{f}(R)$ instead of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(R)$.

Let $\mathbb{K}$ denote an unital commutative ring. Assume that A and B are two $\mathbb{K}$-algebras and assume that M is simultaneously a left A-module and a left B-module. We say that M is an (A, B)-module if the operations satisfy the following identities for $a \in A$, $b \in B$, $c \in \mathbb{K}$ and $m \in M$:

$$
\begin{aligned}
&a \cdot (b \cdot m) = b \cdot (a \cdot m) \\
&(c \cdot a) \cdot (b \cdot m) = (c \cdot b) \cdot (a \cdot m).
\end{aligned}
$$

Then, M is an (A, B)-module if and only if it is endowed with a structure of $A \otimes_{\mathbb{K}} B$-module.

For a ring $\mathscr{R}$ we denote by $\mathscr{R}^{\mathrm{op}}$ the opposite ring of $\mathscr{R}$, whose elements are $\mathscr{R}^{\mathrm{op}} = \{a^{\mathrm{op}} : a \in \mathscr{R}\}$ and the ring operation is given by $a^{\mathrm{op}} b^{\mathrm{op}} := (ba)^{\mathrm{op}}$. Hence, the notion of left $\mathscr{R}$-module (resp. right $\mathscr{R}$-module) coincide with the notion of right $\mathscr{R}^{\mathrm{op}}$-module (resp. left $\mathscr{R}^{\mathrm{op}}$-module).

In the sequel $\mathbb{C}^{\hbar}$ denotes the ring $\mathbb{C}[[\hbar]]$ of formal power series with complex coefficients and $\mathbb{C}^{\hbar,\mathrm{loc}}$ denotes the field $\mathbb{C}((\hbar))$ of Laurent series with complex coefficients.

When the base ring is $\mathbb{C}$ we may omit it. Namely, we shall write $F \otimes G$ and $\mathrm{R}\mathscr{H}om\,(F, G)$ instead of $F \otimes_{\mathbb{C}_X} G$ and $\mathrm{R}\mathscr{H}om_{\mathbb{C}_X}(F, G)$, respectively.

Chapter 1

Background

1.1 Elements of sheaf theory

In this section, we briefly recall some classical constructions for sheaves of $\mathbb{K}_X$-modules, X being a real analytic manifold of dimension n (unless otherwise stated) and $\mathbb{K}$ being a Noetherian unital commutative ring with finite global dimension (i.e. there exists a positive integer $n > 0$ such that any $\mathbb{K}$-module admits an injective resolution of length $\leq n$). We refer to [15] for a detailed study of such constructions and their extensive functorial properties.

Basic aspects. Recall that one canonically associates to any object $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{K}_X)$ its micro-support, denoted by $\mathrm{SS}(F)$, which is a closed involutive subset of the cotangent bundle T^*X. For the reader's convenience we recall the precise definiton:

Definition 1.1.1. Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{K}_X)$ and $p = (x_0; \xi_0) \in T^*X$. Then, $p \in T^*X \backslash \mathrm{SS}(F)$ if there exists an open neighborhood U of p such that for any $x_1 \in X$ and any real function ψ of class C^1 defined in a neighborhood of x_1 verifying $\psi(x_1) = 0$ and $d\psi(x_1) \in U$, we have:

$$(\mathrm{R}\Gamma_{\{x;\psi(x)\geq 0\}}(F))_{x_1} = 0.$$

Recall also that the support of an object $F \in \mathrm{Mod}(\mathbb{K}_X)$ is defined as the complementar in X of the union of the open sets U such that $F|_U = 0$ and it is denoted by $\mathrm{supp}(F)$. Moreover, the support of $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{K}_X)$ is the closed subset of X defined as the adherence of the union $\bigcup_j \mathrm{supp}(H^j(F))$. Denoting by $\pi : T^*X \to X$ the canonical projection, one has $\pi(\mathrm{SS}(F)) = \mathrm{supp}(F)$ for any $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{K}_X)$.

Recall also that a sheaf $F \in \mathrm{Mod}(\mathbb{K}_X)$ is called an $\mathbb{R}$-constructible sheaf if:

(i) the stalk F_x is finitely generated over $\mathbb{K}$ for any $x \in X$;

(ii) $\mathrm{SS}(F)$ is a closed conic subanalytic Lagrangian subset of T^*X.

If, in addition, $\mathrm{SS}(F)$ is $\mathbb{C}^\star$-conic (i.e. invariant for the action of $\mathbb{C}^\star$ on T^*X) then F is said a $\mathbb{C}$-constructible sheaf.

We denote by $\mathrm{Mod}_{\mathbb{R}-\mathrm{c}}(\mathbb{K}_X)$ and $\mathrm{Mod}_{\mathbb{C}-\mathrm{c}}(\mathbb{K}_X)$ the full thick subcategories of $\mathrm{Mod}(\mathbb{K}_X)$ consisting of $\mathbb{R}$-constructible sheaves and $\mathbb{C}$-constructible sheaves, respectively. We also denote by $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{K}_X)$ (resp. $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}-\mathrm{c}}(\mathbb{K}_X)$) the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbb{K}_X)$ consisting of complexes with $\mathbb{R}$-constructible (resp. $\mathbb{C}$-constructible) cohomology modules.

We shall make use of the following particular form of [15, Proposition 5.4.14]:

Proposition 1.1.2. *For $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{K}_X)$ and $G \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{K}_X)$ such that $\mathrm{SS}(F) \cap \mathrm{SS}(G) \subset T^*_X X$, the canonical morphism*

$$\mathrm{R}\mathscr{H}om_{\mathbb{K}_X}(G, \mathbb{K}_X) \overset{\mathrm{L}}{\otimes}_{\mathbb{K}_X} F \to \mathrm{R}\mathscr{H}om_{\mathbb{K}_X}(G, F)$$

is an isomorphism.

Consider a morphism of complex manifolds $f : X \to Y$. In the sequel we shall use the notations $\mathrm{R}f_*$, $\mathrm{R}f_!$, f^{-1} to represent, respectively, the functors of direct image, proper direct image and inverse image associated to f in the topological sense (i.e. in the framework of sheaf theory). We also denote by $f^!$ the right adjoint functor of $\mathrm{R}f_!$, the so-called Poincare-Verdier dual (cf.[15, Theorem 3.1.5]).

Let us denote by ω_X the dualizing sheaf in the category $\mathsf{D}^{\mathrm{b}}(\mathbb{K}_X)$ (cf. [15, Definition 3.1.16]). Since every real manifold is oriented, one has $\omega_X = a_X^!\mathbb{K} \simeq \mathbb{K}_X[n]$ (where the shift n corresponds to the real dimension of the manifold).

We shall use the following duality functors:

$$\begin{aligned} \mathrm{D}'_{\mathbb{K}_X} &: \mathsf{D}^{\mathrm{b}}(\mathbb{K}_X) \to \mathsf{D}^{\mathrm{b}}(\mathbb{K}_X), \quad F \mapsto \mathrm{R}\mathscr{H}om_{\mathbb{K}_X}(F, \mathbb{K}_X) \\ \mathrm{D}_{\mathbb{K}_X} &: \mathsf{D}^{\mathrm{b}}(\mathbb{K}_X) \to \mathsf{D}^{\mathrm{b}}(\mathbb{K}_X), \quad F \mapsto \mathrm{R}\mathscr{H}om_{\mathbb{K}_X}(F, \omega_X). \end{aligned} \tag{1.1.1}$$

It is well-known that they induce functors $\mathrm{D}'_{\mathbb{K}_X}, \mathrm{D}_{\mathbb{K}_X} : \mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{K}_X) \to \mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{K}_X)$ which, for any $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{K}_X)$, satisfy the following microlocal relation:

$$\mathrm{SS}(\mathrm{D}'_{\mathbb{K}_X} F) = \mathrm{SS}(\mathrm{D}_{\mathbb{K}_X} F) = \mathrm{SS}(F)^a.$$

If $X = \{\mathrm{pt}\}$ and $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{K}_{\{\mathrm{pt}\}})$, we use the following notation instead:

$$F^* := \mathrm{D}'_{\mathbb{K}_{\{\mathrm{pt}\}}} F = \mathrm{RHom}_{\mathbb{K}}(F, \mathbb{K}).$$

Sato's specialization. Let $Y \subset X$ be a submanifold of codimension l and consider the normal bundle to Y in X, denoted by $T_Y X$. Recall that $T_Y X$ is defined as the cokernel in the exact sequence below:

$$0 \to TY \to Y \times_X TX \to T_Y X \to 0.$$

Consider an open covering $X = \bigcup_i U_i$ and open embeddings $\varphi_i : U_i \hookrightarrow \mathbb{R}^n$, such that $U_i \cap Y = \varphi_i^{-1}(\{0\}^l \times \mathbb{R}^{n-l})$. Set $x = (x', x'') \in \mathbb{R}^l \times \mathbb{R}^{n-l}$ and

$$V_i = \left\{(x,t) \in \mathbb{R}^n \times \mathbb{R}; (tx', x'' \in \varphi_i(U_i)\right\}.$$

Denote by $t_{V_i} : V_i \to \mathbb{R}$ the projection $(x,t) \to t$, and by $p_{V_i} : V_i \to U_i$ the map defined by the correspondence $(x,t) \mapsto \varphi_i^{-1}(tx', x'')$. One defines the maps $\psi_{ji} : V_i \times_{U_i} (U_i \cap U_j) \to \mathbb{R}^n$ by setting $\psi_{ji}(x,t) = (\psi'_{ji}(x,t), \psi''_{ji}(x,t))$ with:

$$(t\psi'_{ji}(x,t), \psi''_{ji}(x,t)) = \varphi_j \varphi_i^{-1}(tx', x'').$$

Consider the following equivalence relation $\sim$ defined in the disjoint union $\bigsqcup_i V_i$:

$$(x_i, t_i) \in V_i \sim (x_j, t_j) \in V_j \Leftrightarrow t_i = t_j \wedge x_j = \psi_{ji}(x_i, t_i).$$

The quotient space $\widetilde{X}_Y := \bigsqcup_i V_i / \sim$ is a well-defined manifold, the so-called normal deformation of Y in X. It is endowed with two canonical maps $p : \widetilde{X}_Y \to X$ and $t : \widetilde{X}_Y \to \mathbb{R}$ that are locally defined by $p|_{V_i} = p_{V_i}$ and $t|_{V_i} = t_{V_i}$. The following properties hold:

(i) $p^{-1}(X \backslash Y)$ is isomorphic to $(X \backslash Y) \times (\mathbb{R} \backslash \{0\})$;

(ii) $t^{-1}(\mathbb{R} \backslash \{0\})$ is isomorphic to $X \times (\mathbb{R} \backslash \{0\}$;

(iii) $t^{-1}(0)$ is isomorphic to the normal bundle $T_Y X$.

Hence, one identifies $t^{-1}(0)$ with the normal bundle $T_Y X$. Denote by $s : T_Y X \hookrightarrow \widetilde{X}_Y$ the canonical embedding.

Set $\Omega := t^{-1}(\mathbb{R}^+) \overset{j}{\hookrightarrow} \widetilde{X}_Y$ and denote by $\widetilde{p}$ the restriction of p to Ω.

The following diagram summarizes the above data:

$$\begin{array}{ccccc} T_Y X & \overset{s}{\hookrightarrow} & \widetilde{X}_Y & \overset{j}{\hookleftarrow} & \Omega \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle p} & \swarrow{\scriptstyle \widetilde{p}} & \\ Y & \overset{i}{\hookrightarrow} & X. & & \end{array}$$

Definition 1.1.3. Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{K}_X)$ and set

$$\nu_Y^{\mathbb{K}}(F) := s^{-1} \mathrm{R} j_* \widetilde{p}^{-1}(F).$$

The object $\nu_Y^{\mathbb{K}}(F) \in \mathsf{D}^{\mathrm{b}}(\mathbb{K}_{T_Y X})$ is called the Sato's specialization of F along Y.

Moreover, the correspondence $F \mapsto \nu_Y^{\mathbb{K}}(F)$ yields a functor $\nu_Y^{\mathbb{K}}$ from $\mathsf{D}^{\mathrm{b}}(\mathbb{K}_X)$ to $\mathsf{D}^{\mathrm{b}}(\mathbb{K}_{T_Y X})$ that preserves the constructibility of sheaves, that is, it induces functors $\nu_Y^{\mathbb{K}} : \mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{K}_X) \to \mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{K}_{T_Y X})$ and $\nu_Y^{\mathbb{K}} : \mathsf{D}^{\mathrm{b}}_{\mathbb{C}-\mathrm{c}}(\mathbb{K}_X) \to \mathsf{D}^{\mathrm{b}}_{\mathbb{C}-\mathrm{c}}(\mathbb{K}_{T_Y X})$ (cf. Propositions 8.4.12 and 8.5.7 of [15]).

Vanishing cycles and nearby-cycles. Let us assume now that we are in the complex setting: X is a complex analytic manifold of finite dimension and $Y \subset X$ is a complex closed smooth hypersurface of X given by the zero locus of a holomorphic function $f : X \to \mathbb{C}$.

Let $q : \widetilde{\mathbb{C}}^\star \to \mathbb{C}$ denote the universal covering of $\mathbb{C}^\star$ given by $q(z) = e^{2\pi i z}$. Denote by $\widetilde{q} : X \times_{\mathbb{C}} \widetilde{\mathbb{C}}^\star \to X$ the map $\widetilde{q} = \mathrm{id} \times_{\mathbb{C}} q$ and consider the diagram below:

$$\begin{array}{ccccc} & & \widetilde{X}^\star & \longrightarrow & \widetilde{\mathbb{C}}^\star \\ & & \Big\downarrow{\scriptstyle \widetilde{q}} & \square & \Big\downarrow{\scriptstyle q} \\ Y & \xhookrightarrow{\ i\ } & X & \xrightarrow{\ f\ } & \mathbb{C}. \end{array}$$

In this setting, one has the following constructions first introduced by A. Grothendieck and P. Deligne:

Definition 1.1.4. Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{K}_X)$ and set

$$\psi_Y^{\mathbb{K}}(F) := i^{-1}\mathrm{R}\widetilde{q}_*\widetilde{q}^{-1}(F).$$

The object $\psi_Y^{\mathbb{K}}(F) \in \mathsf{D}^{\mathrm{b}}(\mathbb{K}_Y)$ is called the nearby-cycle sheaf associated to $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{K}_X)$.

The mapping cone of the canonical morphism $i^{-1}F \to \psi_Y^{\mathbb{K}}(F)$ is an object $\varphi_Y^{\mathbb{K}}(F) \in \mathsf{D}^{\mathrm{b}}(\mathbb{K}_Y)$, which one calls the vanishing-cycle sheaf associated to F.

For each $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{K}_X)$, one gets the distinguished triangle

$$i^{-1}F \to \psi_Y^{\mathbb{K}}(F) \to \varphi_Y^{\mathbb{K}}(F) \overset{+1}{\to} .$$

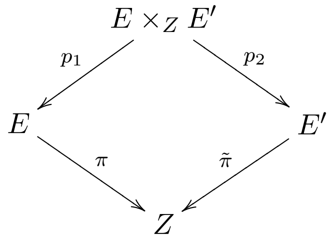
In literature, the morphism $\psi_Y^{\mathbb{K}}(F) \to \varphi_Y^{\mathbb{K}}(F)$ is usually called the *canonical morphism* and denoted by can.

The correspondences $F \mapsto \psi_Y^{\mathbb{K}}(F)$ and $F \mapsto \varphi_Y^{\mathbb{K}}(F)$ set up two functors from $\mathsf{D}^{\mathrm{b}}(\mathbb{K}_X)$ to $\mathsf{D}^{\mathrm{b}}(\mathbb{K}_Y)$, respectively called the nearby-cycle functor and the vanishing cycle functor (of sheaf theory). These functors also preserve the constructibility of sheaves (cf. Theorem 2.2-1 of [22]).

Fourier-Sato transform and microlocalization. Consider the multiplicative group of positive numbers $\mathbb{R}^+$ and suppose now that X is a locally compact topological space endowed with an action of $\mathbb{R}^+$. Denote by $\mathrm{Mod}_{\mathbb{R}^+}(\mathbb{K}_X)$ the full thick subcategory of $\mathrm{Mod}(\mathbb{K}_X)$ consisting of sheaves F such that, for any orbit b of $\mathbb{R}^+$ in X, $F|_b$ is a locally constant sheaf. Denote by $\mathsf{D}^+_{\mathbb{R}^+}(\mathbb{K}_X)$ the full triangulated subcategory of $\mathsf{D}^+(\mathbb{K}_X)$ consisting of complexes F such that $H^j(F) \in \mathrm{Mod}_{\mathbb{R}^+}(\mathbb{K}_X)$ for all $j \in \mathbb{Z}$. An object of $\mathsf{D}^+_{\mathbb{R}^+}(\mathbb{K}_X)$ is called a conic sheaf (or a conic complex of sheaves).

Hereafter $E \overset{\pi}{\to} Z$ denotes a real vector bundle on a real analytic manifold Z. and $E' \overset{\tilde{\pi}}{\to} Z$ denotes its dual bundle. Denote by p_1 and p_2 the canonical projections from

$E \times_Z E'$ to E and E', respectively. We have the following diagram:



Set $P = \{(x, y) \in E \times_Z E' : \langle x, y\rangle \geq 0\}$. Denote by $i : P \hookrightarrow E \times_Z E'$ the closed embedding of P. One has $\mathrm{R}\Gamma_P = i_! i^!$.

Definition 1.1.5. Let $F \in \mathsf{D}^+_{\mathbb{R}^+}(\mathbb{K}_E)$ and set

$$\mathscr{F}^{\mathbb{K}}(F) := \mathrm{R}p_{2*} \circ \mathrm{R}\Gamma_P \circ p_1^{-1}(F).$$

The object $\mathscr{F}^{\mathbb{K}}(F) \in \mathsf{D}^+_{\mathbb{R}^+}(\mathbb{K}_{E'})$ is called the Fourier-Sato transform of F.

Moreover, the correspondence $F \mapsto \mathscr{F}^{\mathbb{K}}(F)$ defines a functor from $\mathsf{D}^+_{\mathbb{R}^+}(\mathbb{K}_E)$ to $\mathsf{D}^+_{\mathbb{R}^+}(\mathbb{K}_{E'})$, the so-called Fourier-Sato functor.

We consider now the case where X is a real analytic manifold, $Y \subset X$ is a submanifold of X, $E = T_Y X$ is the normal bundle to Y in X and $E' = T^*_Y X$ is the conormal bundle to Y in X. Recall that $T^*_Y X$ is defined as being the kernel of the short exact sequence below:

$$0 \to T^*_Y X \to Y \times_X T^*X \to T^*Y \to 0.$$

In such setting, one has the Fourier-Sato functor $\mathscr{F}^{\mathbb{K}} : \mathsf{D}^+_{\mathbb{R}^+}(\mathbb{K}_{T_Y X}) \to \mathsf{D}^+_{\mathbb{R}^+}(\mathbb{K}_{T^*_Y X})$.

It is well-known that the specialization $\nu_Y(F)$ of any $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{K}_X)$ is a conic sheaf, that is, it belongs to $\mathsf{D}^+_{\mathbb{R}^+}(\mathbb{K}_{T_Y X})$ (cf. [15, Theorem 4.2.3]). Thus the following definition makes sense:

Definition 1.1.6. Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{K}_X)$ and set

$$\mu^{\mathbb{K}}_Y(F) := \mathscr{F}^{\mathbb{K}} \circ \nu^{\mathbb{K}}_Y(F).$$

The object $\mu^{\mathbb{K}}_Y(F) \in \mathsf{D}^{\mathrm{b}}(\mathbb{K}_{T^*_Y X})$ is called the geometrical microlocalization of F.

The correspondence $F \mapsto \mu^{\mathbb{K}}_Y$ defines a functor from $\mathsf{D}^{\mathrm{b}}(\mathbb{K}_X)$ to $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}+}(\mathbb{K}_{T^*_Y X})$, which one calls the Sato's microlocalization functor.

The Mittag-Leffler condition. We shall deal in the sequel with projective limits of sheaves of $\mathbb{K}$-modules. For that purpose, recall that the projective limit is, in general, a left exact functor and recall the following notion for projective systems of abelian groups:

Definition 1.1.7. Let $X = \{X_n, \rho_{n,p}\}$ be a projective system of abelian groups. One says that X satisfies the Mittag-Leffler condition if the decreasing sequence $\{\rho_{n,p}(X_p)\}_{p \geq n}$ of subgroups of X_n is stationary for any $n \in \mathbb{N}$.

Proposition 1.1.8. *Let $0 \to X' \to X \to X'' \to 0$ be an exact sequence of projective systems of abelian groups. Assume that X' satisfies the Mittag-Leffler condition. Then the sequence*

$$0 \varprojlim X' \to \varprojlim X \to \varprojlim X'' \to 0$$

is exact.

1.2 A survey on $\mathscr{D}$-modules theory

Basic aspects. Let X denote a complex analytic manifold of complex dimension d_X. Denote by $\mathscr{O}_X$ the sheaf of holomorphic functions on X and by Θ_X the sheaf of holomorphic vector fields on X. The sheaf of holomorphic differential operators on X is the subalgebra of $\mathscr{H}om_{\mathbb{C}_X}(\mathscr{O}_X, \mathscr{O}_X)$ generated by $\mathscr{O}_X$ and by Θ_X and it is denoted by $\mathscr{D}_X$.

Given a local system $(x) = (x_1, ..., x_n)$ of holomorphic coordinates on X, a differential operator $P \in \mathscr{D}_X$ can be written as a finite sum

$$P(x; D_x) = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha; \quad a_\alpha \in \mathscr{O}_X, \quad m \in \mathbb{N}_0,$$

where $\alpha = (\alpha_1, ..., \alpha_n)$, $|\alpha| = \alpha_1 + ... + \alpha_n$ and $D_x^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1} \cdots \frac{\partial^{\alpha_n}}{\partial x_n}$. The integer m doesn't depend on the choice of the local system and it is called the order of the operator P.

The multiplication in $\mathscr{D}_X$ results from the Leibniz rule, thus $\mathscr{D}_X$ is clearly a non-commutative ring. For example, for $\nu \in \Theta_X$ and $a \in \mathscr{O}_X$ one has $[\nu, a] = \nu(a)$. Hence, the notions of left $\mathscr{D}_X$-module and right $\mathscr{D}_X$-module don't coincide. It turns out, however, that one can often choose to treat only left or right $\mathscr{D}_X$-modules, since the study of the remaining case turns out to be similar. Indeed, the functor

$$\mathrm{Mod}(\mathscr{D}_X) \to \mathrm{Mod}(\mathscr{D}_X^{\mathrm{op}}), \quad \mathscr{M} \mapsto \Omega_X \otimes_{\mathscr{O}_X} \mathscr{M},$$

is an equivalence of categories between the category of left $\mathscr{D}_X$-modules and the category of left modules over the opposite ring $\mathscr{D}_X^{\mathrm{op}}$, which coincide with the category of right $\mathscr{D}_X$-modules. We refer to Corollary 1.11 of [14] for details.

Let $\mathscr{M}$ be an $\mathscr{O}_X$-module. In order to construct a structure of left (resp. right) $\mathscr{D}_X$-module on $\mathscr{M}$, it is enough to define a *compatible* left action (resp. right action) of Θ_X on $\mathscr{M}$. More precisely, one has (cf. [14, Lemma 1.7] for details):

Lemma 1.2.1. *Consider a left action of Θ_X on $\mathscr{M}$, say $\Theta_X \times \mathscr{M} \to \mathscr{M}$, $\quad (\nu, m) \mapsto \nu \cdot m$, and suppose that it satisfies the following identities for $\lambda \in \mathbb{C}$, $a \in \mathscr{O}_X$, $\nu, \eta \in \Theta_X$ and $m \in \mathscr{M}$:*

(a) $(a\nu) \cdot m = a(\nu \cdot m)$ *and* $\nu \cdot (\lambda m) = \lambda(\nu \cdot m)$,

(b) $\nu \cdot (am) = a(\nu \cdot m) + v(a)m$,

(c) $[\nu, \eta] \cdot m = \nu \cdot (\eta \cdot m) - \eta \cdot (\nu \cdot m)$.

Then, there exists a unique (left) $\mathscr{D}_X$-module structure on $\mathscr{M}$ that extends the actions of $\mathscr{O}_X$ and Θ_X.

The family formed by the ascending sequence of sets

$$F_m(\mathscr{D}_X) = \{P \in \mathscr{D}_X | \text{P has order } m\}, m \in \mathbb{Z},$$

satisfies:

(i) $\mathscr{D}_X = \bigcup_m F_m(\mathscr{D}_X)$;

(ii) $F_m(\mathscr{D}_X)F_l(\mathscr{D}_X) \subset F_{m+l}(\mathscr{D}_X)$, for all $m, l \in \mathbb{Z}$;

(iii) $[F_m(\mathscr{D}_X), F_l(\mathscr{D}_X)] \subset F_{m+l-1}(\mathscr{D}_X)$, for all $m, l \in \mathbb{Z}$.

This means that $F_m(\mathscr{D}_X)_{m\in\mathbb{Z}}$ is a filtration of the ring $\mathscr{D}_X$, the so-called order filtration. Note that $F_0(\mathscr{D}_X) = \mathscr{O}_X$ and that $F_m(\mathscr{D}_X) = 0$ if $m < 0$. The graded ring defined through such filtration is naturally identified with the symmetric algebra of Θ_X over $\mathscr{O}_X$:

$$\mathrm{Gr}^F(\mathscr{D}_X) := \bigoplus_m \frac{F_{m+1}(\mathscr{D}_X)}{F_m(\mathscr{D}_X)} \simeq \mathscr{O}_X \otimes \mathbb{C}[\xi_1, \dots, \xi_n].$$

Denote by $\pi : T^*X \to X$ the canonical projection. Then, $\pi_X^{-1}\mathrm{Gr}^F(\mathscr{D}_X)$ is a subsheaf of $\mathscr{O}_{T^*X}$.

It is also well-known that any coherent $\mathscr{D}_X$-module $\mathscr{M}$ admits locally a good filtration, that is, an ascending sequence of subsheaves $F_m(\mathscr{M}), m \in \mathbb{Z}$, satisfying the following properties:

(i) $\mathscr{M} = \bigcup_m F_m(\mathscr{M})$;

(ii) $F_m(\mathscr{D}_X)F_l(\mathscr{M}) \subset F_{m+l}(\mathscr{D}_X)$, for all $m, l \in \mathbb{Z}$;

(iii) $\bigoplus_m F_m(\mathscr{D}_X)$ is locally finitely generated as a $\bigoplus_m F_m(\mathscr{D}_X)$-module.

The graded module $\mathrm{Gr}^F(\mathscr{M}) := \bigoplus_m F_{m+1}(\mathscr{M})/F_m(\mathscr{M})$ has a natural structure of coherent $\mathrm{Gr}^F(\mathscr{D}_X)$-module.

Definition 1.2.2. For a coherent $\mathscr{D}_X$-module $\mathscr{M}$ set:

$$\mathrm{char}(\mathscr{M}) := \mathrm{supp}(\mathscr{O}_{T^*X} \otimes_{\pi^{-1}\mathrm{Gr}^F(\mathscr{D}_X)} \pi^{-1}\mathrm{Gr}^F(\mathscr{M})).$$

The object $\mathrm{char}(\mathscr{M}) \subset T^*X$ is called the characteristic variety of $\mathscr{M}$.

Recall that the notion of characteristic variety is independent of the choice of the coherent filtration and extends to the derived category as follows: for an object $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, one sets

$$\mathrm{char}(\mathscr{M}) := \bigcup_{j\in\mathbb{Z}} \mathrm{char}_f(\mathscr{H}^j(\mathscr{M})).$$

Recall also that $\mathrm{char}(\mathscr{M})$ is a closed conic analytic involutive subset of the cotangent bundle T^*X, for any $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$.

Denote by $I_{\mathrm{char}(\mathscr{M})}$ the defining ideal of $\mathrm{char}(\mathscr{M})$ in $\mathrm{Gr}^F(\mathscr{D}_X)$.

Definition 1.2.3. An $\mathscr{O}_X$-module $\mathscr{F}$ is good if for every relatively compact open subset $U \subset X$ there exists a directed family $\{G_i\}_i$ of coherent $\mathscr{O}_U$-submodules of $F|_U$ such that $F|_U = \sum_i G_i$. (Directed family means here that for any i and j there exists k such that $G_i + G_j \subset G_k$).

Definition 1.2.4. A coherent $\mathscr{D}_X$-module $\mathscr{M}$ is good if it is good as an $\mathscr{O}_X$-module.

A coherent $\mathscr{D}_X$-module $\mathscr{M}$ is holonomic if $\mathrm{char}(\mathscr{M})$ is a langragian subvariety of T^*X.

An holonomic $\mathscr{D}_X$-module $\mathscr{M}$ is regular holomic if it locally admits a coherent filtration $F(\mathscr{M})$ satisfying $I_{\mathrm{char}(\mathscr{M})} Gr^F(\mathscr{M}) = 0$.

Remark 1.2.5. It is not known wether any coherent $\mathscr{D}_X$-module is good, but it is well-known that any coherent $\mathscr{D}_X$-module is locally good, which is enough to most applications.

Let $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ denote an exact sequence of coherent $\mathscr{D}_X$-modules. It is well-known that $\mathscr{M}'$ and $\mathscr{M}''$ are holonomic (resp. regular holonomic) if and only if $\mathscr{M}$ is holonomic (resp. regular holonomic). This is a consequence of the additive property of characteristic varieties. Moreover, the same property holds replacing holonomic with good.

One denotes by $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{D}_X)$(resp. $\mathrm{Mod}_{\mathrm{hol}}(\mathscr{D}_X)$; resp. $\mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X)$) the full Serre subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ consisting of good (resp. holonomic; resp. regular holonomic) and by $\mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_X)$ (resp. $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$, resp. $\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$) the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ consisting of complexes with good (resp. holonomic, resp. regular holonomic) cohomology modules.

Duality functors. In $\mathscr{D}_X$-modules theory one makes use of the following duality functor:

$$\underline{\mathrm{D}}_{\mathscr{D}_X} : \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X), \quad \mathscr{M} \mapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1})[d_X].$$

It verifies $\underline{\mathrm{D}}_{\mathscr{D}_X}\underline{\mathrm{D}}_{\mathscr{D}_X}(\mathscr{M}) \simeq \mathscr{M}$. One may also need to use the *prime-dual*:

$$\mathrm{D}'_{\mathscr{D}_X} : \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X), \quad \mathscr{M} \mapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X).$$

Both functors preserve characteristic varieties, that is, for $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, one has $\mathrm{char}(\underline{\mathrm{D}}_{\mathscr{D}_X}(\mathscr{M})) = \mathrm{char}(\mathrm{D}'_{\mathscr{D}_X}(\mathscr{M})) = \mathrm{char}(\mathscr{M})$. Therefore, both functors preserve holonomicity and regular holonomicity.

Direct images and inverse images. Let $f : X \to Y$ be a morphism of complex analytic manifolds. Recall that the object

$$\mathscr{D}_{X\to Y} := \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y$$

has a structure of $(\mathscr{D}_X, f^{-1}\mathscr{D}_Y^{\mathrm{op}})$-bimodule. The right action of $f^{-1}\mathscr{D}_Y$ is induced by the multiplication of $f^{-1}\mathscr{D}_Y$ on itself. The left action of $\mathscr{D}_X$ is induced by the natural action of $\mathscr{O}_X$ together with a compatible action of Θ_X defined as follows: let $v \in \Theta_X$, denote by $f' : TX \to X \times_Y TY$ the differential map and note that f' applied to v defines a unique section $f'(v) \in \mathscr{D}_{X\to Y}$ which can be locally written as a finite sum $\sum_j a_j \otimes w_j$; one sets for any $a \otimes u \in \mathscr{D}_{X\to Y}$:

$$v \cdot (a \otimes u) := v(a) \otimes u + \sum_j a a_j \otimes w_j \circ u.$$

This definition doesn't depend on the choice of $\sum_j a_j \otimes w_j$.

Lemma 1.2.6. (a) *If f is smooth, then $\mathscr{D}_{X\to Y}$ is a coherent $\mathscr{D}_X$-module and a flat $f^{-1}\mathscr{D}_Y^{\mathrm{op}}$-module.*

(b) *If f is an embedding, then $\mathscr{D}_{X\to Y}$ is a coherent $f^{-1}\mathscr{D}_Y^{\mathrm{op}}$-module.*

(c) *$\mathscr{D}_{X\to Y}$ is flat over $\mathscr{O}_X$.*

The object $\mathscr{D}_{X\to Y}$ is commonly called the transfer module, since its bimodular structure allow us to define the direct image and inverse image functors in $\mathscr{D}$-modules theory as follows:

Definition 1.2.7. (a) The functor of direct image by f, denoted by $\underline{f}_*$, is defined by:

$$\begin{array}{rcl} \underline{f}_* : \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{op}}) & \to & \mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y^{\mathrm{op}}), \\ \mathscr{M} & \mapsto & \mathrm{R}f_*(\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X\to Y}). \end{array}$$

(b) The functor of proper direct image by f, denoted by $\underline{f}_!$, is defined by:

$$\begin{array}{rcl} \underline{f}_! : \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{op}}) & \to & \mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y^{\mathrm{op}}), \\ \mathscr{M} & \mapsto & \mathrm{R}f_!(\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X\to Y}). \end{array}$$

(c) The functor of inverse image by f, denoted by $\underline{f}^*$, is defined by:

$$\begin{array}{rcl} \underline{f}^* : \mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y) & \to & \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X), \\ \mathscr{N} & \mapsto & \mathscr{D}_{X\to Y} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} f^{-1}\mathscr{N}. \end{array}$$

The functor of extraordinary inverse image by f, denoted by $\underline{f}^!$, is defined by:

$$\begin{array}{rcl} \underline{f}^! : \mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y) & \to & \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X), \\ \mathscr{M} & \mapsto & \underline{\mathrm{D}}_{\mathscr{D}_X} \underline{f}^*(\underline{\mathrm{D}}_{\mathscr{D}_Y}(\mathscr{M})). \end{array}$$

Remark 1.2.8. Let $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y)$. The following isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ provides an alternative way to define the inverse image of $\mathscr{D}$-modules:

$$\mathscr{O}_X \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{N} \simeq \underline{f}^*\mathscr{N}.$$

Theorem 1.2.9. *Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_X{}^{\mathrm{op}})$ (resp. $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$) and suppose that $f : X \to Y$ is a morphism of complex manifolds which is proper when restricted to the support of $\mathscr{M}$. Then, $\underline{f}_*(\mathscr{M}) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_Y{}^{\mathrm{op}})$ (resp. $\underline{f}_*(\mathscr{M}) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_Y)$).*

Let us consider the canonical morphisms:

$$T^*Y \xleftarrow{f_\pi} X \times_Y T^*Y \xrightarrow{f_d} T^*X. \tag{1.2.1}$$

Definition 1.2.10. A coherent $\mathscr{D}_Y$-module $\mathscr{M}$ is non-characteristic for f if

$$f_\pi^{-1}(\mathrm{char}(\mathscr{M})) \cap \ker f_d \subset X \times_Y T^*_Y Y$$

Denote by $\mathrm{Mod}_{\mathrm{NC(f)}}(\mathscr{D}_{\mathrm{Y}})$ the full Serre subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y)$ consisting of coherent $\mathscr{D}_Y$-modules that are non-characteristic for f and by $\mathsf{D}^{\mathrm{b}}_{\mathrm{NC}(f)}(\mathscr{D}_Y)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y)$ consisting of complexes with non-characteristic cohomology modules.

Theorem 1.2.11. *Let $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y)$ be non-characteristic for a morphism of complex manifolds $f : X \to Y$. Then:*

$$\begin{cases} H^k(\underline{f}^*(\mathscr{M})) = 0, \ \textit{for } k \neq 0; \\ H^0(\underline{f}^*(\mathscr{M})) \textit{is a coherent } \mathscr{D}_X\textit{-module.} \end{cases}$$

Moreover, the characteristic variety of $\mathscr{M}$ and its inverse image are related by the formula $\mathrm{char}(\underline{f}^*(\mathscr{M})) = f_d f_\pi^{-1}\mathrm{char}(\mathscr{M})$.

We recall also the next theorem, proved by Kashiwara in [10]:

Theorem 1.2.12. *Let $\mathscr{M}$ be an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_Y)$ (resp. $\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_Y)$). Then, $\underline{f}^*\mathscr{M}$ is an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$ (resp. $\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$).*

The next theorem due to Kashiwara is called the Cauchy-Kowalewskaia-Kashiwara theorem (we shall speak of CKK theorem, for short) since one can deduce from it the following result on the existence and uniqueness of solutions of a system of PDE's, known as the Cauchy-Kowalewskaia theorem: for given holomorphic functions $h, a_0, ..., a_{m-1}$ on $\mathbb{C}^n$ and for a differential operator $P \in \mathscr{D}_{\mathbb{C}^n}$ of order m, the system of equations:

$$\begin{cases} Pg = h, \\ \partial^j_{x_1} g|_X = a_j, \quad 0 \le j \le m-1. \end{cases}$$

has a unique solution $u \in \mathscr{O}_{\mathbb{C}^n}$.

Theorem 1.2.13. *Let $f : X \to Y$ be a morphism of complex manifolds and $\mathscr{M} \in \mathrm{Mod}_{\mathrm{NC(f)}}(\mathscr{D}_{\mathrm{Y}})$. Then, the canonical morphism*

$$f^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}, \mathscr{O}_Y) \to \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\underline{f}^*(\mathscr{M}), \mathscr{O}_X)$$

is an isomorphism.

Theorems 1.2.11 and 1.2.13 also hold for objects in $\mathsf{D}^{\mathrm{b}}_{\mathrm{NC}(f)}(\mathscr{D}_Y)$.

Theorem 1.2.14. *Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_Y)$. Then, $\underline{f}^!(\mathscr{M}) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$.*

The f-characteristic variety of a coherent $\mathscr{D}$-module. We have already recalled the notion of characteristic variety of a coherent $\mathscr{D}$-module, but, in view of our purpose (namely, in view of Chapter 4), it is convenient to go a step further, recalling a more general construction: the characteristic variety of a coherent $\mathscr{D}_X$-module depending on a morphism $f : X \to Y$ of complex analytic manifolds. We follow the lines of [34].

This construction uses relative $\mathscr{D}$-modules, but we don't give details of this notion since it won't be used again in the sequel. We refer to [33] for a comprehensive study of relative $\mathscr{D}$-modules.

Let S be an analytic manifold. A relative analytic manifold over S is an analytic manifold X endowed with a surjective analytic submersion $\varepsilon : X \to S$. The relative cotangent bundle $T^*X|S$ is defined as the cokernel of the injective dual map $\varepsilon_d : X \times_S T^*S \to T^*X$. Hence, there is an exact sequence

$$0 \to X \times_S T^*S \xrightarrow{\varepsilon_d} T^*X \xrightarrow{\rho} T^*X|S \to 0.$$

One says that $\nu \in \Theta_X$ is a vertical holomorphic vector field of $X|S$ if $\nu(h \circ \varepsilon) = 0$, for any $h \in \mathscr{O}_S$. One denotes by $\Theta_{X|S}$ the sheaf of vertical holomorphic vector fields of $X|S$. The ring of relative differential operators, denoted by $\mathscr{D}_{X|S}$, is defined as the subalgebra of $\mathscr{H}om_{\mathbb{C}_X}(\mathscr{O}_X, \mathscr{O}_X)$ generated by $\Theta_{X|S}$ and $\mathscr{O}_X$.

Similarly to $\mathscr{D}_X$, the ring $\mathscr{D}_{X|S}$ is filtered by the order of the operators. Therefore, the usual techniques apply to define the characteristic variety of $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_{X|S})$. Such characteristic variety is denoted by $\mathrm{char}_{X|S}(\mathscr{M})$ and it is a conic analytic subset of the relative cotangent bundle $T^*X|S$.

Let $f : X \to Y$ be an analytic submersion and consider the associated exact sequence of vector bundles on X:

$$0 \to X \times_Y T^*Y \xrightarrow{f_d} T^*X \xrightarrow{\rho} T^*X|Y \to 0.$$

The f-characteristic variety $\mathrm{char}_f(\mathscr{M})$ of $\mathscr{M}$ with respect to f is the subset of T^*X which coincides on T^*U with $\rho^{-1}\mathrm{char}_{U|Y}(\mathscr{L})$ for any open subset U and any coherent $\mathscr{D}_{U|Y}$-submodule $\mathscr{L}$ of $\mathscr{M}|_U$ which generates $\mathscr{M}|_U$ as a $\mathscr{D}_U$-module, that is, $\mathscr{M}|_U \simeq \mathscr{D}_U \otimes_{\mathscr{D}_{U|Y}} \mathscr{L}$.

More generally, consider a morphism of complex analytic manifolds $f : X \to Y$, not necessarily smooth, and consider the associated graph factorization $X \xrightarrow{i} X \times Y \xrightarrow{q} Y$. One defines the f-characteristic variety of $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ by

$$\mathrm{char}_f(\mathscr{M}) := i_d i_\pi^{-1} \mathrm{char}_q(\underline{i}_!(\mathscr{M})).$$

The two definitions are compatible, that is, they coincide in the case of smooth morphisms. Moreover, the definition extends as usual to the derived category: for an object $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ one sets

$$\mathrm{char}_f(\mathscr{M}) = \bigcup_{j \in \mathbb{Z}} \mathrm{char}_f(\mathscr{H}^j(\mathscr{M})).$$

Recall also that the f-characteristic variety of $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ is a closed conic analytic subvariety of T^*X.

The De Rham functor. For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ set:

$$\mathrm{Sol}_{\mathscr{D}_X}(\mathscr{M}) := \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X).$$

One says that $\mathrm{Sol}_{\mathscr{D}_X}(\mathscr{M})$ is the complex of holomorphic solutions of $\mathscr{M}$. The correspondence $\mathscr{M} \mapsto \mathrm{Sol}_{\mathscr{D}_X}(\mathscr{M})$ defines a functor from $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)^{\mathrm{op}}$ to $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)$ and the study of this functor constitutes one of the central aspects of $\mathscr{D}$-modules theory.

For any morphism of complex manifolds $f : X \to Y$, one has the following microlocal relation (cf. [34, Theorem 2.13]):

$$\mathrm{SS}(\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X \to Y}) \subset \mathrm{char}_f(\mathscr{M}). \tag{1.2.2}$$

When f is the constant map $a_X : X \to \{\mathrm{pt}\}$, then $\mathrm{char}_{a_X}(\mathscr{M})$ coincides with $\mathrm{char}(\mathscr{M})$ and the inclusion (1.2.2) gives the estimative:

$$\mathrm{SS}(\mathrm{Sol}(\mathscr{M})) \subset \mathrm{char}(\mathscr{M}).$$

In fact, one has always the equality $\mathrm{SS}(\mathrm{Sol}(\mathscr{M})) = \mathrm{char}(\mathscr{M})$.

We shall also use the De Rham functor for $\mathscr{D}_X$-modules:

$$\mathrm{DR}_{\mathscr{D}_X} : \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X) \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X), \quad \mathscr{M} \mapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{M}).$$

The solutions and De Rham functors are related by the formula:

$$\mathrm{Sol}_{\mathscr{D}_X}(\mathscr{M}) \simeq \mathrm{D}'_{\mathbb{C}}(\mathrm{DR}_{\mathscr{D}_X}\mathscr{M}).$$

It is also well-known that, for $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$, $\mathrm{Sol}_{\mathscr{D}_X}(\mathscr{M})$ and $\mathrm{DR}_{\mathscr{D}_X}(\mathscr{M})$ are objects of $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}-\mathrm{c}}(\mathbb{C}_X)$. Moreover, $\mathrm{Sol}_{\mathscr{D}_X} : \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}_{\mathbb{C}-\mathrm{c}}(\mathbb{C}_X)$ and $\mathrm{DR}_{\mathscr{D}_X}\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X) \to \mathsf{D}^{\mathrm{b}}_{\mathbb{C}-\mathrm{c}}(\mathbb{C}_X)$ are equivalences of categories. This result is due to Kashiwara [13] and it is known as the Riemann-Hilbert correspondence.

$\mathscr{D}$-modules on fiber bundles. For a complex vector bundle $E \xrightarrow{\pi} Z$ on a complex analytic manifold Z, one denotes by $\mathscr{D}_{[E]}$ the sheaf of homogeneous differential operators over E. Let us briefly recall how this sheaf is constructed. Let $\mathscr{O}_{[E]}$ denote the sheaf of holomorphic functions on E whose fibers have polynomial coefficients. Then, a differential operator $P \in \mathscr{D}_E$ is homogeneous (i.e. P belongs to $\mathscr{D}_{[E]}$) if P is locally written as a finite sum $P = \sum_\alpha a_\alpha \partial^\alpha$, where each a_α belongs to $\mathscr{O}_{[E]}$. The study of the ring $\mathscr{D}_{[E]}$ is similar to the study of $\mathscr{D}_X$ (one can speak of filtrations and graded ring).

Specialization, vanishing cycles and nearby-cycles for $\mathscr{D}$-modules. We recall now the notion of specialization of $\mathscr{D}_X$-modules (along a submanifold) as developed in the work of M. Kashiwara ([11]). For the basic material besides [11], we refer to [20], [22] and [27].

Let $Y \subset X$ be a submanifold of X and denote by I the defining ideal of Y and by $\pi : T_Y X \to Y$ the projection of the normal bundle to Y. One denotes by $V_Y{}^\bullet(\mathscr{D}_X)$ (or by $V^\bullet(\mathscr{D}_X)$ for short, once Y is fixed) the descending V-filtration of $\mathscr{D}_X$ with respect to Y:

$$V^k(\mathscr{D}_X) := \left\{ P \in \mathscr{D}_X : P(I^j) \subset I^{j+k}, \forall j,k \in \mathbb{Z}\ j, j+k \geq 0 \right\}$$

Note that $V^0(\mathscr{D}_X)$ is a subring of $\mathscr{D}_X$ that contains $\mathscr{O}_X$.

Consider the graded ring

$$\mathrm{gr}_V(\mathscr{D}_X) := \bigoplus_{k\in\mathbb{Z}} \frac{V^k(\mathscr{D}_X)}{V^{k+1}(\mathscr{D}_X)}.$$

It is well-known that there is an isomorphism of rings $\mathrm{gr}_V(\mathscr{D}_X) \simeq \pi_* \mathscr{D}_{[T_Y X]}$, where $\mathscr{D}_{[T_Y X]}$ denotes the sheaf of homogeneous differential operators over $T_Y X$.

Let $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$. One says that a family $\{V^\bullet(\mathscr{M})\}_{j\in\mathbb{Z}}$ is a good V-filtration of $\mathscr{M}$ if the following properties hold:

(i) $\mathscr{M} = \bigcup_{j\in\mathbb{Z}} V^j(\mathscr{M})$;

(ii) $V^k(\mathscr{D}_X)V^j(\mathscr{M}) \subset V^{k+j}(\mathscr{M})$, for all $k, j \in \mathbb{Z}$;

(iii) $V^k(\mathscr{D}_X)V^j(\mathscr{M}) = V^{k+j}(\mathscr{M})$ for $j \gg 0$ and $k \geq 0$ or $j \ll 0$ and $k \leq 0$;

(iv) $V^j(\mathscr{M})$ is a coherent $V^0(\mathscr{D}_X)$-module for every $j \in \mathbb{Z}$.

Denote by θ the Euler field on $T_Y X$: in local coordinates $(x_1, ..., x_n)$ on X, one has $\theta = \sum x_i \partial_i$. Hence, the action of θ in I/I^2 coincides with the identity.

Definition 1.2.15. A coherent $\mathscr{D}_X$-module $\mathscr{M}$ is specializable along Y if for every good-V filtration $V^\bullet(\mathscr{M})$ on $\mathscr{M}$ there is locally a non-zero polynomial $b \in \mathbb{C}[s]$ such that

$$b(\theta - k)V^k(\mathscr{M}) \subset V^{k+1}(\mathscr{M}), \quad \forall k \in \mathbb{Z}.$$

b is called a Bernstein-Sato polynomial or a b-function associated to the filtration $V^\bullet$.

In the sequel, when there is no risk of confusion, we often write *specializable* instead of *specializable along* Y, once the submanifold Y is fixed.

Denote by G a section of the canonical morphism $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$ and fix on $\mathbb{C}$ the lexicographical order. Let $\mathscr{M}$ be a specializable $\mathscr{D}_X$-module and denote by $V_G(\mathscr{M})$ the unique V-filtration of $\mathscr{M}$ admitting a b-function whose zeros are contained in G. The existence of this filtration is a classical result of [16]. It is also well-known that given two such sections G and G', then there is an isomorphism of $\mathscr{D}_{T_Y X}$-modules:

$$\mathscr{D}_{T_Y X} \otimes_{\mathscr{D}_{[T_Y X]}} \pi^{-1}\mathrm{gr}_{V_G}(\mathscr{M}) \simeq \mathscr{D}_{T_Y X} \otimes_{\mathscr{D}_{[T_Y X]}} \pi^{-1}\mathrm{gr}_{V_{G'}}(\mathscr{M}).$$

Hence, the following definition doesn't depend on the choice of the section G nor on the corresponding V-filtration:

Definition 1.2.16. Let $\mathscr{M}$ be a specializable $\mathscr{D}_X$-module along a submanifold Y and set

$$\nu_Y(\mathscr{M}) := \mathscr{D}_{T_Y X} \otimes_{\mathscr{D}_{[T_Y X]}} \pi^{-1}\mathrm{gr}_{V_G}(\mathscr{M}).$$

The object $\nu_Y(\mathscr{M}) \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_{T_Y X})$ is called the specialization of $\mathscr{M}$ along Y.

Denote by $\mathrm{Mod}_{\mathrm{sp}}(\mathscr{D}_X)$ the full Serre subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ of specializable $\mathscr{D}_X$-modules along Y (for a proof of the Serre property see for example Proposition 5.1.3 of [20]). Then, the correspondence $\mathscr{M} \mapsto \nu_Y(\mathscr{M})$ determines an exact functor from $\mathrm{Mod}_{\mathrm{sp}}(\mathscr{D}_X)$ to $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_{T_Y X})$.

Let us suppose now that Y is a complex closed smooth hypersurface of X given by the zero locus of a holomorphic function $f : X \to \mathbb{C}$. In this case, one can also associate to a specializable $\mathscr{D}_X$-module $\mathscr{M}$ the following coherent objects (cf. Remark 7.9 of [20]):

Definition 1.2.17. Let $\mathscr{M} \in \mathrm{Mod}_{\mathrm{sp}}(\mathscr{D}_X)$ and denote by $V_G(\mathscr{M})$ the canonical Kashiwara's V-filtration. Set

$$\psi_Y(\mathscr{M}) := \mathrm{gr}^0_{V_G}(\mathscr{M}) = \frac{V^0_G(\mathscr{M})}{V^1_G(\mathscr{M})}$$

and

$$\varphi_Y(\mathscr{M}) := \mathrm{gr}^{-1}_{V_G}(\mathscr{M}) = \frac{V^{-1}_G(\mathscr{M})}{V^0_G(\mathscr{M})}.$$

The objects $\psi_Y(\mathscr{M}), \varphi_Y$ belong to $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y)$ and one calls them the nearby-cycle module of $\mathscr{M}$ along Y and the vanishing-cycle module of $\mathscr{M}$ along Y, respectively.

Moreover, the correspondences $\mathscr{M} \mapsto \psi_Y(\mathscr{M})$ and $\mathscr{M} \mapsto \varphi_Y(\mathscr{M})$ set up two exact functors from $\mathrm{Mod}_{\mathrm{sp}}(\mathscr{D}_X)$ to $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y)$.

Let us recall that Kashiwara constructed in [12, Theorem 1] the following canonical isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{T_Y X})$, for a regular holonomic $\mathscr{D}_X$-module $\mathscr{M}$ (or, more generally, for an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$):

$$\begin{cases} \mathrm{Sol}_{\mathscr{D}_{T_Y X}}(\nu_Y(\mathscr{M})) \xrightarrow{\sim} \nu_Y(\mathrm{Sol}_{\mathscr{D}_X}(\mathscr{M})), \\ \mathrm{DR}_{\mathscr{D}_{T_Y X}}(\nu_Y(\mathscr{M})) \xleftarrow{\sim} \nu_Y(\mathrm{DR}_{\mathscr{D}_X}(\mathscr{M})). \end{cases} \tag{1.2.3}$$

Such theorem has its counterpart when applied to nearby-cycles and vanishing-cycles: when Y is a smooth hypersurface of X and $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$, then the following canonical isomorphisms hold in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_Y)$:

$$\begin{cases} \mathrm{Sol}_{\mathscr{D}_Y}(\psi_Y(\mathscr{M})) \xrightarrow{\sim} \psi_Y(\mathrm{Sol}_{\mathscr{D}_X}(\mathscr{M})), \\ \mathrm{DR}_{\mathscr{D}_Y}(\psi_Y(\mathscr{M})) \xleftarrow{\sim} \psi_Y(\mathrm{DR}_{\mathscr{D}_X}(\mathscr{M})), \end{cases} \tag{1.2.4}$$

and

$$\begin{cases} \mathrm{Sol}_{\mathscr{D}_Y}(\varphi_Y(\mathscr{M})) \xrightarrow{\sim} \varphi_Y(\mathrm{Sol}_{\mathscr{D}_X}(\mathscr{M})), \\ \mathrm{DR}_{\mathscr{D}_Y}(\varphi_Y(\mathscr{M})) \xleftarrow{\sim} \varphi_Y(\mathrm{DR}_{\mathscr{D}_X}(\mathscr{M})). \end{cases} \tag{1.2.5}$$

We refer to [22] for a detailed proof of these classical results, which one calls the comparison theorems. Note that the functors ν_Y, ψ_Y and φ_Y that appear in the right hand side of the isomorphisms are, respectively, the functors of specialization, nearby-cycles and vanishing cycles of sheaf theory (cf. Definitions 1.1.3 and 1.1.4).

Fourier transform and microlocalization for $\mathscr{D}$-modules. Let $E \xrightarrow{\pi} Z$ denote a complex vector bundle on a complex analytic manifold Z and $E' \xrightarrow{\tilde{\pi}} Z$ denote its dual bundle. Denote by $\mathscr{D}_{[E]} \subset \pi_* \mathscr{D}_E$ the sheaf of homogeneous differential operators over E and by θ the Euler field on E.

Definition 1.2.18. A $\mathscr{D}_{[E]}$-left coherent module $\mathscr{N}$ is monodromic if $\mathscr{N}$ is generated by local sections u satisfying $b(\theta)u = 0$ for some non-vanishing $b(s) \in \mathbb{C}[s]$.

Denote by $\mathrm{Mod}_{\mathrm{mon}}(\mathscr{D}_{[E]})$ the full Serre subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_{[E]})$ consisting of monodromic $\mathscr{D}_{[E]}$-modules. We refer to [20] for a study of such objects.

Consider the sheaf $\Omega_{E/Y}$ of relative differential forms to $\pi : E \to Y$ and let us recall the following definition of [2]:

Definition 1.2.19. Let $\mathscr{N} \in \mathrm{Mod}_{\mathrm{mon}}(\mathscr{D}_{[E]})$. The Fourier transform of $\mathscr{N}$ is the monodromic $\mathscr{D}_{[E']}$ denoted by $\mathscr{F}(\mathscr{N})$ and defined by:

$$\mathscr{F}(\mathscr{N}) := \Omega_{E/Y} \otimes_{\pi^{-1}\mathscr{O}_Y} \mathscr{N}.$$

The correspondence $\mathscr{M} \mapsto \mathscr{F}(\mathscr{N})$ defines an exact functor from $\mathrm{Mod}_{\mathrm{mon}}(\mathscr{D}_{[E]})$ to $\mathrm{Mod}_{\mathrm{mon}}(\mathscr{D}_{[E']})$.

Recall that for each $\mathscr{N} \in \mathrm{Mod}_{\mathrm{mon}}(\mathscr{D}_{[E]})$ one has canonical isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{E'})$:

$$\begin{cases} \mathscr{F}(\mathrm{Sol}_{\mathscr{D}_{[E]}}(\mathscr{N})) \simeq \mathrm{Sol}_{\mathscr{D}_{[E']}}(\mathscr{F}(\mathscr{N}))[\text{-codim}Y]; \\ \mathscr{F}(\mathrm{DR}_{\mathscr{D}_{[E]}}(\mathscr{N})) \simeq \mathrm{DR}_{\mathscr{D}_{[E']}}(\mathscr{F}(\mathscr{N}))[\text{-codim}Y]. \end{cases} \tag{1.2.6}$$

The functor$\mathscr{F}^{\mathbb{C}}$ that appear on the left hand side of the isomorphisms is the Fourier-Sato transform of sheaf theory (cf. Definition 1.1.5).

Consider now the case of a complex submanifold $Y \subset X$ and set $E = T_Y X$ and $E' = T^*_Y X$. The composition of the Fourier transform and the specialization defines an exact functor μ_Y from $\mathrm{Mod}_{\mathrm{sp}}(\mathscr{D}_X)$ to $\mathrm{Mod}_{\mathrm{mon}}(\mathscr{D}_{[T^*_Y X]})$, which is called the microlocalization along Y (cf. [27]):

Definition 1.2.20. Let $\mathscr{M} \in \mathrm{Mod}_{\mathrm{sp}}(\mathscr{D}_X)$. The microlocalization of $\mathscr{M}$ is the monodromic $\mathscr{D}_{T^*_Y X}$-module denoted by $\mu_Y(\mathscr{M})$ and defined by

$$\mu_Y(\mathscr{M}) := \mathscr{F}(\nu_Y(\mathscr{M})).$$

For $\mathscr{M} \in \mathrm{Mod}_{\mathrm{sp}}(\mathscr{D}_X)$, one also has the following comparison isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{T^*_Y X})$, where the functor μ_Y appearing in the right hand side represents Sato's microlocalization (Definition 1.1.6):

$$\begin{cases} \mathrm{Sol}_{\mathscr{D}_{T^*_Y X}}(\mu_Y(\mathscr{M})) \simeq \mu_Y(\mathrm{Sol}_{\mathscr{D}_Y}(\mathscr{M}))[\mathrm{codim}Y] \\ \mathrm{DR}_{\mathscr{D}_{T^*_Y X}}(\mu_Y(\mathscr{M})) \simeq \mu_Y(\mathrm{DR}_{\mathscr{D}_Y}(\mathscr{M}))[\mathrm{codim}Y]. \end{cases} \tag{1.2.7}$$

1.3 Algebras of formal deformation

Prestacks and stacks. In the sequel we need to work with prestacks and stacks on topological spaces. We briefly recall such notions following [17]. Reference is made to [18] for further details. Let X be a topological space.

Definition 1.3.1. A prestack $\mathfrak{C}$ on X is defined by the following data:

(i) for each open subset U of X, a category $\mathfrak{C}(U)$,

(ii) for each open inclusion $V \subset U$, a functor $\rho_{VU} : \mathfrak{C}(U) \to \mathfrak{C}(V)$,

(iii) for each open inclusions $W \subset V \subset U$, an isomorphism of functors $\lambda_{WVU} : \rho_{WV} \circ \rho_{VU} \xrightarrow{\sim} \rho_{WU}$,

together with the conditions:

(a) $\rho_{UU} = \mathrm{id}_{\mathfrak{C}(U)}$,

(b) for each open inclusions $U_1 \subset U_2 \subset U_3 \subset U_4$, the diagram below is commutative:

$$\begin{array}{ccc} \rho_{12} \circ \rho_{23} \circ \rho_{34} & \xrightarrow{\lambda_{234}} & \rho_{12} \circ \rho_{24} \\ \downarrow{\scriptstyle \lambda_{123}} & & \downarrow{\scriptstyle \lambda_{124}} \\ \rho_{13} \circ \rho_{34} & \xrightarrow{\lambda_{134}} & \rho_{14}. \end{array}$$

($\rho_{i,j}$ and λ_{ijk} denote $\rho_{U_iU_j}$ and $\lambda_{U_iU_jU_k}$, respectively.)

If $\mathfrak{C}$ is a prestack on X and U is an open subset of X, then the correspondence $U \supset V \mapsto \mathfrak{C}(V)$ clearly defines a prestack on U, denoted by $\mathfrak{C}|_U$.

Definition 1.3.2. Let $\mathfrak{C}$ and $\mathfrak{C}'$ be two prestacks on X. Denote by ρ_{VU}, λ_{WVU} (resp. ρ'_{VU}, λ'_{WVU}) the corresponding functors and morphisms of functors on $\mathfrak{C}$ (resp. $\mathfrak{C}'$). A functor of prestacks $\varphi : \mathfrak{C} \to \mathfrak{C}'$ is the data of:

(i) for each open subset U, a functor $\varphi_U : \mathfrak{C}(U) \to \mathfrak{C}'(U)$,

(ii) for each open inclusion $V \subset U$, an isomorphism of functors $\theta_{VU} : \varphi_V \circ \rho_{VU} \xrightarrow{\sim} \rho'_{VU} \circ \varphi_U$, such that for each open inclusions $W \subset V \subset U$, the diagram below commutes:

$$\begin{array}{ccc} \varphi_W \circ \rho_{WV} \circ \rho_{VU} & \xrightarrow{\lambda_{WVU}} & \varphi_W \circ \rho_{WU} \\ \downarrow{\scriptstyle \theta_{WV}} & & \\ \rho'_{WV} \circ \varphi_V \circ \rho_{VU} & & \downarrow{\scriptstyle \theta_{WU}} \\ \downarrow{\scriptstyle \theta_{VU}} & & \\ \rho'_{WV} \circ \rho'_{VU} \circ \varphi_U & \xrightarrow{\lambda'_{WVU}} & \rho'_{WU} \circ \varphi_U. \end{array}$$

Definition 1.3.3. A stack $\mathfrak{C}$ on X is a prestack on X that satisfies the following conditions:

(a) for any open subset U of X and any $\mathscr{M}, \mathscr{N} \in \mathfrak{C}(U)$, the presheaf $\mathscr{H}om_{\mathfrak{C}|_U}(\mathscr{M}, \mathscr{N})$ is a sheaf on U;

(b) for any open subset $U \subset X$, any open covering $U = \bigcup_{i\in I}$, any family $\mathscr{M}_i \in \mathfrak{C}(U_i)$, any family of isomorphisms $\theta_{ji} : \mathscr{M}_i|_{U_{ji}} \xrightarrow{\sim} \mathscr{M}_j|_{U_{ji}}$ such that:

$$\theta_{ij}|_{U_{ijk}} \circ \theta_{jk}|_{U_{ijk}} = \theta_{ik}|_{U_{ijk}},$$

there exist $F \in \mathfrak{C}(U)$ and isomorphisms $\theta_i : \mathscr{M}|_{U_i} \xrightarrow{\sim} \mathscr{M}_i$ such that

$$\theta_{ij} \circ (\theta_j|_{U_{ij}}) = \theta_i|_{U_{ij}}.$$

A functor of stacks is a functor of the underlying prestacks.

Let $\mathbb{K}$ be a commutative ring and let $\mathfrak{C}$ be a prestack (resp. a stack) on X such that each $\mathfrak{C}(U)$ is a category of $\mathbb{K}$-modules for each open subset $U \subset X$. Then, one says that $\mathfrak{C}$ is a $\mathbb{K}$-linear prestack (resp. a $\mathbb{K}$-linear stack) if, for every pair of open subsets $V \subset U$, the morphism ρ_{VU} is $\mathbb{K}$-linear.

Algebras of formal deformation. In the remaining of this section, X denotes a Hausdorff locally compact topological space and $\mathbb{K}$ denotes a commutative unital ring, unless otherwise stated. We recall notions and results of [19] and we prove also two new useful results, namely Propositions 1.3.13 and 1.3.20.

A family $\mathcal{B}$ of compact subsets of X is said to be a basis of compact subsets of X if, for any $x \in X$ and any open neighborhood U of x, there exists $K \in \mathcal{B}$ such that $x \in \mathrm{Int}(K) \subset U$.

Let $\mathscr{M}$ be a sheaf of $\mathbb{Z}_X[\hbar]$-modules and set:

$$\begin{aligned} \mathscr{M}_n &= \mathrm{Coker}(\mathscr{M} \xrightarrow{\hbar^{n+1}} \mathscr{M}) = \mathscr{M}/\hbar^{n+1}\mathscr{M}, \\ {}_n\mathscr{M} &= \ker(\mathscr{M} \xrightarrow{\hbar^{n+1}} \mathscr{M}). \end{aligned} \tag{1.3.1}$$

Denote by $\rho_{k,n} : \mathscr{M}_n \to \mathscr{M}_k$ the canonical epimorphisms (for $n \geq k$) and denote by $\rho : \mathscr{M} \to \varprojlim_{n\geq 0} \mathscr{M}_n$ the canonical morphism that follows from the universal property of projective limites.

Definition 1.3.4. One says that a $\mathbb{Z}_X[\hbar]$-module $\mathscr{M}$ has no $\hbar$-torsion if ${}_0\mathscr{M} = 0$ and one says that $\mathscr{M}$ is $\hbar$-complete if ρ is an isomorphism.

Let us consider a $\mathbb{K}$-algebra $\mathscr{A}$ on X and a section $\hbar$ of $\mathscr{A}$ contained in the center of $\mathscr{A}$. Set $\mathscr{A}_0 = \mathscr{A}/\hbar\mathscr{A}$. Consider the following conditions:

(i) $\mathscr{A}$ has no $\hbar$-torsion and is $\hbar$-complete,

(ii) $\mathscr{A}_0$ is a left Noetherian ring,

(iii) there exists a basis $\mathcal{B}$ of open subsets of X such that for any $U \in \mathcal{B}$ and any coherent $(\mathscr{A}_0|_U)$-module $\mathscr{F}$ we have $H^n(U;\mathscr{F}) = 0$ for any $n > 0$,

(iv) there exists a basis $\mathcal{B}$ of compact subsets of X and a prestack $U \mapsto \mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_0|_U)$ (U open in X) such that

(a) for any $K \in \mathcal{B}$ and an open subset U such that $K \subset U$, there exists $K' \in \mathcal{B}$ such that $K \subset \mathrm{Int}(K') \subset K' \subset U$,

(b) $U \mapsto \mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_0|_U)$ is a full subprestack of $U \mapsto \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_0|_U)$,

(c) for an open subset U and $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_0|_U)$, if $\mathscr{M}|_V$ belongs to $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_0|_V)$ for any relatively compact open subset V of U, then $\mathscr{M}$ belongs to $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_0|_U)$,

(d) for any U open in X, $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_0|_U)$ is stable by subobjects (and hence, by quotients) in $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_0|_U)$,

(e) for any $K \in \mathcal{B}$, any open set U containing K, any $\mathscr{M} \in \mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_0|_U)$ and any $j > 0$, one has $H^j(K;\mathscr{M}) = 0$,

(f) for any $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_0|_U)$, there exists an open covering $U = \bigcup_i U_i$ such that $\mathscr{M}|_{U_i} \in \mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_0|_{U_i})$,

(g) $\mathcal{A}_0 \in \mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_0)$.

One says that $\mathscr{A}$ is an algebra of formal deformation if $\mathscr{A}$ and $\mathscr{A}_0$ satisfy either Assumption 1.3.5 or Assumption 1.3.6 below:

Assumption 1.3.5. $\mathscr{A}$ and $\mathscr{A}_0$ satisfy conditions (i), (ii) and (iii).

Assumption 1.3.6. $\mathscr{A}$ and $\mathscr{A}_0$ satisfy conditions (i), (ii) and (iv).

In particular, with Assumption 1.3.5 or Assumption 1.3.6, $\mathscr{A}$ and $\mathscr{A}_n$ are left Noetherian rings, for every $n \geq 0$ (cf. Lemma 1.2.3 and Theorems 1.2.5 and 1.3.6 of [19]).

The graded functor. Consider the right exact functor defined by assigning the object $\mathscr{M}/\hbar^{n+1}\mathscr{M} \in \mathrm{Mod}(\mathscr{A}_n)$ to $\mathscr{M} \in \mathrm{Mod}(\mathscr{A})$. Its left derived functor is given by:

$$\begin{aligned} \mathrm{gr}^{\mathrm{n}}_{\hbar} : \mathsf{D}^{\mathrm{b}}(\mathscr{A}) &\to \mathsf{D}^{\mathrm{b}}(\mathscr{A}_n), \\ \mathscr{M} &\mapsto \mathscr{M} \overset{L}{\otimes}_{\mathscr{A}} \mathscr{A}_n. \end{aligned}$$

Remark 1.3.7. Note that the functor $\mathrm{gr}^0_{\hbar}$ was defined and studied in [19] and denoted by $\mathrm{gr}_{\hbar}$. Similarly to Proposition 1.4.3 of [19], $\mathrm{gr}^{\mathrm{n}}_{\hbar}$ satisfies the following commutativity properties for $\mathscr{M}, \mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathscr{A})$ and $\mathscr{L} \in \mathsf{D}^{\mathrm{b}}(\mathscr{A}^{\mathrm{op}})$:

$$\text{(i) } \mathrm{gr}^{\mathrm{n}}_{\hbar}(\mathrm{R}\mathscr{H}om_{\mathscr{A}}(\mathscr{M},\mathscr{N})) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_n}(\mathrm{gr}^{\mathrm{n}}_{\hbar}(\mathscr{M}), \mathrm{gr}^{\mathrm{n}}_{\hbar}(\mathscr{N}));$$

$$\text{(ii) } \mathrm{gr}^{\mathrm{n}}_{\hbar}(\mathscr{L} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} \mathscr{M}) \simeq \mathrm{gr}^{\mathrm{n}}_{\hbar}(\mathscr{L}) \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_n} \mathrm{gr}^{\mathrm{n}}_{\hbar}(\mathscr{M}).$$

It also commutes with topological direct images and inverse images, that is: let $f : X \to Y$ be a morphism of topological spaces and let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathbb{Z}_X[\hbar])$ and $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathbb{Z}_Y[\hbar])$; then by Proposition 1.4.4 of [19] one has:

$$\text{(iii)}\ \mathrm{gr}_\hbar^{\mathrm{n}} \mathrm{R}f_* \mathscr{M} \simeq \mathrm{R}f_* \mathrm{gr}_\hbar^{\mathrm{n}} \mathscr{M};$$
$$\text{(iv)}\ \mathrm{gr}_\hbar^{\mathrm{n}} f^{-1} \mathscr{N} \simeq f^{-1} \mathrm{gr}_\hbar^{\mathrm{n}} \mathscr{N}.$$

Note that if $\mathscr{M}$ is a coherent $\mathscr{A}$-module, then ${}_n\mathscr{M}$ and $\mathscr{M}_n$ are coherent $\mathscr{A}_n$-modules.

Recall that, for each $n \geq 0$, the category $\mathrm{Mod}(\mathscr{A}_n)$ and the full subcategory of $\mathrm{Mod}(\mathscr{A})$ whose objects are those $\mathscr{M}$ such that $\hbar^{n+1}\mathscr{M} \simeq 0$ are equivalent. Moreover:

Lemma 1.3.8 ([19], Lemma 1.2.3)**.** *Let $n \geq 0$. An $\mathscr{A}_n$-module $\mathscr{M}$ is coherent as an $\mathscr{A}_n$-module if and only if it is coherent as an $\mathscr{A}$-module.*

Theorem 1.3.9 ([19], Theorems 1.2.5 and 1.3.6)**.** *Assume that $\mathscr{A}$ is an algebra of formal deformation satisfying Assumption 1.3.5 (resp. Assumption 1.3.6). Then, for any coherent (resp. good) $\mathscr{A}$-module $\mathscr{M}$ and any open set $U \in \mathcal{B}$ (resp. any compact set $K \in \mathcal{B}$), one has $H^j(U;\mathscr{M}) = 0$, for any $j > 0$ (resp. $H^j(K;\mathscr{M}) = 0$, for any $j > 0$).*

Theorem 1.3.10 ([19], Theorem 1.3.6)**.** *An $\mathscr{A}$-module $\mathscr{M}$ is coherent if and only if it is $\hbar$-complete and $\hbar^n\mathscr{M}/\hbar^{n+1}\mathscr{M}$ is a coherent $\mathscr{A}_0$-module for any $n \geq 0$.*

Cohomologically complete objects. Let $\mathscr{R}$ be a sheaf of $\mathbb{Z}[\hbar]$-algebras and set $\mathscr{R}^{\mathrm{loc}} := \mathscr{R} \otimes_{\mathbb{Z}_X[\hbar]} \mathbb{Z}_X[\hbar, \hbar^{-1}]$. The following notion of [19] is the counterpart of $\hbar$-completeness in the derived category setting:

Definition 1.3.11. $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{R})$ is said to be a cohomologically complete object if

$$\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{loc}, \mathscr{M}) \simeq \mathrm{R}\mathscr{H}om_{\mathbb{Z}_X[\hbar]}(\mathbb{Z}_X[\hbar, \hbar^{-1}], \mathscr{M}) = 0.$$

Remark 1.3.12. The category of cohomologically complete objects is clearly a full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathscr{R})$. Namely, if in a distinguished triangle two of the terms are cohomologically complete, then the third one is also cohomologically complete.

Recall that any $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A})$ is cohomologically complete (cf. [19, Theorem 1.6.1]). Recall also that if an $\mathscr{A}$-module $\mathscr{M}$ has no $\hbar$-torsion, then $\mathscr{M}$ is $\hbar$-complete if and only if it is cohomologically complete (cf. [19, Lemma 1..5.4]).

For convenience, we denote by $\mathcal{C}$ the subcategory of cohomologically complete modules of $\mathrm{Mod}(\mathbb{Z}_X[\hbar])$.

Proposition 1.3.13. *$\mathcal{C}$ is a full abelian thick subcategory of* $\mathrm{Mod}(\mathbb{Z}_X[\hbar])$.

Proof. As a particular case of the previous remark, if two terms of a short exact sequence in $\mathrm{Mod}(\mathbb{Z}_X[\hbar])$ are cohomologically complete, then the third is also cohomologically complete. It remains to be proved that $\mathcal{C}$ is closed under kernels and cokernels.

Given a morphism $f : A \to B$ in $\mathcal{C}$, the mapping cone $M(f)$ is cohomologically complete in $\mathsf{D}^{\mathrm{b}}(\mathbb{Z}_X[\hbar])$. From the distinguished triangle

$$\ker f[1] \to M(f) \to \operatorname{coker} f \xrightarrow{+1}$$

we derive a distinguished triangle:

$$\mathrm{R}\mathscr{H}om_{\mathbb{Z}_X[\hbar]}(\mathbb{Z}_X[\hbar,\hbar^{-1}],\ker f[1]) \to \mathrm{R}\mathscr{H}om_{\mathbb{Z}_X[\hbar]}(\mathbb{Z}_X[\hbar,\hbar^{-1}],M(f)) \quad \to$$
$$\mathrm{R}\mathscr{H}om_{\mathbb{Z}_X[\hbar]}(\mathbb{Z}_X[\hbar,\hbar^{-1}],\operatorname{Coker} f) \quad \xrightarrow{+1}$$

and, thus, isomorphisms for each $j \in \mathbb{Z}$:

$$\mathscr{E}xt^{j-1}_{\mathbb{Z}_X[\hbar]}(\mathbb{Z}_X[\hbar,\hbar^{-1}],\operatorname{Coker} f) \simeq \mathscr{E}xt^{j}_{\mathbb{Z}_X[\hbar]}(\mathbb{Z}_X[\hbar,\hbar^{-1}],\ker f[1]). \tag{1.3.2}$$

On the other hand, $\mathbb{Z}[\hbar,\hbar^{-1}]$ is a $\mathbb{Z}[\hbar]$-module with projective dimension ≤ 1, so

$$\begin{cases} \mathscr{E}xt^{j}_{\mathbb{Z}_X[\hbar]}(\mathbb{Z}_X[\hbar,\hbar^{-1}],\operatorname{Coker} f) = 0, \text{for } j \neq 0,1 \\ \mathscr{E}xt^{j}_{\mathbb{Z}_X[\hbar]}(\mathbb{Z}_X[\hbar,\hbar^{-1}],\ker f) = 0, \text{for } j \neq 0,1. \end{cases} \tag{1.3.3}$$

The result is obtained by combining (1.3.2) with (1.3.3). □

The following propositions will be often used in the sequel and exhibit the central role that cohomologically complete objects play in the study of modules over algebras of formal deformation.

Proposition 1.3.14 ([19], Corollary 1.5.7)**.** *Assume that $\mathscr{M} \in \mathrm{Mod}(\mathscr{A})$ is $\hbar$-complete and $\hbar$-torsion free. Assume that there exists a basis $\mathcal{B}$ of open (respectively of compact) subsets Ω such that $H^i(\Omega;\mathscr{M}) = 0$ for $i > 0$. Then $\mathscr{M}$ is cohomologically complete.*

Proposition 1.3.15 ([19], Corollary 1.5.9)**.** *The functor $\mathrm{gr}_\hbar$ is conservative in the category of cohomologically complete objects.*

Proposition 1.3.16 ([19], Proposition 1.5.10)**.** *If $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{A})$ is cohomologically complete, then $\mathrm{R}\mathscr{H}om_{\mathscr{A}}(\mathscr{N},\mathscr{M})$ is cohomologically complete, for any $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathscr{A})$.*

Coherence and flatness criterias.

Theorem 1.3.17 ([19], Theorem 1.6.4)**.** *Let $\mathscr{M} \in \mathsf{D}^{+}(\mathscr{A})$ and assume that $\mathscr{M}$ is cohomologically complete and $\mathrm{gr}_\hbar(\mathscr{M})$ is an object of $\mathsf{D}^{+}_{\mathrm{coh}}(\mathscr{A}_0)$. Then, $\mathscr{M}$ is an object of $\mathsf{D}^{+}_{\mathrm{coh}}(\mathscr{A})$ and, for each $i \in \mathbb{Z}$, one has the isomorphism*

$$H^i(\mathscr{M}) \simeq \varprojlim_{n\geq 0} H^i(\mathrm{gr}^{\mathrm{n}}_\hbar(\mathscr{M})). \tag{1.3.4}$$

Theorem 1.3.18 ([19], Theorem 1.6.5)**.** *Assume that $\mathscr{A}^{\mathrm{op}}/\hbar\mathscr{A}^{\mathrm{op}}$ is a Noetherian ring. Let $\mathscr{M}$ be a cohomologically complete $\mathscr{A}$-module with no $\hbar$-torsion and such that $\mathscr{M}/\hbar\mathscr{M}$ is a flat $\mathscr{A}_0$-module. Then $\mathscr{M}$ is a flat $\mathscr{A}$-module.*

Functorial properties.

Proposition 1.3.19 ([19], Proposition 1.5.12)**.** *Let $f : X \to Y$ be a continuous map between two Hausdorff locally compact spaces and suppose that $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{A})$ is cohomologically complete. Then, $\mathrm{R}f_*\mathscr{M}$ is also cohomologically complete.*

Assume now that $f : X \to Y$ is a morphism of real analytic manifolds and consider the canonical morphisms f_π and f_d (cf. (1.2.1)). Recall that $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{K}_Y)$ is said to be non-characteristic for f if the following transversality condition holds:

$$f_\pi^{-1}(\mathrm{SS}(F)) \cap \ker(f_d) \subset X \times_Y T^*_Y Y.$$

Proposition 1.3.20. *Let $f : X \to Y$ be a morphism of real analytic manifolds and assume that a cohomologically complete object $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathbb{Z}_X[\hbar])$ is non-characteristic for f. Then $f^{-1}\mathscr{M}$ is cohomologically complete.*

Proof. Since $\mathscr{M}$ is non-characteristic for f, one has $f^{-1}\mathscr{M} \otimes \omega_{X|Y} \xrightarrow{\sim} f^!\mathscr{M}$, cf. [15, Prop. 5.4.13 (ii)]. Therefore, $f^{-1}\mathscr{M}$ and $f^!\mathscr{M}$ differ only by a shift and it is enough to prove that $f^!\mathscr{M}$ is cohomologically complete. The result follows from the isomorphism below:

$$\mathrm{R}\mathscr{H}om_{\mathbb{Z}_X[\hbar]}(\mathbb{Z}_X[\hbar, \hbar^{-1}], f^!\mathscr{M}) \simeq f^!\mathrm{R}\mathscr{H}om_{\mathbb{Z}_Y[\hbar]}(\mathbb{Z}_Y[\hbar, \hbar^{-1}], \mathscr{M}).$$

□

1.4 Formal extensions

Let us denote by $\mathbb{C}^\hbar := \mathbb{C}[[\hbar]]$ the ring of formal power series with complex coefficients and by $\mathbb{C}^{\hbar,\mathrm{loc}} := \mathbb{C}((\hbar))$ the ring of Laurent power series with complex coefficients.

In this section we recall results of [3]. We also add some new results, namely Lemma 1.4.5 and Proposition 1.4.8, as well as some complements in the particular case of sheaves of $\mathbb{C}^\hbar$-modules.

Let X denote a topological space and $\mathscr{R}_0$ a sheaf of rings on X. Set $\mathscr{R} := \mathscr{R}_0[[\hbar]] \simeq \Pi_{n\geq 0}\mathscr{R}_0\hbar^n$ and consider the functor:

$$\begin{array}{rcl} (\bullet)^\hbar : \mathrm{Mod}(\mathscr{R}_0) & \to & \mathrm{Mod}(\mathscr{R}) \\ \mathscr{N} & \to & \mathscr{N}^\hbar := \mathscr{N}[[\hbar]] = \varprojlim\limits_{n\geq 0}(\mathscr{N} \otimes_{\mathscr{R}_0} \mathscr{R}/\hbar^{n+1}\mathscr{R}). \end{array} \tag{1.4.1}$$

For an open subset $U \subset X$, a section $f \in \Gamma(U; \mathscr{N}^\hbar)$ can be regarded as a formal power series $f = \sum_{n\geq 0} f_n\hbar^n$ with each $f_n \in \Gamma(U; \mathscr{N})$. The functor $(\bullet)^\hbar$ is left exact and one denotes by $(\bullet)^{\mathrm{R}\hbar}$ its right derived functor:

$$(\bullet)^{\mathrm{R}\hbar} \colon \mathsf{D}^{\mathrm{b}}(\mathscr{R}_0) \to \mathsf{D}^{\mathrm{b}}(\mathscr{R}). \tag{1.4.2}$$

For each $F \in \mathsf{D}^{\mathrm{b}}(\mathscr{R}_0)$, $F^{\mathrm{R}\hbar}$ is called the formal extension of F.

From now on let us assume that $\mathscr{R}_0$ is an $\hbar$-acyclic $\mathbb{C}_X$-algebra, that is,

$$\mathscr{R}_0^{\mathrm{R}\hbar} \simeq \mathscr{R}_0^\hbar = \mathscr{R}.$$

Proposition 1.4.1 ([3], Proposition 2.1)**.** *For any* $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathscr{R}_0)$*, one has*

$$\mathscr{N}^{\mathrm{R}\hbar} \simeq \mathrm{R}\mathscr{H}om_{\mathscr{R}_0}(\mathscr{R}^{\mathrm{loc}}/\hbar\mathscr{R}, \mathscr{N}),$$

where $\mathscr{R}^{\mathrm{loc}}/\hbar\mathscr{R}$ *is regarded as an* $(\mathscr{R}_0, \mathscr{R})$*-bimodule.*

Proposition 1.4.2 ([3], Proposition 2.2)**.** *Let* $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathscr{R}_0)$*. The formal extension* $\mathscr{N}^{\mathrm{R}\hbar}$ *is cohomologically complete.*

Lemma 1.4.3 ([3], Lemma 2.3)**.** *Assume that* $\mathscr{R}_0$ *is an* $\mathscr{S}_0$*-algebra, for* $\mathscr{S}_0$ *a sheaf of commutative rings, and set* $\mathscr{S} = \mathscr{S}_0[[\hbar]]$*. For* $\mathscr{M}, \mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathscr{R}_0)$ *there is an isomorphism in* $\mathsf{D}^{\mathrm{b}}(\mathscr{S})$*:*

$$\mathrm{R}\mathscr{H}om_{\mathscr{R}_0}(\mathscr{M}, \mathscr{N})^{\mathrm{R}\hbar} \simeq \mathrm{R}\mathscr{H}om_{\mathscr{R}_0}(\mathscr{M}, \mathscr{N}^{\mathrm{R}\hbar}).$$

Proposition 1.4.4 ([3], Proposition 2.5)**.** *Let* $\mathscr{M} \in \mathrm{Mod}(\mathscr{R}_0)$ *and suppose that* $\mathcal{B}$ *is either a basis of open subsets of* X *or, assuming that* X *is a locally compact topological space, a basis of compact subsets, such that* $H^j(S; \mathscr{M}) = 0$ *for all* $j > 0$ *and all* $S \in \mathcal{B}$*. Then,* $\mathscr{M}$ *is* $(\bullet)^\hbar$*-acyclic, that is,* $\mathscr{M}^\hbar \simeq \mathscr{M}^{\mathrm{R}\hbar}$*.*

Lemma 1.4.5. *Let* $f : X \to Y$ *be a morphism of topological spaces and assume that* $(f^{-1}\mathscr{R}_0)^\hbar \simeq f^{-1}\mathscr{R}$*. Then, for each* $F \in \mathsf{D}^{\mathrm{b}}(f^{-1}\mathscr{R}_0)$*, we have a canonical morphism in* $\mathsf{D}^{\mathrm{b}}(\mathscr{R})$*:*

$$\mathrm{R}f_!(F^{\mathrm{R}\hbar}) \to \mathrm{R}f_!(F)^{\mathrm{R}\hbar}.$$

Proof. By Proposition 1.4.1, and using (2.6.26) of [15], we get the following chain of morphisms in $\mathsf{D}^{\mathrm{b}}(\mathscr{R})$:

$$\begin{aligned}
\mathrm{R}f_!(F^{\mathrm{R}\hbar}) &\simeq \mathrm{R}f_!\mathrm{R}\mathscr{H}om_{f^{-1}\mathscr{R}_0}(f^{-1}(\mathscr{R}^{\mathrm{loc}}/\hbar\mathscr{R}), F) \\
&\to \mathrm{R}\mathscr{H}om_{\mathscr{R}_0}(\mathrm{R}f_* f^{-1}(\mathscr{R}^{\mathrm{loc}}/\hbar\mathscr{R}), \mathrm{R}f_!F) \\
&\to \mathrm{R}\mathscr{H}om_{\mathscr{R}_0}(\mathscr{R}^{\mathrm{loc}}/\hbar\mathscr{R}, \mathrm{R}f_!F) \\
&\simeq (\mathrm{R}f_!F)^{\mathrm{R}\hbar}.
\end{aligned}$$

□

In view of Proposition 1.4.8 below, let us fix a morphism of complex analytic manifolds $f : X \to Y$ and let $\mathscr{R}_0$ be an $\hbar$-acyclic $\mathbb{C}^\hbar$-algebra on Y.

Remark 1.4.6. There is a canonical morphism of sheaves of $\mathbb{C}^\hbar$-algebras $f^{-1}\mathscr{R} = f^{-1}\mathscr{R}_0^\hbar \to (f^{-1}\mathscr{R}_0)^\hbar$ induced by the morphisms

$$f^{-1}\mathscr{R}_0^\hbar \to f^{-1}(\mathscr{R}_0 \otimes \mathbb{C}_Y^\hbar/\hbar^{n+1}\mathbb{C}_Y^\hbar) \simeq f^{-1}\mathscr{R}_0 \otimes \mathbb{C}_X^\hbar/\hbar^{n+1}\mathbb{C}_X^\hbar,$$

and by the universal property of projective limits. Hence, there is a canonical functor $F \in \mathrm{Mod}((f^{-1}\mathscr{R}_0)^\hbar) \mapsto F \in \mathrm{Mod}(f^{-1}(\mathscr{R}_0^\hbar))$.

Assumption 1.4.7. There exists a basis $\mathcal{B}$ either of open subsets of Y or of compact subsets of Y such that $H^j(S;\mathscr{R}_0) = 0$, for all $j > 0$ and for all $S \in \mathcal{B}$.

Proposition 1.4.8. *Suppose that $f : X \to Y$ is smooth and that $\mathscr{R}_0$ verifies Assumption 1.4.7. Then we have an isomorphism of sheaves of rings $f^{-1}\mathscr{R} \xrightarrow{\sim} (f^{-1}\mathscr{R}_0)^{h}$.*

Proof. Consider the canonical morphism of Remark 1.4.6. Note that the corresponding canonical morphism $\mathrm{gr}_\hbar(f^{-1}\mathscr{R}) \to \mathrm{gr}_\hbar((f^{-1}\mathscr{R}_0)^\hbar)$ is an isomorphism, since both $\mathrm{gr}_\hbar(f^{-1}\mathscr{R})$ and $\mathrm{gr}_\hbar((f^{-1}\mathscr{R}_0)^\hbar)$ are isomorphic to $f^{-1}\mathscr{R}_0$. Hence, in view of Proposition 1.3.15, it suffices to show that both $f^{-1}\mathscr{R}$ and $(f^{-1}\mathscr{R}_0)^\hbar$ are cohomologically complete objects.

The ring $\mathscr{R} \simeq \mathscr{R}_0^{\mathrm{R}\hbar}$ is cohomologically complete and it is also non-characteristic for f, since f is smooth. Hence, $f^{-1}\mathscr{R}$ is cohomologically complete by Proposition 1.3.20.

We shall use Proposition 1.4.4 to prove that $(f^{-1}\mathscr{R}_0)^\hbar$ is isomorphic to $(f^{-1}\mathscr{R}_0)^{\mathrm{R}\hbar}$, thus cohomologically complete by Proposition 1.4.2.

The result is now checked locally, so we can assume that $X = X' \times Y$ and that $f : X \to Y$ is the canonical projection.

Let us consider the case where $\mathcal{B}$ is a basis of open subsets of Y. It is enough to show that there exists a basis $\mathcal{B}'$ of open subsets of $X' \times Y$ such that $H^j(S', f^{-1}\mathscr{R}_0) = 0$ for all $j > 0$ and all $S' \in \mathcal{B}'$.

Consider the basis $\mathcal{B}'$ formed by the open sets $V' \times V \subset X' \times Y$ such that V' is an open ball of X' (hence, contractible) and $V \in \mathcal{B}$. We are in the conditions to apply [15, Proposition 2.7.8]. Hence, for any $j > 0$ one has:

$$H^j(V' \times V, f^{-1}\mathscr{R}_0) \simeq H^j(f^{-1}_{|V'\times V}(V), f^{-1}_{|V'\times V}\mathscr{R}_0|_V) \simeq H^j(V, \mathscr{R}_0|_V) = 0.$$

The case where $\mathcal{B}$ is a basis of compact subsets of X is similar, taking closed balls on X' instead of open balls. □

Assume now that $\mathscr{A} = \mathscr{R}_0[[\hbar]]$ is an algebra of formal deformation. Set also $\mathscr{A}_0 = \mathscr{R}_0$ and assume, in addition, that $\mathscr{A}_0$ is l-syzygic in the following sense: there exists a positive integer l such that any coherent $\mathscr{A}_0$-module can be locally represented by a complex of length l formed by free $\mathscr{A}_0$-modules of finite rank. Then, by Proposition 1.4.4, one has $\mathscr{A} \simeq \mathscr{A}_0^{\mathrm{R}\hbar}$.

One gets the following relation between formal extensions and the graded functor:

Proposition 1.4.9 ([3], Proposition 2.8)**.** *For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_0)$:*

(a) *there is an isomorphism $\mathscr{M}^{\mathrm{R}\hbar} \xrightarrow{\sim} \mathscr{A} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_0} \mathscr{M}$;*

(b) *there is an isomorphism $\mathrm{gr}_\hbar(\mathscr{M}^{\mathrm{R}\hbar}) \simeq \mathscr{M}$.*

Complements on constructible sheaves of $\mathbb{C}[[\hbar]]$-modules. Let X denote a real analytic manifold of dimension d_X.

Remark 1.4.10. As an application of Proposition 1.4.4 (cf. [3, Corollary 2.6]), one knows that $\mathbb{R}$-constructible sheaves are $(\bullet)^{\hbar}$-acyclic. Indeed, given $F \in \mathrm{Mod}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X^{\hbar})$ and $x \in X$, it is well known from the theory of constructible sheaves that F_x is isomorphic to $\mathrm{R}\Gamma(U_x; F_x)$ for some U_x in a fundamental system of open neighborhoods of x. Hence, one can apply Proposition 1.4.4 with $\mathcal{B}$ being the family of these fundamental systems.

For any $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^{\hbar})$, one has $\mathrm{SS}(\mathrm{gr}_{\hbar}(F)) \subset \mathrm{SS}(F)$. Moreover, one has:

Proposition 1.4.11 ([3], Proposition 1.15)**.** *If F is cohomologically complete, then the equalities $\mathrm{SS}(\mathrm{gr}_{\hbar}(F)) = \mathrm{SS}(F)$ and $\mathrm{supp}(F) = \mathrm{supp}(\mathrm{gr}_{\hbar}(F))$ hold.*

Note that the equalities don't hold in general. For example, $\mathbb{C}_X^{\hbar,\mathrm{loc}}$ is not cohomologically complete and $\mathrm{SS}(\mathbb{C}_X^{\hbar,\mathrm{loc}}) = T^*X$, whereas $\mathrm{SS}(\mathrm{gr}_{\hbar}(\mathbb{C}_X^{\hbar,\mathrm{loc}})) = \mathrm{SS}(0) = 0$.

Proposition 1.4.12 ([3], Proposition 1.16)**.** *If $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X^{\hbar})$, then F is cohomologically complete.*

It follows from Proposition 1.4.12 and the study of cohomologically complete objects that the functor

$$\mathrm{gr}_{\hbar} : \mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X^{\hbar}) \to \mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$$

is conservative and preserves the micro-support.

Proposition 1.4.13. *Let $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$. Then we have $F^{\mathrm{R}\hbar} \simeq F \otimes \mathbb{C}_X^{\hbar}$.*

Proof. Let

$$F^{\bullet} : 0 \to \bigoplus_{i \in I_a} \mathbb{C}_{U_{a,i}} \to \cdots \to \bigoplus_{i \in I_b} \mathbb{C}_{U_{b,i}} \to 0,$$

be an almost free resolution of F in the sense of the Appendix of [16]. This means that $F^{\bullet}$ is quasi-isomorphic to F, each family $\{U_{k,i}\}_{k \in I_k}$ is locally finite and each open subset $U_{k,i}$ of X is subanalytic and relatively compact.

Denote by $F^{\bullet,\hbar}$ the complex:

$$F^{\bullet,\hbar} : 0 \to \bigoplus_{i \in I_a} \mathbb{C}^{\hbar}_{U_{a,i}} \to \cdots \to \bigoplus_{i \in I_b} \mathbb{C}^{\hbar}_{U_{b,i}} \to 0.$$

Since $\mathbb{R}$-constructible sheaves are $\hbar$-acyclic, we have the following chain of quasi-isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^{\hbar})$:

$$F^{\mathrm{R}\hbar} \simeq (F^{\bullet})^{\mathrm{R}\hbar} \simeq F^{\bullet,\hbar}.$$

On the other hand, we have $F \otimes \mathbb{C}_X^{\hbar} \simeq F^{\bullet} \otimes \mathbb{C}_X^{\hbar}$ where we set:

$$F^{\bullet} \otimes \mathbb{C}_X^{\hbar} : 0 \to \bigoplus_{i \in I_a} \mathbb{C}_{U_{a,i}} \otimes \mathbb{C}_X^{\hbar} \to \cdots \to \bigoplus_{i \in I_b} \mathbb{C}_{U_{b,i}} \otimes \mathbb{C}_X^{\hbar} \to 0.$$

Therefore, it is enough to note that for each open subanalytic subset U we have a canonical isomorphism:

$$\mathbb{C}_U \otimes \mathbb{C}_X^\hbar \xrightarrow{\sim} \mathbb{C}_U^\hbar.$$

In fact, the stalks $(\mathbb{C}_U^\hbar)_x$ and $(\mathbb{C}_U \otimes \mathbb{C}_X^\hbar)_x$ are both isomorphic to $\mathbb{C}^\hbar$ if $x \in U$, and both vanish if $x \notin U$. □

Corollary 1.4.14. *If $F \in \mathrm{Mod}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$, then $F^\hbar \otimes_{\mathbb{C}_X^\hbar} \bullet$ is an exact functor.*

The next lemma is often used in the sequel.

Lemma 1.4.15. *For $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$ and $G \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)$, there are isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^\hbar)$:*

$$(i) \quad \mathrm{R}\mathscr{H}om_{\mathbb{C}_X^\hbar}(F^{\mathrm{R}\hbar}, G^{\mathrm{R}\hbar}) \simeq (\mathrm{R}\mathscr{H}om(F,G))^{\mathrm{R}\hbar}$$

$$(ii) \quad F^{\mathrm{R}\hbar} \overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X^\hbar} G^{\mathrm{R}\hbar} \simeq F \otimes G^{\mathrm{R}\hbar}.$$

Proof. Applying Proposition 1.4.13 we get the following chain of isomorphisms:

$$\begin{aligned} \mathrm{R}\mathscr{H}om_{\mathbb{C}_X^\hbar}(F^{\mathrm{R}\hbar}, G^{\mathrm{R}\hbar}) &\simeq \mathrm{R}\mathscr{H}om_{\mathbb{C}_X^\hbar}(F \otimes \mathbb{C}_X^\hbar, G^{\mathrm{R}\hbar}) \\ &\simeq \mathrm{R}\mathscr{H}om(F, G^{\mathrm{R}\hbar}). \end{aligned}$$

Then, the isomorphism (i) follows from Lemma 1.4.3.

The proof of (ii) is similar. □

As a particular case of Lemma 1.4.15 (i), one gets $\mathrm{D}'_{\mathbb{C}_X^\hbar} F^\hbar \simeq (\mathrm{D}'_{\mathbb{C}_X} F)^\hbar$ for any $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X^\hbar)$.

Note. The summary of the remaining chapters is the following: we shall use the results listed in Sections 1.3 and 1.4 on algebras of formal deformation and formal extensions to extend the classical functorial properties of $\mathscr{D}_X$-modules (which we recalled in Section 1.2) to the framework of $\mathscr{D}_X[[\hbar]]$-modules, where $\mathscr{D}_X[[\hbar]]$ denotes the formal extension of the ring $\mathscr{D}_X$. This is possible since $\mathscr{D}_X[[\hbar]]$ is itself an algebra of formal deformation, thus the study of $\mathscr{D}_X[[\hbar]]$-modules can often be carried out by reducing the proofs to the classical results on $\mathscr{D}_X$-modules. Such application to the theory of $\mathscr{D}_X[[\hbar]]$-modules is done in Chapters 3 and 4. In Chapter 2 we employ algebraic methods to perform the extension of functors in the framework of algebras of formal deformation.

Chapter 2

Extension of functors for algebras of formal deformation

Throughout this chapter $\mathscr{A}$ denotes an algebra of formal deformation on a complex manifold X of finite dimension.

2.1 The category $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$

Modules of torsion and modules free of torsion. For $\mathscr{M} \in \mathrm{Mod}(\mathscr{A})$ one denotes by $\mathscr{M}_{\hbar-\mathrm{tor}}$ the submodule of $\mathscr{M}$ consisting of sections locally annihilated by some power of $\hbar$. That is, $\mathscr{M}_{\hbar-\mathrm{tor}}$ is the sheaf that results from the increasing union of the kernels $({}_{n}\mathscr{M})_{n\geq 0}$.

Set $\mathscr{M}_{\hbar-\mathrm{tf}} := \mathscr{M}/\mathscr{M}_{\hbar-\mathrm{tor}}$. By definition one has the exact sequence below:

$$0 \to \mathscr{M}_{\hbar-\mathrm{tor}} \to \mathscr{M} \to \mathscr{M}_{\hbar-\mathrm{tf}} \to 0. \tag{2.1.1}$$

If $\mathscr{M}$ is coherent, then the family $\{{}_{n}\mathscr{M}\}_{n\geq 0}$ is locally stationary, so locally there exists $N \geq 1$ such that $\hbar^{N}\mathscr{M}_{\hbar-\mathrm{tor}} = 0$ and both $\mathscr{M}_{\hbar-\mathrm{tor}}$ and $\mathscr{M}_{\hbar-\mathrm{tf}}$ are coherent $\mathscr{A}$-modules. Since $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$ is a thick subcategory of $\mathrm{Mod}(\mathscr{A})$, the converse also holds: if $\mathscr{M}_{\hbar-\mathrm{tor}}$ and $\mathscr{M}_{\hbar-\mathrm{tf}}$ are coherent $\mathscr{A}$-modules, then $\mathscr{M}$ is also a coherent $\mathscr{A}$-module.

One says that $\mathscr{M} \in \mathrm{Mod}(\mathscr{A})$ is an $\hbar$-torsion module if $\mathscr{M}_{\hbar-\mathrm{tor}} \xrightarrow{\sim} \mathscr{M}$. For example, for each $n \geq 0$, $\mathscr{M}_{n}$ is an $\hbar$-torsion module, since $\hbar^{n+1}\mathscr{M}_{n} = 0$.

One says that $\mathscr{M}$ is $\hbar$-torsion free if $\mathscr{M} \xrightarrow{\sim} \mathscr{M}_{\hbar-\mathrm{tf}}$, which is obviously equivalent to having no $\hbar$-torsion. Recall that, by hypothesis, $\mathscr{A}$ has no $\hbar$-torsion.

In the sequel, we shall often use implicitly the fact that, for each $n \geq 0$, the category $\mathrm{Mod}(\mathscr{A}_{n})$ is equivalent to the full subcategory of $\mathrm{Mod}(\mathscr{A})$ whose objects are those $\mathscr{M} \in \mathrm{Mod}(\mathscr{A})$ such that $\hbar^{n+1}\mathscr{M} \simeq 0$.

As a consequence of Lemma 1.3.8, one has:

Corollary 2.1.1. *For $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$, each $\mathscr{M}_{n}$ is coherent as an $\mathscr{A}_{n}$-module.*

Lemma 2.1.2. *Let* $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ *be a short exact sequence in* $\mathrm{Mod}(\mathscr{A})$ *and suppose that* $\mathscr{M}''$ *has no* $\hbar$*-torsion. Then, for each* $n \geq 0$*, the associated sequence of* $\mathscr{A}_n$*-modules,*

$$0 \to \mathscr{M}'_n \to \mathscr{M}_n \to \mathscr{M}''_n \to 0, \tag{2.1.2}$$

is exact both in $\mathrm{Mod}(\mathscr{A})$ *and in* $\mathrm{Mod}(\mathscr{A}_n)$*.*

Proof. Let us fix $n \geq 0$ and apply the functor $\mathrm{gr}^{\mathrm{n}}_{\hbar}$ to $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$. From the resulting distinguished triangle we deduce the following long exact sequence:

$$0 \to_n \mathscr{M}' \to_n \mathscr{M} \to_n \mathscr{M}'' \to \mathscr{M}'_n \to \mathscr{M}_n \to \mathscr{M}''_n \to 0.$$

By assumption we have ${}_n\mathscr{M}'' = 0$, thus the short exact sequence (2.1.2) follows. □

Corollary 2.1.3. *For any* $\mathscr{M} \in \mathrm{Mod}(\mathscr{A})$ *and* $n \geq 0$*, we have the following short exact sequence both in* $\mathrm{Mod}(\mathscr{A})$ *and in* $\mathrm{Mod}(\mathscr{A}_n)$*:*

$$0 \to \mathscr{M}_{\hbar-\mathrm{tor}_n} \to \mathscr{M}_n \to \mathscr{M}_{\hbar-\mathrm{tf}_n} \to 0. \tag{2.1.3}$$

Take $\mathscr{M} \in \mathrm{Mod}(\mathscr{A})$ and $n' \geq n-k$. Consider the morphism $\overline{\hbar^k} : \mathscr{M}_{n'} \to \mathscr{M}_n$ defined by the multiplication by $\hbar^k$. Observe that the action of $\hbar^k$ in $\mathscr{M}_n$ coincides with the composition of the chain of morphisms

$$\mathscr{M}_n \xrightarrow{\overline{\hbar^k}} \mathscr{M}_{n+k} \xrightarrow{\rho_{n,n+k}} \mathscr{M}_n.$$

Lemma 2.1.4. *For each* $n \geq k \geq 1$ *and each* $n' \geq n-k$ *one has an exact sequence in* $\mathrm{Mod}(\mathscr{A})$*:*

$$\mathscr{M}_{n'} \xrightarrow{\overline{\hbar^k}} \mathscr{M}_n \xrightarrow{\rho_{k-1,n}} \mathscr{M}_{k-1} \to 0. \tag{2.1.4}$$

Proof. Clearly, $\ker(\rho_{k-1,n}) = \hbar^k\mathscr{M}/\hbar^{n+1}\mathscr{M} \simeq \overline{\hbar^k}(\mathscr{M}_{n'})$. □

Lemma 2.1.5. *Let* $\mathscr{M}$ *be an* $\hbar$*-complete* $\mathscr{A}$*-module. Then* $\mathscr{M}$ *is* $\hbar$*-torsion free if and only if for every* $n \geq 0$ *the sequence below is exact in* $\mathrm{Mod}(\mathscr{A})$*:*

$$0 \to \mathscr{M}_0 \xrightarrow{\overline{\hbar^n}} \mathscr{M}_n \xrightarrow{\hbar} \mathscr{M}_n. \tag{2.1.5}$$

Proof. If $\mathscr{M}$ is $\hbar$-torsion free, (2.1.5) is clearly exact since, for $m, m' \in \mathscr{M}$, $\hbar^n m = \hbar^{n+1}m'$ entails $m = \hbar m'$.

Conversely, assume that for every $n \geq 0$ we have the exact sequence (2.1.5). Thus, given $(v_n)_n \in \mathscr{M}$ such that $\hbar v_n = 0$ for every $n \geq 0$, it follows that $v_n = \overline{\hbar^n} u_{0n}$ for some (unique) $u_{0n} \in \mathscr{M}_0$. Thus, we may choose $u_n \in \mathscr{M}_n$ such that $v_n = \hbar^n u_n$, $\forall n$. On the other hand $v_n = \rho_{nn'}(v'_n), \forall n' \geq n$, hence $v_n = \rho_{nn'}(\hbar^{n'} u_{n'}) = \hbar^{n'}\rho_{nn'}(u_{n'}) = 0$ since we may take $n' \geq n+1$. □

Stacks. In the sequel, particularly in the applications of chapter 3, we shall deal with subcategories of sheaves whose objects are described by local properties. The convenient language to describe the behaviour of such objects is that of stacks. Moreover, since on each open subset $U \subset X$ we deal with categories of sheaves which are abelian subcategories of modules over some sheaf of rings defined on X, and since the restriction morphisms are the usual restriction of sheaves to open subsets, our stacks are in fact sheaves of categories. A fortiori we deal with $\mathbb{K}$-linear stacks.

Denote by $\mathfrak{Mod}(\mathscr{A})$ and $\mathfrak{Mod}_{\mathrm{coh}}(\mathscr{A})$ the stacks defined on X by the correspondences $U \mapsto \mathrm{Mod}(\mathscr{A}|_U)$ and $U \mapsto \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}|_U)$, respectively.

Given a full substack $\mathfrak{C} : U \mapsto \mathfrak{C}(U)$ of $\mathfrak{Mod}(\mathscr{A})$ of abelian subcategories, we shall consider the following condition defining a full Serre substack $\mathfrak{D}$ of $\mathfrak{C}$ (by a full Serre substack we mean that $\mathfrak{D}(U)$ is a full Serre subcategory of $\mathfrak{C}(U)$ for every open subset $U \subset X$):

Condition 2.1.6. For each open subset $U \subset X$, $\mathscr{M} \in \mathfrak{C}(U)$ belongs to $\mathfrak{D}(U)$ if and only if, for each $x \in U$, there exists a neighborhood $V \subset U$ of x such that, for any submodule $\mathscr{N}$ of $\mathscr{M}$ (and hence for any quotient $\mathscr{N}$ of $\mathscr{M}$ belonging to $\mathfrak{C}(V)$), one has

$$H^j(K;\mathscr{N}) = 0, \text{ for any } j > 0 \text{and for any } V \supset K \in \mathcal{B}. \tag{2.1.6}$$

Note that if $\mathfrak{C}$ coincides with $\mathfrak{Mod}_{\mathrm{coh}}(\mathscr{A})$, then one gets $\mathfrak{D} = \mathfrak{C}$, that is, the condition is automatically fulfilled by coherent modules.

Lemma 2.1.7. *Let $\mathscr{M} \in \mathrm{Mod}(\mathscr{A})$ and suppose that $\mathscr{M}_0$ belongs to $\mathfrak{D}(X)$. Let $(V_i)_i$ be an open covering of X where Condition 2.1.6 is satisfied by $\mathscr{M}_0$. Then, for $K \in \mathcal{B}$ contained in V_i, one has:*

1. $H^j(K;\mathscr{M}_n) = 0, \forall j > 0, n \geq 0$;
2. $H^j(K;\varprojlim_n \mathscr{M}_n) = 0, \forall j > 0$. *In particular, if $\mathscr{M}$ is $\hbar$-complete one also has* $H^j(K;\mathscr{M}) = 0, \forall j > 0$.

Proof. (1) Let us consider, for each $n \in \mathbb{N}$, the exact sequence:

$$\hbar^n\mathscr{M}/\hbar^{n+1}\mathscr{M} \to \mathscr{M}_n \xrightarrow{\rho_{n-1,n}} \mathscr{M}_{n-1} \to 0.$$

Since $\hbar^n\mathscr{M}/\hbar^{n+1}\mathscr{M}$ is the image of the morphism $\overline{\hbar^n} : \mathscr{M}_{n-1} \to \mathscr{M}_n$, it is also a quotient of $\mathscr{M}_{n-1}$. Thus, starting with $\mathscr{M}_0$, the result follows by induction on n.

(2) By (1), when $\mathcal{B}$ is a basis of open sets the statement is clear. When $\mathcal{B}$ is a basis of compact sets, we may consider a fundamental system of compact neighborhoods $\tilde{K} \in \mathcal{B}$ of K in V_i. For any j, we have $H^j(K, \varprojlim_n \mathscr{M}_n) \simeq \varinjlim_{\tilde{K}} H^j(\tilde{K}, \varprojlim_n \mathscr{M}_n)$.

Since the map $H^j(\tilde{K}, \varprojlim_n \mathscr{M}_n) \to H^j(K, \varprojlim_n \mathscr{M}_n)$ factors by

$$H^j(\tilde{K}, \varprojlim_n \mathscr{M}_n) \to H^j(\tilde{K}, \varprojlim_n (\mathscr{M}_n|_{\tilde{K}})) \to H^j(K, \varprojlim_n \mathscr{M}_n)$$

it remains to observe that $H^j(\tilde{K}, \varprojlim_n(\mathscr{M}_n|_{\tilde{K}})) = 0$, for $j > 0$, as a consequence of (1) and of [15, Exercise II.12]. □

Let there be fixed in the sequel a $\mathbb{K}[[\hbar]]$-linear full Serre substack $\mathscr{S}$ of $\mathfrak{Mod}_{\mathrm{coh}}(\mathscr{A})$. By convenience, for each $n \in \mathbb{N}_0$, we denote by $\mathscr{S}_n$ the substack of $\mathfrak{Mod}_{\mathrm{coh}}(\mathscr{A}_n)$ defined by:

$$U \mapsto \mathscr{S}_n(U) := \mathscr{S}(U) \cap \mathrm{Mod}(\mathscr{A}_n|_U).$$

Note that $\mathscr{S}_n(U)$ is a full Serre subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_n|_U)$, for each open subset $U \subset X$ and each $n \in \mathbb{N}_0$.

Convention 2.1.8. Given an open subset $U \subset X$ and an object $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}|_U)$, we shall use the notation $\mathscr{M} \in \mathscr{S}$ (resp. $\mathscr{M} \in \mathscr{S}_n$) to mean that $\mathscr{M} \in \mathscr{S}(U)$ (resp. $\mathscr{M} \in \mathscr{S}_n(U)$), when there is no ambiguity.

The category $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$.

Definition 2.1.9. We denote by $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$ the full subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$ consisting of $\mathscr{A}$-modules $\mathscr{M}$ such that, for each $n \geq 0$, both ${}_n\mathscr{M}$ and $\mathscr{M}_n$ are objects of $\mathscr{S}_n$.

We also denote by $\mathsf{D}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{A})$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathscr{A})$ consisting of objects $\mathscr{M}$ such that, for each $n \geq 0$ and for every $j \in \mathbb{Z}$, the cohomology module $H^j(\mathrm{gr}^{\mathrm{n}}_{\hbar}(\mathscr{M}))$ belongs to $\mathscr{S}_n$.

Since each $\mathscr{M} \in \mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$ is coherent, the sequence $({}_n\mathscr{M})_n$ is locally stationary, in other words, $\mathscr{M}_{\hbar-\mathrm{tor}}$ is locally annihilated by a fixed power $\hbar^N$.

Proposition 2.1.10. (a) *$\mathscr{S}(X)$ is a subcategory of* $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$.

(b) *Let $\mathscr{M} \in \mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$ be an $\hbar$-torsion module. Then $\mathscr{M}$ is also an object of $\mathscr{S}(X)$.*

Proof. (a) Let $\mathscr{M} \in \mathscr{S}(X)$. For each $n \in \mathbb{N}_0$ we have the exact sequences below in $\mathrm{Mod}(\mathscr{A})$:

$$0 \to \hbar^{n+1}\mathscr{M} \to \mathscr{M} \to \mathscr{M}_n \to 0$$

and

$$0 \to_n \mathscr{M} \to \mathscr{M} \to \hbar^{n+1}\mathscr{M} \to 0,$$

thus ${}_n\mathscr{M}$ and $\mathscr{M}_n$ belong to $\mathscr{S}(X)$, since $\mathscr{S}(X)$ is Serre.

(b) We have $\mathscr{M} \simeq \mathscr{M}_{\hbar-\mathrm{tor}}$ hence we can cover X by open subsets U and choose positive integers N_U such that $\hbar^{N_U+1}\mathscr{M}|_U = 0$. Thus $\mathscr{M}|_U \simeq \mathscr{M}_N|_U \in \mathscr{S}_{N_U}(U) \subset \mathscr{S}(U)$, so $\mathscr{M} \in \mathscr{S}(X)$ since $\mathscr{S}$ is a stack. □

Proposition 2.1.11. *Let $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$-module. Then the following properties are equivalent:*

(a) *$\mathscr{M}$ is an object of the category* $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$*;*

(b) $\mathscr{M}_0 \in \mathscr{S}_0$*;*

(c) $\mathscr{M}_n \in \mathscr{S}_n$*, for every* $n \geq 0$*.*

Proof. (a $\Rightarrow$ b) by the definition of $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$.

(b $\Rightarrow$ c) By Lemma 2.1.4, we have the right exact sequence for each $n \geq 0$:

$$\mathscr{M}_{n-1} \xrightarrow{\bar{\hbar}} \mathscr{M}_n \xrightarrow{\rho_{0,n}} \mathscr{M}_0 \to 0.$$

Since $\mathscr{M}_0 \in \mathscr{S}_0$, we can use the Serre property to proceed by induction and conclude that $\mathscr{M}_n \in \mathscr{S}_n$ for every $n \geq 0$.

(c $\Rightarrow$ a) Assume that $\mathscr{M}_n$ belongs to $\mathscr{S}_n$, for every $n \geq 0$, and let us prove that ${}_n\mathscr{M} \simeq_n \mathscr{M}_{\hbar-\mathrm{tor}}$ is an object of $\mathscr{S}_n$. By Corollary 2.1.3 we know that $\mathscr{M}_{\hbar-\mathrm{tor}_n} \in \mathscr{S}_n$ for each $n \geq 0$. The statement being of local nature we may assume the existence of $N \geq 0$ such that $\mathscr{M}_{\hbar-\mathrm{tor}_N} \simeq \mathscr{M}_{\hbar-\mathrm{tor}}$. Thus $\mathscr{M}_{\hbar-\mathrm{tor}}$ belongs to $\mathscr{S}_N$ as an $\mathscr{A}_N$-module which entails, by Proposition 2.1.10, that $\mathscr{M}_{\hbar-\mathrm{tor}} \in \mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$. ☐

Proposition 2.1.12. $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$ *is a Serre subcategory of* $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$*.*

Proof. Consider a short exact sequence in $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$

$$0 \to \mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3 \to 0.$$

It yields the distinguished triangle

$$\mathrm{gr}^{\mathrm{n}}_{\hbar}(\mathscr{M}_1) \to \mathrm{gr}^{\mathrm{n}}_{\hbar}(\mathscr{M}_2) \to \mathrm{gr}^{\mathrm{n}}_{\hbar}(\mathscr{M}_3) \xrightarrow{+1},$$

so $\mathrm{gr}^{\mathrm{n}}_{\hbar}(\mathscr{M}_i) \in \mathsf{D}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{A})$ if it is so for $\mathrm{gr}^{\mathrm{n}}_{\hbar}(\mathscr{M}_j)$ and $\mathrm{gr}^{\mathrm{n}}_{\hbar}(\mathscr{M}_k)$, with $i \neq j, k$, for every $n \in \mathbb{N}_0$.

Therefore it remains to prove that if $\mathscr{M}_2$ is an object of $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$ then $\mathscr{M}_1$ and $\mathscr{M}_3$ also belong to $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$. To prove this we consider the long exact sequence below:

$$0 \to_n \mathscr{M}_1 \to_n \mathscr{M}_2 \to_n \mathscr{M}_3 \to \mathscr{M}_{1,n} \to \mathscr{M}_{2,n} \to \mathscr{M}_{3,n} \to 0.$$

The Serre property of $\mathscr{S}$ entails that ${}_n\mathscr{M}_1, \mathscr{M}_{3,n} \in \mathscr{S}_n$. Then, by Proposition 2.1.11, we also have ${}_n\mathscr{M}_3 \in \mathscr{S}_n$. Finally, $\mathscr{M}_{1,n}$ must also belong to $\mathscr{S}_n$. ☐

Hence, in view of (2.1.1), we have:

Corollary 2.1.13. $\mathscr{M} \in \mathrm{Mod}(\mathscr{A})$ *is an object of* $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$ *if and only if* $\mathscr{M}_{\hbar-\mathrm{tor}}$ *and* $\mathscr{M}_{\hbar-\mathrm{tf}}$ *are objects of* $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$*.*

2.2 The formal extension of a functor

Consider now X and Y two complex manifolds of finite dimension and $\mathscr{A}$ and $\mathscr{A}'$ algebras of formal deformation on X and Y, respectively. For simplicity we still denote by $\hbar$ the fixed section in the center of $\mathscr{A}'$, thus both $\mathscr{A}$ and $\mathscr{A}'$ are $\mathbb{K}[[\hbar]]$-algebras. Let us fix on Y a basis of neighborhoods $\mathcal{B}'$ for which $\mathscr{A}'$ satisfies either Assumption 1.3.5 or Assumption 1.3.6 .

Assume that we are given a $\mathbb{K}[[\hbar]]$-linear functor $F : \mathrm{Mod}(\mathscr{A}) \to \mathrm{Mod}(\mathscr{A}')$. Then, one naturally defines a new functor $F^{\hbar}$, which we may call the formal extension of F:

Definition 2.2.1. $F^{\hbar}$ is the functor from $\mathrm{Mod}(\mathscr{A})$ to $\mathrm{Mod}(\mathscr{A}')$ defined by:

(i) For $\mathscr{M} \in \mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$, we set $F^{\hbar}(\mathscr{M}) := \varprojlim_{n\geq 0} F(\mathscr{M}_n) \in \mathrm{Mod}(\mathscr{A}')$;

(ii) Given a morphism $f : \mathscr{M} \to \mathscr{N}$ in $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$, its image

$$F^{\hbar}(f) : F^{\hbar}(\mathscr{M}) \to F^{h}(\mathscr{N})$$

is the morphism associated to the family of morphisms

$$F(\mathscr{M}_n) \xrightarrow{F(f_n)} F(\mathscr{N}_n), \quad (n \geq 0)$$

where $\mathscr{M}_n \xrightarrow{f_n} \mathscr{N}_n$ is induced by f.

Our goal now is to discuss the properties of $F^{\hbar}$ when we start with a functor F that is defined as a functor of prestacks (in a sense to be clarified) and study the restriction of $F^{\hbar}$ to a subcategory of the form $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$, where $\mathscr{S}$ denotes a full Serre substack of $\mathfrak{Mod}_{\mathrm{coh}}(\mathscr{A})$ as in the preceding section. In the sequel we shall assume that $\mathscr{S}$ satisfies the following:

Assumption 2.2.2. For each open subset $U \subset X$, $\mathscr{S}(U) = \cup_n \mathscr{S}_n(U)$.

Hence, if $\mathscr{M} \in \mathrm{Mod}(\mathscr{A}|_U)$ belongs to $\mathscr{S}(U)$, then $\mathscr{M}$ is an $\hbar$-torsion module.

According to the notations addopted in the preceding section, given a substack $\mathscr{S}'$ of $\mathfrak{Mod}(\mathscr{A}')$ and $n \geq 0$, we shall denote by $\mathscr{S}'_n$ the substack:

$$\mathscr{S}'_n : V \mapsto \mathscr{S}'_n(V) = \mathscr{S}'(V) \cap \mathrm{Mod}(\mathscr{A}'_n|_V).$$

From now on we consider fixed a full abelian substack $\mathfrak{C}'$ of $\mathfrak{Mod}(\mathscr{A}')$ as well as a full Serre substack $\mathscr{S}'$ of $\mathfrak{C}'$.

Assumption 2.2.3. Henceforward we assume that $\mathscr{S}'$ plays the role of $\mathfrak{D}$ in Condition 2.1.6 with respect to $\mathfrak{C}'$ and $\mathcal{B}'$.

Assumption 2.2.4. We fix a functor Φ from the category $\mathrm{Op}(X)$ of open subsets of X to the category $\mathrm{Op}(Y)$ satisfying the following conditions:

- $\Phi(X) = Y$;
- For any open subset $\Omega \subset X$ and any open covering $(U_i)_i$ of Ω, $(\Phi(U_i))_i$ is an open covering of $\Phi(\Omega)$.

Let us denote by $\Phi^*\mathscr{S}'$ the prestack defined by assigning to each open subset $U \subset X$ the subcategory $\Phi^*\mathscr{S}'(U) := \mathscr{S}'(\Phi(U))$ of $\mathrm{Mod}(\mathscr{A}'|_{\Phi(U)})$, the restriction morphism associated to $U \supset V$ being the sheaf restriction from $\mathrm{Mod}(\mathscr{A}'|_{\Phi(U)})$ to $\mathrm{Mod}(\mathscr{A}'|_{\Phi(V)})$.

Let now F be a $\mathbb{K}[[\hbar]]$-linear functor of prestacks

$$F : \mathscr{S} \to \Phi^*\mathscr{S}'.$$

In particular, to each $U \in \mathrm{Op}(X)$, F determines a $\mathbb{K}[[\hbar]]$-linear functor of categories $F(U) : \mathscr{S}(U) \to \mathscr{S}'(\Phi(U))$ that is compatible with the restriction morphisms in $\mathrm{Op}(X)$.

Whenever there is no ambiguity, we may write F instead of $F(X)$.

According to the preceding conventions, if $\mathscr{M} \in \mathscr{S}$ and $\hbar^{n+1}\mathscr{M} = 0$, then we also have $\hbar^{n+1}F(\mathscr{M}) = 0$. Therefore $F|_{\mathscr{S}_n}$ takes values in $\mathscr{S}'_n$.

Let $U \in \mathrm{Op}(X)$ and consider $\mathscr{M} \in \mathrm{Mod}_{\mathscr{S}(U)}(\mathscr{A}|_U)$. For $n \geq k \geq 0$ denote by $F(U)(\rho_{k,n})$ the image of the epimorphism $\rho_{k,n} : \mathscr{M}_n \to \mathscr{M}_k$ by the functor $F(U)$. Note that the family $(F(U)(\mathscr{M}_n), F(U)(\rho_{k,n}))_{k,n}$ is a projective system of $\mathscr{A}'|_{\Phi(U)}$-modules. By Definition 2.2.1 we get a functor:

$$\begin{aligned} F(U)^{\hbar} : \mathrm{Mod}_{\mathscr{S}(U)}(\mathscr{A}|_U) &\to \mathrm{Mod}(\mathscr{A}'|_{\Phi(U)}), \\ \mathscr{M} &\mapsto F(U)^{\hbar}(\mathscr{M}) := \varprojlim_{n\geq 0} F(U)(\mathscr{M}_n). \end{aligned}$$

Recall also that the functor $\varprojlim$ on the category of projective systems of $\mathrm{Mod}(\mathscr{A}')$ commutes with restrictions to open subsets, hence, if we start with $\mathscr{M} \in \mathrm{Mod}(\mathscr{A}')$, for each open subset $U \subset X$, we have

$$F^{\hbar}(\mathscr{M})|_{\Phi(U)} \simeq (F|_U)^{\hbar}(\mathscr{M}|_U).$$

Proposition 2.2.5. *Let $\mathscr{M} \in \mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$ be an $\hbar$-torsion module. Then, we have $F^{\hbar}(\mathscr{M}) \simeq F(\mathscr{M})$ in $\mathrm{Mod}(\mathscr{A}')$.*

Proof. In accordance with Proposition 2.1.10, $\mathscr{M} \in \mathscr{S}$, hence we have a natural morphism $F(\mathscr{M}) \to F^{\hbar}(\mathscr{M})$. We shall see that this morphism is locally an isomorphism. We can cover X by open subsets $U \subset X$ and consider a family of positive integers N_U such that $\hbar^{n+1}\mathscr{M}|_U = 0$. By the assumptions, $\{\Phi(U)\}$ is an open covering of Y. Since, for each $n \geq N_U$, $\mathscr{M}_n|_U \simeq \mathscr{M}_{N_U}|_U \simeq \mathscr{M}_U$, we obtain:

$$F^{\hbar}(\mathscr{M})|_{\Phi(U)} := \varprojlim_{n\geq 0} F(\mathscr{M}_n)|_{\Phi(U)} \simeq F(U)(\mathscr{M}_{N_U}|_U) \simeq F(U)(\mathscr{M}|_U) \simeq F(\mathscr{M})|_{\Phi(U)},$$

which ends the proof. □

In particular, by the Assumption 2.2.2 on $\mathscr{S}$, we conclude:

$$F^{\hbar}|_{\mathscr{S}} \simeq F.$$

Remark 2.2.6. The conditions required on the functor Φ are necessary to prove Proposition 2.2.5, which is a key result in the sequel. Such conditions would also be necessary if, with our machinery in hand, we went on constructing a stack $\mathfrak{Mod}_{\mathscr{S}}(\mathscr{A})$ defined by the correspondence $U \mapsto \mathrm{Mod}_{\mathscr{S}}(\mathscr{A})(U)$, the category $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})(U)$ being defined in U in a similar way to Definition 2.1.9. Indeed one might define $F^{\hbar}$ not only as a morphism of categories but as a functor of prestacks $\mathfrak{Mod}_{\mathscr{S}}(\mathscr{A}) \to \Phi^*\mathfrak{Mod}(\mathscr{A}')$. However, in view of the applications, it is enough to work with $F^{\hbar}$ defined as a morphism of categories cf. Definition 2.2.1.

2.3 Extension of a right exact functor

Assumption 2.3.1. From now on we assume that $F(X) : \mathscr{S}(X) \to \mathscr{S}'(X)$ is a right exact functor. Recall that when there is no risk of confusion we denote $F(X)$ simply by F.

Lemma 2.3.2. *Let $\mathscr{M} \in \mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$, $y \in Y$ and suppose that $K \in \mathcal{B}'$ is contained in a neighborhood V of y satisfying Condition 2.1.6 with respect to $F(\mathscr{M}_0)$. Then, one has $H^j(K; F(\mathscr{M}_n)) = 0$, for any $j > 0$ and $n \in \mathbb{N}_0$.*

Proof. In accordance with the right exactness of $F(X)$, for each $n \geq 1$ the sequence

$$F(h^n\mathscr{M}/h^{n+1}\mathscr{M}) \to F(\mathscr{M}_n) \xrightarrow{F(\rho_{n-1,n})} F(\mathscr{M}_{n-1}) \to 0$$

is exact. On the other hand, $F(h^n\mathscr{M}/h^{n+1}\mathscr{M})$ is the image of the morphism $F(\overline{\hbar^n}) : F(\mathscr{M}_{n-1}) \to F(\mathscr{M}_n)$, thus it is a quotient of $F(\mathscr{M}_{n-1})$.

The proof then proceeds by induction using the hypothesis on $F(\mathscr{M}_0)$. □

Denote by $\varrho_n : F^{\hbar}(\mathscr{M}) \to F(\mathscr{M}_n)$ the canonical projection.

Lemma 2.3.3. *For each $n \geq 0$, the sequence*

$$F^{\hbar}(\mathscr{M}) \xrightarrow{\hbar^{n+1}} F^{\hbar}(\mathscr{M}) \xrightarrow{\varrho_n} F(\mathscr{M}_n) \to 0. \tag{2.3.1}$$

is exact in $\mathrm{Mod}(\mathscr{A}')$.

Proof. By Lemma 2.1.4 we have, for any $N \geq n$, the exact sequence:

$$F(\mathscr{M}_N) \xrightarrow{F(\overline{\hbar^{n+1}})} F(\mathscr{M}_N) \xrightarrow{F(\rho_{n,N})} F(\mathscr{M}_n) \to 0.$$

Then, considering sufficiently small Ω in a basis $\mathcal{B}'$ in the conditions of Assumption 1.3.5 or Assumption 1.3.6, it follows that, for any $N \geq n$, the sequence

$$\Gamma(\Omega; F(\mathscr{M}_N)) \xrightarrow{F(\hbar^{n+1})} \Gamma(\Omega; F(\mathscr{M}_N)) \xrightarrow{F(\rho_{n,N})} \Gamma(\Omega; F(\mathscr{M}_n)) \to 0 \tag{2.3.2}$$

is also exact. In this way we obtain an exact sequence of projective systems satisfying Mittag-Leffler's condition, so, applying the functor $\varprojlim_N$ we obtain an exact sequence:

$$\varprojlim_N \Gamma(\Omega; F(\mathscr{M}_N)) \to \varprojlim_N \Gamma(\Omega; F(\mathscr{M}_N)) \to \Gamma(\Omega; F(\mathscr{M}_n)) \to 0. \tag{2.3.3}$$

If $\mathcal{B}'$ is a basis of open sets, this immediately entails the exactness of (2.3.1).

If $\mathcal{B}'$ is a basis of compact sets, we prove the exactness in the stalks: let $y \in Y$ and let us consider a fundamental system of open neighborhoods $\{\Omega_l\}_{l\in\mathbb{N}}$ of y and a fundamental system of compact neighborhoods $\{K_l\}_{l\in\mathbb{N}}$ of y, with each $K_l \in \mathcal{B}'$ and satisfying

$$K_{l+1} \subset \Omega_l \subset \operatorname{Int}(K_l). \tag{2.3.4}$$

Then, by applying $\varinjlim_l$ to the sequence obtained by replacing in (2.3.3) Ω by K_l, we obtain an exact sequence:

$$F^{\hbar}(\mathscr{M})_y \xrightarrow{\hbar^{n+1}} F^{\hbar}(\mathscr{M})_y \xrightarrow{\varrho_n} F(\mathscr{M}_n)_y \to 0,$$

as desired. □

Note that the exact sequence (2.3.1) entails that, for each $n \geq 0$, we have an isomorphism $F^{\hbar}(\mathscr{M})_n \xrightarrow{\sim} F(\mathscr{M}_n)$. Moreover, these isomorphisms are compatible with the transition morphisms. Hence, we have:

Corollary 2.3.4. *Let $\mathscr{M} \in \operatorname{Mod}_{\mathscr{S}}(\mathscr{A})$. Then*

$$F^{\hbar}(\mathscr{M}) := \varprojlim_{n\geq 0} F(\mathscr{M}_n) \simeq \varprojlim_{n\geq 0} F^{\hbar}(\mathscr{M})_n,$$

that is, $F^{\hbar}(\mathscr{M})$ is an $\hbar$-complete $\mathscr{A}'$-module.

As a consequence of Corollary 2.3.4 together with Lemma 2.1.7 we get:

Proposition 2.3.5. *Let $\mathscr{M} \in \operatorname{Mod}_{\mathscr{S}}(\mathscr{A})$. Then $F^{\hbar}(\mathscr{M})$ satisfies the vanishing condition (2.1.6) on Condition 2.1.6 for sufficiently small $K \in \mathcal{B}'$.*

Theorem 2.3.6. *The functor $F^{\hbar}$ is right exact.*

Proof. Let $\mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ be an exact sequence in $\operatorname{Mod}_{\mathscr{S}}(\mathscr{A})$. Then we have an exact sequence of projective systems of objects in $\mathscr{S}'$:

$$F(\mathscr{M}'_n) \to F(\mathscr{M}_n) \to F(\mathscr{M}''_n) \to 0 \quad (n \geq 0). \tag{2.3.5}$$

Thus, for every sufficiently small set Ω in a basis $\mathcal{B}'$ in the conditions of Assumption 1.3.5 or Assumption 1.3.6, we get an exact sequence of projective systems

$$\Gamma(\Omega; F(\mathscr{M}'_n)) \to \Gamma(\Omega; F(\mathscr{M}_n)) \to \Gamma(\Omega; F(\mathscr{M}''_n)) \to 0, \tag{2.3.6}$$

where each system satisfies Mittag-Leffler's condition. The proof then proceeds by the same argument as in Lemma 2.3.3. □

Corollary 2.3.7. *For $\mathscr{M} \in \mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$ the sequence below is exact in* $\mathrm{Mod}(\mathscr{A}')$*:*

$$F^\hbar(\mathscr{M}_{\hbar-\mathrm{tor}}) \to F^\hbar(\mathscr{M}) \to F^\hbar(\mathscr{M}_{\hbar-\mathrm{tf}}) \to 0. \tag{2.3.7}$$

Proposition 2.3.8. *Let us assume that $\mathscr{S}'$ is a subcategory of* $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}')$*. Then, for every $\mathscr{M} \in \mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$, $F^\hbar(\mathscr{M})$ belongs to $Mod_{\mathscr{S}'}(\mathscr{A}')$.*

Proof. In view of Proposition 2.1.11 it is enough to prove that $F^\hbar(\mathscr{M})_0$ belongs to $\mathscr{S}'_0$ and that $F^\hbar(\mathscr{M})$ is a coherent $\mathscr{A}'$-module. Note that the first part follows from the isomorphism $F^\hbar(\mathscr{M})_0 \simeq F(\mathscr{M}_0)$.

By Theorem 1.3.10 and Corollary 2.3.4, the second part is reduced to prove that $\hbar^n F^\hbar(\mathscr{M})/\hbar^{n+1}F^\hbar(\mathscr{M})$ is a coherent $\mathscr{A}'_0$-module.

The result follows, since $\hbar^n F^\hbar(\mathscr{M})/\hbar^{n+1}F^\hbar(\mathscr{M}) = \hbar^n F^\hbar(\mathscr{M})_n \simeq \hbar^n F(\mathscr{M}_n) \in \mathscr{S}'_n$. □

Proposition 2.3.9. *Consider the case where each stack $\mathscr{S}_n$ coincides with the stack $\mathfrak{Mod}_{\mathrm{coh}}(\mathscr{A}_n)$. Assume in addition that $F^\hbar(\mathscr{A})$ is $\hbar$-torsion free. Then:*

1. *$F^\hbar(\mathscr{A})$ is cohomologically complete.*
2. *For any $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$, $F^\hbar(\mathscr{M})$ is cohomologically complete.*

Proof. Let us start by noticing that the assumption on each $\mathscr{S}_n$ implies that $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$ coincides with $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$.

(1) The statement follows by Proposition 1.3.14, together with Corollary 2.3.4 and Proposition 2.3.5.

(2) Let us consider a local presentation of $\mathscr{M}$, say

$$\mathscr{A}^{\oplus N} \to \mathscr{A}^{\oplus L} \to \mathscr{M} \to 0,$$

for some $N, L \in \mathbb{N}$. We get an exact sequence

$$F^\hbar(\mathscr{A})^{\oplus N} \to F^\hbar(\mathscr{A})^{\oplus L} \to F^\hbar(\mathscr{M}) \to 0,$$

and the result follows by Proposition 1.3.13. □

2.4 Extension of an exact functor

Let us keep the notations as well as the setting of the preceding section. In addition, we assume:

Assumption 2.4.1. $F(U) : \mathscr{S}(U) \to \mathscr{S}'(U)$ is an exact functor for each open subset $U \subset X$.

By Lemma 2.1.5 we have a family of exact sequences:

$$0 \to F(\mathscr{M}_0) \xrightarrow{F(\overline{\hbar^n})} F(\mathscr{M}_n) \xrightarrow{\hbar} F(\mathscr{M}_n), \forall n.$$

Since $F(\mathscr{M}_k)$ is isomorphic to $F^{\hbar}(\mathscr{M})_k$ for each $k \geq 0$, the above exact sequences entail the following ones:

$$0 \to F^{\hbar}(\mathscr{M})_0 \xrightarrow{\overline{\hbar^n}} F^{\hbar}(\mathscr{M})_n \xrightarrow{\hbar} F^{\hbar}(\mathscr{M})_n, \forall n.$$

Thus, again by Lemma 2.1.5, we conclude:

Corollary 2.4.2. *If $\mathscr{M} \in \mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$ is $\hbar$-torsion free, then so is $F^{\hbar}(\mathscr{M})$.*

Our main goal in this section is to prove that the exactness of each $F(U)$ implies the exactness of the formal extension functor $F^{\hbar}$. We need some auxiliar results.

Lemma 2.4.3. *For each $\mathscr{M} \in \mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$ we have the short exact sequence in $\mathrm{Mod}(\mathscr{A}')$:*

$$0 \to F^{\hbar}(\mathscr{M}_{\hbar-\mathrm{tor}}) \to F^{\hbar}(\mathscr{M}) \to F^{\hbar}(\mathscr{M}_{\hbar-\mathrm{tf}}) \to 0. \tag{2.4.1}$$

Proof. Applying the exactness of F to (2.1.3), we obtain an exact sequence of projective systems of objects in $\mathscr{S}'$:

$$0 \to F((\mathscr{M}_{\hbar-\mathrm{tor}})_n) \to F(\mathscr{M}_n) \to F((\mathscr{M}_{\hbar-\mathrm{tf}})_n) \to 0. \tag{2.4.2}$$

Thus, for every sufficiently small set Ω in a basis $\mathcal{B}'$ in the conditions of Assumption 1.3.5 or Assumption 1.3.6, we get an exact sequence of projective systems:

$$0 \to \Gamma(\Omega; F((\mathscr{M}_{\hbar-\mathrm{tor}})_n)) \to \Gamma(\Omega; F(\mathscr{M}_n)) \to \Gamma(\Omega; F((\mathscr{M}_{\hbar-\mathrm{tf}})_n)) \to 0. \tag{2.4.3}$$

The result follows by a similar argument to that used in the proof of Lemma 2.3.3. □

Corollary 2.4.4. *For every $\mathscr{M} \in \mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$ and $n \geq 0$ we have the following isomorphism both in $\mathrm{Mod}(\mathscr{A}')$ and $\mathrm{Mod}(\mathscr{A}'_n)$:*

$$_n(F^{\hbar}(\mathscr{M})) \simeq F(_n\mathscr{M}).$$

Proof. Fix $n \geq 0$. By Proposition 2.2.5, $F^{\hbar}(\mathscr{M}_{\hbar-\mathrm{tor}}) \simeq F(\mathscr{M}_{\hbar-\mathrm{tor}})$ in $\mathrm{Mod}(\mathscr{A}')$. Then, Corollary 2.4.2 and Lemma 2.4.3 together with the exactness of F imply the chain of isomorphisms:

$$_nF^{\hbar}(\mathscr{M}) \simeq_n F^{\hbar}(\mathscr{M}_{\hbar-\mathrm{tor}}) \simeq_n F(\mathscr{M}_{\hbar-\mathrm{tor}}) \simeq F(_n\mathscr{M}_{\hbar-\mathrm{tor}}) \simeq F(_n\mathscr{M}).$$

□

Lemma 2.4.5. *For any $\mathscr{M} \in \mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$, $F^{\hbar}(\mathscr{M})$ is cohomologically complete.*

Proof. Let us first consider the case where $\mathscr{M}$ is $\hbar$-torsion free. It is enough to see that $F^{\hbar}(\mathscr{M})$ is in the conditions of Proposition 1.3.14. In fact, we already know that $F^{\hbar}(\mathscr{M})$ is $\hbar$-torsion free, $\hbar$-complete and that it satisfies the cohomological condition, cf. Corollaries 2.4.2 and 2.3.4 and Proposition 2.3.5, respectively. Therefore, the result holds in this case.

Consider now the case where $\mathscr{M}$ is an $\hbar$-torsion module. Note that the result is of local nature, thus we can cover Y by open subsets of the form $\Phi(U)$ and consider integers N_U such that $\hbar^{N_U} F^{\hbar}(\mathscr{M})|_{\Phi(U)} \simeq \hbar^{N_U} F(\mathscr{M}|_U) = 0$. Since $\mathscr{A}'^{\mathrm{loc}} \simeq \hbar^{N_U} \mathscr{A}'^{\mathrm{loc}}$, it follows that in $\Phi(U)$ we have:

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}'}(\mathscr{A}'^{\mathrm{loc}}, F^{\hbar}(\mathscr{M})) = 0.$$

Hence $F^{\hbar}(\mathscr{M})$ is cohomologically complete.

Finally, the general case follows by Proposition 1.3.13 and Lemma 2.4.3. □

The following result summarizes the relation between $F^{\hbar}$ and F:

Corollary 2.4.6. *For every $\mathscr{M} \in \mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$ and $n \geq 0$, we have a family of isomorphisms both in $\mathrm{Mod}(\mathscr{A}')$ and $\mathrm{Mod}(\mathscr{A}'_n)$:*

$$H^j(\mathrm{gr}^{\mathrm{n}}_{\hbar}(F^{\hbar}(\mathscr{M}))) \simeq F(H^j(\mathrm{gr}^{\mathrm{n}}_{\hbar}(\mathscr{M}))), \forall j \in \mathbb{Z}. \tag{2.4.4}$$

We have now the tools to prove the exactness of $F^{\hbar}$:

Theorem 2.4.7. *The formal extension functor $F^{\hbar} : \mathrm{Mod}_{\mathscr{S}}(\mathscr{A}) \to \mathrm{Mod}(\mathscr{A}')$ is an exact functor.*

Proof. A short exact sequence in $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$, say

$$0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0,$$

yields a sequence of cohomologically complete objects in $\mathrm{Mod}(\mathscr{A}')$.

$$0 \to F^{\hbar}(\mathscr{M}') \to F^{\hbar}(\mathscr{M}) \to F^{\hbar}(\mathscr{M}'') \to 0. \tag{2.4.5}$$

We apply the conservativism of $\mathrm{gr}_{\hbar}$ when restricted to cohomologically complete objects to conclude that (2.4.5) is exact. In fact, by the exactness of F and by the properties stated in Corollary 2.4.6, we know that the long sequence of cohomology below is exact:

$$0 \to {}_0F^{\hbar}(\mathscr{M}') \to {}_0F^{\hbar}(\mathscr{M}) \to {}_0F^{\hbar}(\mathscr{M}'') \to F^{\hbar}(\mathscr{M}')_0 \to F^{\hbar}(\mathscr{M})_0 \to F^{\hbar}(\mathscr{M}'')_0 \to 0.$$

□

2.5 Unicity of extensions

Let us now discuss the unicity of the extensions of the functors treated in the previous sections.

Consider Serre substacks $\mathscr{S}$ and $\mathscr{S}'$ and a functor $\Phi : \mathrm{Op}(X) \to \mathrm{Op}(Y)$ as above. Let $G : \mathscr{S} \to \Phi^*\mathscr{S}'$ be a $\mathbb{K}[[\hbar]]$-linear functor. Let $\mathcal{G}$ be a functor from $\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$ to $\mathrm{Mod}(\mathscr{A}')$ such that $\mathcal{G}|_{\mathscr{S}}$ takes values in $\mathscr{S}'$.

Definition 2.5.1. We say that $\mathcal{G}$ extends $G(X)$ if $\mathcal{G}|_{\mathscr{S}(X)}$ and G are isomorphic functors.

In particular, if $\mathcal{G}$ extends $G(X)$, then the natural morphisms $\mathcal{G}(\mathscr{M}) \to \mathcal{G}(\mathscr{M}_n) \simeq G(\mathscr{M}_n)$ define a morphism of functors:

$$\mathcal{G}(\cdot) \to G^{\hbar}(\cdot).$$

Proposition 2.5.2. *Consider the case where each $\mathscr{S}_n$ coincides with the stack $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_n)$. Assume that $G(X)$ is right exact. Then, up to isomorphism, $G^{\hbar}$ is the unique right exact functor of the form $\mathcal{G} : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}) \to \mathrm{Mod}(\mathscr{A}')$ that extends $G(X)$ and verifies $\mathcal{G}(\mathscr{A}) \xrightarrow{\sim} G^{\hbar}(\mathscr{A})$.*

Proof. Recall that, with the hypothesis, $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$ coincides with $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$.

First of all, it is clear that $G^{\hbar}$ satisfies the statement.

Suppose that $\mathcal{G}$ is another right exact functor that extends $G(X)$ and verifies $\mathcal{G}(\mathscr{A}) \xrightarrow{\sim} G^{\hbar}(\mathscr{A})$. Taking a local presentation of $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$, say,

$$\mathscr{A}^{\oplus N} \to \mathscr{A}^{\oplus L} \to \mathscr{M} \to 0,$$

and applying $\mathcal{G}$ and $G^{\hbar}$, one gets the diagram below with exact rows:

$$\begin{array}{ccccccccc}
\mathcal{G}(\mathscr{A})^{\oplus N} & \longrightarrow & \mathcal{G}(\mathscr{A})^{\oplus L} & \longrightarrow & \mathcal{G}(\mathscr{M}) & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow & & \downarrow & & \downarrow \\
(G^{\hbar}(\mathscr{A}))^{\oplus N} & \longrightarrow & (G^{\hbar}(\mathscr{A}))^{\oplus L} & \longrightarrow & G^{\hbar}(\mathscr{M}) & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}$$

The statement then follows by the Five Lemma. □

Proposition 2.5.3. *Consider the case where $G(U) : \mathscr{S}(U) \to \mathscr{S}'(U)$ is an exact functor for any $U \in Op(X)$. Then, up to isomorphism, $G^{\hbar}$ is the unique functor $\mathcal{G}$ that extends $G(X)$, takes values in the category of cohomologically complete objects and verifies ${}_0\mathcal{G}(\mathscr{M}) \simeq \mathcal{G}({}_0\mathscr{M})$ and $\mathcal{G}(\mathscr{M})_0 \simeq \mathcal{G}(\mathscr{M}_0)$.*

Proof. Clearly, $G^{\hbar}$ satisfies the statement.

On the other hand, consider a functor $\mathcal{G}$ which extends $G(X)$, takes values in the category of cohomologically complete $\mathscr{A}'$-modules and commutes with ${}_n(\cdot)$ and $(\cdot)_n$. Then, applying the conservativism of $\mathrm{gr}_{\hbar}$ to the morphism $\mathcal{G} \to G^{\hbar}$, one concludes the isomorphism $\mathcal{G} \simeq G^{\hbar}$. □

We can now sum up the above discussion and state the main result of this chapter:

Theorem 2.5.4. *Let X and Y be complex manifolds, let $\mathscr{A}$ (resp. $\mathscr{A}'$) be an algebra of formal deformation on X (resp. on Y), let $\mathscr{S}$ (resp. $\mathscr{S}'$) be a full Serre substack of $\mathfrak{Mod}_{\mathrm{coh}}(\mathscr{A})$ (resp. a full Serre substack of a full substack $\mathfrak{C}'$ of abelian categories of $\mathfrak{Mod}(\mathscr{A}')$) and let be given a functor $\Phi : Op(X) \to Op(Y)$ in the conditions of Assumption 2.2.4. Assume that $\mathscr{S}$ satisfies assumption 2.2.2 and that $\mathscr{S}'$ satisfies assumption 2.2.3 with respect to $\mathfrak{C}'$. Let $F : \mathscr{S} \to \Phi^*\mathscr{S}'$ be a $\mathbb{K}[[\hbar]]$-linear functor and assume that $F(X)$ is right exact. Then:*

1. $F^\hbar : \mathrm{Mod}_{\mathscr{S}}(\mathscr{A}) \to \mathrm{Mod}(\mathscr{A}')$ *is a canonical right exact* $\mathbb{K}[[\hbar]]$*-linear extension of* F*;*

2. *when* $\mathfrak{C}'$ *coincides with* $\mathfrak{Mod}_{\mathrm{coh}}(\mathscr{A}')$*, then* $F^\hbar$ *takes values in* $Mod_{\mathscr{S}'}(\mathscr{A}')$*;*

3. *if, for each open subset* $U \subset X$*,* $F(U)$ *is exact, then* $F^\hbar$ *is also exact; moreover, up to isomorphism it is the unique extension of* F *that takes values in the category of cohomologically complete objects and commutes with* ${}_0(\cdot)$ *and* $(\cdot)_0$*.*

Remark 2.5.5. $F^\hbar$ is canonical in the following sense: keeping the preceding notations, if we are given a functor $H : \mathscr{S} \to \tilde{\mathscr{S}}$, a functor $\tilde{H} : \mathscr{S}' \to \tilde{\mathscr{S}}'$, and a morphism θ of functors $\Phi, \Psi : \mathrm{Op}(X) \to \mathrm{Op}(Y)$, we derive a functor of prestacks $\tilde{H}^* : \Phi^*\mathscr{S}' \to \Psi^*\tilde{\mathscr{S}}'$, together with an extension $H^\hbar : \mathrm{Mod}_{\mathscr{S}}(\mathscr{A}) \to \mathrm{Mod}_{\tilde{\mathscr{S}}}(\mathscr{A}')$. If moreover $F : \mathscr{S} \to \Phi^*\mathscr{S}'$, $\tilde{F} : \tilde{\mathscr{S}} \to \Psi^*\tilde{\mathscr{S}}'$ are given, one may define a morphism $F \to \tilde{F}$ as a morphism of functors $\tilde{H}^* \circ F \to \tilde{F} \circ H$ (cf. diagram below):

$$\begin{array}{ccc} \mathscr{S} & \xrightarrow{F} & \Phi^*\mathscr{S}' \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle \tilde{H}^*} \\ \tilde{\mathscr{S}} & \xrightarrow{\tilde{F}} & \Psi^*\tilde{\mathscr{S}}' \end{array}$$

In this situation, we get a morphism of functors $F^\hbar \to \tilde{F}^\hbar \circ H^\hbar$ (cf. diagram below):

$$\begin{array}{ccccc} \mathscr{S} & \hookrightarrow & \mathrm{Mod}_{\mathscr{S}}(\mathscr{A}) & \xrightarrow{F^\hbar} & \mathrm{Mod}(\mathscr{A}') \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H^\hbar} & & \downarrow{\scriptstyle \sim} \\ \tilde{\mathscr{S}} & \hookrightarrow & \mathrm{Mod}_{\tilde{\mathscr{S}}}(\mathscr{A}) & \xrightarrow{\tilde{F}^\hbar} & \mathrm{Mod}(\mathscr{A}') \end{array}$$

Chapter 3

Application to $\mathscr{D}[[\hbar]]$-modules

3.1 Basic aspects of $\mathscr{D}_X[[\hbar]]$-modules

Let X be a complex manifold of dimension d_X and consider the rings $\mathscr{D}_X^\hbar = \mathscr{D}_X[[\hbar]]$ and $\mathscr{O}_X^\hbar = \mathscr{O}_X[[\hbar]]$, which are, respectively, the images of $\mathscr{D}_X$ and $\mathscr{O}_X$ by the functor (1.4.1).

On a subset $U \subset X$ the sections $P \in \Gamma(U; \mathscr{D}_X[[\hbar]])$ are operators $\sum_{n\geq 0} P_n \hbar^n$, where each P_n belongs to $\Gamma(U; \mathscr{D}_X)$, and the multiplication of such operators is given by

$$\left(\sum_{n\geq 0} P_n \hbar^n\right)\left(\sum_{i\geq 0} Q_i \hbar^i\right) = \sum_{n\geq 0}\sum_{i\geq 0} P_n Q_i \hbar^{n+i}.$$

Hence, $\hbar$ belongs to the center of $\mathscr{D}_X^\hbar$. Similarly, $\mathscr{O}_X^\hbar$ is endowed with a natural multiplication and also with a canonical structure of left $\mathscr{D}_X^\hbar$-module.

Recall that one says that an open subset U of X (respectively a compact subset K of X) is an open Stein subset of X (respectively a compact Stein subset of X) if it satisfies the so-called Theorems A and B of Cartan, that is, if $H^j(U, \mathscr{F})$ (resp. $H^j(K, \mathscr{F})$) vanishes for any $\mathscr{F} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X)$ and any $j \geq 1$. It is well-known that each $x \in X$ admits an open neighborhood that is Stein, as well as a compact neighborhood that is Stein. Hence, the family of open Stein subsets of X (resp. compact Stein subsets of X) constitutes an open basis of X (resp. a compact basis of X). The following remarks are based on well-known results on Stein subsets:

Remark 3.1.1. (i) The ring $\mathscr{O}_X^\hbar$ satisfies Assumption 1.3.5 if one takes for $\mathcal{B}$ the family of open Stein subsets of X. Here $\mathscr{A}_0$ is the $\mathbb{C}$-algebra $\mathscr{O}_X$. Using the same basis $\mathcal{B}$ and taking $\mathscr{R}_0 = \mathscr{O}_X$ in Proposition 1.4.4, one concludes that coherent $\mathscr{O}_X$-modules are $\hbar$-acyclic (cf. Corollary 2.6 of [3]).

(ii) The ring $\mathscr{D}_X^\hbar$ (and also the ring $\mathscr{D}_X^{\hbar\,\mathrm{op}}$) satisfies Assumption 1.3.6 if one takes for $\mathcal{B}$ the family of compact Stein subsets of X. In this case, $\mathscr{A}_0$ is the $\mathbb{C}$-algebra $\mathscr{D}_X$ and one considers the prestack of good $\mathscr{D}_X$-modules in the sense of Definition 1.2.4. Using the same basis $\mathcal{B}$ and taking $\mathscr{R}_0 = \mathscr{D}_X$ in Proposition 1.4.4, one concludes that coherent $\mathscr{D}_X$-modules are $\hbar$-acyclic (cf. Corollary 2.6 of [3]).

While in $\mathscr{D}$-modules theory Lemma 1.2.1 provides a natural way to construct $\mathscr{D}_X$-modules from $\mathscr{O}_X$-modules, we remark that in the $\hbar$-setting the situation is not so straigthforward. One has only the following lemma that requires an additional condition, namely, the $\hbar$-completion of the object. This condition is not always trivially checked, since projective limits do not behave in a nice way under some sheaf operations.

Lemma 3.1.2. *Let $\mathscr{M} \in \mathrm{Mod}(\mathbb{C}_X^\hbar)$ be an $\hbar$-complete object and assume that there is a well-defined structure of $\mathscr{D}_X$-module on $\mathscr{M}$. Then, there is a well-defined structure of $\mathscr{D}_X^\hbar$-module on $\mathscr{M}$.*

Proof. Let U be an open subset of X, $\sum P_n\hbar^n \in \Gamma(U;\mathscr{D}_X^\hbar)$ and $m \in \Gamma(U;\mathscr{M})$. For each n, $m_n := P_n \cdot m$ is a well-defined section in $\Gamma(U;\mathscr{M})$. On the other hand, by the $\hbar$-completeness of $\mathscr{M}$, $\sum_n m_n\hbar^n$ is also a well-defined section of $\Gamma(U;\mathscr{M})$. Hence, the action of $\mathscr{D}_X^\hbar$ is naturally defined by

$$\left(\sum_{n\geq 0} P_n\hbar^n\right) \cdot m = \sum_{n\geq 0}(P_n \cdot m_n)\hbar^n.$$

□

Coherence and flatness criteria for $\mathscr{D}_X^\hbar$-modules. By remark 3.1.1, the results on algebras of formal deformation and the results on formal extensions apply to the study of $\mathscr{D}_X^\hbar$-modules. In particular, objects of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ are cohomologically complete and the functor

$$\mathrm{gr}_\hbar : \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^\hbar) \to \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$$

is conservative. Moreover, as particular formulations of Theorems 1.3.17 and 1.3.18, one has, respectively:

Theorem 3.1.3. *Let $\mathscr{M} \in \mathsf{D}^+(\mathscr{D}_X^\hbar)$ and assume that $\mathscr{M}$ is cohomologically complete and $\mathrm{gr}_\hbar(\mathscr{M}) \in \mathsf{D}^+_{\mathrm{coh}}(\mathscr{D}_X)$. Then, $\mathscr{M} \in \mathsf{D}^+_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ and for each $i \in \mathbb{Z}$ one has the isomorphism*

$$H^i(\mathscr{M}) \xrightarrow{\sim} \varprojlim_n H^i(\mathscr{D}_X^\hbar/\hbar^{n+1}\mathscr{D}_X^\hbar \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar} \mathscr{M}).$$

Theorem 3.1.4. *Let $\mathscr{M}$ be a cohomologically complete $\mathscr{D}_X^\hbar$-module, with no $\hbar$-torsion and such that $\mathscr{M}/\hbar\mathscr{M}$ is a flat $\mathscr{D}_X$-module. Then $\mathscr{M}$ is a flat $\mathscr{D}_X^\hbar$-module.*

Since the ring $\mathscr{D}_X$ is d_X-syzygic, one has the following result (cf. Propositon 1.4.9):

Proposition 3.1.5. *For $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$:*

(a) *there is an isomorphism $\mathscr{N}^{\mathrm{R}\hbar} \xrightarrow{\sim} \mathscr{D}_X^\hbar \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{N}$;*

(b) *there is an isomorphism $\mathrm{gr}_\hbar(\mathscr{N}^\hbar) \simeq \mathscr{N}$.*

One also has analogous formulas of those stated in Lemma 1.4.15:

Lemma 3.1.6. *Let* $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ *and* $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ *(resp.* $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar\,\mathrm{op}})$*). We have the following isomorphism in* $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\hbar}_X)$*:*

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}^{\hbar}_X}(\mathscr{M}^{\mathrm{R}\hbar}, \mathscr{N}^{\mathrm{R}\hbar}) \simeq (\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N}))^{\mathrm{R}\hbar}$$

(resp. $\mathscr{M}^{\mathrm{R}\hbar} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}^{\hbar}_X} \mathscr{N}^{\mathrm{R}\hbar} \simeq \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{N}^{\mathrm{R}\hbar}$*.)*

Left and right $\mathscr{D}^{\hbar}_X$-modules. In the sequel we state results either for left or right $\mathscr{D}^{\hbar}_X$-modules, but in each situation one can also get similar results for the remaining case. Indeed, the category $\mathrm{Mod}(\mathscr{D}^{\hbar}_X)$ of left $\mathscr{D}^{\hbar}_X$-modules and the category $\mathrm{Mod}(\mathscr{D}_X^{\hbar\,\mathrm{op}})$ of right $\mathscr{D}^{\hbar}_X$-modules are equivalent to each other. This equivalence is constructed similarly to the $\mathscr{D}_X$-modules case: the ring $\mathscr{D}_X^{\hbar\,\mathrm{op}}$ is identified with the ring $\Omega_X \otimes_{\mathscr{O}_X} \mathscr{D}^{\hbar}_X \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1}$ (cf. [14, Proposition 1.10]) and one has the following equivalence of categories by [14, Proposition 1.9]:

$$\mathrm{Mod}(\mathscr{D}^{\hbar}_X) \to \mathrm{Mod}(\Omega_X \otimes_{\mathscr{O}_X} \mathscr{D}^{\hbar}_X \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1}), \quad \mathscr{M} \mapsto \Omega_X \otimes_{\mathscr{O}_X} \mathscr{M}.$$

Hence, we shall consider only left or right $\mathscr{D}^{\hbar}_X$-modules depending on the problem that we are treating.

We use the following duality functors:

$$\begin{aligned}
&\underline{\mathrm{D}}_{\mathscr{D}^{\hbar}_X} : \mathsf{D}^{\mathrm{b}}((\mathscr{D}^{\hbar}_X)^{\mathrm{op}}) \to \mathsf{D}^{\mathrm{b}}((\mathscr{D}^{\hbar}_X)^{\mathrm{op}}), \quad \mathscr{M} \mapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}^{\hbar}_X}(\mathscr{M}, \Omega_X[d_X] \otimes_{\mathscr{O}_X} \mathscr{D}^{\hbar}_X)\\
&\mathrm{D}'_{\mathscr{D}^{\hbar}_X} : \mathsf{D}^{\mathrm{b}}((\mathscr{D}^{\hbar}_X)^{\mathrm{op}}) \to \mathsf{D}^{\mathrm{b}}((\mathscr{D}^{\hbar}_X)), \quad \mathscr{M} \mapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}^{\hbar}_X}(\mathscr{M}, \mathscr{D}^{\hbar}_X).
\end{aligned}$$

Note that both $\underline{\mathrm{D}}_{\mathscr{D}^{\hbar}_X}$ and $\mathrm{D}'_{\mathscr{D}^{\hbar}_X}$ preserve coherence.

Lemma 3.1.7. *Let* $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}^{\hbar}_X)$ *be a cohomologically complete object and let* $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar\,\mathrm{op}})$*. Then* $\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}^{\hbar}_X} \mathscr{N}$ *is cohomologically complete.*

Proof. Since $\mathscr{M}$ is coherent, the following isomorphisms hold:

$$\begin{aligned}
\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}^{\hbar}_X} \mathscr{N} &\simeq \mathrm{D}'_{\mathscr{D}^{\hbar}_X} \mathrm{D}'_{\mathscr{D}^{\hbar}_X} \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}^{\hbar}_X} \mathscr{N}\\
&\simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}^{\hbar}_X}(\mathrm{D}'_{\mathscr{D}^{\hbar}_X} \mathscr{M}, \mathscr{D}^{\hbar}_X) \overset{\mathrm{L}}{\otimes}_{\mathscr{D}^{\hbar}_X} \mathscr{N}\\
&\simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}^{\hbar}_X}(\mathrm{D}'_{\mathscr{D}^{\hbar}_X} \mathscr{M}, \mathscr{N}).
\end{aligned}$$

The conclusion follows from Proposition 1.3.16. ☐

Characteristic variety and holonomic modules.

Definition 3.1.8. The characteristic variety of $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ is defined by

$$\mathrm{char}_\hbar(\mathscr{M}) := \mathrm{char}(\mathrm{gr}_\hbar(\mathscr{M})).$$

The characteristic variety of coherent $\mathscr{D}^\hbar_X$-modules was first studied in [3]. The following result proved in loc. cit. states useful properties of the characteristic variety. We include the proof for the reader's convenience, since it constitutes a nice example of the techniques applied when studying $\mathscr{D}^\hbar$-modules.

Proposition 3.1.9 ([3], Proposition 3.6)**.** (i) *If* $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}^\hbar_X)$ *is an* $\hbar$*-torsion module, one has* $\mathrm{char}_\hbar(\mathscr{M}) = \mathrm{char}_\hbar(\mathscr{M}_0) = \mathrm{char}_\hbar({}_0\mathscr{M})$.

(ii) *For* $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}^\hbar_X)$ *one has* $\mathrm{char}_\hbar(\mathscr{M}) = \mathrm{char}(\mathscr{M}_0)$.

(iii) *The characteristic variety* $\mathrm{char}_\hbar$ *is additive both on* $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}^\hbar_X)$ *and on* $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}^\hbar_X)$.

Proof. (i) By definition one has $\mathrm{char}_\hbar(\mathscr{M}) = \mathrm{char}(\mathscr{M}_0) \cup \mathrm{char}({}_0\mathscr{M})$. Then it is enough to prove that $\mathrm{char}(\mathscr{M}_0) = \mathrm{char}({}_0\mathscr{M})$. Since the problem is local, one assume that $\hbar^N\mathscr{M} = 0$ for some $N \in \mathbb{N}$ and one proceeds by induction.

The case $N = 1$ follows from $\mathscr{M} \simeq \mathscr{M}_0 \simeq_0 \mathscr{M}$.

Assuming that the result holds for $N-1$ and considering the short exact sequence $0 \to \hbar\mathscr{M} \to \mathscr{M} \to \mathscr{M}_0 \to 0$, note that $\hbar\mathscr{M}$ is an $\hbar$-torsion module such that $\hbar^{N-1}(\hbar\mathscr{M}) = 0$. Thus, by hypothesis, one has $\mathrm{char}_\hbar({}_0(\hbar\mathscr{M})) = \mathrm{char}_\hbar((\hbar\mathscr{M})_0)$. The result follows from the associated long exact sequence

$$0 \to_0 (\hbar\mathscr{M}) \to_0 \mathscr{M} \to \mathscr{M}_0 \to (\hbar\mathscr{M})_0 \to 0$$

and by the additivity of char.

(ii) Again, it is enough to prove that $\mathrm{char}({}_0\mathscr{M}) \subset \mathrm{char}(\mathscr{M}_0)$. Recall that one has ${}_0(\mathscr{M}_{\hbar-\mathrm{tor}}) \simeq_0 \mathscr{M}$ and also the short exact sequence $0 \to (\mathscr{M}_{\hbar-\mathrm{tor}})_0 \to \mathscr{M}_0 \to (\mathscr{M}_{\hbar-\mathrm{tf}})_0 \to 0$. Hence, one gets

$$\mathrm{char}({}_0\mathscr{M}) = \mathrm{char}({}_0(\mathscr{M}_{\hbar-\mathrm{tor}})) = \mathrm{char}((\mathscr{M}_{\hbar-\mathrm{tor}})_0) \subset \mathrm{char}(\mathscr{M}_0)$$

where the second equality follows from (i) and the inclusion follows from the additivity of char.

(iii) Let $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ be a short exact sequence. Applying the functor $\mathrm{gr}_\hbar$ one obtains the long exact cohomology sequence:

$${}_0(\mathscr{M}'') \to \mathscr{M}'_0 \to \mathscr{M}_0 \to (\mathscr{M}'')_0 \to 0.$$

Such sequence gives, by additivity of char and by (ii), the following identities:

$$\begin{aligned}
&\mathrm{char}_\hbar(\mathscr{M}'') \subset \mathrm{char}_\hbar(\mathscr{M}),\\
&\mathrm{char}_\hbar(\mathscr{M}) \subset \mathrm{char}_\hbar(\mathscr{M}') \cup \mathrm{char}_\hbar(\mathscr{M}''),\\
&\mathrm{char}_\hbar(\mathscr{M}') \subset \mathrm{char}({}_0(\mathscr{M}'')) \cup \mathrm{char}_\hbar(\mathscr{M}) \subset \mathrm{char}_\hbar(\mathscr{M}'') \cup \mathrm{char}_\hbar(\mathscr{M}) \subset \mathrm{char}_\hbar(\mathscr{M}).
\end{aligned}$$

□

Proposition 3.1.10 ([3], Proposition 3.8). *Let $\mathscr{M} \in \mathrm{Mod}(\mathscr{D}_X^\hbar)$ be an $\hbar$-torsion module. Then $\mathscr{M}$ is coherent as a $\mathscr{D}_X^\hbar$-module if and only if it is coherent as a $\mathscr{D}_X$-module, and in this case one has* $\mathrm{char}_\hbar(\mathscr{M}) = \mathrm{char}(\mathscr{M})$.

Definition 3.1.11. An object $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ is good (resp. holonomic; resp. regular holonomic) if $\mathrm{gr}_\hbar(\mathscr{M})$ is an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_X)$ (resp. an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$; resp. an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$).

The full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ consisting of good objects (resp. holonomic objects; resp. regular holonomic objects) is denoted by $\mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_X^\hbar)$ (resp. $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X^\hbar)$; resp. $\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X^\hbar)$).

Remark 3.1.12. Note that Theorem 3.1.3 can be rewritten replacing the coherence property by goodness or holonomicity, that is, a cohomologically complete object $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^\hbar)$ is an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_X^\hbar)$ (resp. an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X^\hbar)$) if and only if $\mathrm{gr}_\hbar(\mathscr{M})$ is an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_X)$ (resp. an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$).

Example 3.1.13. (a) $\mathscr{D}_X^\hbar$ and $\mathscr{O}_X^\hbar$ are objects of $\mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_X^\hbar)$ and $\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X^\hbar)$. Note that $\mathrm{char}_\hbar(\mathscr{O}_X^\hbar) = T^*_X X$, while $\mathrm{char}_\hbar(\mathscr{D}_X^\hbar) = T^*X$.

(b) The sheaf $\Omega_X^\hbar \simeq \Omega_X^{\mathrm{R}\hbar}$, formal extension of Ω_X, is an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_X^{\hbar\,\mathrm{op}})$ and $\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X^{\hbar\,\mathrm{op}})$.

(c) More generally, if $\mathscr{N}$ is a regular holonomic $\mathscr{D}_X$-module, then the equality $\mathrm{gr}_\hbar(\mathscr{N}^\hbar) = \mathscr{N}$ implies that $\mathscr{N}^\hbar$ is a regular holonomic $\mathscr{D}_X^\hbar$-module.

(d) Let $\mathscr{N}$ be a coherent $\mathscr{D}_X$-module. Consider $\mathscr{N}$ as a $\hbar$-torsion coherent $\mathscr{D}_X^\hbar$-module with the action being given by: $\hbar \cdot n = 0$ for any section n of $\mathscr{N}$. Then, one has $\mathrm{gr}_\hbar(\mathscr{N}) = \mathscr{N} \oplus \mathscr{N}[1]$ and one concludes that $\mathscr{N}$ is holonomic as a $\mathscr{D}_X$-module if and only if it is holonomic as a $\mathscr{D}_X^\hbar$-module.

(e) Let $P = \sum_{n\geq 0} P_n \hbar^n \in \mathscr{D}_X^\hbar$, with $P_n \in \mathscr{D}_X$, for each $n \geq 0$. Set $\mathscr{M} := \mathscr{D}_X^\hbar / \mathscr{D}_X^\hbar P$. By Proposition 3.1.9 one has $\mathrm{char}_\hbar(\mathscr{M}) = \mathrm{char}(\mathscr{M}_0)$. On the other hand, we have $\mathscr{M}_0 \simeq \mathscr{D}_X / \mathscr{D}_X P_0$. Hence, the characteristic variety of $\mathscr{M}$ coincides with the characteristic variety of the coherent $\mathscr{D}_X$-module $\mathscr{D}_X / \mathscr{D}_X P_0$ and the holonomicity of $\mathscr{M}$ depends only on P_0.

Consider the *solutions* functor and the De Rham functor for $\mathscr{D}^\hbar$-modules:

$$\begin{aligned}
&\mathrm{Sol}_{\mathscr{D}_X^\hbar} : \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^\hbar), \quad \mathscr{M} \mapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathscr{M}, \mathscr{O}_X^\hbar),\\
&\mathrm{DR}_{\mathscr{D}_X^\hbar} : \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^\hbar) \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^\hbar), \quad \mathscr{M} \mapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathscr{O}_X^\hbar, \mathscr{M}),
\end{aligned}$$

Recall that the equality $\mathrm{SS}(\mathrm{Sol}_{\mathscr{D}_X^\hbar}(\mathscr{M})) = \mathrm{char}_\hbar(\mathscr{M})$ is proved in [3, Theorem 3.13]. Moreover, an exhaustive study of holonomic $\mathscr{D}_X^\hbar$-modules is performed in loc. cit., namely, it is proved that $\mathrm{Sol}_{\mathscr{D}_X^\hbar}$ and $\mathrm{DR}_{\mathscr{D}_X^\hbar}$ apply holonomic objects on $\mathbb{C}$-constructible objects and that the following diagram is commutative:

$$\begin{array}{ccc} \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X^\hbar)^{\mathrm{op}} & \xrightarrow{\mathrm{DR}_{\mathscr{D}_X^\hbar}} & \mathsf{D}^{\mathrm{b}}_{\mathbb{C}-\mathrm{c}}(\mathbb{C}_X^\hbar)^{\mathrm{op}} \\ {\scriptstyle \mathrm{Sol}_{\mathscr{D}_X^\hbar}}\downarrow & \swarrow {\scriptstyle \mathrm{D}'_{\mathbb{C}_X^\hbar}} & \\ \mathsf{D}^{\mathrm{b}}_{\mathbb{C}-\mathrm{c}}(\mathbb{C}_X^\hbar). & & \end{array}$$

Furthermore, the Riemann-Hilbert correspondence is generalized to the $\hbar$-setting, that is, $\mathrm{Sol}_{\mathscr{D}_X^\hbar} : \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X^\hbar)^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}_{\mathbb{C}-\mathrm{c}}(\mathbb{C}_X^\hbar)$ is proved to be an equivalence of categories (cf. [3, Theorem 5.4]).

Remark 3.1.14. Let $n \geq 0$. Recall that the algebra $\mathscr{D}_{X,n}^\hbar = \mathscr{D}_X^\hbar / \hbar^{n+1}\mathscr{D}_X^\hbar$ has a canonical structure of $\hbar$-torsion $\mathscr{D}_X^\hbar$-module. Moreover, $\mathrm{Mod}(\mathscr{D}_{X\,n}^\hbar)$ is equivalent to the full subcategory of $\mathrm{Mod}(\mathscr{D}_X^\hbar)$ consisting on $\mathscr{D}_X^\hbar$-modules $\mathscr{M}$ such that $\hbar^{n+1}\mathscr{M} = 0$. On the other hand, $\mathscr{D}_X^\hbar$ induces on $\mathscr{D}_{X,n}^\hbar$ a left and a right structure of free $\mathscr{D}_X$-module of finite rank $(n+1)$. Therefore, $\mathscr{D}_{X,n}^\hbar$ becomes a $(\mathscr{D}_X^\hbar, \mathscr{D}_X)$-bimodule. These algebraic compatible structures are implicitly used in the sequel.

Remark 3.1.15. In the remaining of this chapter we shall extend to $\mathscr{D}^\hbar$-modules functors that are defined on full Serre subcategories of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ whose objects are characterized by local properties. As we shall see, since these full subcategories are the data of full Serre substacks, the functors we are interested in define linear functors to which the general results of Chapter 2 can be applied. In particular, according with our previous notations, the category $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ equals the category $\mathrm{Mod}_{\mathscr{S}}(\mathscr{D}_X^\hbar)$ when the full Serre substack $\mathscr{S}$ is given by:

$$U \mapsto \mathscr{S}(U) = \bigcup_n \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_{X,n}^\hbar)(U).$$

3.2 Transfer module and inverse image

From now on, $f : X \to Y$ denotes a morphism of complex manifolds of dimensions d_X and d_Y, respectively.

Consider the right exact functor

$$\begin{array}{rcl} f^* : \mathrm{Mod}(\mathscr{D}_Y) & \to & \mathrm{Mod}(\mathscr{D}_X) \\ \mathscr{M} & \mapsto & f^*(\mathscr{M}) = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{M}, \end{array} \tag{3.2.1}$$

from which one derives the inverse image functor for $\mathscr{D}$-modules. If $\mathscr{M} \in \mathrm{Mod}(\mathscr{D}_{Y,n}^\hbar)$, then $f^*(\mathscr{M})$ is naturally endowed with a canonical structure of $\mathscr{D}_{X,n}^\hbar$-module or, equivalently, a canonical structure of $\hbar$-torsion $\mathscr{D}_X^\hbar$-module. Therefore, the functor (3.2.1) can be regarded as a right exact functor from $\bigcup_n \mathrm{Mod}(\mathscr{D}_{Y,n}^\hbar)$ to $\bigcup_n \mathrm{Mod}(\mathscr{D}_{X,n}^\hbar)$. Our purpose in this section is to use the machinery of Chapter 2 to extend f^* to the framework of $\mathscr{D}^\hbar$-modules and to study the resulting functor.

Denote by $\mathrm{Mod}_{\mathrm{pc}}(\mathscr{D}_X^\hbar)$ the full abelian subcategory of $\mathrm{Mod}(\mathscr{D}_X^\hbar)$ consisting on pseudocoherent $\mathscr{D}_X^\hbar$-modules.

Lemma 3.2.1. *Let $\mathscr{M}$ be an object of $\bigcup_n \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_{Y,n}^\hbar)$. Then, $f^*(\mathscr{M})$ is an object of $\mathrm{Mod}_{\mathrm{pc}}(\mathscr{D}_X^\hbar)$ that satisfies the vanishing condition 2.1.6 with respect to the basis $\mathcal{B}$ of Stein compact subsets of X.*

Proof. Let $n \geq 0$ and suppose that $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_{Y,n}^\hbar)$.

By [21], it is known that $f^*(\mathscr{M})$ is a pseudocoherent $\mathscr{D}_X$-module which satisfies the following property:

(i) In a suitable neighborhood of each $x \in X$, it is an inductive limit of good $\mathscr{D}_X$-submodules.

Recall that any coherent $\mathscr{D}_X$-module is locally good and that any pseudocoherent submodule of a good $\mathscr{D}_X$-module is itself good.

Since inductive limits commute with cohomology on compact sets, it follows that $f^*(\mathscr{M})$ satisfies (2.1.6). Note also that condition (i) is closed for quotients and hence for submodules in the abelian category of pseudocoherent modules. Indeed, given $\tilde{\mathscr{M}}$ a pseudocoherent module satisfying (i) and given a pseudocoherent submodule $\tilde{\mathscr{N}}$ of $\tilde{\mathscr{M}}$, the quotient

$$\tilde{\mathscr{M}}/\tilde{\mathscr{N}}$$

is pseudocoherent. If in an open set Ω we have $\tilde{\mathscr{M}}|_\Omega \simeq \varinjlim_\alpha \mathscr{M}_\alpha$, for given good submodules $\mathscr{M}_\alpha$ of $\tilde{\mathscr{M}}$, since their images in $\tilde{\mathscr{M}}/\tilde{\mathscr{N}}$ are locally finitely generated, hence coherent, hence good, it follows that each $\tilde{\mathscr{N}} \cap \mathscr{M}_\alpha$ is good. By the exactness of inductive limits we get that

$$\tilde{\mathscr{N}}|_\Omega \simeq \varinjlim_\alpha \tilde{\mathscr{N}} \cap \mathscr{M}_\alpha.$$

This ends the proof. □

Let us now establish the setting that allows us to regard f^* as a morphism of suitable prestacks.

Denote by $\mathscr{S}$ be the full Serre substack $U \mapsto \bigcup_n \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_{Y,n})(U)$ of $\mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_X^h)$.

Let $\mathfrak{Mod}_{\mathrm{pc}}(\mathscr{D}_X^\hbar)$ be the full abelian substack of pseudocoherent $\mathscr{D}_X^\hbar$-modules, that is, the stack defined by the correspondence $U \mapsto \mathrm{Mod}_{\mathrm{pc}}(\mathscr{D}_X^\hbar|_U)$. Let $\mathscr{S}'$ denote the full Serre substack of $\mathfrak{Mod}_{\mathrm{pc}}(\mathscr{D}_X^\hbar)$ defined by the correspondence $U \mapsto \mathscr{S}'(U)$, where $\mathscr{S}'(U)$ is the full Serre subcategory of $\mathrm{Mod}_{\mathrm{pc}}(\mathscr{D}_X^\hbar|_U)$ consisting of modules that satisfy the vanishing condition 2.1.6 with respect to the basis of Stein compact subsets of X.

Consider the functor $\Phi : \mathrm{Op}(X) \to \mathrm{Op}(Y)$ given by $\Phi(U) = f^{-1}(U)$ together with the inclusions $U \supset V \mapsto \Phi(U) \supset \Phi(V)$. Clearly such Φ satisfies Assumption 2.2.4.

The inverse image of $\mathscr{D}$-modules can be regarded as a functor of prestacks $f^* : \mathscr{S} \to \Phi^*\mathscr{S}'$ which satisfies the conditions of Theorem 2.5.4 for $\mathscr{S}$, $\mathscr{S}'$ and Φ defined as above. Hence, the construction of Chapter 2 gives a functor

$$\begin{aligned} f^{*,\hbar} : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y^\hbar) &\to \mathrm{Mod}(\mathscr{D}_X^\hbar), \\ \mathscr{M} &\mapsto f^{*,\hbar}(\mathscr{M}) = \varprojlim_{n\geq 0} \left(\mathscr{O}_X \otimes_{f^{-1}(\mathscr{O}_Y)} f^{-1}\mathscr{M}_n \right), \end{aligned} \tag{3.2.2}$$

that is a right exact extension of f^*.

Let us consider the Serre subcategory $\mathrm{Mod}_{\mathrm{NC(f)}}(\mathscr{D}_{\mathrm{Y}})$ of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y)$ consisting of $\mathscr{D}_Y$-modules that are non-characteristic for f (see Definition 1.2.10).

Given $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_{Y,n}^\hbar)$ we shall say that $\mathscr{M}$ is non-characteristic for f if it is non-characteristic for f when endowed with the $\mathscr{D}_Y$-module structure explained in Remark 3.1.14. Denote by $\mathrm{Mod}_{\mathrm{NC(f)}}(\mathscr{D}_{\mathrm{Y,n}}^\hbar)$ the full Serre subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y^\hbar)$ consisting of such modules. Moreover $\cup_{n\geq 0}\mathrm{Mod}_{\mathrm{NC(f)}}(\mathscr{D}_{\mathrm{Y,n}}^\hbar)$ is also a full Serre subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y^\hbar)$. We obtain a full Serre substack $\mathfrak{Mod}_{\mathrm{NC}}(\mathscr{D}_Y^\hbar)$ of $\mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_Y^\hbar)$ by assigning to each open subset $U \subset X$ the subcategory $\cup_{n\geq 0}\mathrm{Mod}_{\mathrm{NC(f)}}(\mathscr{D}_{\mathrm{Y,n}}^{\mathrm{h}}|_{\mathrm{U}})$.

Definition 3.2.2. We say that $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y^\hbar)$ is non-characteristic for f if $\mathrm{gr}_\hbar(\mathscr{M}) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{NC}}(\mathscr{D}_Y)$.

In particular, $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y^\hbar)$ is non-characteristic for f if ${}_0\mathscr{M}$ and $\mathscr{M}_0$ are both non-characteristic in the usual sense. We denote by $\mathrm{Mod}_{\mathrm{NC(f)}}(\mathscr{D}_{\mathrm{Y}}^\hbar)$ the full abelian subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y^\hbar)$ consisting on non-characteristic modules for f, and by $\mathsf{D}^{\mathrm{b}}_{\mathrm{NC}}(\mathscr{D}_Y^\hbar)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y^\hbar)$ consisting of non-characteristic objects for the morphism f.

Let us now consider $\mathscr{S} = \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_Y^\hbar)$. The corresponding category $\mathrm{Mod}_{\mathscr{S}}(\mathscr{D}_Y^\hbar)$ coincides with $\mathrm{Mod}_{\mathrm{NC(f)}}(\mathscr{D}_{\mathrm{Y}}^\hbar)$ and, as an application of Proposition 2.1.12, $\mathrm{Mod}_{\mathrm{NC(f)}}(\mathscr{D}_{\mathrm{Y}}^\hbar)$ is a Serre subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y^\hbar)$. Proposition 2.1.11 entails also that, for $\mathscr{M} \in \mathrm{Mod}_{\mathrm{NC(f)}}(\mathscr{D}_{\mathrm{Y}}^\hbar)$, $\mathscr{M}_n$ is non-characteristic for f both as $\mathscr{D}_Y$-module and $\mathscr{D}_Y^\hbar$-module.

Recall that f^* restricted to $\mathrm{Mod}_{\mathrm{NC(f)}}(\mathscr{D}_{\mathrm{X}})$ becomes an exact functor that takes values in $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ (cf. Theorem 1.2.11). In the stacks setting, we can regard f^* as a morphism of prestacks, $f^* : \mathscr{S} \to \Phi^*\mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$, that verifies the conditions of Theorem 2.5.4 to the extension of exact functors. Therefore, we have an exact functor

$$f^{*,\hbar} : \mathrm{Mod}_{\mathrm{NC}(f)}(\mathscr{D}_Y^\hbar) \to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^\hbar), \mathscr{M} \mapsto f^{*,\hbar}(\mathscr{M})$$

that extends f^* and coincides with the restriction of the functor (3.2.2) to $\mathrm{Mod}_{\mathrm{NC}(f)}(\mathscr{D}_Y^\hbar)$.

Example 3.2.3. For any morphism $f : X \to Y$, $\mathscr{O}_Y^\hbar \in \mathrm{Mod}_{\mathrm{NC(f)}}(\mathscr{D}_{\mathrm{Y}}^\hbar)$ and we have

$f^{*,\hbar}(\mathscr{O}_Y^\hbar) \simeq \mathscr{O}_X^\hbar$. Indeed, one has

$$\begin{aligned} f^{*,\hbar}(\mathscr{O}_Y^\hbar) &= \varprojlim_{n\geq 0} \Big(\mathscr{O}_X \otimes_{f^{-1}(\mathscr{O}_Y)} f^{-1}(\mathscr{O}_Y^\hbar/\hbar^{n+1}\mathscr{O}_Y^\hbar)\Big) \\ &\simeq \varprojlim_{n\geq 0} \Big(\mathscr{O}_X \otimes_{f^{-1}(\mathscr{O}_Y)} f^{-1}(\mathscr{O}_Y \otimes_{\mathbb{C}_Y} (\mathbb{C}_Y^\hbar/\hbar^{n+1}\mathbb{C}_Y^\hbar))\Big) \\ &\simeq \varprojlim_{n\geq 0} \Big(\mathscr{O}_X \otimes_{\mathbb{C}_X} (\mathbb{C}_X^\hbar/\hbar^{n+1}\mathbb{C}_X^\hbar)\Big) \simeq \mathscr{O}_X^\hbar. \end{aligned}$$

Set

$$\mathscr{D}_{X\to Y,\hbar} := f^{*,\hbar}(\mathscr{D}_Y^\hbar) = \varprojlim_n (\mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}(\mathscr{D}_Y^\hbar/\hbar^{n+1}\mathscr{D}_Y^\hbar)).$$

(This object was denoted by $\mathcal{K}$ in [25].) Since each component of the projective limit has a canonical structure of $\hbar$-torsion $(\mathscr{D}_X^\hbar, f^{-1}\mathscr{D}_Y^\hbar)$-bimodule, then the projective limit $\mathscr{D}_{X\to Y,\hbar}$ is also a $(\mathscr{D}_X^\hbar, f^{-1}\mathscr{D}_Y^\hbar)$-bimodule. The object $\mathscr{D}_{X\to Y,\hbar}$ plays the role of the transfer module in the framework of $\mathscr{D}^\hbar$-modules.

Notation 3.2.4. From now on, when there is no risk of confusion and assuming that the morphism $f : X \to Y$ is fixed, we shall use the following abbreviations:

$$\begin{aligned} \mathscr{K}_\hbar &:= \mathscr{D}_{X\to Y,\hbar} \\ \mathscr{K} &:= \mathscr{D}_{X\to Y}. \end{aligned}$$

Since $f^{-1}(\mathscr{D}_Y^\hbar/\hbar^{n+1}\mathscr{D}_Y^\hbar)$ is isomorphic to $f^{-1}(\mathscr{D}_Y) \otimes_{\mathbb{C}_X} \mathbb{C}_X^\hbar/\hbar^{n+1}\mathbb{C}_X^\hbar$ for each $n \geq 0$, we conclude:

Lemma 3.2.5. *$\mathscr{K}_\hbar$ is isomorphic to $(\mathscr{D}_{X\to Y})^\hbar = \mathscr{K}^\hbar$ as $(\mathscr{D}_X^\hbar, f^{-1}(\mathscr{D}_Y^\hbar))$-bimodules. In particular, $\mathscr{K}_\hbar$ is $\hbar$-complete.*

Then, as a consequence of Lemma 1.2.6 and the coherence and flatness criteria for $\mathscr{D}^\hbar$-modules, we have:

Lemma 3.2.6. (a) *If f is a smooth morphism, then $\mathscr{K}_\hbar$ is a coherent left $\mathscr{D}_X^\hbar$-module and it is flat over $f^{-1}(\mathscr{D}_Y^{\hbar\,\mathrm{op}})$.*

(b) *If f is an embedding of a submanifold, then $\mathscr{K}_\hbar$ is a coherent $f^{-1}(\mathscr{D}_Y^{\hbar\,\mathrm{op}})$-module and it is flat over $\mathscr{D}_X^\hbar$.*

(c) *The transfer module $\mathscr{K}_\hbar$ is flat over $\mathscr{O}_X^\hbar$.*

Proposition 3.2.7. *For a given morphism of complex manifolds $f : X \to Y$, we have:*

(i) *$\mathscr{K}_\hbar$ is cohomologically complete.*

(ii) *For each $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y^\hbar)$, $f^{*,\hbar}(\mathscr{M})$ is cohomologically complete.*

(iii) *For* $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y^\hbar)$*, one has an isomorphism in* $\mathrm{Mod}(\mathscr{D}_X^\hbar)$*:*

$$\mathscr{K}_\hbar \otimes_{f^{-1}(\mathscr{D}_Y^\hbar)} f^{-1}\mathscr{M} \simeq f^{*,\hbar}(\mathscr{M}). \tag{3.2.3}$$

(iv) *For each* $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y^\hbar)$*,* $\mathscr{K}_\hbar \overset{\mathrm{L}}{\otimes}_{f^{-1}(\mathscr{D}_Y^\hbar)} f^{-1}\mathscr{M}$ *is cohomologically complete.*

Proof. (i) Follows by Proposition 2.3.9 since $\mathscr{K}_\hbar = f^{*,\hbar}(\mathscr{D}_Y^\hbar) \simeq \mathscr{D}^\hbar_{X\to Y}$ is $\hbar$-torsion free.

(ii) Follows also by Proposition 2.3.9.

To prove (iii) note that $\mathscr{M} \mapsto \mathscr{K} \otimes_{f^{-1}(\mathscr{D}_X^\hbar)} f^{-1}(\mathscr{M})$ is a right exact functor that extends f^* in the sense of Definition 2.5.1. Hence, the result follows by Proposition 2.5.2.

(iv) Let now be given $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y^\hbar)$ and a local free resolution of $\mathscr{M}$: $\mathscr{D}_Y^{\hbar,\bullet} \overset{\mathrm{qis}}{\to} \mathscr{M}$. Applying $\mathscr{K}_\hbar \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_Y^\hbar} f^{-1}(\cdot)$, we get a quasi-isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^\hbar)$:

$$\mathscr{K}_\hbar^\bullet \overset{\mathrm{qis}}{\to} \mathscr{K}_\hbar \overset{\mathrm{L}}{\otimes}_{f^{-1}(\mathscr{D}_Y^\hbar)} f^{-1}\mathscr{M}$$

The result follows since each term of $\mathscr{K}_\hbar^\bullet$ is cohomologically complete and so is the object $\mathscr{K}_\hbar^\bullet$ itself, by Proposition 1.3.13. □

The left hand side of (3.2.3) defines a left derivable right exact functor, say I_f on $\mathrm{Mod}(\mathscr{D}_Y^\hbar)$ which coincides with the functor $f^{*,\hbar}$ when restricted to $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y^\hbar)$, as we have shown in Proposition 3.2.7. Since any $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y^\hbar)$ admits locally a free resolution, hence an I_f-projective resolution, we may say without ambiguity that the left derived functor of $f^{*,\hbar}$ is given by the left derived functor of the functor I_f restricted to $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y^\hbar)$.

Definition 3.2.8. The functor of inverse image by f in the $\hbar$-setting, denoted by $\underline{f}^{*,\hbar}$, is defined by:

$$\begin{aligned} \underline{f}^{*,\hbar} : \mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y^\hbar) &\to \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^\hbar), \\ \mathscr{M} &\mapsto \underline{f}^{*,\hbar}(\mathscr{M}) = \mathscr{K}_\hbar \overset{\mathrm{L}}{\otimes}_{f^{-1}(\mathscr{D}_Y^\hbar)} f^{-1}\mathscr{M}. \end{aligned}$$

We remark that in our previous results we have generalized the classical coherence criteria of inverse images (cf. Theorem 1.2.11) to the $\hbar$-setting. Note that the commutativity of $\mathrm{gr}_\hbar$ with topological direct images and inverse images gives also the estimative

$$\mathrm{char}_\hbar(\underline{f}^{*,\hbar}(\mathscr{M})) = f_d f_\pi^{-1} \mathrm{char}_\hbar(\mathscr{M}).$$

Proposition 3.2.9. *Let* $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_Y^\hbar)$*. Then* $\underline{f}^{*,\hbar}(\mathscr{M}) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X^\hbar)$*.*

Proof. The result follows from the analogous property for $\mathscr{D}$-modules and from the formula

$$\mathrm{gr}_\hbar(\underline{f}^{*,\hbar}(\mathscr{M})) \simeq \mathscr{K} \overset{\mathrm{L}}{\otimes}_{f^{-1}(\mathscr{D}_Y)} f^{-1}\,\mathrm{gr}_\hbar(\mathscr{M}).$$

□

We can now generalize the Cauchy-Kowalewskaia-Kashiwara theorem to the $\hbar$-setting:

Theorem 3.2.10. *Assume that $\mathscr{M}$ belongs to $\mathsf{D}^{\mathrm{b}}_{\mathrm{NC}(f)}(\mathscr{D}_Y^\hbar)$. Then one has a natural isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^\hbar)$:*

$$f^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{D}_Y^\hbar}(\mathscr{M},\mathscr{O}_Y^\hbar) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\underline{f}^{*,\hbar}(\mathscr{M}),\mathscr{O}_X^\hbar). \tag{3.2.4}$$

Proof. There is a canonical basis change morphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^\hbar)$ (cf. Exercise II.24 of [15]):

$$f^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{D}_Y^\hbar}(\mathscr{M},\mathscr{O}_Y^\hbar) \to \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathscr{K}_\hbar\overset{\mathrm{L}}{\otimes}_{f^{-1}(\mathscr{D}_Y^\hbar)}f^{-1}\mathscr{M},\mathscr{K}_\hbar\overset{\mathrm{L}}{\otimes}_{f^{-1}(\mathscr{D}_Y^\hbar)}f^{-1}\mathscr{O}_Y^\hbar).$$

By Propositions 1.3.16 and 1.3.20, the objects in the left hand side and in the right hand side are both cohomologically complete. Besides, by Example 3.2.3 and by (3.2.3), we have $\mathscr{K}_\hbar\otimes_{f^{-1}(\mathscr{D}_Y^\hbar)}f^{-1}\mathscr{O}_Y^\hbar \simeq \mathscr{O}_X^\hbar$.

The result then follows by the CKK theorem for $\mathscr{D}$-modules by applying the conservativism of $\mathrm{gr}_\hbar$. □

Similarly to the $\mathscr{D}$-modules case, we can introduce the following notion:

Definition 3.2.11. The functor of extraordinary inverse image by f in the $\hbar$-setting, denoted by $\underline{f}^{!,\hbar}$, is defined by:

$$\begin{aligned}\underline{f}^{!,\hbar}:\mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y^\hbar) &\to \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^\hbar),\\ \mathscr{M} &\mapsto \underline{\mathrm{D}}_{\mathscr{D}_X^\hbar}(\underline{f}^{*,\hbar}(\underline{\mathrm{D}}_{\mathscr{D}_Y^\hbar}(\mathscr{M}))).\end{aligned}$$

By Proposition 3.2.9 we also get:

Corollary 3.2.12. *Let $\mathscr{M}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_Y^\hbar)$. Then, $\underline{f}^{!,\hbar}(\mathscr{M})\in\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X^\hbar)$.*

3.3 Direct image

We shall treat now the direct image functor for right $\mathscr{D}^\hbar$-modules but the results are easily adapted to the case of left $\mathscr{D}^\hbar$-modules.

Let $f: X\to Y$ be a morphism of complex manifolds and let $\mathscr{K}_\hbar$ be the corresponding transfer module. The $(\mathscr{D}_X^\hbar, f^{-1}(\mathscr{D}_Y^\hbar))$-bimodule structure on $\mathscr{K}_\hbar$ allows us to introduce the following notion in analogy with the theory of $\mathscr{D}$-modules:

Definition 3.3.1. The functors of direct image by f and proper direct image by f, in the $\hbar$-setting, are denoted by $\underline{f}_{*,\hbar}$ and $\underline{f}_{!,\hbar}$, respectively, and defined by:

$$\underline{f}_{*,\hbar}:\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^{\hbar\,\mathrm{op}})\to\mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y^{\hbar\,\mathrm{op}}),\quad \underline{f}_{*,\hbar}(\mathscr{M}):=\mathrm{R}f_*(\mathscr{M}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar}\mathscr{K}_\hbar),$$

$$\underline{f}_{!,\hbar}:\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^{\hbar\,\mathrm{op}})\to\mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y^{\hbar\,\mathrm{op}}),\quad \underline{f}_{!,\hbar}(\mathscr{M}):=\mathrm{R}f_!(\mathscr{M}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar}\mathscr{K}_\hbar).$$

Lemma 3.3.2. *Let $\mathscr{M}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar\,\mathrm{op}})$. Then $\underline{f}_{*,\hbar}(\mathscr{M})$ is cohomologically complete.*

Proof. The canonical isomorphisms

$$\mathscr{M}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar}\mathscr{K}_\hbar \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathrm{D}'_{\mathscr{D}_X^\hbar}\mathscr{M},\mathscr{D}_X^\hbar)\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar}\mathscr{K}_\hbar \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathrm{D}'_{\mathscr{D}_X^\hbar}\mathscr{M},\mathscr{K}_\hbar).$$

entail that $\mathscr{M}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar}\mathscr{K}_\hbar$ is cohomologically complete by Lemma 3.1.7. Finally, we conclude that $\underline{f}_{*,\hbar}(\mathscr{M})$ is cohomologically complete by Proposition 1.3.19. □

We are now able to extend to $\mathscr{D}^\hbar$-modules the classical coherence and holonomicity criteria for direct images of $\mathscr{D}$-modules (cf. Theorem 1.2.9):

Theorem 3.3.3. *Suppose that* $\mathscr{M}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_X^{\hbar\,\mathrm{op}})$ *(resp.* $\mathscr{M}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X^{\hbar\,\mathrm{op}})$*) and that f is proper on* $\mathrm{supp}(\mathscr{M})$*. Then,*

$$\underline{f}_{*,\hbar}(\mathscr{M})\in\mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_Y^{\hbar\,\mathrm{op}})$$

(resp. $\underline{f}_{*,\hbar}(\mathscr{M})\in\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_Y^{\hbar\,\mathrm{op}})$*.)*

Proof. Since by assumption $\mathrm{gr}_\hbar(\underline{f}_{*,\hbar}(\mathscr{M}))$ is an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_Y{}^{\mathrm{op}})$ (resp. $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_Y{}^{\mathrm{op}})$), the conclusion follows by applying Theorem 3.1.3 to the object $\underline{f}_{*,\hbar}(\mathscr{M})\in\mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y^{\hbar\,\mathrm{op}})$. □

Let us now discuss the direct image functor in view of our results regarding the extension of functors through projective limits. By Lemma 3.2.5, we have an isomorphism of bimodules: $\mathscr{K}_\hbar\simeq\mathscr{K}^\hbar$. It follows that, for each $n\geq 0$, we also have an isomorphism of $\hbar$-torsion $(\mathscr{D}_X^\hbar, f^{-1}(\mathscr{D}_Y^{\hbar\,\mathrm{op}}))$-bimodules:

$$\mathscr{K}_{\hbar n}\simeq\mathscr{D}_{X,n}^\hbar\otimes_{\mathscr{D}_X}\mathscr{K}.$$

So, for $\mathscr{M}\in\mathrm{Mod}(\mathscr{D}_X^{\hbar\,\mathrm{op}})$, we get natural isomorphisms in $\mathrm{Mod}(f^{-1}(\mathscr{D}_{Y\,n}^\hbar))$:

$$\mathscr{M}_n\otimes_{\mathscr{D}_X^\hbar}\mathscr{K}_\hbar\simeq\mathscr{M}_n\otimes_{\mathscr{D}_X^\hbar}\mathscr{K}_{\hbar n}\simeq(\mathscr{M}_n\otimes_{\mathscr{D}_X^\hbar}\mathscr{D}_{X,n}^\hbar)\otimes_{\mathscr{D}_X}\mathscr{K}\simeq\mathscr{M}_n\otimes_{\mathscr{D}_X}\mathscr{K}. \tag{3.3.1}$$

Since projective limits commute with direct images, (3.3.1) entails the following morphism in $\mathrm{Mod}(\mathscr{D}_Y^{\hbar\,\mathrm{op}})$:

$$f_*(\mathscr{M}\otimes_{\mathscr{D}_X^\hbar}\mathscr{K}_\hbar)\to\varprojlim_{n\geq 0}f_*(\mathscr{M}_n\otimes_{\mathscr{D}_X}\mathscr{K}). \tag{3.3.2}$$

Let us discuss the conditions that ensure that (3.3.2) is an isomorphism.

Lemma 3.3.4. *If* $\mathscr{M}\in\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar\,\mathrm{op}})$ *is such that* $\mathscr{M}\otimes_{\mathscr{D}_X^\hbar}\mathscr{K}_\hbar$ *is $\hbar$-complete then (3.3.2) is an isomorphism.*

Proof. If $\mathscr{M}\otimes_{\mathscr{D}_X^\hbar}\mathscr{K}_\hbar$ is $\hbar$-complete then we have isomorphisms in $\mathrm{Mod}(f^{-1}(\mathscr{D}_Y^\hbar)_n)$

$$\mathscr{M}\otimes_{\mathscr{D}_X^\hbar}\mathscr{K}_\hbar\simeq\varprojlim_{n\geq 0}(\mathscr{M}\otimes_{\mathscr{D}_X^\hbar}\mathscr{K}_\hbar)_n\simeq\varprojlim_{n\geq 0}(\mathscr{M}_n\otimes_{\mathscr{D}_X^\hbar}\mathscr{K}_\hbar)\simeq\varprojlim_{n\geq 0}(\mathscr{M}_n\otimes_{\mathscr{D}_X}\mathscr{K}),$$

and the result follows since f_* commutes with projective limits. □

Lemma 3.3.5. *For any* $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar\,\mathrm{op}})$, $\mathscr{M} \otimes_{\mathscr{D}_X^\hbar} \mathscr{K}_\hbar$ *is cohomologically complete.*

Proof. By Proposition 3.2.7, $\mathscr{K}_\hbar$ is cohomologically complete. On the other hand we can choose a local presentation of $\mathscr{M}$ by locally free $\mathscr{D}_X^\hbar$-modules to which we apply the right exact functor $\cdot \otimes_{\mathscr{D}_X^\hbar} \mathscr{K}_\hbar$. Hence, $\mathscr{M} \otimes_{\mathscr{D}_X^\hbar} \mathscr{K}_\hbar$ is locally the cokernel of a $\mathbb{C}_X^\hbar$-linear morphism of cohomologically complete modules and the result follows by Proposition 1.3.13. □

Corollary 3.3.6. *Let* $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar\,\mathrm{op}})$. *Then (3.3.2) is an isomorphism in each one of the following cases:*

(i) $\mathscr{M}$ *is an* $\hbar$*-torsion module;*

(ii) $\mathscr{M} \otimes_{\mathscr{D}_X^\hbar} \mathscr{K}_\hbar$ *is* $\hbar$*-torsion free.*

Proof. By Lemma 3.3.4 it is enough to prove that in both cases $\mathscr{M} \otimes_{\mathscr{D}_X^\hbar} \mathscr{K}_\hbar$ is $\hbar$-complete.

(i) If $\mathscr{M}$ is an $\hbar$-torsion module and since the result is local, we may assume that there exists some $N > 0$ such that $\hbar^N \mathscr{M} = 0$. This entails that $\hbar^N(\mathscr{M} \otimes_{\mathscr{D}_X^\hbar} \mathscr{K}_\hbar) \simeq 0$ and, in particular, that $\mathscr{M} \otimes_{\mathscr{D}_X^\hbar} \mathscr{K}_\hbar$ is $\hbar$-complete.

(ii) An $\hbar$-torsion free module is $\hbar$-complete if and only if it is cohomologically complete. Hence, the result follows by Lemma 3.3.5. □

The following result is eventually well known but we find useful to give here a proof:

Lemma 3.3.7. *Let* $F \in \mathrm{Mod}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X^\hbar)$. *Then* F *is* $\hbar$*-complete.*

Proof. We shall prove that the natural morphism $F \to \varprojlim_{n \geq 0} F_n$ is an isomorphism.

By the triangulation theorem (Proposition 8.2.5 of [15]) we may assume that F is a constructible sheaf on the realization of a finite simplicial complex (S, Δ) (we refer [15] for the notation). For each n, $F_n = \mathrm{Coker}(F \xrightarrow{\hbar^{n+1}} F)$ is also constructible on (S, Δ). It follows that there exists a locally finite open covering $\{U(\sigma)\}_{\sigma \in \Delta}$ of S such that, for each $\sigma \in \Delta$ and $x \in |\sigma|$, one has $\Gamma(U(\sigma); F) \simeq F_x$ and $\Gamma(U(\sigma); F_n) \simeq (F_n)_x$, for every $n \in \mathbb{N}$.

As a finitely generated $\mathbb{C}^\hbar$-module, F_x is $\hbar$-complete and hence

$$\Gamma(U(\sigma); F) \simeq F_x \simeq \varprojlim_{n \geq 0} (F_x)_n \simeq \varprojlim_{n \geq 0} (F_n)_x \simeq \varprojlim_{n \geq 0} \Gamma(U(\sigma); F_n) \simeq \Gamma(U(\sigma); \varprojlim_{n \geq 0} F_n),$$

and the result follows. □

Corollary 3.3.8. *If* $\mathscr{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathscr{D}_X^{\hbar\,\mathrm{op}})$, *then* $\mathscr{M} \otimes_{\mathscr{D}_X^\hbar} \mathscr{O}_X^\hbar$ *is* $\hbar$*-complete. In particular, when* $f = a_X : X \to \{pt\}$ *is the constant map, then* (3.3.2) *is an isomorphism for every holonomic* $\mathscr{D}_X^\hbar$*-module* $\mathscr{M}$.

Proof. Note that in this case we have $\mathscr{K}_\hbar \simeq \mathscr{O}_X^\hbar$. Hence,

$$\mathscr{M} \otimes_{\mathscr{D}_X^\hbar} \mathscr{K}_\hbar \simeq H^0(\mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathrm{D}'_{\mathscr{D}_X^\hbar}\mathscr{M}, \mathscr{O}_X^\hbar))$$

is an $\mathbb{R}$-constructible module (cf. [3, Th.3.13]) and the result follows by Lemma 3.3.7. □

Example 3.3.9. Consider the constant map $f : X \to \{\mathrm{pt}\}$. As a particular case of Corollary 3.3.8, we get that (3.3.2) is an isomorphism if X is a complex line and $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ has a discrete support. Indeed, the support of $\mathscr{M}$ coincides with the support of $\mathscr{M}_0$, so, if $\mathrm{supp}(\mathscr{M})$ is discrete, then $\mathscr{M}$ is holonomic.

Remark 3.3.10. As a matter of fact we didn't find a counter-example for the conjecture that if $\mathscr{M}$ is $\mathscr{D}_X^\hbar$-coherent, then $\mathscr{M} \otimes_{\mathscr{D}_X^\hbar} \mathscr{K}_\hbar$ is always $\hbar$-complete. Of course, such a counter-example, to exist, should firstly occur in the smooth case.

An interesting particular case of morphism (3.3.2) is that of a closed embedding $f = i : X \hookrightarrow Y$. In this case, $\mathscr{K}$ is flat over $\mathscr{D}_X$ and the direct image in the $\mathscr{D}$-modules framework turns out to be an exact functor:

$$\underline{i}_* : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X{}^{\mathrm{op}}) \to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y{}^{\mathrm{op}}), \quad \mathscr{M} \mapsto \underline{i}_*(\mathscr{M}) := i_*(\mathscr{M} \otimes_{\mathscr{D}_X} \mathscr{K}).$$

Consider the full Serre substacks $\mathscr{S} = \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ and $\mathscr{S}' = \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_Y)$. Choose for the functor $\Phi : \mathrm{Op}(X) \to \mathrm{Op}(Y)$ the data $U \mapsto \varphi(U) := X \backslash (Y \backslash U)$ which clearly satisfies Assumption 2.2.4. We can regard $\underline{i}_*$ as a functor from $\mathscr{S}$ to $\Phi^*\mathscr{S}'$ and we are in the conditions to apply Theorem 2.5.4 to extend $\underline{i}_*$ as an exact functor

$$\begin{aligned} \underline{i}_*^\hbar : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar\,\mathrm{op}}) &\to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y^{\hbar\,\mathrm{op}}), \\ \mathscr{M} &\mapsto \underline{i}_*^\hbar(\mathscr{M}) := \varprojlim_{n \geq 0} i_*(\mathscr{M}_n \otimes_{\mathscr{D}_X} \mathscr{K}). \end{aligned}$$

On the other hand, since $\mathscr{K}_\hbar$ is flat over $\mathscr{D}_X^\hbar$ and i_* is exact, we have an exact functor:

$$\begin{aligned} \underline{i}_{*,\hbar} : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar\,\mathrm{op}}) &\to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y^{\hbar\,\mathrm{op}}), \\ \mathscr{M} &\mapsto \underline{i}_{*,\hbar}(\mathscr{M}) \simeq i_*(\mathscr{M} \otimes_{\mathscr{D}_X^\hbar} \mathscr{K}_\hbar). \end{aligned} \tag{3.3.3}$$

Proposition 3.3.11. *The functors $\underline{i}_{*,\hbar}$ and $\underline{i}_*^\hbar$ are isomorphic.*

Proof. The result folows from Proposition 2.5.3, since (3.3.3) is an extension of $\underline{i}_* : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X{}^{\mathrm{op}}) \to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y{}^{\mathrm{op}})$ that takes values in the category of coherent modules (thus, cohomologically complete modules) and commutes with ${}_0(\bullet)$ and $(\bullet)_0$. □

This implies, in particular, that (3.3.2) is an isomorphism in the case of a closed embedding $i : X \hookrightarrow Y$, for any $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar\,\mathrm{op}})$.

3.4 Specialization, vanishing cycles and nearby-cycles

Specialization along a submanifold. Let Y denote a submanifold of a complex manifold X of finite dimension and denote by $\pi : T_Y X \to Y$ the canonical projection. Fix a section G of the canonical morphism $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$ to which all canonical V-filtrations mentioned below will refer.

Given $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_{X,n}^\hbar)$ we shall say that $\mathscr{M}$ is specializable along Y if it is so when endowed with the structure of $\mathscr{D}_X$-module explained in Remark 3.1.14. In this way we obtain a Serre subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ which we shall denote by $\mathrm{Mod}_{\mathrm{sp}}(\mathscr{D}_{X,n}^\hbar)$. Moreover, $\mathscr{S} = \cup_{n\geq 0}\mathrm{Mod}_{\mathrm{sp}}(\mathscr{D}_{X,n}^\hbar)$ is also a Serre subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ and, in fact, we obtain a full Serre substack $\mathscr{S}$ of $\mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ by assigning to each open subset $U \subset X$ the full Serre subcategory $\mathscr{S}(U) = \cup_{n\geq 0}\mathrm{Mod}_{\mathrm{sp}}(\mathscr{D}_{X,n}^\hbar|_U)$. For such $\mathscr{S}$, let us consider the corresponding auxiliar subcategory $\mathrm{Mod}_{\mathscr{S}}(\mathscr{D}_X^\hbar)$.

The results of Section 2.1 entail that a given $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ is an object of $\mathrm{Mod}_{\mathscr{S}}(\mathscr{D}_X^\hbar)$ if and only if ${}_0\mathscr{M}$ and $\mathscr{M}_0$ are specializable $\mathscr{D}_X$-modules, and also that $\mathrm{Mod}_{\mathscr{S}}(\mathscr{D}_X^\hbar)$ is a Serre subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$.

Definition 3.4.1. We say that a coherent $\mathscr{D}_X^\hbar$-module $\mathscr{M}$ is specializable along Y if $\mathscr{M} \in \mathrm{Mod}_{\mathscr{S}}(\mathscr{D}_X^\hbar)$.

Notation 3.4.2. Assuming that the manifold Y is fixed, we denote, alternatively, by $\mathrm{Mod}_{\mathrm{sp}}(\mathscr{D}_X^\hbar)$ the Serre subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ consisting of specializable $\mathscr{D}_X^\hbar$-modules (along Y).

Example 3.4.3. Similarly to the $\mathscr{D}$-modules case, if $\mathscr{M}$ is a coherent $\mathscr{D}_X^\hbar$-module such that $\mathrm{supp}(\mathscr{M}) \subset Y$, then $\mathscr{M}$ is specializable along Y. Indeed, one has:

$$\mathrm{supp}(\mathrm{gr}_\hbar(\mathscr{M})) = \mathrm{supp}({}_0\mathscr{M}) \cup \mathrm{supp}(\mathscr{M}_0) \subset \mathrm{supp}(\mathscr{M}) \subset Y.$$

Hence ${}_0\mathscr{M}$ and $\mathscr{M}_0$ are specializable along Y.

Consider the functor of specialization along Y, $\nu_Y : \mathrm{Mod}_{\mathrm{sp}}(\mathscr{D}_X) \to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_{T_Y X})$, an exact functor whose definition we have recalled in Section 1.2. Let us establish the setting that allows us to apply the results of Chapter 2 to extend ν_Y. Set

$$\mathscr{S} = \bigcup_{n\geq 0} \mathfrak{Mod}_{\mathrm{sp}}(\mathscr{D}_{X,n}^\hbar), \quad \mathscr{S}' = \bigcup_{n\geq 0} \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_{T_Y X,n}^\hbar).$$

Note that $\mathscr{S}$ and $\mathscr{S}'$ are full Serre substacks of $\mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ and $\mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_{T_Y X}^\hbar)$, respectively. Let $\Phi : \mathrm{Op}(X) \to \mathrm{Op}(T_Y X)$ be the functor defined by $U \mapsto \Phi(U) = \pi^{-1}(U \cap Y)$. Then, the specialization of $\mathscr{D}$-modules can be regarded as a morphism of prestacks $\nu_Y : \mathscr{S} \to \Phi^*\mathscr{S}'$ which verifies the conditions of Theorem 2.5.4. Therefore, ν_Y extends as an exact functor

$$\begin{aligned} \nu_Y^\hbar : \mathrm{Mod}_{\mathrm{sp}}(\mathscr{D}_X^\hbar) &\to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_{T_Y X}^\hbar) \\ \mathscr{M} &\mapsto \nu_Y^\hbar(\mathscr{M}) := \varprojlim_{n\geq 0} \nu_Y(\mathscr{M}_n). \end{aligned}$$

Definition 3.4.4. We say that $\nu_Y^\hbar$ is the functor of specialization (along Y) for $\mathscr{D}^\hbar$-modules. For $\mathscr{M} \in \mathrm{Mod}_{\mathrm{sp}}(\mathscr{D}_X^\hbar)$, the coherent $\mathscr{D}_{T_Y X}^\hbar$-module $\nu_Y^\hbar(\mathscr{M})$ is called the specialization of $\mathscr{M}$ (along Y).

Propositions 2.1.10 and 2.2.5 entail the following result:

Corollary 3.4.5. *Let $\mathscr{M}$ be an $\hbar$-torsion $\mathscr{D}_X^\hbar$-module. Then $\mathscr{M}$ is specializable as a $\mathscr{D}_X^\hbar$-module if and only if $\mathscr{M}$ is specializable in the $\mathscr{D}_X$-modules sense. Moreover, if $\mathscr{M}$ is specializable then $\nu_Y^\hbar(\mathscr{M}) \simeq \nu_Y(\mathscr{M})$ in $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_{T_Y X}^\hbar)$.*

By Proposition 2.1.11 we have the following characterization:

Corollary 3.4.6. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X^\hbar$-module. Then the following properties are equivalent:*

1. *$\mathscr{M}$ is a specializable $\mathscr{D}_X^\hbar$-module;*
2. *$\mathscr{M}_0$ is a specializable $\mathscr{D}_X$-module;*
3. *$\mathscr{M}_n$ is specializable as a $\mathscr{D}_X$-module, for each $n \geq 0$.*

Remark 3.4.7. Let $\mathscr{M}$ be a specializable $\mathscr{D}_X^\hbar$-module. Regarding $\mathrm{gr}_\hbar(\mathscr{M})$ as an object of $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$, we have a specializable complex in the sense of [20]. Since

$$\mathrm{gr}_\hbar \nu_Y^\hbar(\mathscr{M}) \simeq \varprojlim_{n\geq 0} \mathrm{gr}_\hbar \nu_Y(\mathscr{M}_n)$$

and, for each n, by construction, $\mathrm{gr}_\hbar \nu_Y(\mathscr{M}_n)$ is isomorphic to $\nu_Y \,\mathrm{gr}_\hbar(\mathscr{M}_n)$, (considering the specialization for complexes as in [20]), we get a morphism

$$\mathrm{gr}_\hbar \nu_Y^\hbar(\mathscr{M}) \to \nu_Y(\mathrm{gr}_\hbar \mathscr{M}).$$

Theorem 2.5.4 asserts that this morphism is an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{T_Y X})$.

Remark 3.4.8. Recall that given an exact sequence of specializable $\mathscr{D}_X$-modules,

$$0 \to \mathscr{M}_1 \to \mathscr{M} \to \mathscr{M}_2 \to 0,$$

if $b_i(s)$ is a Bernstein-Sato polynomial for the canonical V-filtration on $\mathscr{M}_i$, $i = 1, 2$ then $b_1(s) \cdot b_2(s)$ is a Bernstein-Sato polynomial for the canonical V-filtration on $\mathscr{M}$. (This standard technique is used, fore example, in [22, Proposition 4.2]).

Remark 3.4.9. Since the ring $\mathscr{D}_X^\hbar$ is not filtered neither by the order nor by V-filtrations, the notion of Bernstein polynomial for a specializable $\mathscr{D}_X^\hbar$-module does not make sense in general. However, we have the following result:

Proposition 3.4.10. *Let $\mathscr{M}$ be a specializable $\mathscr{D}_X^\hbar$-module with no $\hbar$-torsion. Assume that $b(s)$ is a Bernstein polynomial for the canonical V-filtration on $\mathscr{M}_0$ as a specializable $\mathscr{D}_X$-module. Then, $b_n(s) := (b(s))^{n+1}$ is a Bernstein polynomial of $\mathscr{M}_n$ for the canonical V-filtration.*

Proof. The exact sequence

$$0 \to \mathscr{M}_0 \xrightarrow{\overline{\hbar}} \mathscr{M}_1 \xrightarrow{\rho_{0,1}} \mathscr{M}_0 \to 0,$$

together with Remark 3.4.8, entails that, if $b(s)$ is a Bernstein polynomial for the canonical V-filtration on $\mathscr{M}_0$, then $(b(s))^2$ is a Bernstein polynomial for the canonical V-filtration on $\mathscr{M}_1$, and we proceed by induction applying the same argument to the exact sequence

$$0 \to \mathscr{M}_{n-1} \xrightarrow{\hbar} \mathscr{M}_n \xrightarrow{\rho_{0,n}} \mathscr{M}_0 \to 0.$$

□

In the examples below we assume $X = \mathbb{C}^m$, for some $m \in \mathbb{N}$, with coordinates $(t, x_1, ..., x_{m-1})$, and $Y = \{(t, x_1, ..., x_{m-1}) \in \mathbb{C}^m : t = 0\}$.

Example 3.4.11. Let $\mathscr{M}$ be a $\mathscr{D}_X^\hbar$-module with one generator, let us say $\mathscr{M} \simeq \mathscr{D}_X^\hbar/\mathcal{J}$, for a coherent ideal $\mathcal{J}$. Then we have a chain of isomorphisms of $\mathscr{D}_X$-modules,

$$\mathscr{M}_n \simeq \frac{\mathscr{D}_X^\hbar}{\hbar^{n+1}\mathscr{D}_X^\hbar + \mathcal{J}} \simeq \frac{\oplus_{i=0,\dots,n}\mathscr{D}_X\hbar^i}{\tilde{\mathcal{J}}_n},$$

where $\tilde{\mathcal{J}}_n$ is the submodule of $\oplus_{i=0,\dots,n}\mathscr{D}_X\hbar^i$ given by

$$\tilde{\mathcal{J}}_n = \frac{\mathcal{J}}{\hbar^{n+1}\mathscr{D}_X^\hbar \cap \mathcal{J}}.$$

Suppose that $\mathscr{M} = \mathscr{D}_X^\hbar/\mathscr{D}_X^\hbar b(t\partial_t)$, where $b(s)$ is a polynomial in $\mathbb{C}^\hbar[s]$, that is, $b(s)$ is of the form:

$$b(s) = \sum_{i=0}^{m} a_i(\hbar)s^i, \;\; m \in \mathbb{N}, \qquad \text{with } a_i(\hbar) := \sum_{j\geq 0} a_{ij}\hbar^j \in \mathbb{C}^\hbar, \text{ for } i = 0, ..., m.$$

Set $b_0(s) = \sum_{i=0}^{m} a_{i0}s^i$.

Since $\mathscr{M}_0 \simeq \mathscr{D}_X/\mathscr{D}_X b_0(t\partial_t)$, $\mathscr{M}$ is specializable if and only if $b_0(s)$ is a non-zero polynomial in $\mathbb{C}[s]$. We shall calculate particular cases in the following examples:

Example 3.4.12. Let $\mathscr{M} = \mathscr{D}_X^\hbar/\mathscr{D}_X^\hbar(\hbar t\partial_t + 1)$. Then, $\mathscr{M}$ has no $\hbar$-torsion and clearly $\mathscr{M}_0 = 0$. Hence, the exact sequences $0 \to \mathscr{M}_{n-1} \to \mathscr{M}_n \to \mathscr{M}_0 \to 0$ give $\mathscr{M}_n = 0$ for every n, which entails $\nu_Y^\hbar(\mathscr{M}) = 0$.

Example 3.4.13. Assume that $\mathcal{J} = \mathscr{D}_X^\hbar(t\partial_t - \hbar)$. Then

$$\tilde{\mathcal{J}}_n \simeq \{P_0 t\partial_t + \sum_{i=1}^{n}(P_i t\partial_t - P_{i-1})\hbar^i : P_i \in \mathscr{D}_X\}.$$

Therefore $\mathscr{M}_n$ can be identified with the cokernel of the $\mathscr{D}_X$-linear morphism from $\mathscr{D}_X^{n+1}$ to itself given by the right multiplication by the matrix

$$A_n = \begin{bmatrix} t\partial_t & -1 & 0 & ... & 0 & 0 \\ 0 & t\partial_t & -1 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & t\partial_t & -1 \\ 0 & 0 & 0 & ... & 0 & t\partial_t \end{bmatrix}.$$

Denoting by $u_{1,n}, ..., u_{n+1,n}$, respectively, the classes of the elements of the canonical basis of $\mathscr{D}_X^{n+1}$ in $\mathscr{M}_n$, we obtain a system of generators for $\mathscr{M}_n$ satisfying

$$(t\partial_t)u_{1,n} = 0,\ (t\partial_t)u_{k,n} = u_{k-1,n},$$

for $k = 2, ..., n+1$. Classically one derives an isomorphism

$$\mathscr{D}_X/\mathscr{D}_X(t\partial_t)^{n+1} \to \mathscr{M}_n$$

defined by

$$1 \operatorname{mod} \mathscr{D}_X(t\partial_t)^{n+1} \mapsto u_{n+1,n}.$$

Therefore, denoting by (x, τ) the associated coordinates in $T_Y X$, we obtain $\nu_Y(\mathscr{M}_n) \simeq \mathscr{D}_{T_Y X}/\mathscr{D}_{T_Y X}(\tau\partial_\tau)^{n+1}$.

Since $t\partial_t$ acts by multiplication by $\hbar$ in $\mathscr{M}_n$, the action of $\hbar$ in $\nu_Y(\mathscr{M}_n)$ coincides with the multiplication by $\tau\partial_\tau$ hence, as a $\mathscr{D}^\hbar_{T_Y X}$-module,

$$\nu_Y(\mathscr{M}_n) \simeq \frac{\mathscr{D}^\hbar_{T_Y X}}{\mathscr{D}^\hbar_{T_Y X}(\tau\partial_\tau - \hbar) + \hbar^{n+1}\mathscr{D}^\hbar_{T_Y X}}$$

and it follows that

$$\nu^\hbar_Y(\mathscr{M}) = \varprojlim_{n\geq 0} \nu_Y(\mathscr{M}_n) \simeq \frac{\mathscr{D}^\hbar_{T_Y X}}{\mathscr{D}^\hbar_{T_Y X}(\tau\partial_\tau - \hbar)}.$$

Vanishing cycles and nearby-cycles of $\mathscr{D}^\hbar_X$-modules along a submanifold Y. Assume now that Y is a complex closed smooth hypersurface of X given by the zero locus of a holomorphic function $f : X \to \mathbb{C}$. We can treat the extension of the functors of vanishing cycles and nearby-cycles of $\mathscr{D}$-modules theory similarly to the case of specialization. Hence, from the exact functors $\psi_Y, \varphi_Y : \mathrm{Mod}_{\mathrm{sp}}(\mathscr{D}_X) \to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_Y)$, we obtain exact functors

$$\begin{array}{rccc} \psi^\hbar_Y : & \mathrm{Mod}_{\mathrm{sp}}(\mathscr{D}^\hbar_X) & \to & \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}^\hbar_Y) \\ & \mathscr{M} & \mapsto & \psi^\hbar_Y(\mathscr{M}) := \varprojlim\limits_{n\geq 0} \psi_Y(\mathscr{M}_n), \end{array}$$

$$\begin{array}{rccc} \varphi^\hbar_Y : & \mathrm{Mod}_{\mathrm{sp}}(\mathscr{D}^\hbar_X) & \to & \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}^\hbar_Y) \\ & \mathscr{M} & \mapsto & \varphi^\hbar_Y(\mathscr{M}) := \varprojlim\limits_{n\geq 0} \varphi_Y(\mathscr{M}_n). \end{array}$$

Definition 3.4.14. The functors $\psi^\hbar_Y$ and $\varphi^\hbar_Y$ are, respectively, the functors of vanishing cycles and nearby-cycles (along Y) for $\mathscr{D}^\hbar$-modules. If $\mathscr{M} \in \mathrm{Mod}_{\mathrm{sp}}(\mathscr{D}^\hbar_X)$, we say that $\psi^\hbar_Y(\mathscr{M})$ is the vanishing cycle of $\mathscr{M}$ (along Y) and $\varphi^\hbar_Y(\mathscr{M})$ is the nearby-cycle of $\mathscr{M}$ (along Y).

Remark that Propositions 2.2.5 and 2.3.5 and Corollaries 2.4.4 and 2.4.6 apply to $\psi_Y^\hbar$ and $\varphi_Y^\hbar$.

Example 3.4.15. Keeping the notations of Example 3.4.13, we infer from the results of [27] that, for each $n \geq 0$, $\psi_Y(\mathscr{M}_n)$ is quasi-isomorphic to the complex $\nu_Y(\mathscr{M}_n) \xrightarrow{\tau-1} \nu_Y(\mathscr{M}_n)$, that is,

$$\psi_Y(\mathscr{M}_n) \simeq \frac{\mathscr{D}_{T_Y X}}{\mathscr{D}_{T_Y X}(\tau - 1) + \mathscr{D}_{T_Y X}(\tau\partial_\tau)^{n+1}}.$$

Thus

$$\psi_Y^\hbar(\mathscr{M}) \simeq \frac{\mathscr{D}^\hbar_{T_Y X}}{\mathscr{D}^\hbar_{T_Y X}(\tau - 1) + \mathscr{D}^\hbar_{T_Y X}(\tau\partial_\tau - \hbar)},$$

in other words $\psi_Y^\hbar(\mathscr{M})$ is quasi-isomorphic to the complex $\nu_Y^\hbar(\mathscr{M}) \xrightarrow{\tau-1} \nu_Y^\hbar(\mathscr{M})$.

The holonomic case. Let us consider the Serre subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ consisting of holonomic (resp. regular holonomic) $\mathscr{D}_X$-modules. By Proposition 2.1.11, if $\mathscr{M}$ is a holonomic (resp. regular holonomic) $\mathscr{D}_X^\hbar$-module, then each $\mathscr{M}_n$ is holonomic (resp. regular holonomic) for the $\mathscr{D}_X$-module structure of $\mathscr{D}^\hbar_{X,n}$ given in Remark 3.1.14.

Recall that every holonomic (resp. regular holonomic) $\mathscr{D}_X$-module is specializable along any submanifold Y and that its specialization is also holonomic (resp. regular holonomic). We clearly have the analogous result in the $\hbar$-setting:

Proposition 3.4.16. (a) *Any holonomic $\mathscr{D}_X^\hbar$-module $\mathscr{M}$ is specializable along any submanifold Y. Moreover $\nu_Y^\hbar(\mathscr{M})$ is a holonomic $\mathscr{D}^\hbar_{T_Y X}$-module. If $\mathscr{M}$ is regular holonomic, so is $\nu_Y^\hbar(\mathscr{M})$.*

(b) *Let Y be a smooth hypersurface of X. If $\mathscr{M}$ is holonomic (resp. regular holonomic), then $\psi_Y^\hbar(\mathscr{M})$ and $\varphi_Y^\hbar(\mathscr{M})$ are holonomic (resp. regular holonomic) as $\mathscr{D}_Y^\hbar$-modules.*

3.5 Fourier transform and microlocalization

Let $E \xrightarrow{\pi} Z$ denote a complex vector bundle on a complex analytic manifold Z and $E' \xrightarrow{\tilde{\pi}} Z$ denote its dual bundle. Let us consider the Fourier transform $\mathscr{F} : \mathrm{Mod}_{\mathrm{mon}}(\mathscr{D}_{[E]}) \to \mathrm{Mod}_{\mathrm{mon}}(\mathscr{D}_{[E']})$, an exact functor whose definition we recalled in Section 1.2.

We employ a similar procedure to that of the previous sections: denote by $\mathscr{S}$ the full Serre substack of $\mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}^\hbar_{[E]})$ given by

$$U \mapsto \mathscr{S}(U) = \cup_{n \geq 0} \mathrm{Mod}_{\mathrm{mon}}(\mathscr{D}^\hbar_{[E],n}|_U),$$

considering the structure of free $\mathscr{D}_{[E]}$-module on $\mathscr{D}^\hbar_{[E],n}$; similarly, denote by $\mathscr{S}'$ the full Serre substack of $\mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}^\hbar_{[E']})$

$$V \mapsto \mathscr{S}'(V) = \cup_{n \geq 0} \mathrm{Mod}_{\mathrm{mon}}(\mathscr{D}^\hbar_{[E'],n}|_V).$$

For such $\mathscr{S}$ and $\mathscr{S}'$, consider the associated full Serre subcategories $\mathrm{Mod}_{\mathscr{S}}(\mathscr{D}^{\hbar}_{[E]})$ and $\mathrm{Mod}_{\mathscr{S}'}(\mathscr{D}^{\hbar}_{[E']})$ cf. Definition 2.1.9. Denote these categories instead by $\mathrm{Mod}_{\mathrm{mon}}(\mathscr{D}^{\hbar}_{[E]})$ and $\mathrm{Mod}_{\mathrm{mon}}(\mathscr{D}^{\hbar}_{[E']})$, respectively.

Definition 3.5.1. We say that a coherent $\mathscr{D}^{\hbar}_{[E]}$-module $\mathscr{N}$ is monodromic if $\mathscr{N}$ is an object of $\mathrm{Mod}_{\mathrm{mon}}(\mathscr{D}^{\hbar}_{[E]})$.

Consider the functor $\Phi : \mathrm{Op}(E) \to \mathrm{Op}(E')$ defined by $\Phi(U) := \tilde{\pi}^{-1}\pi(U)$.

We can regard $\mathscr{F}$ as a morphism of prestacks $\mathscr{F} : \mathscr{S} \to \Phi\mathscr{S}'$ and, again, our results of Chapter 2 allow us to extend $\mathscr{F}$ as an exact functor

$$\begin{array}{rcl} \mathscr{F}^{\hbar} : \mathrm{Mod}_{\mathrm{mon}}(\mathscr{D}^{\hbar}_{[E]}) & \to & \mathrm{Mod}_{\mathrm{mon}}(\mathscr{D}^{\hbar}_{[E']}) \\ \mathscr{N} & \mapsto & \mathscr{F}^{\hbar}(\mathscr{N}) := \varprojlim_{n\geq 0}\mathscr{F}(\mathscr{N}_n) = \varprojlim_{n\geq 0}(\Omega_{E/Y} \otimes_{\pi^{-1}\mathscr{O}_Y} \mathscr{N}_n). \end{array}$$

Definition 3.5.2. For $\mathscr{N} \in \mathrm{Mod}_{\mathrm{mon}}(\mathscr{D}^{\hbar}_{[E]})$, we say that $\mathscr{F}^{\hbar}(\mathscr{N})$ is the Fourier transform of $\mathscr{M}$.

Proposition 3.5.3. *For $\mathscr{N} \in \mathrm{Mod}_{\mathrm{mon}}(\mathscr{D}^{\hbar}_{[E]})$, we have functorial isomorphisms:*

$$\mathscr{F}^{\hbar}(\mathscr{N}) \simeq \Omega_{E|Y} \otimes_{\pi^{-1}\mathscr{O}_Y} \mathscr{N}.$$

Proof. By the unicity condition of Theorem 2.5.4, it is enough to note that $\mathscr{N} \to \Omega_{E|Y} \otimes_{\pi^{-1}\mathscr{O}_Y} \mathscr{N}$ is an extension of $\mathscr{F}$ that takes values in the category of cohomologically complete objects and verifies the formulas:

$$\begin{aligned} (\Omega_{E|Y} \otimes_{\pi^{-1}\mathscr{O}_Y} \mathscr{N})_0 &\simeq \Omega_{E|Y} \otimes_{\pi^{-1}\mathscr{O}_Y} \mathscr{N}_0, \\ {}_0(\Omega_{E|Y} \otimes_{\pi^{-1}\mathscr{O}_Y} \mathscr{N}) &\simeq \Omega_{E|Y} \otimes_{\pi^{-1}\mathscr{O}_Y} {}_0\mathscr{N}. \end{aligned}$$

□

Remark that Propositions 2.2.5 and 2.3.5, Theorem 2.4.7 and Corollaries 2.4.4 and 2.4.6 apply to the functor $\mathscr{F}^{\hbar}$.

Consider the case of a complex manifold of finite dimension X and a submanifold Y and take $E = T_Y X$ and $E' = T^*_Y X$.

Lemma 3.5.4. *Let $\mathscr{M}$ be a specializable $\mathscr{D}^{\hbar}_X$-module. Then $\nu^{\hbar}_Y(\mathscr{M})$ is a monodromic $\mathscr{D}^{\hbar}_{T_Y X}$-module.*

Proof. This is a consequence of the analogous property for $\mathscr{D}$-modules and the formulas $\nu^{\hbar}_Y(\mathscr{M})_0 \simeq \nu_Y(\mathscr{M}_0)$ and $\nu^{\hbar}_Y(\mathscr{M}) \simeq \nu_Y({}_0\mathscr{M})$. □

Definition 3.5.5. The functor of microlocalization for $\mathscr{D}^{\hbar}_X$-modules along a submanifold Y is given by:

$$\begin{array}{rcl} \mu^{\hbar}_Y : \mathrm{Mod}_{\mathrm{sp}}(\mathscr{D}^{\hbar}_X) & \to & \mathrm{Mod}_{\mathrm{mon}}(\mathscr{D}^{\hbar}_{[T^*_Y X]}) \\ \mathscr{M} & \mapsto & \mu^{\hbar}_Y(\mathscr{M}) := \mathscr{F}^{\hbar}(\nu^{\hbar}_Y(\mathscr{M})). \end{array}$$

If $\mathscr{M}$ is a specializable $\mathscr{D}^{\hbar}_X$-module along Y, then we say that $\nu^{\hbar}_Y(\mathscr{M})$ is the microlocalization of $\mathscr{M}$ along Y.

Similarly to Proposition 3.5.3, the unicity criteria of Theorem 2.5.4 gives:

Proposition 3.5.6. *We have functorial isomorphisms for $\mathscr{M} \in \mathrm{Mod}_{\mathrm{sp}}(\mathscr{D}_X^\hbar)$:*

$$\mu_Y^\hbar(\mathscr{M}) \simeq \varprojlim_{n\geq 0} \mu_Y(\mathscr{M}_n).$$

3.6 Comparison theorems

As we have recalled in Section 1.2, M. Kashiwara constructed in [12] a canonical isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{T_Y X})$ for $\mathscr{M} \in \mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X)$:

$$\mathrm{DR}_{\mathscr{D}_{T_Y X}}(\nu_Y(\mathscr{M})) \xleftarrow{\sim} \nu_Y(\mathrm{DR}_{\mathscr{D}_X}(\mathscr{M})).$$

Recall that on the left hand side one has the specialization of $\mathscr{D}$-modules theory, whereas on the right hand side one has Sato's specialization for sheaves.

Moreover, setting

$$\theta_1 = \nu_Y \circ \mathrm{DR}_{\mathscr{D}_X} : \mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X) \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{T_Y X}),$$
$$\theta_2 = \mathrm{DR}_{\mathscr{D}_{T_Y X}} \circ \nu_Y : \mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X) \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{T_Y X}),$$

Kashiwara's construction gives a natural transformation $\theta_1 \overset{\Psi_K}{\to} \theta_2$. In particular, if the modules are provided with an action of $\mathbb{C}^\hbar$, then we get a commutative diagram in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{T_Y X})$ for each $\lambda \in \mathbb{C}^\hbar$:

$$\begin{array}{ccc}
\nu_Y(\mathrm{DR}_{\mathscr{D}_X}(\mathscr{M})) & \xrightarrow{\lambda} & \nu_Y(\mathrm{DR}_{\mathscr{D}_X}(\mathscr{M})) \\
\downarrow{\scriptstyle \Psi_K(\mathscr{M})} & & \downarrow{\scriptstyle \Psi_K(\mathscr{M})} \\
\mathrm{DR}_{\mathscr{D}_{T_Y X}}(\nu_Y(\mathscr{M})) & \xrightarrow{\lambda} & \mathrm{DR}_{\mathscr{D}_{T_Y X}}(\nu_Y(\mathscr{M}))
\end{array}$$

As a consequence, we can regard Ψ_K as a $\mathbb{C}^\hbar$-linear natural transformation.

We shall now generalize this isomorphism to the $\hbar$-setting, as well as the other comparison isomorphisms mentioned in Section 1.2. We need the following lemmas:

Lemma 3.6.1. (a) *If $g : X \to Z$ is a morphism of complex manifolds and $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^\hbar)$, then $\mathrm{R}g_*^{\mathbb{C}^\hbar} F \simeq \mathrm{R}g_*^{\mathbb{C}} F$ and $\mathrm{R}g_!^{\mathbb{C}^\hbar} F \simeq \mathrm{R}g_!^{\mathbb{C}} F$ in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_Z)$.*

(b) *For every $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^\hbar)$, one has $\nu_Y^{\mathbb{C}^\hbar}(F) \simeq \nu_Y^{\mathbb{C}}(F)$ and, if Y is a smooth hypersurface of X, we also have $\psi_Y^{\mathbb{C}^\hbar}(F) \simeq \psi_Y^{\mathbb{C}}(F)$ and $\varphi_Y^{\mathbb{C}^\hbar}(F) \simeq \varphi_Y^{\mathbb{C}}(F)$.*

Proof. (a) Let $I^\bullet$ be a flabby resolution of F in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^\hbar)$. Then each I^j is also flabby in $\mathrm{Mod}(\mathbb{C}_X)$. Hence, both $\mathrm{R}g_*^{\mathbb{C}^\hbar} F$ and $\mathrm{R}g_*^{\mathbb{C}} F$ are quasi-isomorphic to $g_*(I^\bullet)$. Similarly, using a c-soft resolution of F instead, we get $\mathrm{R}g_!^{\mathbb{C}^\hbar} F \simeq \mathrm{R}g_!^{\mathbb{C}} F$.

(b) follows as a consequence of (a) and the definition of the functors. □

Remark 3.6.2. Given $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^\hbar)$, since $\mathrm{gr}_\hbar$ commutes with inverse and direct images, we conclude that $\mathrm{gr}_\hbar(\nu_Y^{\mathbb{C}^\hbar}(F)) \simeq \nu_Y^{\mathbb{C}}(\mathrm{gr}_\hbar(F))$, $\mathrm{gr}_\hbar(\psi_Y^{\mathbb{C}^\hbar}(F)) \simeq \psi_Y^{\mathbb{C}}(\mathrm{gr}_\hbar(F))$ and $\mathrm{gr}_\hbar(\varphi_Y^{\mathbb{C}^\hbar}(F)) \simeq \varphi_Y^{\mathbb{C}}(\mathrm{gr}_\hbar(F))$.

Henceforth we use the simplified notations ν_Y, ψ_Y, φ_Y for the specialization, the nearby-cycle and the vanishing-cycle functors on sheaves of $\mathbb{C}^\hbar$-modules, respectively.

Let E denote a complex vector bundle on a complex analytic manifold and let E' denote its dual bundle. Recall that we denote by $\mathscr{F}^{\mathbb{K}}$ the Fourier-Sato transform for sheaves of $\mathbb{K}_X$-modules.

Lemma 3.6.3. *Let $F \in \mathsf{D}^+_{\mathbb{R}^+}(\mathbb{C}_E^\hbar)$. Then, there is an isomorphism $\mathscr{F}^{\mathbb{C}^\hbar}(F) \simeq \mathscr{F}^{\mathbb{C}}(F)$ in $\mathsf{D}^+_{\mathbb{R}^+}(\mathbb{C}_{E'})$.*

Proof. The result follows by Lemma 3.6.1 and the definition of the Fourier-Sato transform (see Definition 1.1.5). Note that $i^!$ and p_1^{-1} are exact functors. □

By Lemmas 3.6.1 and 3.6.3, one concludes:

Lemma 3.6.4. *For each $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^\hbar)$ the objects $\mu_Y^{\mathbb{C}^\hbar}(F)$ and $\mu_Y^{\mathbb{C}}(F)$ are isomorphic in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{T_Y^*X})$.*

We state now our first comparison theorem for $\mathscr{D}^\hbar$-modules and we give a detailed proof:

Theorem 3.6.5. *Let $\mathscr{M} \in \mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X^\hbar)$. We have the following canonical isomorphisms in $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}-\mathrm{c}}(\mathbb{C}_{T_Y X}^\hbar)$:*

(i) $\mathrm{DR}_{\mathscr{D}^\hbar_{T_Y X}}(\nu_Y^\hbar(\mathscr{M})) \xleftarrow{\sim} \nu_Y(\mathrm{DR}_{\mathscr{D}^\hbar_X}(\mathscr{M}))$;

(ii) $\mathrm{Sol}_{\mathscr{D}^\hbar_{T_Y X}}(\nu_Y^\hbar(\mathscr{M})) \xrightarrow{\sim} \nu_Y(\mathrm{Sol}_{\mathscr{D}^\hbar_X}(\mathscr{M}))$.

Let $\mathscr{M} \in \mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X^\hbar)$. To construct the isomorphism

$$\mathrm{DR}_{\mathscr{D}^\hbar_{T_Y X}}(\nu_Y^\hbar(\mathscr{M})) \xleftarrow{\sim} \nu_Y(\mathrm{DR}_{\mathscr{D}^\hbar_X}(\mathscr{M}))$$

we need to replace the image by the De Rham functor with the De Rham complex of a $\mathscr{D}^\hbar$-module.

Denote by Ω_X^j the sheaf of holomorphic forms of degree j on X. Denote by d the usual exterior derivative and consider the De Rham complex of X:

$$\Omega_X^\bullet : 0 \to \Omega_X^0 \xrightarrow{d} \Omega_X^1 \to \cdots \to \Omega_X^{n-1} \xrightarrow{d} \Omega_X^{d_X} \to 0.$$

Then, one has the isomorphism $\mathrm{DR}_{\mathscr{D}^\hbar_X}(\mathscr{M}) \simeq \Omega_X^\bullet \otimes_{\mathscr{O}_X} \mathscr{M}$ that holds in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^\hbar)$. Indeed, each $\Omega_X^j \otimes_{\mathscr{O}_X} \mathscr{M}$ has a natural structure of a $\mathscr{D}_X^\hbar$-module and the derivatives turn out to be $\mathbb{C}_X^\hbar$-linear.

By definition, the projective limit $\varprojlim_{n\geq 0}(\Omega^\bullet_X \otimes_{\mathscr{O}_X} \mathscr{M}_n)$ is given by the complex

$$0 \to \varprojlim_{n\geq 0}(\Omega^0_X \otimes_{\mathscr{O}_X} \mathscr{M}_n) \to \cdots \to \varprojlim_{n\geq 0}(\Omega^{d_X}_X \otimes_{\mathscr{O}_X} \mathscr{M}_n) \to 0, \tag{3.6.1}$$

and we have the following auxilar lemma:

Lemma 3.6.6. *Let $\mathscr{M}$ be a coherent $\mathscr{D}^\hbar_X$-module. Then $\varprojlim_{n\geq 0}(\Omega^\bullet_X \otimes_{\mathscr{O}_X} \mathscr{M}_n)$ is isomorphic to $\Omega^\bullet_X \otimes_{\mathscr{O}_X} \mathscr{M}$ in $\mathsf{C}^{\mathrm{b}}(\mathbb{C}^\hbar_X)$.*

Proof. Note that $\mathscr{M}$ is $\hbar$-complete. The natural morphism

$$\Omega^j_X \otimes_{\mathscr{O}_X} \mathscr{M} \to \varprojlim_{n\geq 0}(\Omega^j_X \otimes_{\mathscr{O}_X} \mathscr{M}_n)$$

is an isomorphism of sheaves of $\mathbb{C}^\hbar$-modules for each j, since Ω^j_X is locally a free $\mathscr{O}_X$-module of finite rank and the projective limit is additive. Clearly these isomorphisms are compatible with the derivatives, hence the complex (3.6.1) is isomorphic to $\Omega^\bullet_X \otimes_{\mathscr{O}_X} \mathscr{M}$, which proves the lemma. □

Proof of Theorem 3.6.5 (i). For each $n \geq 0$ there is a natural morphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^\hbar_{T_Y X})$:

$$\nu_Y(\mathrm{DR}_{\mathscr{D}^\hbar_X}(\mathscr{M})) \to \nu_Y(\mathrm{DR}_{\mathscr{D}^\hbar_X}(\mathscr{M}_n)).$$

Since $\mathscr{M}_n$ is an $\hbar$-torsion regular holonomic $\mathscr{D}^\hbar_X$-module, we have the following chain of isomorphisms that hold both in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{T_Y X})$ and in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^\hbar_{T_Y X})$:

$$\nu_Y(\mathrm{DR}_{\mathscr{D}^\hbar_X}(\mathscr{M}_n)) \simeq \nu_Y(\mathrm{DR}_{\mathscr{D}_X}(\mathscr{M}_n)) \simeq \mathrm{DR}_{\mathscr{D}_{T_Y X}}(\nu_Y(\mathscr{M}_n)) \simeq \mathrm{DR}_{\mathscr{D}^\hbar_{T_Y X}}(\nu^\hbar_Y(\mathscr{M}_n)).$$

The second isomorphism is given by Kashiwara's comparison theorem for $\mathscr{D}$-modules, which we have mentioned before. Hence, we get canonical morphisms in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^\hbar_{T_Y X})$,

$$\nu_Y(\mathrm{DR}_{\mathscr{D}^\hbar_X}(\mathscr{M})) \to \mathrm{DR}_{\mathscr{D}^\hbar_{T_Y X}}(\nu^\hbar_Y(\mathscr{M}_n)), \tag{3.6.2}$$

which entail morphisms:

$$\nu_Y(\mathrm{DR}_{\mathscr{D}^\hbar_X}(\mathscr{M})) \to \Omega^\bullet_{T_Y X} \otimes_{\mathscr{O}_{T_Y X}} \nu^\hbar_Y(\mathscr{M}_n).$$

So we obtain a morphism in $\mathsf{C}^{\mathrm{b}}(\mathbb{C}^\hbar_{T_Y X})$:

$$\nu_Y(\mathrm{DR}_{\mathscr{D}^\hbar_X}(\mathscr{M})) \to \varprojlim_{n\geq 0}(\Omega^\bullet_{T_Y X} \otimes_{\mathscr{O}_{T_Y X}} \nu^\hbar_Y(\mathscr{M}_n)).$$

Finally, the morphism (i) follows from Lemma 3.6.6 as the composition of the sequence of morphisms below:

$$\nu_Y(\mathrm{DR}_{\mathscr{D}^\hbar_X}(\mathscr{M})) \to \Omega^\bullet_{T_Y X} \otimes_{\mathscr{O}_{T_Y X}} \nu^\hbar_Y(\mathscr{M}) \underset{qis}{\longrightarrow} \mathrm{DR}_{\mathscr{D}^\hbar_{T_Y X}}(\nu^\hbar_Y(\mathscr{M})).$$

Let us now prove that (i) is an isomorphism. Note that the objects $\nu_Y(\mathrm{DR}_{\mathscr{D}_X^\hbar}(\mathscr{M}))$ and $\mathrm{DR}_{\mathscr{D}_X^\hbar}(\nu_Y^\hbar(\mathscr{M}))$ both belong to $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}-\mathrm{c}}(\mathbb{C}^\hbar_{T_YX})$. Therefore, it is enough to prove that we obtain an isomorphism when we apply $\mathrm{gr}_\hbar$ to (i). We have on one hand $\mathrm{gr}_\hbar(\nu_Y(\mathrm{DR}_{\mathscr{D}_X^\hbar}(\mathscr{M}))) \simeq \nu_Y(\mathrm{DR}_{\mathscr{D}_X}(\mathrm{gr}_\hbar(\mathscr{M})))$. Since $\mathrm{gr}_\hbar(\mathscr{M}) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$ we conclude that

$$\nu_Y(\mathrm{DR}_{\mathscr{D}_X}(\mathrm{gr}_\hbar \mathscr{M})) \simeq \mathrm{DR}_{\mathscr{D}_{T_YX}} \nu_Y(\mathrm{gr}_\hbar \mathscr{M}).$$

On the other hand we have, by Remark 3.4.7,

$$\mathrm{gr}_\hbar \mathrm{DR}_{\mathscr{D}^\hbar_{T_YX}}(\nu_Y^\hbar(\mathscr{M})) \simeq \mathrm{DR}_{\mathscr{D}_{T_YX}} \mathrm{gr}_\hbar(\nu_Y^\hbar(\mathscr{M})) \simeq \mathrm{DR}_{\mathscr{D}_{T_YX}} \nu_Y(\mathrm{gr}_\hbar \mathscr{M}).$$

This is enough to conclude that (i) is an isomorphism.
(ii) To end the proof we remark that (ii) follows by the following chain of isomorphisms:

$$\begin{aligned}
\mathrm{Sol}_{\mathscr{D}^\hbar_{T_YX}}(\nu_Y^\hbar(\mathscr{M})) &\simeq \mathrm{D}'_{\mathbb{C}^\hbar_{T_YX}}(\mathrm{DR}_{\mathscr{D}^\hbar_{T_YX}}(\nu_Y^\hbar(\mathscr{M}))) \\
&\xrightarrow{\sim} \mathrm{D}'_{\mathbb{C}^\hbar_{T_YX}} \nu_Y(\mathrm{DR}_{\mathscr{D}_X^\hbar}(\mathscr{M})) \\
&\simeq \nu_Y(\mathrm{D}'_{\mathbb{C}^\hbar_{T_YX}}(\mathrm{DR}_{\mathscr{D}_X^\hbar}(\mathscr{M}))) \\
&\simeq \nu_Y(\mathrm{Sol}_{\mathscr{D}_X^\hbar}(\mathscr{M})).
\end{aligned}$$

The first and fourth isomorphisms result from (3.1.1), the second results from applying the contravariant functor $\mathrm{D}'_{\mathbb{C}^\hbar_{T_YX}}$ to (i) and the third follows by Proposition 8.4.13 of [15].

□

Remark 3.6.7. An alternative proof of the isomorphisms can be given by considering separately the cases of an $\hbar$-torsion free module and a module of $\hbar$-torsion. The general conclusion would follow from the short exact sequence $0 \to \mathscr{M}_{\hbar-\mathrm{tor}} \to \mathscr{M} \to \mathscr{M}_{\hbar-\mathrm{tf}} \to 0$.

Using exactly the same technique and arguments of the proof of Theorem 3.6.5, we have the following comparison theorems:

Theorem 3.6.8. *Let Y be a smooth hypersurface of X and $\mathscr{M} \in \mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X^\hbar)$. There are canonical isomorphisms in $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}-\mathrm{c}}(\mathbb{C}^\hbar_Y)$:*

$$\begin{cases}\mathrm{Sol}_{\mathscr{D}_Y^\hbar}(\psi_Y^\hbar(\mathscr{M})) \xrightarrow{\sim} \psi_Y(\mathrm{Sol}_{\mathscr{D}_X^\hbar}(\mathscr{M})) \\ \mathrm{DR}_{\mathscr{D}_Y^\hbar}(\psi_Y^\hbar(\mathscr{M})) \xleftarrow{\sim} \psi_Y(\mathrm{DR}_{\mathscr{D}_X^\hbar}(\mathscr{M})),\end{cases}$$

and

$$\begin{cases}\mathrm{Sol}_{\mathscr{D}_Y^\hbar}(\varphi_Y^\hbar(\mathscr{M})) \xrightarrow{\sim} \varphi_Y(\mathrm{Sol}_{\mathscr{D}_X^\hbar}(\mathscr{M})) \\ \mathrm{DR}_{\mathscr{D}_Y^\hbar}(\varphi_Y^\hbar(\mathscr{M})) \xleftarrow{\sim} \varphi_Y(\mathrm{DR}_{\mathscr{D}_X^\hbar}(\mathscr{M})).\end{cases}$$

Theorem 3.6.9. *Let $E \xrightarrow{\pi} Z$ denote a complex vector bundle on a complex analytic manifold Z and $E' \xrightarrow{\tilde{\pi}} Z$ denote its dual bundle. For each $\mathscr{N} \in \mathrm{Mod}_{\mathrm{mon}}(\mathscr{D}^{\hbar}_{[E]})$, we have natural isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\hbar}_{E'})$:*

$$\begin{cases} \mathscr{F}(\mathrm{Sol}_{\mathscr{D}^{\hbar}_{[E]}}(\mathscr{N})) \simeq \mathrm{Sol}_{\mathscr{D}^{\hbar}_{[E']}}(\mathscr{F}^{\hbar}(\mathscr{N}))[-\mathrm{codim}Y] \\ \mathscr{F}(\mathrm{DR}_{\mathscr{D}^{\hbar}_{[E]}}(\mathscr{N})) \simeq \mathrm{DR}_{\mathscr{D}^{\hbar}_{[E']}}(\mathscr{F}^{\hbar}(\mathscr{N}))[-\mathrm{codim}Y]. \end{cases}$$

Corollary 3.6.10. *Let X be a complex manifold and Y a submanifold. For $\mathscr{M} \in \mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}^{\hbar}_X)$, we have natural isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\hbar}_{T^*_Y X})$:*

$$\begin{cases} \mathrm{Sol}_{\mathscr{D}^{\hbar}_{T^*_Y X}}(\mu^{\hbar}_Y(\mathscr{M})) \simeq \mu_Y(\mathrm{Sol}_{\mathscr{D}^{\hbar}_X}(\mathscr{M}))[\mathrm{codim}Y] \\ \mathrm{DR}_{\mathscr{D}^{\hbar}_{T^*_Y X}}(\mu^{\hbar}_Y(\mathscr{M})) \simeq \mu_Y(\mathrm{DR}_{\mathscr{D}^{\hbar}_X}(\mathscr{M}))[\mathrm{codim}Y]. \end{cases}$$

Chapter 4

Elliptic pairs over $\mathbb{C}[[\hbar]]$

In this chapter $f : X \to Y$ denotes a morphism of complex analytic manifolds of finite dimensions d_X and d_Y, respectively.

4.1 Elliptic pairs: definition and motivation

The $\mathscr{D}$-modules case. Elliptic pairs over $\mathbb{C}$ were studied by P. Schapira and J.P. Schneiders in [34] as a generalization of the notion of elliptic system. For this purpose, the authors introduce the notion of f-characteristic variety that we recalled in Section 1.2.

Definition 4.1.1. Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}({\mathscr{D}_X}^{\mathrm{op}})$ and $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$. One says that $(\mathscr{M}, F)$ is an f-elliptic pair over $\mathbb{C}$ if $\mathrm{char}_f(\mathscr{M}) \cap \mathrm{SS}(F) \subset T^*_X X$.

The functorial properties of f-elliptic pairs were studied in detail in [34], where the authors prove theorems of regularity, finiteness and duality for these objects. Let us recall such results:

Theorem 4.1.2 ([34])**.** (a) *Let $(\mathscr{M}, F)$ be an f-elliptic pair over $\mathbb{C}$. Then, the natural morphism below is an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)$:*

$$F \overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X} (\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{K}) \to \mathrm{R}\mathscr{H}om_{\mathbb{C}_X}(\mathrm{D}'_{\mathbb{C}_X} F, \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{K}).$$

(b) *Let $(\mathscr{M}, F)$ be an f-elliptic pair over $\mathbb{C}$ and suppose that $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_X^{\mathrm{op}})$ and that f is proper when restricted to the support of $(\mathscr{M}, F)$. Then, $\underline{f}_!(\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X} F)$ is an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_Y^{\mathrm{op}})$ and the canonical morphism*

$$\underline{f}_!(\mathrm{D}'_{\mathbb{C}_X} F \overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X} \underline{\mathrm{D}}_{\mathscr{D}_X}(\mathscr{M})) \to \underline{\mathrm{D}}_{\mathscr{D}_Y}(\underline{f}_!(F \overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X} \mathscr{M}))$$

is an isomorphism in $\mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_Y^{\mathrm{op}})$.

Note. The main purpose of this chapter is to extend the results stated in Theorem 4.1.2 to $\mathscr{D}^\hbar$-modules. Namely, we are able to prove regularity and finiteness theorems in the $\hbar$-setting and we are also able to construct a duality morphism and the corresponding duality theorem in the smooth case. Similarly to the results of Chapter 3, the main idea behind the proofs consists in a reduction to the case of $\mathscr{D}$-modules.

The $\mathscr{D}[[\hbar]]$-modules case.

Definition 4.1.3. The f-characteristic variety of an object $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}^\hbar_X)$ is denoted by $\mathrm{char}_{f,\hbar}(\mathscr{M})$ and defined by

$$\mathrm{char}_{f,\hbar}(\mathscr{M}) := \mathrm{char}_f(\mathrm{gr}_\hbar(\mathscr{M})).$$

Lemma 4.1.4. *For any $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}^\hbar_X)$, we have $\mathrm{SS}(\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}^\hbar_X} \mathscr{K}_\hbar) \subset \mathrm{char}_{f,\hbar}(\mathscr{M})$.*

Proof. The object $\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}^\hbar_X} \mathscr{K}_\hbar$ is cohomologically complete by Lemma 3.3.2. Hence,

$$\mathrm{SS}(\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}^\hbar_X} \mathscr{K}_\hbar) = \mathrm{SS}(\mathrm{gr}_\hbar(\mathscr{M}) \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{K}).$$

Therefore, the result is reduced to the estimative (1.2.2) applied to $\mathrm{gr}_\hbar(\mathscr{M}) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$. □

Definition 4.1.5. Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}^{\hbar\,\mathrm{op}}_X)$ and $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}^\hbar_X)$. We say that $(\mathscr{M}, F)$ is an f-elliptic pair over $\mathbb{C}^\hbar$ if $\mathrm{char}_{f,\hbar}(\mathscr{M}) \cap \mathrm{SS}(F) \subset T^*_X X$. If in addition $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}^{\hbar\,\mathrm{op}}_X)$, then we say that $(\mathscr{M}, F)$ is a good f-elliptic pair over $\mathbb{C}^\hbar$. The support of the f-elliptic pair $(\mathscr{M}, F)$ is the intersection $\mathrm{supp}(\mathscr{M}) \cap \mathrm{supp}(F)$.

Since $\mathrm{gr}_\hbar$ preserves the micro-support of $\mathbb{R}$-constructible sheaves and the characteristic variety of coherent $\mathscr{D}^\hbar$-modules, the following result is obvious:

Proposition 4.1.6. *A pair $(\mathscr{M}, F)$ is an f-elliptic pair over $\mathbb{C}^\hbar$ if and only if $(\mathrm{gr}_\hbar(\mathscr{M}), \mathrm{gr}_\hbar(F))$ is an f-elliptic pair over $\mathbb{C}$.*

If $(\mathscr{M}, F)$ is an f-elliptic pair over $\mathbb{C}^\hbar$, then $(\underline{\mathrm{D}}_{\mathscr{D}^\hbar_X} \mathscr{M}, \mathrm{D}'_{\mathbb{C}^\hbar_X} F)$ is also an f-elliptic pair over $\mathbb{C}^\hbar$, the dual elliptic pair of $(\mathscr{M}, F)$.

Recall that we denote by a_X the constant map $X \to \{\mathrm{pt}\}$. For short, we say that an a_X-elliptic pair is an elliptic pair.

Assume that X is the complexification of a real analytic manifold M.

Definition 4.1.7. We say that a coherent $\mathscr{D}^\hbar_X$-module $\mathscr{M}$ is an elliptic $\mathscr{D}^\hbar_X$-module if $(\mathscr{M}, \mathbb{C}^\hbar_M)$ is an elliptic pair over $\mathbb{C}^\hbar$. We say that an operator $P \in \mathscr{D}^\hbar_X$ is an elliptic operator if $\mathscr{D}^\hbar_X / \mathscr{D}^\hbar_X P$ is an elliptic $\mathscr{D}^\hbar_X$-module.

In other words, $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ is elliptic if $\mathrm{char}_\hbar(\mathscr{M}) \cap T^*_M X \subset T^*_X X$ and, by Example 3.1.13(e), we deduce that $P \in \mathscr{D}_X^\hbar$ is elliptic if and only if it is locally written as $P = P_0 + \hbar P'$ for some $P' \in \mathscr{D}_X^\hbar$ and P_0 an elliptic operator in the classical sense. Take a system of local coordinates $(x;\eta)$ on T^*X. Recall that P_0 is elliptic if its principal symbol $\sigma(P_0)$ satisfies $\sigma(P_0)((x;i\eta)) \neq 0$ for $\eta \neq 0$. Take, for example, $X = \mathbb{C}^n$, $M = \mathbb{R}^n$ and denote by Δ the Laplace operator. Then, $P = \Delta + \hbar P'$ is elliptic for any $P' \in \mathscr{D}_X^\hbar$.

The meaning of elliptic pairs on the real analytic setting illustrates why one can regard the theory of elliptic pairs (over $\mathbb{C}^\hbar$) as a natural generalization of the theory of elliptic systems on real analytic manifolds. We shall return to this particular setting later in the applications.

4.2 Theorems for elliptic pairs over $\mathbb{C}[[\hbar]]$

4.2.1 Regularity theorems

General regularity theorem.

Theorem 4.2.1. *Let $(\mathscr{M}, F)$ be an f-elliptic pair over $\mathbb{C}^\hbar$. Then, the natural morphism below is an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^\hbar)$:*

$$F \overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X^\hbar} (\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar} \mathscr{K}_\hbar) \to \mathrm{R}\mathscr{H}om_{\mathbb{C}_X^\hbar}(\mathrm{D}'_{\mathbb{C}_X^\hbar} F, \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar} \mathscr{K}_\hbar) \tag{4.2.1}$$

Proof. The morphism (4.2.1) is induced by the isomorphism $F \simeq \mathrm{D}'_{\mathbb{C}_X^\hbar} \mathrm{D}'_{\mathbb{C}_X^\hbar} F$ and by the canonical morphism:

$$\mathrm{D}'_{\mathbb{C}_X^\hbar} \mathrm{D}'_{\mathbb{C}_X^\hbar} F \overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X^\hbar} (\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar} \mathscr{K}_\hbar) \to \mathrm{R}\mathscr{H}om_{\mathbb{C}_X^\hbar}(\mathrm{D}'_{\mathbb{C}_X^\hbar} F, \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar} \mathscr{K}_\hbar).$$

Note that $\mathrm{D}'_{\mathbb{C}_X^\hbar} F$ has $\mathbb{R}$-constructible cohomology and $\mathrm{SS}(\mathrm{D}'_{\mathbb{C}_X^\hbar} F) = \mathrm{SS}(F)^a$, where a denotes the opposite map on T^*X. The transversality condition on the pair $(\mathscr{M}, F)$ together with Lemma 4.1.4 entails:

$$\mathrm{SS}(\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar} \mathscr{K}_\hbar) \cap \mathrm{SS}(\mathrm{D}'_{\mathbb{C}_X^\hbar} F)^a \subset T^*_X X.$$

The conclusion follows by Proposition 1.1.2. □

Remark 4.2.2. We can rewrite the isomorphism (4.2.1) in terms of left $\mathscr{D}_X^\hbar$-modules, i.e., assuming that $(\mathscr{M}, F)$ is an f-elliptic pair, with $\mathscr{M}$ being a left $\mathscr{D}_X^\hbar$-module; we get the following isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^\hbar)$:

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathscr{M}, \mathrm{D}'_{\mathbb{C}_X^\hbar} F \overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X^\hbar} \mathscr{K}_\hbar) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathscr{M}, \mathrm{R}\mathscr{H}om_{\mathbb{C}_X^\hbar}(F, \mathscr{K}_\hbar)).$$

Regularity theorems in the formal extension case. We want to refine the regularity property in the cases where F (resp. $\mathscr{M}$) are formal extensions of objects in $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$ (resp. $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$). Let us start with some auxiliar results.

Lemma 4.2.3. *Let $F, G \in \mathrm{Mod}(\mathbb{C}_X)$. There is a natural morphism in $\mathrm{Mod}(\mathbb{C}^\hbar_X)$:*

$$F \otimes G^\hbar \to (F \otimes G)^\hbar. \tag{4.2.2}$$

Proof. It is enough to note that there is a projective system of morphisms

$$\begin{aligned} F \otimes G^\hbar &\to F \otimes (G \otimes (\mathbb{C}^\hbar/\hbar^{n+1}\mathbb{C}^\hbar)) \\ &\xrightarrow{\sim} (F \otimes G) \otimes (\mathbb{C}^\hbar/\hbar^{n+1}\mathbb{C}^\hbar). \end{aligned}$$

Then, the morphism (4.2.2) follows from the universal property of projective limits. □

Lemma 4.2.4. *Let $F, G \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)$. Then there is a natural bifunctorial morphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^\hbar_X)$:*

$$F \otimes G^{\mathrm{R}\hbar} \to (F \otimes G)^{\mathrm{R}\hbar}. \tag{4.2.3}$$

Proof. (i) First, fix $F \in \mathrm{Mod}(\mathbb{C}_X)$ and consider the two functors from $\mathrm{Mod}(\mathbb{C}_X)$ to $\mathrm{Mod}(\mathbb{C}^\hbar_X)$: $\theta_1 \colon G \mapsto F \otimes G^\hbar$ and $\theta_2 \colon G \mapsto (F \otimes G)^\hbar$. By Lemma 4.2.3, there is a morphism of functors $u \colon \theta_1 \to \theta_2$. Since both functors are left exact, this morphism u extends to the derived functors and we get the morphism (4.2.3) for a fixed $F \in \mathrm{Mod}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$.
(ii) Now let us fix $G \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)$ and consider the two functors from $\mathrm{Mod}(\mathbb{C}_X)$ to $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^\hbar_X)$: $\lambda_1 \colon F \mapsto F \otimes G^{\mathrm{R}\hbar}$ and $\lambda_2 \colon F \mapsto (F \otimes G)^{\mathrm{R}\hbar}$. By (i) there exists a morphism of functors $v \colon \lambda_1 \to \lambda_2$. Both functors extend naturally to the category of bounded complexes $\mathrm{C}^{\mathrm{b}}(\mathrm{Mod}(\mathbb{C}_X))$ and send complexes quasi-isomorphic to zero to objects isomorphic to zero in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^\hbar_X)$. Hence, both functors extend to $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)$ as well as the morphism of functors v. □

Lemma 4.2.5. *For $F, G \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)$, we have a commutative diagram in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^\hbar_X)$:*

$$\begin{array}{ccc} \mathrm{D}'_{\mathbb{C}_X} F \otimes G^{\mathrm{R}\hbar} & \longrightarrow & (\mathrm{D}'_{\mathbb{C}_X} F \otimes G)^{\mathrm{R}\hbar} \\ & \searrow & \downarrow \\ & & \mathrm{R}\mathscr{H}om\,(F, G)^{\mathrm{R}\hbar}, \end{array} \tag{4.2.4}$$

such that the morphism in the horizontal arrow is the one given by Lemma 4.2.4.

Proof. The oblique arrow is the composition of two canonical morphisms:

$$\mathrm{R}\mathscr{H}om\,(F, \mathbb{C}) \otimes G^{\mathrm{R}\hbar} \to \mathrm{R}\mathscr{H}om\,(F, G^{\mathrm{R}\hbar}) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om\,(F, G)^{\mathrm{R}\hbar}.$$

The vertical arrow is defined by applying the functor $(\,\bullet\,)^{\mathrm{R}\hbar}$ to the canonical morphism $\mathrm{R}\mathscr{H}om\,(F, \mathbb{C}) \otimes G \to \mathrm{R}\mathscr{H}om\,(F, G)$. The commutativity of the diagram is obvious. □

Remark 4.2.6. Assuming that $G \in \mathrm{Mod}(\mathscr{D}^\hbar_X)$ in Lemma 4.2.3, then the morphism (4.2.2) is $\mathscr{D}^\hbar_X$-linear. Similarly, if $G \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}^\hbar_X)$, then the morphism (4.2.3) is a morphism in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}^\hbar_X)$ and the diagram in Lemma 4.2.5 is a commutative diagram in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}^\hbar_X)$. In particular, we shall use in the sequel the following canonical morphism in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}^\hbar_X)$, for $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)$:

$$F \otimes \mathscr{K}_\hbar \to (F \otimes \mathscr{K})^{\mathrm{R}\hbar}. \tag{4.2.5}$$

Theorem 4.2.7. *Let $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$ and suppose that $(\mathscr{M}, F^\hbar)$ is an f-elliptic pair over $\mathbb{C}^\hbar$. Then there is a commutative diagram of isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^\hbar_X)$:*

$$\begin{array}{ccc} \mathrm{R}\mathscr{H}om_{\mathscr{D}^\hbar_X}(\mathscr{M}, \mathrm{D}'_{\mathbb{C}^\hbar_X} F^\hbar \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^\hbar_X} \mathscr{K}_\hbar) & \xrightarrow{\sim} & \mathrm{R}\mathscr{H}om_{\mathscr{D}^\hbar_X}(\mathscr{M}, (\mathrm{D}'_{\mathbb{C}_X} F \otimes \mathscr{K})^{\mathrm{R}\hbar}) \\ & \searrow^{\sim} & \downarrow{\scriptstyle\sim} \\ & & \mathrm{R}\mathscr{H}om_{\mathscr{D}^\hbar_X}(\mathscr{M}, \mathrm{R}\mathscr{H}om(F, \mathscr{K})^{\mathrm{R}\hbar}). \end{array} \tag{4.2.6}$$

Proof. First note that we obtain the diagram (4.2.6) by applying $\mathrm{R}\mathscr{H}om_{\mathscr{D}^\hbar_X}(\mathscr{M}, \bullet)$ to the commutative diagram provided by Lemma 4.2.5:

$$\begin{array}{ccc} \mathrm{D}'_{\mathbb{C}_X} F \otimes \mathscr{K}_\hbar & \longrightarrow & (\mathrm{D}'_{\mathbb{C}_X} F \otimes \mathscr{K})^{\mathrm{R}\hbar} \\ & \searrow & \downarrow \\ & & \mathrm{R}\mathscr{H}om(F, \mathscr{K})^{\mathrm{R}\hbar}. \end{array}$$

We also use the fact that $\mathrm{D}'_{\mathbb{C}^\hbar_X} F^\hbar \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^\hbar_X} \mathscr{K}_\hbar$ is isomorphic to $\mathrm{D}'_{\mathbb{C}_X} F \otimes \mathscr{K}_\hbar$.

Note that the oblique arrow is an isomorphism since it is the composition of two canonical isomorphisms:

$$\begin{aligned} \mathrm{R}\mathscr{H}om_{\mathscr{D}^\hbar_X}(\mathscr{M}, \mathrm{D}'_{\mathbb{C}^\hbar_X} F^\hbar \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^\hbar_X} \mathscr{K}_\hbar) &\xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}^\hbar_X}(\mathscr{M}, \mathrm{R}\mathscr{H}om_{\mathbb{C}^\hbar_X}(F^\hbar, \mathscr{K}_\hbar)) \\ &\xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}^\hbar_X}(\mathscr{M}, \mathrm{R}\mathscr{H}om(F, \mathscr{K})^{\mathrm{R}\hbar}). \end{aligned}$$

The first one results from applying the regularity theorem 4.2.1 to the f-elliptic pair $(\mathscr{M}, F^\hbar)$ (see also Remark 4.2.2) and the second isomorphism results from Lemma 1.4.15.

The vertical arrow is also an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^\hbar_X)$.

In fact, the objects on both sides of the vertical arrow are cohomologically complete by Propositions 1.3.16 and 1.4.2. Hence, since $\mathrm{gr}_\hbar$ is conservative on cohomologically complete objects, it is enough to prove that the canonical morphism below is an isomorphism:

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathrm{gr}_\hbar(\mathscr{M}), \mathrm{D}'_{\mathbb{C}_X} F \otimes \mathscr{K}) \to \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathrm{gr}_\hbar(\mathscr{M}), \mathrm{R}\mathscr{H}om(F, \mathscr{K})). \tag{4.2.7}$$

In fact, $(\mathrm{gr}_\hbar(\mathscr{M}), F)$ is an f-elliptic pair over $\mathbb{C}$ and the morphism (4.1.4) is precisely the regularity isomorphism for elliptic pairs over $\mathbb{C}$ applied to $(\mathrm{gr}_\hbar(\mathscr{M}), F)$ (cf. Theorem 4.1.2).

We have proved that the oblique and vertical arrows in diagram (4.2.6) are isomorphisms. The commutativity of the diagram allow us to conclude that the horizontal arrow is also an isomorphism. □

Theorem 4.2.7 and the functorial properties of $(\bullet)^{\mathrm{R}\hbar}$ give the next corollary.

Corollary 4.2.8. *Let* $(\mathscr{M}, F)$ *be an* f*-elliptic pair over* $\mathbb{C}$*. Then,* $(\mathscr{M}^{\hbar}, F^{\hbar})$ *is an* f*-elliptic pair over* $\mathbb{C}^{\hbar}$ *and there are canonical isomorphisms in* $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\hbar}_X)$*:*

$$\begin{aligned}\mathrm{R}\mathscr{H}om_{\mathscr{D}^{\hbar}_X}(\mathscr{M}^{\hbar}, \mathrm{D}'_{\mathbb{C}^{\hbar}_X} F^{\hbar} \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^{\hbar}_X} \mathscr{K}_{\hbar}) &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathrm{D}'_{\mathbb{C}_X} F \otimes \mathscr{K})^{\mathrm{R}\hbar} \\ &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathrm{R}\mathscr{H}om_{\mathbb{C}_X}(F, \mathscr{K}))^{\mathrm{R}\hbar}.\end{aligned}$$

4.2.2 Finiteness theorem

Theorem 4.2.9. *Let* $(\mathscr{M}, F)$ *be a good* f*-elliptic pair over* $\mathbb{C}^{\hbar}$ *and suppose that* f *is proper when restricted to the support of* $(\mathscr{M}, F)$*. Then,* $\underline{f}_{!,\hbar}(\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^{\hbar}_X} F)$ *is an object of* $\mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}^{\hbar\,\mathrm{op}}_Y)$*.*

Proof. Note that by Theorem 3.1.3 it suffices to prove that

(i) $\underline{f}_{!,\hbar}(F \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^{\hbar}_X} \mathscr{M})$ is cohomologically complete;

(ii) $\mathrm{gr}_{\hbar}(\underline{f}_{!,\hbar}(F \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^{\hbar}_X} \mathscr{M}))$ belongs to $\mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}^{\hbar\,\mathrm{op}}_Y)$.

Set $\mathscr{L} := (F \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^{\hbar}_X} \mathscr{M}) \overset{\mathrm{L}}{\otimes}_{\mathscr{D}^{\hbar}_X} \mathscr{K}_{\hbar}$. The regularity theorem 4.2.1 yields the isomorphism:

$$\mathscr{L} \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}^{\hbar}_X}(\mathrm{D}'_{\mathbb{C}^{\hbar}_X} F, \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}^{\hbar}_X} \mathscr{K}_{\hbar}).$$

Since $\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}^{\hbar}_X} \mathscr{K}_{\hbar}$ is cohomologically complete, $\mathscr{L}$ is also cohomologically complete in view of Proposition 1.3.16. Finally, $\underline{f}_{!,\hbar}(F \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^{\hbar}_X} \mathscr{M}) = \mathrm{R}f_!(\mathscr{L})$ is cohomologically complete by Proposition 1.3.19, since $\mathrm{R}f_*(\mathscr{L})$ and $\mathrm{R}f_!(\mathscr{L})$ coincide under the hypothesis on the support.

On the other hand, note that $(\mathrm{gr}_{\hbar}(\mathscr{M}), \mathrm{gr}_{\hbar}(F))$ is an f-elliptic pair over $\mathbb{C}$ that satisfies the conditions of the finiteness theorem for elliptic pairs over $\mathbb{C}$. Hence,

$$\mathrm{gr}_{\hbar}(\underline{f}_{!,\hbar}(F \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^{\hbar}_X} \mathscr{M})) \simeq \underline{f}_!(\mathrm{gr}_{\hbar}(F) \otimes \mathrm{gr}_{\hbar}(\mathscr{M}))$$

is an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_Y{}^{\mathrm{op}})$ cf. Theorem 4.1.2. □

4.2.3 Duality theorem

Assumption 4.2.10. The remaining results of this section are obtained assuming that $f : X \to Y$ is a smooth morphism of complex manifolds.

Proposition 4.2.11. *If $f : X \to Y$ is smooth, then $f^{-1}\mathscr{D}_Y^\hbar \simeq (f^{-1}\mathscr{D}_Y)^\hbar$.*

Proof. This is a particular case of Proposition 1.4.8 choosing for $\mathcal{B}$ the family of compact Stein subsets of Y. □

The rings $f^{-1}\mathscr{D}_Y^\hbar$ and $(f^{-1}\mathscr{D}_Y)^\hbar$ are not necessarily isomorphic when f is not smooth.

Example 4.2.12. Let $i : Y \hookrightarrow X$ be the embedding of a closed submanifold of dimension $d_Y < d_X$. Given an open subset V of Y, a section $Q \in \Gamma(V; i^{-1}\mathscr{D}_X^\hbar)$ can be regarded as a formal series $Q = \sum Q_n|_V \hbar^n$, with $Q_n \in \Gamma(U; \mathscr{D}_X)$, for some open subset U of X such that $U \cap Y = V$. On the other hand, a section $R \in \Gamma(V; (i^{-1}\mathscr{D}_X)^\hbar)$ can be regarded as a formal series $R = \sum R_n|_V \hbar^n$, with $R_n \in \Gamma(U_n; \mathscr{D}_X)$, for some family $(U_n)_{n\geq 0}$ of open subsets of X such that $U_n \cap Y = V$, for each $n \geq 0$. Hence, the canonical morphism $i^{-1}\mathscr{D}_X^\hbar \to (i^{-1}\mathscr{D}_X)^\hbar$ is a monomorphism but not an isomorphism.

The trace morphism in the $\hbar$-setting. If $\mathscr{M}$ is a right $(\mathscr{D}_X, \mathscr{D}_X)$-bimodule, its direct image as a bimodule is the object in the derived category $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y^{\mathrm{op}} \otimes \mathscr{D}_Y^{\mathrm{op}})$ defined by:

$$\underline{\underline{f}}_!(\mathscr{M}) := \mathrm{R}f_!((\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{K}) \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{K}).$$

In particular $\Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{D}_X$ is a right $(\mathscr{D}_X, \mathscr{D}_X)$-bimodule and one has the so-called differential trace morphism in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y^{\mathrm{op}} \otimes \mathscr{D}_Y^{\mathrm{op}})$ (cf. Proposition 5.10 of [34]):

$$\mathrm{tr}_f : \underline{\underline{f}}_!(\Omega_X[d_X] \otimes_{\mathscr{O}_X} \mathscr{D}_X) \to \Omega_Y[d_Y] \otimes_{\mathscr{O}_Y} \mathscr{D}_Y. \tag{4.2.8}$$

Similarly, we define the direct image of a right $(\mathscr{D}_X^\hbar, \mathscr{D}_X^\hbar)$-bimodule as the object in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y^{\hbar\,\mathrm{op}} \otimes_{\mathbb{C}^\hbar} \mathscr{D}_Y^{\hbar\,\mathrm{op}})$ given by:

$$\underline{\underline{f}}_{!,\hbar}(\mathscr{M}) := \mathrm{R}f_!((\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar} \mathscr{K}_\hbar) \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar} \mathscr{K}_\hbar).$$

We want to construct now a version of the trace morphism in the $\hbar$-framework for a smooth morphism.

Recall that in the smooth case $\mathscr{K}_\hbar$ is a coherent left $\mathscr{D}_X^\hbar$-module. Hence, for any coherent right $\mathscr{D}_X^\hbar$-module $\mathscr{M}$, one has:

$$\mathscr{M} \otimes_{\mathscr{D}_X^\hbar} \mathscr{K}^\hbar \simeq \mathscr{M} \otimes_{\mathscr{D}_X^\hbar} (\mathscr{D}_X^\hbar \otimes_{\mathscr{D}_X} \mathscr{K}) \simeq \mathscr{M} \otimes_{\mathscr{D}_X} \mathscr{K}. \tag{4.2.9}$$

By the isomorphisms in (4.2.9), the object $\mathscr{M} \otimes_{\mathscr{D}_X} \mathscr{K}$ has a structure of $f^{-1}(\mathscr{D}_Y^{\hbar\,\mathrm{op}})$-module. Passing to the derived category and applying $\mathrm{R}f_!$, one concludes that $\underline{f}_!(\mathscr{M})$ is an object of $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y^{\hbar\,\mathrm{op}})$ and that it is isomorphic to $\underline{f}_{!,\hbar}(\mathscr{M})$ for any $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar\,\mathrm{op}})$. In other words, the direct image of $\mathscr{M}$ in the $\hbar$-setting coincides with its direct image as a $\mathscr{D}$-module. We use this fact in the following proofs.

Lemma 4.2.13. *Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^{\mathrm{op}})$. There is a morphism $\underline{f}_{!,\hbar}(\mathscr{M}^\hbar) \to \underline{f}_!(\mathscr{M})^{\mathrm{R}\hbar}$ in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y^{\hbar\,\mathrm{op}})$.*

Proof. First note that we have a chain of isomorphisms in $\mathsf{D}^{\mathrm{b}}(f^{-1}\mathscr{D}_Y^{\hbar\,\mathrm{op}})$:

$$\begin{aligned}\mathscr{M}^\hbar \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{K} &\simeq \mathscr{M}^\hbar \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathrm{D}'_X \mathrm{D}'_X \mathscr{K} \\ &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathrm{D}'_X\mathscr{K}, \mathscr{M}_\hbar) \\ &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathrm{D}'_X\mathscr{K}, \mathscr{M})^{\mathrm{R}\hbar}.\end{aligned}$$

The first and second isomorphisms follow from the coherence of $\mathscr{K}$ over $\mathscr{D}_X$. The third follows from Lemma 1.4.3.

Applying $\mathrm{R}f_!$ we get the following isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y^{\mathrm{op}})$:

$$\underline{f}_{!,\hbar}(\mathscr{M}^\hbar) \simeq \mathrm{R}f_!((\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{K})^{\mathrm{R}\hbar}).$$

Finally, Lemma 1.4.5 entails the following morphism in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y^{\hbar\,\mathrm{op}})$:

$$\underline{f}_{!,\hbar}(\mathscr{M}^\hbar) \to \mathrm{R}f_!((\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{K}))^{\mathrm{R}\hbar} = \underline{f}_!(\mathscr{M})^{\mathrm{R}\hbar}.$$

$\square$

Similarly to the case of direct image, note that if $\mathscr{M}$ is a right $(\mathscr{D}_X^\hbar, \mathscr{D}_X^\hbar)$-bimodule then $\underline{\underline{f}}_!(\mathscr{M})$ is an object of $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y^{\hbar\,\mathrm{op}} \otimes_{\mathbb{C}^\hbar} \mathscr{D}_Y^{\hbar\,\mathrm{op}})$ isomorphic to $\underline{\underline{f}}_{!,\hbar}(\mathscr{M})$. Hence, the trace morphism in the $\hbar$-setting follows from the classical one as we show in the next result:

Proposition 4.2.14. *The morphism f induces a differential trace morphism in the derived category $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y^{\hbar\,\mathrm{op}} \otimes_{\mathbb{C}^\hbar} \mathscr{D}_Y^{\hbar\,\mathrm{op}})$:*

$$\mathrm{tr}_{f,\hbar} : \underline{\underline{f}}_{!,\hbar}(\Omega_X[d_X] \otimes_{\mathscr{O}_X} \mathscr{D}_X^\hbar) \to \Omega_Y[d_Y] \otimes_{\mathscr{O}_Y} \mathscr{D}_Y^\hbar.$$

Proof. We have the following isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y^{\hbar\,\mathrm{op}} \otimes_{\mathbb{C}^\hbar} \mathscr{D}_Y^{\hbar\,\mathrm{op}})$:

$$\begin{aligned}\underline{\underline{f}}_{!,\hbar}(\Omega_X[d_X] \otimes_{\mathscr{O}_X} \mathscr{D}_X^\hbar) &\simeq \underline{f}_{!,\hbar}(\Omega_X[d_X] \otimes_{\mathscr{O}_X} \mathscr{K}_\hbar) \\ &\simeq \underline{f}_{!,\hbar}((\Omega_X[d_X] \otimes_{\mathscr{O}_X} \mathscr{K})^\hbar)\end{aligned}$$

The first one results from the associativity of tensor products. The second one results from the isomorphism

$$\Omega_X[d_X] \otimes_{\mathscr{O}_X} \mathscr{K}_\hbar \simeq (\Omega_X[d_X] \otimes_{\mathscr{O}_X} \mathscr{K})^\hbar.$$

The morphism $\mathrm{tr}_{f,\hbar}$ is then constructed composing the morphisms below:

$$\underline{f}_{!,\hbar}((\Omega_X[d_X] \otimes_{\mathscr{O}_X} \mathscr{K})^\hbar) \to \underline{f}_!(\Omega_X[d_X] \otimes_{\mathscr{O}_X} \mathscr{K})^{\mathrm{R}\hbar} \to \Omega_Y[d_Y] \otimes_{\mathscr{O}_Y} \mathscr{D}_Y^\hbar.$$

The first morphism is an application of Lemma 4.2.13 which, in this case, preserves the bimodule structures involved. The second morphism is the formal extension of the classical trace morphism (4.2.8). $\square$

Remark 4.2.15. Consider the absolute case $f = a_X : X \to \{\mathrm{pt}\}$. In this case one has $\mathscr{K}_\hbar \simeq \mathscr{O}_X^\hbar$ and we get the following isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^\hbar)$:

$$\begin{aligned}\underline{\underline{a_X}}_{!,\hbar}(\Omega_X[d_X] \otimes_{\mathscr{O}_X} \mathscr{D}_X^\hbar) &\simeq \mathrm{R}a_{X!}(\Omega_X[d_X] \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{O}_X^\hbar) \\ &\simeq \mathrm{R}a_{X!}(\mathbb{C}_X^\hbar[2d_X]) \\ &\simeq \mathrm{R}\Gamma_c(X;\mathbb{C}_X^\hbar[2d_X]) \simeq \mathrm{R}\Gamma_c(X;\omega_X^\hbar).\end{aligned}$$

Thus, the trace morphism $\mathrm{tr}_{a_X,\hbar}$ coincides with the morphism $\mathrm{R}\Gamma_c(X,\omega_X^\hbar) \to \mathbb{C}^\hbar$ induced by the natural transformation $\mathrm{R}a_{X!}a_X^! \to \mathrm{id}$.

The duality morphism.

Proposition 4.2.16. *For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^{\hbar\,\mathrm{op}})$ there is a canonical morphism in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y^{\hbar\,\mathrm{op}})$:*

$$\underline{f}_{!,\hbar}(\underline{\mathrm{D}}_{\mathscr{D}_X^\hbar}\mathscr{M}) \to \underline{\mathrm{D}}_{\mathscr{D}_Y^\hbar}(\underline{f}_{!,\hbar}\mathscr{M}). \tag{4.2.10}$$

Proof. First note that there is a basis change morphism in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y^{\hbar\,\mathrm{op}})$:

$$\begin{aligned}&\underline{f}_{!,\hbar}(\underline{\mathrm{D}}_{\mathscr{D}_X^\hbar}(\mathscr{M})) \simeq \mathrm{R}f_!(\mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathscr{M},\Omega_X[d_X] \otimes_{\mathscr{O}_X} \mathscr{K}_\hbar)) \\ &\to \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y^\hbar}(\mathrm{R}f_!(\mathscr{M}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar}\mathscr{K}_\hbar), \mathrm{R}f_!((\Omega_X[d_X] \otimes_{\mathscr{O}_X} \mathscr{K}_\hbar)\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar}\mathscr{K}_\hbar)) \\ &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y^\hbar}(\underline{f}_{!,\hbar}(\mathscr{M}), \underline{\underline{f}}_{!,\hbar}(\Omega_X[d_X] \otimes_{\mathscr{O}_X} \mathscr{D}_X^\hbar)).\end{aligned}$$

We obtain (4.2.10) composing the above morphism with the morphism induced by Proposition 4.2.14 on the second term of the $\mathrm{R}\mathscr{H}om$. □

Corollary 4.2.17. *For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^{\hbar\,\mathrm{op}})$ and $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^\hbar)$, there is a canonical morphism in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y^{\hbar\,\mathrm{op}})$:*

$$\underline{f}_{!,\hbar}(\mathrm{D}'_{\mathbb{C}_X^\hbar}F\overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X^\hbar}\underline{\mathrm{D}}_{\mathscr{D}_X^\hbar}(\mathscr{M})) \to \underline{\mathrm{D}}_{\mathscr{D}_Y^\hbar}(\underline{f}_{!,\hbar}(F\overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X^\hbar}\mathscr{M})). \tag{4.2.11}$$

Proof. Note that there is a canonical morphism in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^{\hbar\,\mathrm{op}})$:

$$\mathrm{D}'_{\mathbb{C}_X^\hbar}F\overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X^\hbar}\underline{\mathrm{D}}_{\mathscr{D}_X^\hbar}(\mathscr{M}) \to \underline{\mathrm{D}}_{\mathscr{D}_X^\hbar}(F\overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X^\hbar}\mathscr{M}). \tag{4.2.12}$$

The morphism (4.2.11) is then obtained as a composition of morphisms:

$$\begin{aligned}\underline{f}_{!,\hbar}(\mathrm{D}'_{\mathbb{C}_X^\hbar}F\overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X^\hbar}\underline{\mathrm{D}}_{\mathscr{D}_X^\hbar}(\mathscr{M})) &\to \underline{f}_{!,\hbar}(\underline{\mathrm{D}}_{\mathscr{D}_X^\hbar}(F\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^\hbar}\mathscr{M})) \\ &\to \underline{\mathrm{D}}_{\mathscr{D}_Y^\hbar}(\underline{f}_{!,\hbar}(F\overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X^\hbar}\mathscr{M})),\end{aligned}$$

where the first arrow results from (4.2.12) and the second one results from the duality morphism (4.2.10) applied to the object $F\overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X^\hbar}\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y^{\hbar\,\mathrm{op}})$. □

Duality theorem.

Theorem 4.2.18. *Let f be a smooth morphism, $(\mathscr{M}, F)$ be a good f-elliptic pair over $\mathbb{C}^\hbar$ and suppose that f is proper when restricted to the support of $(\mathscr{M}, F)$. Then, the canonical morphism*

$$\underline{f}_{!,\hbar}(\mathrm{D}'_{\mathbb{C}^\hbar_X} F \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^\hbar_X} \underline{\mathrm{D}}_{\mathscr{D}^\hbar_X}(\mathscr{M})) \to \underline{\mathrm{D}}_{\mathscr{D}^\hbar_Y}(\underline{f}_{!,\hbar}(F \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^\hbar_X} \mathscr{M})) \tag{4.2.13}$$

is an isomorphism in $\mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}^{\hbar\,\mathrm{op}}_Y)$.

Proof. Set $\mathscr{L}_1 = \underline{f}_{!,\hbar}(\mathrm{D}'_{\mathbb{C}^\hbar_X} F \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^\hbar_X} \underline{\mathrm{D}}_{\mathscr{D}^\hbar_X}(\mathscr{M}))$ and $\mathscr{L}_2 = \underline{\mathrm{D}}_{\mathscr{D}^\hbar_Y}(\underline{f}_{!,\hbar}(F \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^\hbar_X} \mathscr{M}))$.

First note that $(\mathscr{M}, F)$ and $(\underline{\mathrm{D}}_{\mathscr{D}^\hbar_X} \mathscr{M}, \mathrm{D}'_{\mathbb{C}^\hbar_X} F)$ are f-elliptic pairs that verify the conditions of the finiteness theorem 4.2.9. Therefore, $\mathscr{L}_1$ and $\mathscr{L}_2$ are objects of $\mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}^{\hbar\,\mathrm{op}}_Y)$. Hence, by the conservativism of $\mathrm{gr}_\hbar$ on coherent objects, it suffices to check that the induced morphism $\mathrm{gr}_\hbar(\mathscr{L}_1) \to \mathrm{gr}_\hbar(\mathscr{L}_2)$ is an isomorphism in $\mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}^{\mathrm{op}}_Y)$. Consider the isomorphisms:

$$\begin{aligned} \mathrm{gr}_\hbar(\mathscr{L}_1) &\simeq \underline{f}_!(\mathrm{D}'_{\mathbb{C}_X} \mathrm{gr}_\hbar(F) \otimes \underline{\mathrm{D}}_{\mathscr{D}_X}(\mathrm{gr}_\hbar(\mathscr{M}))) \\ \mathrm{gr}_\hbar(\mathscr{L}_2) &\simeq \underline{\mathrm{D}}_{\mathscr{D}_Y}(\underline{f}_!(\mathrm{gr}_\hbar(F) \otimes_{\mathbb{C}_X} \mathrm{gr}_\hbar(\mathscr{M}))). \end{aligned}$$

By the construction of the duality morphism, $\mathrm{gr}_\hbar(\mathscr{L}_1) \to \mathrm{gr}_\hbar(\mathscr{L}_2)$ is precisely the duality morphism for f-elliptic pairs over $\mathbb{C}$ applied to the pair $(\mathrm{gr}_\hbar(\mathscr{M}), \mathrm{gr}_\hbar(F))$. We conclude that $\mathrm{gr}_\hbar(\mathscr{L}_1) \to \mathrm{gr}_\hbar(\mathscr{L}_2)$ is an isomorphism in $\mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}^{\hbar\,\mathrm{op}}_Y)$ since the pair $(\mathrm{gr}_\hbar \mathscr{M}, \mathrm{gr}_\hbar F)$ is in the conditions of the duality theorem for elliptic pairs over $\mathbb{C}$ (cf. Theorem 4.1.2). □

4.3 Examples and applications

4.3.1 Finiteness and duality for $\mathscr{D}_X[[\hbar]]$-modules

For each $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}^{\hbar\,\mathrm{op}}_X)$, the pair $(\mathscr{M}, \mathbb{C}^\hbar_X)$ forms an f-elliptic pair over $\mathbb{C}^\hbar$ since $\mathrm{SS}(\mathbb{C}^\hbar_X) = T^*_X X$. Applying Theorems 4.2.9 and 4.2.18 in this particular case, we recover Theorem 3.3.3 and we obtain a duality theorem for $\mathscr{D}^\hbar_X$-modules in the smooth case:

Corollary 4.3.1. *Assume that $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}^{\hbar\,\mathrm{op}}_X)$ and that f is proper when restricted to $\mathrm{supp}(\mathscr{M})$. Then, the objects $\underline{f}_{!,\hbar}(\underline{\mathrm{D}}_{\mathscr{D}^\hbar_X} \mathscr{M})$ and $\underline{\mathrm{D}}_{\mathscr{D}^\hbar_Y}(\underline{f}_{!,\hbar}(\mathscr{M}))$ belong to $\mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}^{\hbar\,\mathrm{op}}_Y)$. Assume in addition that f is smooth. Then, there is an isomorphism in $\mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}^{\hbar\,\mathrm{op}}_Y)$:*

$$\underline{f}_{!,\hbar}(\underline{\mathrm{D}}_{\mathscr{D}^\hbar_X} \mathscr{M}) \xrightarrow{\sim} \underline{\mathrm{D}}_{\mathscr{D}^\hbar_Y}(\underline{f}_{!,\hbar}(\mathscr{M})).$$

4.3.2 Finiteness and duality for $\mathscr{O}_X[[\hbar]]$-modules

Since the ring $\mathscr{O}^\hbar_X$ satisfies Assumption 1.3.5, all the machinery developed to study deformation-quantization algebras also apply to the study of $\mathscr{O}^\hbar_X$-modules.

Recall that Gauert's direct image theorem ([5]), states that if f is proper when restricted to the support of $\mathscr{F} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$, then the direct image $\mathrm{R}f_!(\mathscr{F})$ is an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_Y)$.

We can generalize Grauert's theorem to the $\hbar$-framework. Such generalization follows from the classical case and from the coherence criteria (Theorem 1.3.17), remarking also that $\mathrm{R}f_!(\mathscr{F})$ is cohomologically complete if f is proper on $\mathrm{supp}(\mathscr{F})$. We omit further details, since the proof goes like the one of Theorem 4.2.9.

Theorem 4.3.2. *Let $\mathscr{F} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}^\hbar_X)$ and assume that f is proper when restricted to* $\mathrm{supp}(\mathscr{F})$*. Then* $\mathrm{R}f_!(\mathscr{F})$ *is an object of* $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}^\hbar_Y)$*.*

Recall also the classical results of [30] regarding the relative duality theorem for $\mathscr{O}_X$-modules: first, the authors construct a relative trace morphism $\mathrm{R}f_!(\Omega_X[d_X]) \to \Omega_Y[d_Y]$ in the derived category $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_Y)$; then, the trace morphism and Grauert's theorem are used to construct a duality morphism

$$\mathrm{R}f_!(\mathrm{R}\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \Omega_X[d_X])) \to \mathrm{R}\mathscr{H}om_{\mathscr{O}_Y}(\mathrm{R}f_!(\mathscr{F}), \Omega_Y[d_Y]), \tag{4.3.1}$$

for any $\mathscr{F} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$; finally, it is proved that (4.3.1) is an isomorphism if f is proper on $\mathrm{supp}(\mathscr{F})$.

Assume that one has $f^{-1}\mathscr{O}^\hbar_X \xrightarrow{\sim} (f^{-1}\mathscr{O}_X)^\hbar$. Under such hypothesis, the trace morphism extends to the $\hbar$-framework using Lemma 1.4.5:

$$\mathrm{R}f_!(\Omega^\hbar_X[d_X]) \to \mathrm{R}f_!(\Omega_X[d_X])^{\mathrm{R}\hbar} \to \Omega_Y[d_Y]^{\mathrm{R}\hbar} \simeq \Omega^\hbar_Y[d_Y]. \tag{4.3.2}$$

Moreover, we construct a canonical duality morphism in $\mathsf{D}^{\mathrm{b}}(\mathscr{O}^\hbar_Y)$

$$\begin{aligned}\mathrm{R}f_!(\mathrm{R}\mathscr{H}om_{\mathscr{O}^\hbar_X}(\mathscr{F}, \Omega^\hbar_X[d_X])) &\to \mathrm{R}\mathscr{H}om_{\mathscr{O}^\hbar_Y}(\mathrm{R}f_!(\mathscr{F}), \mathrm{R}f_!(\Omega^\hbar_X[d_X])) &(4.3.3)\\ &\to \mathrm{R}\mathscr{H}om_{\mathscr{O}^\hbar_Y}(\mathrm{R}f_*(\mathscr{F}), \Omega^\hbar_Y[d_Y]), &(4.3.4)\end{aligned}$$

and the relative duality theorem of [30] extends to the $\hbar$-framework:

Theorem 4.3.3. *Let $\mathscr{F} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}^\hbar_X)$. Suppose that $f^{-1}\mathscr{O}^\hbar_Y \xrightarrow{\sim} (f^{-1}\mathscr{O}_Y)^\hbar$ and that f is proper when restricted to* $\mathrm{supp}(\mathscr{F})$*. Then* (4.3.3) *is an isomorphism.*

To prove this result one uses the conservativism of $\mathrm{gr}_\hbar$ on cohomologically complete objects. We omit again the details of the proof since it repeats arguments already employed.

Note that Theorem 4.3.3 holds in the smooth case:

Proposition 4.3.4. *If $f : X \to Y$ is a smooth morphism of complex manifolds, then $f^{-1}\mathscr{O}^\hbar_X \xrightarrow{\sim} (f^{-1}\mathscr{O}_X)^\hbar$.*

Proof. This is a particular case of Proposition 1.4.8, choosing $\mathcal{B}$ as the family of open Stein subsets of Y. □

Grauert's theorem and the relative duality theorem for $\mathscr{O}_X$-modules are both recovered as particular cases of the finiteness and duality results of [34] applied to elliptic pairs of the form $(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{D}_X, \mathbb{C}_X)$, $\mathscr{F}$ being a coherent $\mathscr{O}_X$-module. This is a consequence of the following two properties:

(i) an $\mathscr{O}_X$-module $\mathscr{F}$ is coherent over $\mathscr{O}_X$ if and only if $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ is coherent over $\mathscr{D}_X$;

(ii) for any $\mathscr{F} \in \mathsf{D}^{\mathrm{b}}(\mathscr{O}_X)$ one has $\underline{f}_!(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{D}_X) \xrightarrow{\sim} \mathrm{R}f_!(\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{D}_Y$ in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y)$.

Note that for a given $\mathscr{F} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}^\hbar_X)$, the pair $(\mathscr{F} \otimes_{\mathscr{O}^\hbar_X} \mathscr{D}^\hbar_X, \mathbb{C}^\hbar_X)$ is an f-elliptic pair over $\mathbb{C}^\hbar$. Moreover, the following counterpart of (i) in the $\hbar$-setting is clear:

(i') a cohomologically complete $\mathscr{O}^\hbar_X$-module $\mathscr{F}$ is coherent over $\mathscr{O}^\hbar_X$ if and only if $\mathscr{F} \otimes_{\mathscr{O}^\hbar_X} \mathscr{D}^\hbar_X$ is coherent over $\mathscr{D}^\hbar_X$.

However, since $(\,\bullet\,)^\hbar$ doesn't behave in a nice way under inverse images or tensor products, the relation between $\underline{f}_{!,\hbar}(\mathscr{F} \otimes_{\mathscr{O}^\hbar_X} \mathscr{D}^\hbar_X)$ and $\mathrm{R}f_!(\mathscr{F}) \otimes_{\mathscr{O}^\hbar_Y} \mathscr{D}^\hbar_Y$ is not clear. In other words, it is not obvious in what conditions Theorems 4.2.9 and 4.2.18 applied to f-elliptic pairs of the form $(\mathscr{F} \otimes_{\mathscr{O}^\hbar_X} \mathscr{D}^\hbar_X, \mathbb{C}^\hbar_X)$ give Theorems 4.3.2 and 4.3.3. Note that this is true if $f = a_X$ is the constant map.

Remark 4.3.5. Recall that $\mathscr{O}^\hbar_X$ is the natural local model of the deformation-quantization algebras $\mathscr{A}_X$ studied in [19]. Therefore, Theorems 4.3.2 and 4.3.3 are closely related to the finiteness and duality results for kernels of $\mathscr{A}_X$-modules proved in loc.cit.. More precisely, the case $f = a_X$ is contained in Corollaries 3.2.3 and 3.3.4 of loc.cit..

4.3.3 The absolute case: global solutions of $\mathscr{D}_X[[\hbar]]$-modules

From now on we shall consider the absolute case $f = a_X : X \to \{\mathrm{pt}\}$. If we rewrite the morphism of Corollary 4.2.17 for $f = a_X$ and for left $\mathscr{D}^\hbar_X$-modules, we get, for each pair $(\mathscr{M}, F)$, the morphism below in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^\hbar)$:

$$\mathrm{R}\Gamma_c(X; \Omega^\hbar_X[d_X] \overset{\mathrm{L}}{\otimes}_{\mathscr{D}^\hbar_X} \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^\hbar_X} F) \to (\mathrm{R}\Gamma_c(X; \mathrm{R}\mathscr{H}om_{\mathscr{D}^\hbar_X}(\mathscr{M}, \mathrm{R}\mathscr{H}om_{\mathbb{C}^\hbar_X}(F, \mathscr{O}^\hbar_X))))^* \tag{4.3.5}$$

Note that the morphism (4.3.5) is induced by the morphism of Remark 4.2.15.

Moreover, as a particular case of Theorems 4.2.9 and 4.2.18 we obtain the following corollary:

Corollary 4.3.6. *Let $(\mathscr{M}, F)$ be a good elliptic pair over $\mathbb{C}^\hbar$ with compact support. Then, the complexes*

$$\mathrm{R}\Gamma_c(X; \Omega^\hbar_X[d_X] \overset{\mathrm{L}}{\otimes}_{\mathscr{D}^\hbar_X} \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^\hbar_X} F)$$
$$\mathrm{R}\Gamma_c(X; \mathrm{R}\mathscr{H}om_{\mathscr{D}^\hbar_X}(\mathscr{M}, \mathrm{R}\mathscr{H}om_{\mathbb{C}^\hbar_X}(F, \mathscr{O}^\hbar_X)))$$

have finitely generated cohomology over $\mathbb{C}^\hbar$ and the morphism (4.3.5) *is an isomorphism in $\mathsf{D}^{\mathrm{b}}_f(\mathbb{C}^\hbar)$.*

One can regard the complex $\mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathscr{M}, \mathrm{R}\mathscr{H}om_{\mathbb{C}_X^\hbar}(F, \mathscr{O}_X^\hbar))$ as the complex of solutions of $\mathscr{M}$ on the generalized sheaf of formal power series of holomorphic functions associated to F. In the last subsection we give an application in a case where the sheaf F is not the constant sheaf $\mathbb{C}_X^\hbar$.

Recall that in [3] the authors prove the constructibility of the complex $\mathrm{Sol}_{\mathscr{D}_X^\hbar}(\mathscr{M}) = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathscr{M}, \mathscr{O}_X^\hbar)$, when $\mathscr{M}$ is an holonomic $\mathscr{D}_X^\hbar$-module (cf. [3, Theorem 3.13]). Applying Corollary 4.3.6 to elliptic pairs of the form $(\mathscr{M}, \mathbb{C}_X^\hbar)$, we also obtain a result concerning the functor $\mathrm{Sol}_{\mathscr{D}_X^\hbar}$:

Corollary 4.3.7. *Let $\mathscr{M}$ be a good $\mathscr{D}_X^\hbar$-module with compact support. Then,*

$$\mathrm{R}\Gamma_c(X; \Omega_X[d_X] \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M}) \to (\mathrm{R}\Gamma_c(X; \mathrm{Sol}_{\mathscr{D}_X^\hbar}(\mathscr{M})))^*$$

is an isomorphism in $\mathsf{D}^{\mathrm{b}}_f(\mathbb{C}^\hbar)$.

The next example illustrates Corollary 4.3.7 and is based on a formula due to M. Kashiwara (cf. [3, Example 8.5]).

Example 4.3.8. Consider $X = \mathbb{C}$, $P = x - \hbar\partial_x$ and $\mathscr{M} = \mathscr{D}_X^\hbar / \mathscr{D}_X^\hbar P$. Note that $\mathrm{supp}(\mathscr{M}) = \{0\}$. One knows that $\mathscr{M} \simeq \mathscr{D}_\mathbb{C}^\hbar / \mathscr{D}_\mathbb{C}^\hbar x$. Hence, standard computations lead us to the following quasi-isomorphisms:

$$\mathrm{R}\Gamma(\mathbb{C}; \mathrm{Sol}_{\mathscr{D}_X^\hbar}(\mathscr{M})) \simeq \mathrm{R}\Gamma(\mathbb{C}; \Omega_\mathbb{C}^\hbar[1] \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_\mathbb{C}^\hbar} \mathscr{M}) \simeq \mathbb{C}^\hbar.$$

Remark 4.3.9. For each $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X^\hbar)$, the pair $(\mathscr{O}_X^\hbar, F)$ is an elliptic pair over $\mathbb{C}^\hbar$, since $\mathrm{char}_\hbar(\mathscr{O}_X^\hbar) = T_X^*X$. Moreover, Corollary 4.3.6 applied to elliptic pairs of this form gives a well-known finiteness and duality result for $\mathbb{R}$-constructible sheaves of $\mathbb{C}^\hbar$-modules.

A refinement in the holonomic case.

Lemma 4.3.10. (a) *For $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^\hbar)$ and V an open subset of X, one has*

$$\mathrm{R}\Gamma(V; \mathrm{D}_{\mathbb{C}_X^\hbar} F) \simeq (\mathrm{R}\Gamma_c(V; F))^*.$$

(b) *If $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X^\hbar)$ and V is a relatively compact open subanalytic subset of X, then both $\mathrm{R}\Gamma(V; \mathrm{D}_{\mathbb{C}_X^\hbar} F)$ and $\mathrm{R}\Gamma_c(V; F)$ have finitely generated cohomology over $\mathbb{C}^\hbar$ and the isomorphism in (a) holds in $\mathsf{D}^{\mathrm{b}}_f(\mathbb{C}^\hbar)$.*

Proof. The isomorphism in (a) follows from the formula [15, (3.1.8)] with $A = \mathbb{C}^\hbar$. The finiteness property in (b) is a consequence of [15, Corollary 8.4.11]. ☐

Given $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X^\hbar)$, we can apply Lemma 4.3.10 with F replaced by $\mathrm{Sol}_{\mathscr{D}_X^\hbar}(\mathscr{M})$:

Proposition 4.3.11. *Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}^{\hbar}_X)$ and V be a relatively compact open subanalytic subset of X. Then, $\mathrm{R}\Gamma_c(V; \mathrm{Sol}_{\mathscr{D}^{\hbar}_X}(\mathscr{M}))$ is an object in $\mathsf{D}^{\mathrm{b}}_f(\mathbb{C}^{\hbar})$ and, moreover, there is an isomorphism in $\mathsf{D}^{\mathrm{b}}_f(\mathbb{C}^{\hbar})$:*

$$\mathrm{R}\Gamma(V; \Omega^{\hbar}_X[d_X] \overset{\mathrm{L}}{\otimes}_{\mathscr{D}^{\hbar}_X} \mathscr{M}) \simeq (\mathrm{R}\Gamma_c(V; \mathrm{Sol}_{\mathscr{D}^{\hbar}_X}(\mathscr{M})))^*.$$

Proof. This is a direct application of Lemma 4.3.10, using also the isomorphism:

$$\mathrm{D}_{\mathbb{C}^{\hbar}_X} \mathrm{Sol}_{\mathscr{D}^{\hbar}_X}(\mathscr{M}) \simeq \Omega^{\hbar}_X[d_X] \overset{\mathrm{L}}{\otimes}_{\mathscr{D}^{\hbar}_X} \mathscr{M},$$

which follows from [3, Theorem 3.15]. □

Let us illustrate Proposition 4.3.11 with some examples:

Example 4.3.12. Consider $X = \mathbb{C}$, $P = x\partial_x - \lambda\hbar$, $\lambda \in \mathbb{C}^{\star}$, and $\mathscr{M} = \mathscr{D}^{\hbar}_X / \mathscr{D}^{\hbar}_X P$. Let V be an open simply connected subset of $\mathbb{C}$. In this situation, we have $\mathrm{R}\Gamma(V; \mathrm{Sol}_{\mathscr{D}^{\hbar}_X}(\mathscr{M})) \simeq \mathbb{C}^{\hbar}[-1]$ if $0 \notin V$, and $\mathrm{R}\Gamma(V; \mathrm{Sol}_{\mathscr{D}^{\hbar}_X}(\mathscr{M})) \simeq \mathbb{C}$ otherwise. Let us give some details of the computations.

First note that an equation of the form $Pu = g$ is equivalent to the following infinite system of linear holomorphic differential equations:

$$\begin{cases} x\partial_x u_0 = g_0, \\ x\partial_x u_n - \lambda u_{n-1} = g_n, \quad n \geq 1. \end{cases} \tag{4.3.6}$$

If $0 \notin V$, then the solutions of the homogeneous equation $Pu = 0$ are given recursively by

$$u_n = \sum_{j=0}^{n} a_{n-j} \lambda^j \frac{\log(x)^j}{j!} \hbar^j, \quad a_0, \dots, a_n \in \mathbb{C}.$$

Hence, each solution is determined by the family of constants $(a_n)_{n\geq 0}$ and we get $\ker(P) \simeq \mathbb{C}^{\hbar}$. On the other hand, $\mathrm{Coker}(P) = 0$ since, recursively, the Cauchy theorem guarantees that, for each $n \geq 0$, there is a solution $u_n \in \Gamma(V; \mathscr{O}_X)$ of the nth-equation and then the formal series $u = \sum_n u_n \hbar^n \in \Gamma(V; \mathscr{O}^{\hbar}_X)$ solves $Pu = g$.

If $0 \in V$, it is obvious that $\ker(P) = 0$. On the other hand, $\mathrm{Coker}(P) \simeq \mathbb{C}$. Indeed, there is a solution of the nth-equation of (4.3.6) if and only if we have $g_0(0) \neq 0$. Moreover $f = \sum f_n \hbar^n \in \Gamma(V; \mathscr{O}^{\hbar}_X)$ and $g = \sum g_n \hbar^n \in \Gamma(V; \mathscr{O}^{\hbar}_X)$ determine the same element in $\mathrm{Coker}(P)$ if and only if $f_0(0) = g_0(0)$, and the conclusion follows.

Example 4.3.13. Consider $X = \mathbb{C}$, $P = x^2\partial_x + \lambda\hbar$, $\lambda \in \mathbb{C}^{\star}$, and $\mathscr{M} = \mathscr{D}^{\hbar}_X / \mathscr{D}^{\hbar}_X P$. Let V be an open simply connected subset of $\mathbb{C}$. Then, we have $\mathrm{R}\Gamma(V; \mathrm{Sol}_{\mathscr{D}^{\hbar}_X}(\mathscr{M})) \simeq \mathbb{C}^{\hbar}[-1]$ if $0 \notin V$, and $\mathrm{R}\Gamma(V; \mathrm{Sol}_{\mathscr{D}^{\hbar}_X}(\mathscr{M})) \simeq \mathbb{C}^2$ otherwise. We omit the computations, since they are similar to those of Example 4.3.12. Let us just remark that in the case $0 \notin V$, the solutions $u = (u_n)_{n\geq 0} \in \Gamma(V; \mathscr{O}^{\hbar}_X)$ of $Pu = 0$ are recursively given by:

$$u_n = \sum_{j=0}^{n} a_{n-j} \frac{\lambda^j}{j! x^j} \hbar^j, \quad a_0, \dots, a_n \in \mathbb{C}.$$

Example 4.3.14. Consider $X = \mathbb{C}^2$, $P = (x - \lambda\hbar)\partial_y$, with $\lambda \in \mathbb{C}$, $\mathscr{M} = \mathscr{D}_X^\hbar / \mathscr{D}_X^\hbar P$ and let V be an open subset of X. The finiteness property doesn't hold in this non-holonomic case. Indeed, the kernel of $\Gamma(V; \mathscr{O}_{\mathbb{C}^2}^\hbar \xrightarrow{P} \Gamma(V; \mathscr{O}_{\mathbb{C}^2}^\hbar)$ consists on formal power series of holomorphic functions which don't depend on y and the family of such series is not finitely generated over $\mathbb{C}^\hbar$.

4.3.4 Hyperfunctions with $\hbar$-parameter

Let X be the complexification of a real analytic manifold M and let $\mathscr{A}_M := \mathbb{C}_M \otimes \mathscr{O}_X$ denote the sheaf of real analytic functions on M. In the $\hbar$-framework we can consider two distinct sheaves of real analytic manifolds with $\hbar$-parameter, denoted by $\mathscr{A}_{M,\hbar}$ and $\mathscr{A}_M^\hbar$:

$$\begin{aligned} \mathscr{A}_{M,\hbar} &:= \mathbb{C}_M^\hbar \otimes_{\mathbb{C}_X^\hbar} \mathscr{O}_X^\hbar \simeq \mathbb{C}_M \otimes \mathscr{O}_X^\hbar, \\ \mathscr{A}_M^\hbar &= (\mathbb{C}_M \otimes \mathscr{O}_X)^\hbar. \end{aligned}$$

One has the isomorphism $\mathscr{A}_M^\hbar \simeq \mathscr{A}_M^{\mathrm{R}\hbar}$, since $\mathscr{A}_M$ verifies the hypothesis of Proposition 1.4.4 taking for $\mathcal{B}$ the family of all open subsets of M. Hence, both $\mathscr{A}_{M,\hbar}$ and $\mathscr{A}_M^\hbar$ are concentrated in degree 0 and we can identify them to usual sheaves.

Similarly to Example 4.2.12, one concludes that there is a natural monomorphism $\mathscr{A}_{M,\hbar} \hookrightarrow \mathscr{A}_M^\hbar$ in $\mathrm{Mod}(\mathscr{D}_X^\hbar)$ which is not an isomorphism. Moreover, we remark that this morphism is a particular case of the morphism constructed in Lemma 4.2.4.

Let us also consider the c-soft sheaf of hyperfunctions on M, defined by $\mathscr{B}_M := \mathrm{R}\mathscr{H}om_{\mathbb{C}_X}(\mathbb{C}_M, \mathscr{M})$. Remark that $\mathscr{B}_M$ is $\hbar$-acyclic by Proposition 1.4.4 and by the fact that c-soft objects are $\Gamma(K; \_)$-injective, for any compact $K \subset X$. Thus, the formal extension of $\mathscr{B}_M$ verifies the following isomorphisms:

$$\begin{aligned} \mathscr{B}_M^{\mathrm{R}\hbar} \simeq \mathscr{B}_M^\hbar &= (\mathrm{R}\mathscr{H}om\,(\mathrm{D}'_{\mathbb{C}_X}\mathbb{C}_M, \mathscr{O}_X))^\hbar \\ &\simeq \mathrm{R}\mathscr{H}om\,(\mathrm{D}'_{\mathbb{C}_X}\mathbb{C}_M, \mathscr{O}_X^\hbar), \\ &\simeq \mathrm{R}\mathscr{H}om_{\mathbb{C}_X^\hbar}(\mathrm{D}'_{\mathbb{C}_X^\hbar}\mathbb{C}_M^\hbar, \mathscr{O}_X^\hbar). \end{aligned}$$

We say that $\mathscr{B}_M^\hbar$ is the sheaf of hyperfunctions on M with $\hbar$-parameter.

As a particular case of Lemma 4.2.5, we get the following commutative diagram of morphisms in $\mathrm{Mod}(\mathscr{D}_X^\hbar)$:

$$\begin{array}{ccc} \mathscr{A}_{M,\hbar} & \hookrightarrow & \mathscr{A}_M^\hbar \\ & \searrow & \downarrow \\ & & \mathscr{B}_M^\hbar \end{array}$$

Let $\mathscr{M}$ be a coherent $\mathscr{D}_X^\hbar$-module. Recall that $\mathscr{M}$ is elliptic on M if $(\mathscr{M}, \mathbb{C}_M^\hbar)$ is an elliptic pair over $\mathbb{C}^\hbar$, cf. Definition 4.1.7. Applying our main results to elliptic pairs of the form $(\mathscr{M}, \mathbb{C}_M^\hbar)$, we obtain the following regularity, finiteness and duality property for elliptic $\mathscr{D}_X^\hbar$-modules.

Corollary 4.3.15. *Let $\mathscr{M}$ be an elliptic $\mathscr{D}_X^\hbar$-module on M.*

(a) *There is a commutative diagram of isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^\hbar)$:*

$$\begin{array}{ccc} \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathscr{M}, \mathscr{A}_{M,\hbar}) & \xrightarrow{\sim} & \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathscr{M}, \mathscr{A}_M^\hbar) \\ & \searrow^{\sim} & \downarrow{\scriptstyle\sim} \\ & & \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathscr{M}, \mathscr{B}_M^\hbar). \end{array}$$

(b) *If M is compact and $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_X^\hbar)$, then $\mathrm{R}\Gamma(M; \Omega_M^\hbar \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_M^\hbar} \mathscr{M})$ is the dual of*

$$\mathrm{R}\Gamma(M; \mathrm{R}\mathscr{H}om_{\mathscr{D}_M^\hbar}(\mathscr{M}, \mathscr{B}_{M,\hbar}))$$

and both objects belong to $\mathsf{D}^{\mathrm{b}}_f(\mathbb{C}^\hbar)$.

Bibliography

[1] E. Andronikof, *Microlocalisation tempérée,* Mémoires Soc. Math. France, **57**, (1994).

[2] J.L. Brylinski, B. Malgrange and J.L.Verdier *Transformation de Fourier géométrique I, II,* C.R. Acad. Sci., **297**, (1983) p. 55-58 and **303**, (1986) p. 193-198.

[3] A. D'Agnolo, S. Guillermou and P. Schapira, *Regular holonomic $\mathscr{D}[[\hbar]]$-modules,* Publ. RIMS, Kyoto Univ. **47** no. 1, (2011), p. 221-255.

[4] A. D'Agnolo and P. Polesello, *Stacks of twisted modules and integral transforms,* Geometric aspects of Dwork theory, Vol. I, II, Walter de Gruyter GmbH & Co. KG, Berlin (2004).

[5] H. Grauert, *Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen,* Inst. Hautes Études Sci. Publ. Math, **5** (1960), p. 233-292.

[6] L. Hörmander,*An Introduction to Complex Analysis in Several Variables,* North Holland Mathematical Library, (1973).

[7] R. Hotta and M. Kashiwara, *The invariant holonomic systems on a semisimple Lie Algebra,* Invent. Math., **75**, (1984) p. 327-358.

[8] R. Hotta, K. Takeuchi and T. Tanisaki, *$\mathscr{D}$-Modules, Perverse Sheaves, and Representation Theory,* Progress in Mathematics, **236**, Birkhäuser (2008).

[9] C. Houzel and P. Schapira, *Images directes des modules différentiels,* C. R. Acad. Sci. Paris **298** (1984), p. 461-464.

[10] M. Kashiwara, *On the holonomic systems of linear differential equations II,* Invent. Math., **49**, (1978) p. 121-135.

[11] ———, *Systems of microdifferential equations,* Progress in Mathematics, **34**, Birkhäuser (1983).

[12] ———, *Vanishing cycle sheaves and holonomic systems of differential equations,* Lect. Notes in Math., **1016**, Springer-Verlag (1983).

[13] ———, *The Riemman-Hilbert problem for holonomic systems,* Publ. RIMS, Kyoto Univ. **20** (1984), p. 319-365.

[14] ———, *$\mathscr{D}$-modules and Microlocal Calculus,* Translations of Mathematical Monographs, **217** American Math. Soc. (2003).

[15] M. Kashiwara and P. Schapira, *Sheaves On Manifolds,* Grundlehren der Math. Wiss. **292** Springer-Verlag (1990).

[16] ———, *Moderate and formal cohomology associated with constructible sheaves,* Mem. Soc. Math. France **64** (1996).

[17] ———, *Ind-Sheaves,* Astérisque **271** Soc. Math. France, (2001).

[18] ———, *Categories and Sheaves,* Grundlehren der Math. Wiss. **332** Springer-Verlag (2005).

[19] ———, *Deformation Quantization Modules,* Astérisque **345** Soc. Math. France, (2012).

[20] Y. Laurent and B. Malgrange, *Cycles proches, spécialisation et $\mathscr{D}$-modules,* Annales de l'institut Fourier, **45**, n5 (1995), 1353-1405.

[21] Y. Laurent and P. Schapira, *Images inverses des modules différentiels,* Compositio Math. **61** (1987), 229–251.

[22] P. Maisonobe and Z. Mebkhout, *Le théorème de comparaison pour les cycles évanescents,* Séminaires et Congrès, **8**, Soc. math. France (2004), 311-389.

[23] B. Malgrange, *Transformation de Fourier geométrique,* Séminaire Bourbaki, **692**, (1987-88) p. 133-150.

[24] A.R. Martins and T. Monteiro Fernandes, *Formal extension of the Whitney functor and duality,* to Rendiconti del Seminario Matematico della Università di Padova, **126** (2011).

[25] A.R. Martins, T.Monteiro Fernandes and D. Raimundo, *Extension of functors for algebras of formal deformation,* submitted. `arXiv:1201.0536v2`

[26] Z. Mebkhout, *Le formalisme des six opérations de Grothendieck pour les $\mathscr{D}_X$-modules cohérents,* Travaux em Cours, **35**, Hermann (1989).

[27] T. Monteiro Fernandes, *Microlocalisation of $\mathscr{D}$-modules along a submanifold,* Bull. Soc. math. France, **123** (1995), p. 293-327.

[28] P. Polesello and P. Schapira, *Stacks of quantization-deformation modules over complex symplectic manifolds,* Int. Math. Res. Notices **49**, p. 2637-2664 (2004).

[29] D. Raimundo, *Regularity for elliptic pairs over* $\mathbb{C}[[\hbar]]$, accepted for publication in Rendiconti del Seminario Matematico della Università di Padova. `arxiv:1003.4873v4`

[30] J. P. Ramis, G. Ruget, and J. L. Verdier, *Dualité relative en geométrie analytique complexe,* Invent. Math. **13** (1971), p. 261-283.

[31] C. Sabbah, *De Rham and Dolbeault cohomology of $\mathscr{D}$-modules,* (Lecture Notes, Oberwolfach, 2007), math.polytechnique.fr/∼sabbah.

[32] M. Sato, T. Kawai, and M. Kashiwara, *Hyperfunctions and pseudo-differential equations*, Hyperfunctions and Pseudo-differential Equations (H. Komatsu, ed.), Proceedings Katata 1971, Lecture Notes in Mathematics, **287**, Springer (1973), p. 265-529.

[33] P. Schapira, *Microdifferential Systems in the Complex Domain,* Grundlehren der Math. Wiss. **269** Springer-Verlag (1985).

[34] P. Schapira and J-P. Schneiders, *Elliptic pairs I: relative finiteness and duality,* Astérisque Soc. Math. France, **224** (1994).

List of notations

Sheaf theory.

$\mathrm{SS}(F)$: micro-support of a sheaf F
ω_X: the dualizing sheaf
$\widetilde{X}_Y$: the normal deformation of X on Y
$\mathrm{Mod}_{\mathbb{R}-\mathrm{c}}(\mathbb{K}_X)$: abelian category of $\mathbb{R}$-constructible sheaves
$\mathsf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{K}_X)$: derived category of $\mathbb{R}$-constructible sheaves
$\mathrm{Mod}_{\mathbb{C}-\mathrm{c}}(\mathbb{K}_X)$: abelian category of $\mathbb{C}$-constructible sheaves
$\mathsf{D}^{\mathrm{b}}_{\mathbb{C}-\mathrm{c}}(\mathbb{K}_X)$: derived category of $\mathbb{C}$-constructible sheaves
$\mathrm{Mod}_{\mathbb{R}^+}(\mathbb{K}_X)$: abelian category of conic sheaves
$\mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbb{K}_X)$: derived category of conic sheaves
$\nu_Y^{\mathbb{K}}$: Sato's specialization functor for sheaves of $\mathbb{K}$-modules
$\psi_Y^{\mathbb{K}}$: nearby-cycle functor for sheaves of $\mathbb{K}$-modules
$\varphi_Y^{\mathbb{K}}$: vanishing-cycle functor for sheaves of $\mathbb{K}$-modules
$\mathscr{F}^{\mathbb{K}}$: Fourier-Sato functor for sheaves of $\mathbb{K}$-modules
$\mu_Y^{\mathbb{K}}$: microlocalization functor for sheaves of $\mathbb{K}$-modules

Algebras of formal deformation.

$\mathrm{gr}^{\mathrm{n}}_\hbar$: the n-graded functor for algebras of formal deformation
$\mathrm{gr}_\hbar$: the graded functor for algebras of formal deformation (coincides with $\mathrm{gr}^0_\hbar$)
$\mathbb{C}^\hbar$: the ring of formal power series on $\hbar$ with complex coefficients
$\mathbb{C}^{\hbar,\mathrm{loc}}$: the field of formal Laurent series on $\hbar$ with complex coefficients
$(\bullet)^{\mathrm{R}\hbar}$: the functor of formal extension
$\mathrm{Mod}_{\mathscr{S}}(\mathscr{A})$: denotes an auxiliar subcategory of $\mathrm{Mod}(\mathscr{A})$ for $\mathscr{A}$ an algebra of formal deformation and for $\mathscr{S}$ a full Serre subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$
${}_n\mathscr{M}$: the kernel of the morphism $\hbar^{n+1}:\mathscr{M}\to\mathscr{M}$, for $\mathscr{M}\in\mathrm{Mod}(\mathscr{A})$
$\mathscr{M}_n$: the cokernel of the morphism $\hbar^{n+1}:\mathscr{M}\to\mathscr{M}$, for $\mathscr{M}\in\mathrm{Mod}(\mathscr{A})$
$\mathscr{M}_{\hbar-\mathrm{tor}}$: the increasing union of the kernels ${}_n\mathscr{M}$
$\mathscr{M}_{\hbar-\mathrm{tf}}$: the cokernel of $0\to\mathscr{M}_{\hbar-\mathrm{tor}}\to\mathscr{M}$

$\mathscr{D}$-modules and $\mathscr{D}^\hbar$-modules.

For each of the following notations we use a natural counterpart in the $\hbar$-setting, replacing $\mathscr{D}_X$ for $\mathscr{D}^\hbar_X$ or adding an $\hbar$ to the classical notation:

$\mathscr{O}_X$: sheaf of holomorphic functions on X
Θ_X: sheaf of holomorphic vector fields on X
$\mathscr{D}_X$: sheaf of holomorphic differential operators on X
$\mathscr{D}_{X|S}$: sheaf of relative differential operators
$\mathscr{D}_{[E]}$: sheaf of homogeneous differential operators on a complex vector bundle E
$\mathscr{A}_M$: sheaf of analytic functions on a real analytic manifold M

$\mathscr{B}_M$: sheaf of hyperfunctions on a real analytic manifold M
Ω_X: sheaf of holomorphic forms of maximal degree on X
$\mathrm{char}(\mathscr{M})$: characteristic variety of a coherent $\mathscr{D}$-module $\mathscr{M}$
$\mathrm{char}_f(\mathscr{M})$: f-characteristic variety of a coherent $\mathscr{D}$-module $\mathscr{M}$

$\mathrm{Mod}_{\mathrm{gd}}(\mathscr{D}_X)$: Serre subcategory of good $\mathscr{D}_X$-modules
$\mathrm{Mod}_{\mathrm{hol}}(\mathscr{D}_X)$: Serre subcategory of holonomic $\mathscr{D}_X$-modules
$\mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X)$: Serre subcategory of regular holonomic $\mathscr{D}_X$-modules
$\mathrm{Mod}_{\mathrm{NC(f)}}(\mathscr{D}_{\mathrm{X}})$: Serre subcategory of non-characteristic $\mathscr{D}_X$-modules for a morphism f
$\mathrm{Mod}_{\mathrm{mon}}(\mathscr{D}_X)$: Serre subcategory of monodromic $\mathscr{D}_X$-modules
$\mathrm{Mod}_{\mathrm{sp}}(\mathscr{D}_X^{\hbar})$: Serre subcategory of specializable $\mathscr{D}$-modules
$\mathsf{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_X)$: triangulated subcategory of complexes of $\mathscr{D}_X$-modules with good cohomology
$\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$: triangulated subcategory of complexes of $\mathscr{D}_X$-modules with holonomic cohomology
$\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$: triangulated subcategory of complexes of $\mathscr{D}_X$-modules with regular holonomic cohomology
$\mathsf{D}^{\mathrm{b}}_{\mathrm{NC}}(\mathscr{D}_X)$: triangulated subcategory of complexes of $\mathscr{D}_X$-modules with non-characteristic cohomology
$\mathsf{D}^{\mathrm{b}}_{\mathrm{mon}}(\mathscr{D}_X)$: triangulated subcategory of complexes of $\mathscr{D}_X$-modules with monodromic cohomology

$\mathscr{D}_{X\to Y}$: transfer module for $\mathscr{D}$-modules (also denoted by $\mathscr{K}$)
$\underline{f}_*$: direct image functor for $\mathscr{D}$-modules
$\underline{f}_!$: proper direct image functor for $\mathscr{D}$-modules
$\underline{f}^*$: inverse image functor for $\mathscr{D}$-modules
$\underline{f}^!$: extraordinary inverse image functor for $\mathscr{D}$-modules
$\underline{\underline{f}}_!$: direct image for $(\mathscr{D}^{\mathrm{op}}, \mathscr{D}^{\mathrm{op}})$-bimodules
$\mathrm{Sol}_{\mathscr{D}_X}$: *Solutions* functor for $\mathscr{D}$-modules
$\mathrm{DR}_{\mathscr{D}_X}$: De Rham functor for $\mathscr{D}$-modules

ν_Y: specialization functor for $\mathscr{D}$-modules
φ_Y: nearby-cycle functor for $\mathscr{D}$-modules
ψ_Y: vanishing-cycle functor for $\mathscr{D}$-modules
$\mathscr{F}$: Fourier transform for $\mathscr{D}$-modules
μ_Y: microlocalization for $\mathscr{D}$-modules

Index