ISEG ECONÓMICAS

Departamento de Economia

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WP 1/1999/DE

WORKING PAPERS

ISSN Nº 0874-4548

UNIVERSIDADE TÉCNICA DE LISBOA

Instituto Superior de Economia e Gestão

Aggregation and Persistence in a Macromodel

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ABSTRACT: This paper shows that the behaviour of an otherwise conventional model of real business cycles (RBCs) in which heterogeneous individual firms are subject to temporary technology shocks will be characterised by long memory and nonlinearity. We start from microfoundations, using a standard RBC model of monopolistic competition. We then derive the fundamental intertemporal equilibrium path of the economy and we study analytically the time series properties of GDP.

We show that the resulting stochastic process is radically different from the process followed by the firms' productivities, which are conventional stable log-linear Auto-Regressive (AR) processes. This new process is nonlinear, more persistent than any stable AR and yet is mean-reverting (unlike unit-root processes). In our model, small temporary shocks can lead to large fluctuations and/or persistence at the macro level, without requiring large shocks or unit roots at the microeconomic level. Within our model, common shocks are more potent than idiosyncratic ones. The process is also characterised by long cycles which have random lengths and which are asymmetric. Increased monopoly power will tend to reduce the amplitude and increase the persistence of business cycles.

KEYWORDS: Auto-Regressive (AR) process, Autocovariance functions, Autocorrelation functions, Heterogeneous (non-representative) firms, Long memory processes, Monopolistic Competition, Real Business Cycle (RBC).

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*We thank Jean-Pascal Bénassy, Mick Devereux and Paul Newbold for their comments. The first author gratefully acknowledges ESRC (UK) grant R000236627.

1 Introduction

This paper shows that the behaviour of an otherwise conventional model of real business cycles (RBCs) in which heterogeneous individual firms are subject to temporary technology shocks will be characterised by long memory and nonlinearity. Hence, two of the most counterfactual aspects of current RBC models, i.e. the lack of persistence and the need for large aggregate shocks,[1] are successfully dealt with (*inter alia*) solely through going beyond the assumption of a representative firm. Earlier work on the effect on long memory of aggregation over heterogeneous entities includes Robinson (1978), Granger (1980), Forni and Reichlin (1998), Linden (1998), Lippi and Zaffaroni (1998). Furthermore, dissatisfaction with the simple unit-root hypothesis in Auto-Regressive (AR) macroeconomic models, and the need for alternatives such as long memory models, has been highlighted in the work of Diebold and Rudebusch (1989), Rudebusch (1993), Diebold and Senhadji (1996). In AR models, memory decays exponentially (integrated of order 0) or is infinite (integrated of order 1), but nothing in between. Long memory models fill this gap, and our work will give rise to them, from microfoundations.

Our approach differs from existing studies of long memory in that we start with a fully specified general equilibrium model, rather than assuming a specific time series model for the macroeconomy. We then characterize the type of stochastic process and features that arise from solving the model, and we find that the result is an unconventional nonlinear time series process.

More specifically, in Section 2, we modify a standard RBC model of monopolistic competition of Devereux, Head and Lapham (1993, 1996) in which we assume that the productivity of firms are subject to AR idiosyncratic as well as common (economy-wide) shocks. We then derive the fundamental intertemporal equilibrium path of the economy and we study analytically the time series properties of GDP. We want to avoid dependence of our result on increasing returns, as was the case for Devereux et al. (1993, 1996) and pointed out by Bénassy (1996). For this purpose, our specification for the aggregation of firms' output displays no return to variety. We also want to avoid the effect of entry and exit of firms since these phenomena have already been described. The simplest way to do it is to fix exogenously the number of firms. Monopoly firms *à la* Dixit and Stiglitz (1977) exhibit positive variable profits. Models with an endogenous number of monopoly firms require the introduction of a fixed set-up cost or some other feature to prevent infinite entry. With an exogenous number of firms, we

[1] The internal propagation mechanism of RBCs has been characterised as weak, and they require unrealistically large economy-wide technology shocks to account for the variations in the Solow residuals. See for instance Rotemberg and Woodford (1996) and Muellbauer (1997).

can dispense with this fixed cost which also simplifies considerably the analysis.

In Section 3, we show that the resulting stochastic process is radically different from the process followed by the firms' productivities, which are conventional stable log-linear AR processes. This new process is nonlinear, more persistent than any stable AR and yet is mean-reverting (unlike unit-root processes). In our model, small temporary shocks can lead to large fluctuations and/or persistence at the macro level, without requiring large shocks or unit roots at the microeconomic level. Within our model, common shocks are more potent than idiosyncratic ones. The process is also characterised by long cycles which have random lengths and which are asymmetric. Increased monopoly power will tend to reduce the amplitude and increase the persistence of business cycles.

The derivations of the time series properties are collected in an Appendix, which also contains solutions to technical problems that are of independent interest.

2 Framework

2.1 Aggregation and derived demand

The productive sector of the economy is composed of N infinitely-lived monopolistically competitive firms. Let $q_{n,t}$ be the output of firm n at time t. A final good industry, operating under perfect competition, uses this specialised inputs to produce a final good according to the standard CES aggregation function, see for instance Bénassy (1996),

$$(1) \qquad Y_t \equiv \left[\frac{1}{N^{1-\rho}} \sum_{n=1}^{N} q_{n,t}^{\rho} \right]^{1/\rho}, \qquad \rho \in (0,1),$$

where Y_t is the aggregate output of the final good industry. Since this production function exhibits constant returns to scale in the inputs q's, the number of firms operating in the final good sector is irrelevant by the perfect competition assumption. Notice that the elasticity of substitution between any two products is $1/(1-\rho)$. Also notice that our specification of this CES aggregation displays no return to variety N, see Bénassy (1996).

The aggregate output Y_t can be used either for consumption or for investment purposes. Investment in period t increases the capital stock of period $t+1$, i.e. with a one period lag. In order to be able to derive a closed form solution for the intertemporal equilibrium of our economy, we need to assume, as in Devereux et al. (1993, 1996), a 100% depreciation rate on capital. Hence, the stock of capital in period $t+1$ is equal to the investment of period t.

Cost minimization gives rise to the derived demand for good n, $q_{n,t}$, as a function of its own price $p_{n,t}$ and of the aggregate price p_t

$$(2) \qquad q_{n,t} \equiv \left[\frac{p_t}{p_{n,t}}\right]^{1/(1-\rho)} \frac{Y_t}{N}, \text{ where } \frac{1}{p_t^{\nu}} \equiv \frac{1}{N}\sum_{n=1}^{N} \frac{1}{p_{n,t}^{\nu}},$$

with $\nu \equiv \rho/(1-\rho) \in \mathcal{R}_+$.[2]

2.2 Production sector setting

2.2.1 Cost minimization

We assume that good n is produce according to a standard Cobb-Douglas production function which takes capital and labour as inputs

$$q_{n,t} = \theta_{n,t} K_{n,t}^{\gamma} L_{n,t}^{1-\gamma}, \qquad \gamma \in (0,1)$$

where, for firm n, $\theta_{n,t}$ is the technical efficiency and $K_{n,t}$, $L_{n,t}$ are the inputs of labour and capital used up by the firm. In contrast to Devereux et al. (1993, 1996), who assumed that all firms use identical technologies, we assume that each firm n is characterised by its own technology level $\theta_{n,t}$. Individual productivities follow some autoregressive process to be specified in the next section. An example of such a process would be a generalization of the geometric AR(1) process in Devereux et al. (1993, 1996)

$$\log \theta_{n,t} = \alpha_n \log \theta_{n,t-1} + \epsilon_{n,t}$$

where the shocks $\epsilon_{n,t}$ can incorporate a firm-specific component in addition to the usual economy-wide component, and can be made to have a nonzero mean. For more details, see Section 3.

The cost minimization problem of every firm is

$$\min_{K_{n,t}, L_{n,t}} \quad w_t L_{n,t} + i_t K_{n,t}, \quad \text{subject to } \theta_{n,t} K_{n,t}^{\gamma} L_{n,t}^{1-\gamma} = q_{n,t},$$

where w_t and i_t are the wage rate and rental rate, respectively. Capital fully depreciates in each period. Cost minimization implies that every firm will chose

[2]The implication of the CES formula is that the aggregate price (p_t^{ν}) has to be calculated as a harmonic mean of the individual prices ($p_{n,t}^{\nu}$) rather than as the more traditional arithmetic mean. Suppose there was a sizable variance across prices in one single period. A price index based on an arithmetic mean would be dominated by the higher prices in that period whereas a price index based on a harmonic mean is dominated by the low prices. Hence, the harmonic mean formula simply reflects the fact that firms which charge higher prices have a smaller market share and contribute less to the aggregate price level than firms which charge a lower price.

capital and labour to equalise its capital-labour ratio, k_t, to the common optimal factor share ratio

$$k_t \equiv \frac{K_{n,t}}{L_{n,t}} = \frac{\gamma}{1-\gamma} \frac{w_t}{i_t}. \tag{3}$$

Since the derived demand for labour and capital by an intermediate input firm producing $q_{n,t}$ is

$$K_{n,t} = \frac{k_t^{1-\gamma} q_{n,t}}{\theta_{n,t}} \quad \text{and} \quad L_{n,t} = \frac{q_{n,t}}{\theta_{n,t} k_t^{\gamma}}, \tag{4}$$

then the firm's unit-cost $z_{n,t}$ is

$$z_{n,t} = \frac{w_t}{(1-\gamma)\, \theta_{n,t}\, k_t^{\gamma}}. \tag{5}$$

2.2.2 Pricing

The productive process implies that, given aggregate demand Y_t and the unit cost given by (5), firm n will chose a price which solves

$$\max_{p_{n,t}, q_{n,t}} p_{n,t}\, q_{n,t} - z_{n,t}\, q_{n,t}, \quad \text{s.t. } q_{n,t} = \left[\frac{p_t}{p_{n,t}}\right]^{1/(1-\rho)} \frac{Y_t}{N}.$$

The optimal price charged by the firm is

$$p_{n,t} = \frac{z_{n,t}}{\rho}$$

which implies a constant mark-up of $(1-\rho)/\rho = \nu^{-1}$. This mark-up is a measure of the monopoly power of the firm. Hence, from (2),

$$\begin{aligned} p_{n,t} &= \frac{w_t}{(1-\gamma)\rho\, k_t^{\gamma}} \frac{1}{\theta_{n,t}}, \\ p_t &= \frac{w_t}{(1-\gamma)\rho\, k_t^{\gamma}} \frac{1}{\theta_t}; \\ \text{where } \theta_t^{\nu} &\equiv \frac{1}{N} \sum_{n=1}^{N} \theta_{n,t}^{\nu}. \end{aligned} \tag{6}$$

Defining the real wage and the real rental as

$$\widetilde{w}_t = \frac{w_t}{p_t} \quad \text{and} \quad \widetilde{i}_t = \frac{i_t}{p_t},$$

we have, from (3),

$$\widetilde{w}_t = (1-\gamma)\rho\, \theta_t\, k_t^{\gamma} \quad \text{and} \quad \widetilde{i}_t = \frac{\gamma\, \rho\, \theta_t}{k_t^{1-\gamma}}. \tag{7}$$

The demand for the intermediate input is, from (2),

$$q_{n,t} = \left[\frac{\theta_{n,t}}{\theta_t}\right]^{1/(1-\rho)} \frac{Y_t}{N}. \tag{8}$$

By (4) and (8), the individual demands for labour are

$$L_{n,t} = \frac{\theta^{\nu}_{n,t}}{\theta_t^{1/(1-\rho)}} \frac{1}{k_t^{\gamma}} \frac{Y_t}{N} \Longrightarrow L_t \equiv \sum_{n=1}^{N} L_{n,t} = \frac{Y_t}{\theta_t\, k_t^{\gamma}} \Longleftrightarrow Y_t = \theta_t\, k_t^{\gamma}\, L_t. \tag{9}$$

The supply of labour is inelastic and equal to L in each period. Because capital depreciates fully in each period, the supply of capital is given by the savings of the previous period

$$L_t = L \text{ and } K_t = s_{t-1}\, Y_{t-1}$$

where K_t is both the aggregate investment in period $t-1$ and the aggregate stock of capital in period t, and s_{t-1} is the savings rate in period $t-1$. Hence, the capital-labour ratio is predetermined as

$$k_t = \frac{s_{t-1}\, Y_{t-1}}{L}. \tag{10}$$

2.3 Consumer optimization

There is a representative agent who inelastically supplies labour in quantity L, owns all the firms and all of the outstanding capital. The consumer's problem is

$$\max_{c_t} \mathcal{E}_0 \left[\sum_{t=0}^{\infty} \delta^t \log C_t\right], \text{ s.t. } A_{t+1} = \widetilde{i}_t A_t + \widetilde{w}_t L - C_t$$

where $\mathcal{E}_\tau\,[.]$ is the expectation operator taken with respect to the information available at time τ, A_τ are the assets in period τ consisting of the stock of capital K_τ and of firms' ownership. Firm's ownership is valuable as profits are strictly positive in each period. When all shocks are idiosyncratic, it is customary to assume that the number of firms is large enough to justify valuation by means of the present value of future expected profits. When there is a common shock, the value of a firm will be given by the stock market equilibrium. In any case, the assumption of a fixed number of firms implies that firms' ownership does not affect aggregate output, and the representative consumer specification enables us to compute the equilibrium path of GDP from the Euler equation without having to calculate share prices.

In each period t, the prevailing national income identity implies that the capital stock in the next period is predetermined, and (9) implies that

$$Y_t = \theta_t\, k_t^{\gamma} L = C_t + K_{t+1} \Longrightarrow K_{t+1} = \theta_t\, K_t^{\gamma}\, L^{1-\gamma} - C_t.$$

So, we have

$$\begin{aligned} y_t &= \theta_t\, k_t^\gamma \\ k_{t+1} &= \theta_t\, k_t^\gamma - c_t = s_t\, \theta_t\, k_t^\gamma \\ c_t &= (1 - s_t)\, \theta_t\, k_t^\gamma \end{aligned}$$

where $y_t \equiv Y_t/L$ and $c_t \equiv C_t/L$ are output per capita and consumption per capita, respectively. The Euler equation is

$$(11) \qquad \frac{1}{c_t} = \mathcal{E}_t \left[\frac{\delta\, \tilde{i}_{t+1}}{c_{t+1}} \right] \Longrightarrow \frac{1}{(1 - s_t)\, \theta_t\, k_t^\gamma} = \mathcal{E}_t \left[\frac{\delta\, \gamma\, \rho}{\theta_t\, (1 - s_{t+1})\, s_t\, k_t^\gamma} \right].$$

2.4 Dynamic equilibrium

From (11), an equilibrium associated with a deterministic savings rate exists which will be referred to as the *fundamental path* of the economy. It is given by

$$\frac{1}{\theta_t\, (1 - s_t)\, k_t^\gamma} = \frac{\delta\, \gamma\, \rho}{\theta_t\, (1 - s_{t+1})\, s_t\, k_t^\gamma} \Longrightarrow 1 - s_{t+1} = \delta\, \gamma \rho \frac{1 - s_t}{s_t}.$$

This last difference equation corresponds to the unstable dynamics

$$s_{t+1} - s_t = 1 + s^* - \frac{s^*}{s_t} - s_t, \text{ where } s^* \equiv \delta\, \gamma\, \rho,$$

so the unique savings rate associated with the fundamental path is constant over time

$$s_t = s^*, \qquad \forall t = 0, \dots, \infty.$$

The dynamic equations of the economy are

$$\begin{aligned} k_{t+1} &= s^*\, y_t \\ y_t &= \theta_t\, k_t^\gamma \\ c_t &= (1 - s^*)\, y_t \end{aligned}$$

which yields the dynamic equation for GDP per capita

$$(12) \qquad y_t = \theta_t\, (s^*)^\gamma\, y_{t-1}^\gamma.$$

The dynamic properties of the process $\{y_t\}$ will be determined by those of $\{\theta_t\}$, where θ_t is defined in (6). We now turn to the derivation of the implied time series properties of GNP.

3 The effect of heterogeneity on the Time Series properties

In AR models, memory decays exponentially or is infinite, but nothing in between. Macroeconomic processes require a more elaborate characterization, and we shall show that the economic model outlined in the previous section gives rise to a process which is very different from ARs.

Why are the published results in the time series literature on aggregation not applicable here? There are a number of complications that our economic model gives rise to:

1. The aggregation is not linear (arithmetic mean) or log-linear (geometric mean), either of which could be handled by current techniques. Here, we have a hybrid made of the arithmetic sum of geometric (not arithmetic) ARs. This results in a highly nonlinear process, for which linear models (e.g. ARIMA; see Abadir and Taylor (1999) for precise definitions) will not capture all the salient features.

2. Dependence between the various firms (hence AR components of the aggregate) complicates the setup, and the time series properties have not been worked out in the general case, except in Granger (1980) where the setup is linear.

3. Due to the new results of Abadir and Talmain (1998) on nonlinear transformations of time series, we are able to characterize our nonlinear process.

These complications are not avoided by non-CES aggregation. Any other method which sums geometric ARs (i.e. which sums the *levels* of technology shocks whose *logarithms* follow a linear process) will run into similar nonlinearities. This aggregation is essential in economics, where one often aggregates levels of variates which are strictly positive (hence not representable by ARs in levels which are unrestricted and can become negative).

Let the individual sequence $\{\theta_{n,t}\}_{t=1}^{T}$ be generated by the geometric AR process

$$\log\theta_{n,t} = \alpha_n \log\theta_{n,t-1} + \epsilon_{n,t}, \tag{13}$$

where $|\alpha_n| < 1$ and $\{\epsilon_{n,t}\} \sim \mathrm{IN}(\mu, \sigma^2)$ is a sequence of Independent Normal variates with mean μ and variance σ^2. We condition upon $\theta_{n,0} = 1$, since the model is unaffected by the value of $\theta_{n,0}$. To see this, rewrite the process as

$$\log\frac{\theta_{n,t}}{\theta_{n,0}} = \alpha_n \log\frac{\theta_{n,t-1}}{\theta_{n,0}} + \left((\alpha_n - 1)\log\theta_{n,0} + \epsilon_{n,t}\right),$$

where $\{\tilde{\epsilon}_{n,t}\} \equiv \{(\alpha_n - 1)\log\theta_{n,0} + \epsilon_{n,t}\} \sim \mathrm{IN}(\mu + (\alpha_n - 1)\log\theta_{n,0}, \sigma^2) \equiv \mathrm{IN}(\tilde{\mu}, \sigma^2)$, so that any other value of $\theta_{n,0}$ can be absorbed into redefined $\{\epsilon_{n,t}\}$ and μ without changing the structure of (13).

Let $\tau \in \mathcal{N} \cup \{0\}$ and $\nu \in \mathcal{R}_+$. Then, the autocorrelation function of $\{\theta^{\nu}_{n,t}\}$ for any given n is

$$(14) \qquad r_{t,t+\tau} \equiv \frac{v_{t,t+\tau}}{\sqrt{v_{t,t}v_{t+\tau,t+\tau}}} = \frac{\exp\left(\frac{\nu^2\sigma^2\alpha_n^{\tau}(1-\alpha_n^{2t})}{1-\alpha_n^2}\right) - 1}{\sqrt{\exp\left(\frac{\nu^2\sigma^2(1-\alpha_n^{2t})}{1-\alpha_n^2}\right) - 1}\sqrt{\exp\left(\frac{\nu^2\sigma^2\left(1-\alpha_n^{2(t+\tau)}\right)}{1-\alpha_n^2}\right) - 1}},$$

where $v_{t,t+\tau}$ is the autocovariance function of $\{\theta^{\nu}_{n,t}\}$; see the Appendix for details. For large t (or small α_n^t), this function behaves as

$$r_{t,t+\tau} \simeq \frac{\exp\left(\frac{\nu^2\sigma^2}{1-\alpha_n^2}\alpha_n^{\tau}\right) - 1}{\exp\left(\frac{\nu^2\sigma^2}{1-\alpha_n^2}\right) - 1}$$

which does not depend on t, because the process is *asymptotically* stationary.[3] At least three comments are in order for this function, comparing it to that of the well-known asymptotic autocorrelation function α_n^{τ} of $\{\log(\theta_{n,t})\}$ (i.e. the autocorrelation function of an arithmetic AR):

1. As $\tau \to \infty$, this function declines exponentially as α_n^{τ} (by expanding the exponential in the numerator).

2. More generally, a small-σ expansion (expanding exp[.] for $\sigma \to 0$) of the function will reproduce the asymptotic autocorrelation function α_n^{τ} of $\{\log(\theta_{n,t})\}$. The same is true for $\nu \to 0$. The general rule that gives rise to such results is derived in Abadir and Talmain (1998).

3. For non-trivial σ, the autocorrelation function depends on σ, which actually carries the same weight as ν, in contrast to the usual arithmetic AR where σ has no effect at all on autocorrelations. This will have important implications for the effect of σ on autocorrelation functions when analysing the nonlinear CES aggregation. It is a feature that does not arise in the traditional literature on aggregation in time series.

[3]The chosen definition of the autocorrelation function is preferred to the more common one $v_{t,t+\tau}/v_{t,t}$, which is less transparent for the nonstationary series to be dealt with here.

Now, recall that the aggregate θ_t is given by (6). Suppose that

$$X_t = \frac{1}{N}\sum_{n=1}^{N}\theta_{n,t}^{\nu}, \tag{15}$$

where $\{\theta_{n,t}\}_{t=1}^{T}$ are generated by

$$\log\theta_{n,t} = \alpha_n \log\theta_{n,t-1} + u_{n,t} + \beta_n e_t, \tag{16}$$

with

$$\begin{aligned} |\alpha_n| &< 1, \\ \{u_{n,t}\} &\sim \mathrm{IN}\left(\mu, \omega_n^2\right) \quad \text{and} \quad \mathcal{E}\left[u_{n,t}\left(u_{k,s}-\mu\right)\right] = 0,\ \forall k \neq n, \\ \{e_t\} &\sim \mathrm{IN}\left(0,1\right) \quad \text{and} \quad \mathcal{E}\left[u_{n,t}e_s\right] = 0. \end{aligned}$$

The u's are idiosyncratic shocks, whereas the e's are common shocks whose impact is transmitted to individual series via the scaling parameters given by the β's. By definition and without loss of generality, the two types of shocks are independent of each other. Furthermore, there is no loss of generality in assuming a unit variance for e_t, since any other shock $\tilde{\beta}_n\tilde{e}_t$ where $\{\tilde{e}_t\} \sim \mathrm{IN}(0,\psi^2)$ can be written as

$$\tilde{\beta}_n\tilde{e}_t = \tilde{\beta}_n\psi e_t \equiv \beta_n e_t. \tag{17}$$

It is also assumed that the common drift of the AR processes is $\mu \in \mathcal{R}_+$.

The underlying probability measure is defined over time (t) and space (n), and in the latter case, it is the one from which the individual parameters α_n, ω_n^2 and β_n are drawn. One may specify a joint density for α, ω^2 and β, but there is no reason to believe that they interact in a systematic way: α_n is independent of the errors, with the idiosyncratic and common errors being mutually independent by definition. It is therefore enough to specify the marginal densities of α, ω^2 and β. For simplicity, we will assume that $\alpha \in (0,1)$, $\xi \equiv \omega^2 \in \mathcal{R}_+$ and $\beta \in \mathcal{R}_+$ are continuous variates with density functions $f_\alpha(\alpha)$, $f_\xi(\xi)$ and $f_\zeta(\zeta)$, respectively. We will further assume that:

1. the variate $\alpha \in (0,1)$ is distributed according to the Beta density $f_\alpha(\alpha) = \alpha^{g_\alpha - 1}(1-\alpha)^{h_\alpha - 1}/\mathrm{B}(g_\alpha, h_\alpha)$;

2. the variate $\xi \equiv \omega^2 \in \mathcal{R}_+$ is distributed according to a special (the power in the exponential is 2) generalized Gamma density $f_\xi(\xi) = 2h_\omega^{g_\omega}\xi^{2g_\omega - 1}\exp\left[-h_\omega\xi^2\right]/\Gamma(g_\omega)$; and

3. the variate $\zeta \equiv \beta^2 \in \mathcal{R}_+$ is distributed according to a special (the power in the exponential is 2) generalized Gamma density $f_\zeta(\zeta) = 2h_\beta^{g_\beta}\zeta^{2g_\beta - 1}\exp\left[-h_\beta\zeta^2\right]/\Gamma(g_\beta)$;

where $\Gamma(.)$ is the Gamma (generalized factorial) function, $\mathrm{B}(g,h) \equiv \Gamma(g)\Gamma(h)/\Gamma(g+h)$ is the Beta function, the parameters $g_\bullet, h_\bullet$ are all positive and further:

- $h_\alpha \in (0,1]$, such that we exclude the unrealistic case of $h_\alpha > 1$ where AR roots close to unity are almost ruled out; and
- $g_\omega, g_\beta \in \left(\frac{1}{2}, \infty\right)$, such that we exclude the unrealistic case of $g_\omega, g_\beta \leq \frac{1}{2}$ where $\omega = 0$ and $\beta = 0$ are the most 'likely' values (mode of the density).

The first assumed density is typical in the literature on aggregation; e.g. see Granger (1980). The next two assumptions are required because of the relevance of the variance in our geometric AR setting (unlike in the literature on arithmetic AR's). They are reasonable, because one would expect the average variance of idiosyncratic shocks (ω_n^2) and/or of the amplification of common shocks (β_n^2) to be finite, and that the likelihood of survival (existence) of firms to decline rapidly (e.g. exponentially) as the size of the risk they are exposed to becomes larger; both after a possible mode of the density near (but not at) zero. In a model of economic equilibrium, μ should be fixed over n, as assumed. Otherwise the firm with the largest μ_n would grow to the extent of taking over the economy.

To illustrate that the Generalized Gamma is not an unreasonable assumption, we include Figures 1 and 2 where we plot $f_\xi(\xi)$ and $f_\zeta(\zeta)$ for some parameters values and compare them to the histograms for the published data. The source is the Risk Measurement Services of LBS (1988), where a listing of the 2,150 UK corporations' beta and specific risk for 1988 is found. The standard deviation of the idiosyncratic risk is simply the product of the published specific risk and of the standard deviation of the common risk.

The way we have defined β_n in (16)-(17) shows that the variance ψ^2 of the economy-wide shock need not be forced to take unrealistically large levels in practice. Representative-firm models force $\beta_n = \beta$ for all firms n (i.e. no variance), so that the variance ψ has to do all the work of accounting for the variance in βe_t (or $\tilde{\beta}\tilde{e}_t$), while in reality the existing heterogeneity of firms will already contribute to the variance of our factor $\beta_n e_t$ (or $\tilde{\beta}_n \tilde{e}_t$).[4]

Now, the question of interest is whether the autocorrelation function of the aggregate series $\{X_t\} \equiv \{N^{-1}\sum_{n=1}^{N} \theta_{n,t}^{\nu}\}$ decays more slowly than the rate in (14) belonging to any of its typical components $\theta_{n,t}^{\nu}$, and by how much. Here, we depart from the proofs used in the literature on the aggregation of time series. The standard approach has so far been to derive the spectrum of $\left\{\theta_{n,t}^{\nu}\right\}$ for the special case $\nu = 1$, then approximating the spectrum of the aggregate series, from which one finally approximates its autocorrelation function. Instead, we take the

[4]It can be shown that the variance of β_n is inversely proportional to $\sqrt{h_\beta}$, namely $\sqrt{h_\beta}\mathrm{Var}(\beta_n) = \left(\Gamma\left(g_\beta + \frac{1}{2}\right)/\Gamma(g_\beta)\right) - \left(\Gamma\left(g_\beta + \frac{1}{4}\right)/\Gamma(g_\beta)\right)^2$.

more direct approach of deriving the autocorrelation function of $\{X_t\}$ from those of $\{\theta^{\nu}_{n,t}\}$, and this for any $\nu \in \mathcal{R}_+$.[5]

Theorem 1 *The autocovariance function of* $\{X_t\}$ *is*

$$\begin{aligned}
V_{t,t+\tau} &\equiv \mathcal{E}\left[X_t X_{t+\tau}\right] - \mathcal{E}\left[X_t\right]\mathcal{E}\left[X_{t+\tau}\right] \\
&\simeq \frac{16\pi^2}{\left((t-1)(t+\tau-1)\nu^2\mu^2 t(t+\tau)\right)^{h_\alpha}}\left(\frac{\Gamma(g_\alpha+h_\alpha)}{\Gamma(g_\alpha)\Gamma(g_\omega)\Gamma(g_\beta)}\right)^2 \\
&\quad \left(\frac{\nu^4 t(t+\tau)}{16h_\omega}\right)^{2g_\omega-1}\left(\frac{\nu^4 t(t+\tau)}{16h_\beta}\right)^{2g_\beta-1} \\
&\quad \exp\left(\nu\mu(2t+\tau)+\frac{\nu^4}{16}\left(t^2+(t+\tau)^2\right)\left(\frac{1}{h_\omega}+\frac{1}{h_\beta}\right)\right)\left(\exp\left(\frac{\nu^4 t\sqrt{t(t+\tau)}}{4h_\beta}\right)-1\right),
\end{aligned}$$

and its corresponding autocorrelation function is

$$R_{t,t+\tau} \equiv \frac{V_{t,t+\tau}}{\sqrt{V_{t,t}V_{t+\tau,t+\tau}}} \simeq \frac{\exp\left(\frac{\nu^4 t\sqrt{t(t+\tau)}}{4h_\beta}\right)-1}{\sqrt{\exp\left(\frac{\nu^4 t^2}{4h_\beta}\right)-1}\sqrt{\exp\left(\frac{\nu^4 (t+\tau)^2}{4h_\beta}\right)-1}}.$$

The approximations that are reported in this theorem are known as leading-term approximations. They give the dominant term in the expansions of functions. For more details on such issues, see Abadir (1995). For the other terms, it is possible to obtain infinite series from the Appendix, but these do not affect the analysis to follow in the body of this paper. By a standard lag-polynomial factorization in time series analysis, the leading-term approximations given in the previous section are unaffected if the microeconomic processes followed by the technology shocks are higher-order stable ARMA processes, as long as the absolute value of the largest AR root is not too close to the next largest one.

It is possible to use spectral analysis, which is in a one-to-one relation with autocovariances, subject to extra caution since the spectra of nonstationary series are not time-invariant. Here, we simply infer the amplitude of cycles from the autocovariance function, and the frequency of cycles from the autocorrelation function. Cycles arise because some of the shocks (ϵ's) are negative, while others are positive.

Recalling that $X_t \equiv \theta^{\nu}_t$, with $\nu \equiv \rho/(1-\rho) \in \mathcal{R}_+$ the inverse of mark-up (monopoly power), we have the following remarks on the theorem:

[5]We needed a general result for any such ν. The cost of such a generalization is that we assumed Normality of $\epsilon_{n,t}$ (but not of the aggregate shock) as opposed to a general distribution with finite first four moments (which is required for the standard analysis of spectral estimation to go through).

1. Clearly, the (autoco)variance of the series changes over time, as the long-memory process that we have for $\{X_t\}$ is not strictly stationary. One can only talk about 'the' (autoco)variance of the series with reference to some point in time. The remarks to follow will either presume conditioning on a fixed point in time, or tracing the evolution as time passes.

2. What are the effects of μ, ν, $g_\bullet$ and $h_\bullet$ on the amplitude of the series? Here, μ affects the leading term of the autocovariance in an exponential manner. This means that economies which are growing faster than others will experience an amplification of their business cycles. The parameters ν^{-1} (mark-up, or degree of monopoly power), h_β^{-1} (extent of common shock and heterogeneity of firms) and h_ω^{-1} (extent of idiosyncratic shock) have an important influence. The effect of h_β^{-1} and h_ω^{-1} on the diffusion (amplification) of common shocks is similar to ν^4. Finally, of the two shocks in our model, the common shock is the more potent, as is seen from the last exponential term of $V_{t,t+\tau}$ in the theorem. The spread of the two shocks is also positively related to g_β and g_ω, with both increasing the autocovariance of the series albeit in a weaker way than the scaling parameters h_β^{-1} and h_ω^{-1}. Notice that the parameters of the distribution of α_n have little relative impact, so long as roots close to unity are not excluded (i.e. given our earlier assumption of $h_\alpha \in (0,1]$), with the impact of roots close to zero (i.e. g_α) being virtually non-existent.

3. What are the effects of μ, ν, $g_\bullet$ and $h_\bullet$ on the memory of the series? Surprisingly, μ does not affect the leading term of the autocorrelation, whereas ν^{-1} (mark-up, or degree of monopoly power) and h_β^{-1} (extent of common shock and heterogeneity of firms) have an important influence. The effect of h_β^{-1} on the persistence of common shocks is similar to ν^4. Other parameters have a lesser effect on memory, if at all.

4. We have talked about the effect of parameters in our setting of heterogeneous firms. Now we need to compare our theorem's result with representative-firm models. When the latter is hit by stable technology shocks, we can use (14). When it faces a geometric random walk (unit root)

$$\log x_t = \log x_{t-1} + \epsilon_t,$$

the procedure leading to (14) and given in the first part of the Appendix would yield

$$\begin{aligned} v_{t,t+\tau}^{(1)} &= \exp\left[\nu\left(\mu + \frac{\nu\sigma^2}{2}\right)(2t+\tau)\right]\left(\exp\left[\nu^2\sigma^2 t\right] - 1\right), \\ r_{t,t+\tau}^{(1)} &= \sqrt{\frac{\exp\left[\nu^2\sigma^2 t\right] - 1}{\exp\left[\nu^2\sigma^2(t+\tau)\right] - 1}}, \end{aligned}$$

where the superscript $\bullet^{(1)}_{\bullet}$ refers to the unit root case. Note the linearity of the exponentials in ν^2 and t, and the absence of a $t+\tau$ term in the numerator's exponential. Exponential terms appear, because we are dealing with a geometric random walk instead of an arithmetic one. We now compare our theorem to representative-firm models.

5. As $t \to \infty$ or $\nu \to \infty$, the binomial expansion gives

$$\sqrt{1+\frac{\tau}{t}} \simeq 1+\frac{\tau}{2t}-\frac{\tau^2}{4t^2},$$

so that

$$\begin{aligned} R_{t,t+\tau} &\simeq \exp\left[-\frac{\nu^4}{8h_\beta}\left(\frac{3\tau}{2}+t\right)\tau\right] \\ r^{(1)}_{t,t+\tau} &\simeq \exp\left[-\frac{\nu^2\sigma^2}{2}\tau\right], \end{aligned}$$

where the decay rate of our theorem's $R_{t,t+\tau}$ is faster than the one for the unit-root process, $r^{(1)}_{t,t+\tau}$, as either t, τ or ν increase.[6]

6. As $\tau \to \infty$, one may analyse the memory features of the process as we consider points that are further apart in time, and we have

$$\begin{aligned} R_{t,t+\tau} &\simeq \left(\exp\left[\frac{\nu^4 t^2}{4h_\beta}\right]-1\right)^{-\frac{1}{2}} \exp\left[-\frac{\nu^4}{8h_\beta}\left((t+\tau)^2-2t\sqrt{t\,(t+\tau)}\right)\right] \\ r^{(1)}_{t,t+\tau} &\simeq \sqrt{1-\exp\left[-\nu^2\sigma^2 t\right]}\exp\left[-\frac{\nu^2\sigma^2}{2}\tau\right] \end{aligned}$$

where the decay rate of $R_{t,t+\tau}$ in terms of τ is again faster than for $r^{(1)}_{t,t+\tau}$.

7. As $\nu \to 0$, the rate of decay of the memory of $\{X_t\}$ is slower than the corresponding α_n^τ of the stable ($|\alpha_n| < 1$) AR of (14), whatever measure of autocorrelation is adopted, as we have

$$\begin{aligned} R_{t,t+\tau} &\simeq \sqrt{\frac{t}{t+\tau}} \\ r^{(1)}_{t,t+\tau} &\simeq \sqrt{\frac{t}{t+\tau}}. \end{aligned}$$

[6]Direct comparison of the nonstationary's $R_{t,t+\tau}$ with the stationary's $r_{t,t+\tau}$ of (14) for extreme values requires writing $|\alpha|^\tau = \exp[\tau \log|\alpha|]$ then doing some manipulations. Long-memory processes like $\{X_t\} \equiv \{N^{-1}\sum_{n=1}^N \theta^\nu_{n,t}\}$ will have more memory than the stable components $\{\theta^\nu_{n,t}\}$ anyway, and looking at autocovariances (amplitudes) or $V_{t,t+\tau}/V_{t,t}$ may be more instructive when comparing across the stationarity divide.

Notice that we recover the behaviour of the arithmetic random walk in *both* cases, even though there are no unit roots in any of the components of the aggregate X_t. The resulting effect is that our model will generate *seemingly* random-walk behaviour when $\nu \to 0$, though the model has generally less memory than a random walk. This may help interpret the findings of a near-unit-root in some macroeconomic series when (log-)linear models are fitted.

8. The parameter $\nu \equiv \rho/(1-\rho) \in \mathcal{R}_+$ is the inverse of the mark-up, and is inversely related to monopoly power. The case $\nu \to 0$ where monopoly power increases leads to the aggregate series having very long memory, to the point of being confounded with a unit root (Remark 7). This contrasts with perfect competition where $\nu \to \infty$ (Remark 5) leading to less memory in the aggregate series, and a clear distinction from the permanent memory of a unit root. Unlike (log-)linear models, our nonlinear model can generate both types of behaviour, depending on ν, and provides a rich framework for analysing economies with different characteristics.

We now need a result linking the memory of $\{X_t\} \equiv \{N^{-1}\sum_{n=1}^{N}\theta_{n,t}^{\nu}\}$ to the required one for the GDP $\{y_t\}$, by means of $X_t \equiv \theta_t^{\nu}$ and (12).

Corollary 2 *The leading terms of the autocorrelation functions of* $\{\log X_t\}$ *and* $\{\log y_t\}$ *coincide, when* $|\gamma| < 1$.

The Cobb-Douglas parameter γ has almost no impact on the memory of y_t, as it is swamped by the long-memory impact of aggregation.

We have given the autocorrelation function of $\{X_t\}$ in our theorem. For large t or τ, it is possible to obtain an explicit expression for the autocorrelation of $\{\log X_t\}$. This is done by the method in Abadir and Talmain (1998). Here, it implies doing a large-h_β, h_ω (i.e. small-variance) expansion, which downplays higher order terms of the exponential ($\log^{-1}$) expansion. Applied to either of the preceding remarks 5 or 6, the result is slow decay of memory. The main interest (for a given ν) is in the formula of Remark 6, which yields the memory features of the process as we consider points that are further apart in time; i.e. large τ. For $\{\log y_t\}$, we get

$$R_{t,t+\tau} \simeq \frac{2\sqrt{h_\beta}}{\nu^2 t\left(1+\frac{\tau^2}{h_\beta}\right)^{\frac{\nu^4}{8}}} \tag{18}$$

which shows that memory increases with large h_β (small sectoral heterogeneity, i.e. little diversity) and small ν (large monopoly power). In our setting, the effect of h_α is not as powerful as in the available time series models of linear aggregation of arithmetic ARs. CES aggregation downplays the effect of h_α. In contrast, the effects of ν and h_β are much more important.

4 Concluding remarks

Our findings can be summarized by the following properties:

1. Unconventional results that are reminiscent of functional limit theory in statistics: the autocorrelation of sums of variates will be greater than the autocorrelation of the components, thus producing long memory. Intuitively, this arises because of the increased likelihood of correlation of one of the components with another component at a later date.

2. The economic model gives rise to long-memory, and to a nonlinear process. A single-sector model with one AR will have exponential decay of memory and will not give rise to long memory (except for a unit root, which leads to infinite memory). CES aggregation means that linear ARIMA macromodels will not pick up the nonlinearity. We have shown how the autocorrelation function of log-linear models is much less affected by the variance of the shocks than our nonlinear model. This means that the usual assumption of linearity is not innocuous, not even as a first-order approximation.

3. Persistence, endogenous cycles and overreaction: small temporary shocks in our model lead to long memory, without requiring unit roots. There is no need for large shocks at the microeconomic level in order to generate large macroeconomic fluctuations. This is the first magnification effect of aggregation. Both the common and idiosyncratic shocks matter and are amplified at the macro level, though the former is more potent than the latter. Their size is positively related to the amplitude of the aggregate shock, but negatively related to the aggregate memory (temporary shocks are more easily reversed). The decay of memory is however much slower than the usual exponential rate for stable ARs.

4. The second magnification effect of aggregation is that a larger growth rate causes some overreaction of the economy: economies which are growing faster than others will experience an amplification of their business cycles.

5. The third magnification effect of aggregation. The effect of a higher degree of mark-up (monopolistic power) in the economy is longer memory, but the amplitude of the cycle is reduced. With higher monopoly power, firms do not adjust their output so much after shocks, so there are less fluctuations but more persistence. As monopoly power increases, our nonlinear model will generate behaviour that *seems* increasingly like a random walk, even though there are no unit roots in any of the components of the aggregate, and though the model has generally less memory than a random walk. By extension, in a multi-sectoral model, longer memory (monopolistic) sectors would dominate the long-term time series properties. This is in line with the findings of Blundell, Griffith and Van Reenen (1993).

6. There is eventual mean-reversion in the cycles generated by our model, unlike in infinite-memory unit-root models. The length of the cycle is random, and the process is not periodic.

7. Mean-reversion, coupled with long memory, implies that an economy can get 'stuck' in a mode for a while, and have asymmetric business cycles for that duration (e.g. post-war expansion, recent recessions). Mean-reversion acts as an attractor: there is a slow tendency to the long-term trend, with occasional bursts away from it.

Our paper has characterized the time series process that arises from the economic model, and has explored its properties and implications. It has however not derived the optimal (if any) estimation and inference procedures for dealing with it, as this goes beyond the scope of this paper. One possible estimation procedure would be as follows. Normalize the scaling factor in (18) such that $R_{t,t} = 1$, and estimate the autocorrelation function $(1+\tau^2/h_\beta)^{-\nu^4/8}$ from the data by (say) nonlinear least squares, where the parameters ν and h_β can be identified. One can then infer ψ and the remaining parameters by the method illustrated before our theorem.

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Appendix

Derivation of (14). The variate $x_t \equiv \theta_{n,t}$ can be written as

$$x_t = \exp\left[\sum_{j=0}^{t-1} \alpha^j \epsilon_{t-j}\right],$$

where we use $\alpha \equiv \alpha_n$ and $\epsilon_t \equiv \epsilon_{n,t}$ as shorthand in the first proof of this Appendix. Then,

$$\mathcal{E}\left[(x_t x_{t+\tau})^\nu\right] = \mathcal{E}\left[\exp\left[\nu\left((1+\alpha^\tau)\sum_{j=0}^{t-1}\alpha^j\epsilon_{t-j} + \sum_{j=0}^{\tau-1}\alpha^j\epsilon_{t+\tau-j}\right)\right]\right],$$

where empty sums are taken to be zero, by convention. By independence of the sequence $\{\epsilon_t\}$, and by the moment generating function (MGF) of the distribution $\mathrm{N}(\mu,\sigma^2)$

$$\mathcal{E}\left[\exp\left[b\epsilon_t\right]\right] = \exp\left[b\mu + \frac{b^2\sigma^2}{2}\right],$$

we get

$$
\begin{aligned}
& \mathcal{E}\left[(x_t x_{t+\tau})^\nu\right] \\
= & \exp\left[\nu\mu\left((1+\alpha^\tau)\sum_{j=0}^{t-1}\alpha^j+\sum_{j=0}^{\tau-1}\alpha^j\right)+\frac{\nu^2\sigma^2}{2}\left((1+\alpha^\tau)^2\sum_{j=0}^{t-1}\alpha^{2j}+\sum_{j=0}^{\tau-1}\alpha^{2j}\right)\right] \\
= & \exp\left[\nu\mu\frac{(1+\alpha^\tau)(1-\alpha^t)+1-\alpha^\tau}{1-\alpha}+\frac{\nu^2\sigma^2}{2}\frac{(1+\alpha^\tau)^2(1-\alpha^{2t})+1-\alpha^{2\tau}}{1-\alpha^2}\right] \\
= & \exp\left[\frac{\nu\mu(2-\alpha^t(1+\alpha^\tau))}{1-\alpha}+\frac{\nu^2\sigma^2(1+\alpha^\tau)(2-\alpha^{2t}(1+\alpha^\tau))}{2(1-\alpha^2)}\right].
\end{aligned}
$$

Finally, by a similar method,

$$
\begin{aligned}
\mathcal{E}\left[x_t^\nu\right] &= \mathcal{E}\left[\exp\left[\nu\sum_{j=0}^{t-1}\alpha^j\epsilon_{t-j}\right]\right] \\
&= \exp\left[\nu\mu\sum_{j=0}^{t-1}\alpha^j+\frac{\nu^2\sigma^2}{2}\sum_{j=0}^{t-1}\alpha^{2j}\right] \\
&= \exp\left[\frac{\nu\mu(1-\alpha^t)}{1-\alpha}+\frac{\nu^2\sigma^2(1-\alpha^{2t})}{2(1-\alpha^2)}\right],
\end{aligned}
$$

so that the autocovariance function of x_t^ν is

$$
\begin{aligned}
v_{t,t+\tau} &\equiv \mathcal{E}\left[(x_t x_{t+\tau})^\nu\right]-\mathcal{E}\left[x_t^\nu\right]\mathcal{E}\left[x_{t+\tau}^\nu\right] \\
&= \exp\left[\frac{\nu\mu(2-\alpha^t(1+\alpha^\tau))}{1-\alpha}+\frac{\nu^2\sigma^2(1+\alpha^\tau)(2-\alpha^{2t}(1+\alpha^\tau))}{2(1-\alpha^2)}\right] \\
&\quad -\exp\left[\frac{\nu\mu(2-\alpha^t(1+\alpha^\tau))}{1-\alpha}+\frac{\nu^2\sigma^2(2-\alpha^{2t}(1+\alpha^{2\tau}))}{2(1-\alpha^2)}\right] \\
&= \exp\left[\frac{\nu\mu(2-\alpha^t(1+\alpha^\tau))}{1-\alpha}+\frac{\nu^2\sigma^2(2-\alpha^{2t}(1+\alpha^{2\tau}))}{2(1-\alpha^2)}\right] \\
&\quad \times\left(\exp\left[\frac{\nu^2\sigma^2\alpha^\tau(1-\alpha^{2t})}{1-\alpha^2}\right]-1\right)
\end{aligned}
$$

and its autocorrelation function is as stated in (14). Q.E.D.

Proof of Theorem 1. It can be seen that the results displayed earlier in this Appendix go through for any $\theta_{n,t}$, with the parameter equivalencies

$$
\begin{aligned}
\alpha &\leftrightarrow \alpha_n, \\
\{\epsilon_t\} &\leftrightarrow \{u_{n,t}+\beta_n e_t\}, \\
\sigma^2 &\leftrightarrow \omega_n^2+\beta_n^2,
\end{aligned}
$$

so that

$$
\begin{aligned}
\theta_{n,t} &= \exp\left[\sum_{j=0}^{t-1} \alpha_n^j \left(u_{n,t-j} + \beta_n e_{t-j}\right)\right], \\
\mathcal{E}\left[(\theta_{n,t}\theta_{n,t+\tau})^\nu\right] &= \exp\left[\frac{\nu\mu\left(2-\alpha_n^t\left(1+\alpha_n^\tau\right)\right)}{1-\alpha_n} + \frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1+\alpha_n^\tau\right)\left(2-\alpha_n^{2t}\left(1+\alpha_n^\tau\right)\right)}{2\left(1-\alpha_n^2\right)}\right], \\
\mathcal{E}\left[\theta_{n,t}^\nu\right] &= \exp\left[\frac{\nu\mu\left(1-\alpha_n^t\right)}{1-\alpha_n} + \frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1-\alpha_n^{2t}\right)}{2\left(1-\alpha_n^2\right)}\right].
\end{aligned}
$$

The penultimate expression is readily generalizable to

$$
\begin{aligned}
\mathcal{E}\left[(\theta_{n,t}\theta_{k,t+\tau})^\nu\right] &= \mathcal{E}\left[\exp\left[\nu\sum_{j=0}^{t-1}\alpha_n^j\left(u_{n,t-j}+\beta_n e_{t-j}\right) + \nu\sum_{j=0}^{t+\tau-1}\alpha_k^j\left(u_{k,t+\tau-j}+\beta_k e_{t+\tau-j}\right)\right]\right] \\
&= \mathcal{E}\left[\exp\left[\nu\sum_{j=0}^{t-1}\alpha_n^j u_{n,t-j} + \nu\sum_{j=0}^{t+\tau-1}\alpha_k^j u_{k,t+\tau-j}\right.\right. \\
&\qquad \left.\left. +\nu\sum_{j=0}^{t-1}\left(\beta_n\alpha_n^j+\beta_k\alpha_k^{j+\tau}\right)e_{t-j} + \nu\beta_k\sum_{j=0}^{\tau-1}\alpha_k^j e_{t+\tau-j}\right]\right] \\
&= \exp\left[\nu\mu\left(\sum_{j=0}^{t-1}\alpha_n^j + \sum_{j=0}^{t+\tau-1}\alpha_k^j\right) + \frac{\nu^2\omega_n^2}{2}\sum_{j=0}^{t-1}\alpha_n^{2j} + \frac{\nu^2\omega_k^2}{2}\sum_{j=0}^{t+\tau-1}\alpha_k^{2j}\right. \\
&\qquad \left. +\frac{\nu^2}{2}\sum_{j=0}^{t-1}\left(\beta_n\alpha_n^j+\beta_k\alpha_k^{j+\tau}\right)^2 + \frac{\nu^2\beta_k^2}{2}\sum_{j=0}^{\tau-1}\alpha_k^{2j}\right] \\
&= \exp\left[\nu\mu\left(\frac{1-\alpha_n^t}{1-\alpha_n}+\frac{1-\alpha_k^{t+\tau}}{1-\alpha_k}\right) + \frac{\nu^2\omega_n^2\left(1-\alpha_n^{2t}\right)}{2\left(1-\alpha_n^2\right)} + \frac{\nu^2\omega_k^2\left(1-\alpha_k^{2t+2\tau}\right)}{2\left(1-\alpha_k^2\right)}\right. \\
&\qquad +\frac{\nu^2}{2}\left(\frac{\beta_n^2\left(1-\alpha_n^{2t}\right)}{1-\alpha_n^2} + \frac{\beta_k^2\alpha_k^{2\tau}\left(1-\alpha_k^{2t}\right)}{1-\alpha_k^2} + \frac{2\beta_n\beta_k\alpha_k^\tau\left(1-\alpha_n^t\alpha_k^t\right)}{1-\alpha_n\alpha_k}\right) \\
&\qquad \left. +\frac{\nu^2\beta_k^2\left(1-\alpha_k^{2\tau}\right)}{2\left(1-\alpha_k^2\right)}\right] \\
&= \exp\left[\nu\mu\left(\frac{1-\alpha_n^t}{1-\alpha_n}+\frac{1-\alpha_k^{t+\tau}}{1-\alpha_k}\right) + \frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1-\alpha_n^{2t}\right)}{2\left(1-\alpha_n^2\right)}\right. \\
&\qquad \left. +\frac{\nu^2\left(\omega_k^2+\beta_k^2\right)\left(1-\alpha_k^{2t+2\tau}\right)}{2\left(1-\alpha_k^2\right)} + \frac{\nu^2\beta_n\beta_k\alpha_k^\tau\left(1-\alpha_n^t\alpha_k^t\right)}{1-\alpha_n\alpha_k}\right],
\end{aligned}
$$

when $k \neq n$.

For the autocovariance function of $\{X_t\}$, we need

$$\begin{aligned}
&\mathcal{E}\left[X_t X_{t+\tau}\right] \\
= \;& \mathcal{E}\left[\left(\frac{1}{N}\sum_{n=1}^{N}\theta_{n,t}^{\nu}\right)\left(\frac{1}{N}\sum_{n=1}^{N}\theta_{n,t+\tau}^{\nu}\right)\right] \\
= \;& \frac{1}{N^2}\sum_{n=1}^{N}\mathcal{E}\left[\theta_{n,t}^{\nu}\theta_{n,t+\tau}^{\nu}+\sum_{k\neq n}\theta_{n,t}^{\nu}\theta_{k,t+\tau}^{\nu}\right] \\
= \;& \frac{1}{N^2}\sum_{n=1}^{N}\left(\exp\left[\frac{\nu\mu\left(2-\alpha_n^t\left(1+\alpha_n^\tau\right)\right)}{1-\alpha_n}+\frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1+\alpha_n^\tau\right)\left(2-\alpha_n^{2t}\left(1+\alpha_n^\tau\right)\right)}{2\left(1-\alpha_n^2\right)}\right]\right. \\
& +\sum_{k\neq n}\exp\left[\nu\mu\left(\frac{1-\alpha_n^t}{1-\alpha_n}+\frac{1-\alpha_k^{t+\tau}}{1-\alpha_k}\right)+\frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1-\alpha_n^{2t}\right)}{2\left(1-\alpha_n^2\right)}\right. \\
& \left.\left.+\frac{\nu^2\left(\omega_k^2+\beta_k^2\right)\left(1-\alpha_k^{2t+2\tau}\right)}{2\left(1-\alpha_k^2\right)}+\frac{\nu^2\beta_n\beta_k\alpha_k^\tau\left(1-\alpha_n^t\alpha_k^t\right)}{1-\alpha_n\alpha_k}\right]\right).
\end{aligned}$$

We also require

$$\begin{aligned}
\mathcal{E}\left[X_t\right] &= \frac{1}{N}\sum_{n=1}^{N}\mathcal{E}\left[\theta_{n,t}^{\nu}\right] \\
&= \frac{1}{N}\sum_{n=1}^{N}\exp\left[\frac{\nu\mu\left(1-\alpha_n^t\right)}{1-\alpha_n}+\frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1-\alpha_n^{2t}\right)}{2\left(1-\alpha_n^2\right)}\right]
\end{aligned}$$

for the autocovariance function of $\{X_t\}$ to be derived as

$$V_{t,t+\tau}\equiv\mathcal{E}\left[X_tX_{t+\tau}\right]-\mathcal{E}\left[X_t\right]\mathcal{E}\left[X_{t+\tau}\right].$$

For large N, the operators

$$\frac{1}{N}\sum_{n=1}^{N}\simeq\mathcal{E}_n$$

are exchangeable (by the Law of Large Numbers), with $\mathcal{E}_n$ denoting the expectation taken with respect to the distribution of parameters of the individual series, and

$$
\begin{aligned}
(19) \qquad & V_{t,t+\tau} \\
\simeq \ & \frac{1}{N}\mathcal{E}_n \exp\left[\frac{\nu\mu\left(2-\alpha_n^t\left(1+\alpha_n^\tau\right)\right)}{1-\alpha_n}\right. \\
& \left. + \frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1+\alpha_n^\tau\right)\left(2-\alpha_n^{2t}\left(1+\alpha_n^\tau\right)\right)}{2\left(1-\alpha_n^2\right)}\right] \\
& + \mathcal{E}_n\mathcal{E}_k \exp\left[\nu\mu\left(\frac{1-\alpha_n^t}{1-\alpha_n}+\frac{1-\alpha_k^{t+\tau}}{1-\alpha_k}\right)+\frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1-\alpha_n^{2t}\right)}{2\left(1-\alpha_n^2\right)}\right. \\
& \left. + \frac{\nu^2\left(\omega_k^2+\beta_k^2\right)\left(1-\alpha_k^{2t+2\tau}\right)}{2\left(1-\alpha_k^2\right)} + \frac{\nu^2\beta_n\beta_k\alpha_k^\tau\left(1-\alpha_n^t\alpha_k^t\right)}{1-\alpha_n\alpha_k}\right] \\
& - \mathcal{E}_n \exp\left[\frac{\nu\mu\left(1-\alpha_n^t\right)}{1-\alpha_n}+\frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1-\alpha_n^{2t}\right)}{2\left(1-\alpha_n^2\right)}\right] \\
& \quad \mathcal{E}_n \exp\left[\frac{\nu\mu\left(1-\alpha_n^{t+\tau}\right)}{1-\alpha_n}+\frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1-\alpha_n^{2t+2\tau}\right)}{2\left(1-\alpha_n^2\right)}\right] \\
\simeq \ & \mathcal{E}_n\left[\exp\left(\frac{\nu\mu\left(1-\alpha_n^t\right)}{1-\alpha_n}+\frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1-\alpha_n^{2t}\right)}{2\left(1-\alpha_n^2\right)}\right)\mathcal{E}_k\left[\exp\left(\frac{\nu\mu\left(1-\alpha_k^{t+\tau}\right)}{1-\alpha_k}\right.\right.\right. \\
& \left.\left.\left. + \frac{\nu^2\left(\omega_k^2+\beta_k^2\right)\left(1-\alpha_k^{2t+2\tau}\right)}{2\left(1-\alpha_k^2\right)} + \frac{\nu^2\beta_n\beta_k\alpha_k^\tau\left(1-\alpha_n^t\alpha_k^t\right)}{1-\alpha_n\alpha_k}\right)\right]\right] \\
& - \mathcal{E}_n\left[\exp\left(\frac{\nu\mu\left(1-\alpha_n^t\right)}{1-\alpha_n}+\frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1-\alpha_n^{2t}\right)}{2\left(1-\alpha_n^2\right)}\right)\right] \\
& \quad \mathcal{E}_n\left[\exp\left(\frac{\nu\mu\left(1-\alpha_n^{t+\tau}\right)}{1-\alpha_n}+\frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1-\alpha_n^{2t+2\tau}\right)}{2\left(1-\alpha_n^2\right)}\right)\right] \\
\equiv \ & \widetilde{V}_{t,t+\tau} \equiv G_{t,t+\tau} - H_t H_{t+\tau},
\end{aligned}
$$

for $\exists\beta_n \neq 0$. It is seen that a non-trivial common stochastic shock ($\exists\beta_n \neq 0$) introduces a lot more persistence in the aggregate series than when all $\beta_n = 0$ (in which case $V_{t,t+\tau} \simeq 0$ above). For β's small relative to ω's, one may even get frequent negative autocovariances as opposed to long cyclical behaviour. The latter is what we need to analyse now.

Returning to formula (19) for $\widetilde{V}_{t,t+\tau}$, one may remark the following two aspects. First, one can substitute

$$\mathcal{E}_n[.] = \mathcal{E}_{\omega_n^2}\left[\mathcal{E}_{\beta_n^2}\left[\mathcal{E}_{\alpha_n}[.]\right]\right] = \mathcal{E}_{\omega_n^2,\beta_n^2}\left[\mathcal{E}_{\alpha_n}[.]\right]$$

and similarly for $\mathcal{E}_k[.]$. Second, the main value of the integrals (expectations) comes from $\alpha \to 1$, and one may exploit this property to solve an otherwise intractable multiple integral. Let us take one component at a time from $\widetilde{V}_{t,t+\tau}$ of (19), starting with the easiest (latter) one for expository purposes:

$$\begin{aligned}
H_t &\equiv \mathcal{E}_n\left[\exp\left(\frac{\nu\mu\left(1-\alpha_n^t\right)}{1-\alpha_n}+\frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1-\alpha_n^{2t}\right)}{2\left(1-\alpha_n^2\right)}\right)\right] \\
&= \mathcal{E}_{\omega_n^2,\beta_n^2}\left[\int_0^1 \alpha^{g_\alpha-1}\left(1-\alpha\right)^{h_\alpha-1}\exp\left(\frac{\nu\mu\left(1-\alpha^t\right)}{1-\alpha}+\frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1-\alpha^{2t}\right)}{2\left(1-\alpha^2\right)}\right)\frac{\mathrm{d}\alpha}{\mathrm{B}\left(g_\alpha,h_\alpha\right)}\right] \\
&\simeq \mathcal{E}_{\omega_n^2,\beta_n^2}\left[\int_0^1 \alpha^{g_\alpha-1}\left(1-\alpha\right)^{h_\alpha-1}\exp\left(\nu\mu t\alpha^{t-1}+\frac{\nu^2\left(\omega_n^2+\beta_n^2\right)t}{2}\alpha^{2(t-1)}\right)\frac{\mathrm{d}\alpha}{\mathrm{B}\left(g_\alpha,h_\alpha\right)}\right] \\
&= \int_0^1 \alpha^{g_\alpha-1}\left(1-\alpha\right)^{h_\alpha-1}\exp\left(\nu\mu t\alpha^{t-1}\right)\mathcal{E}_{\omega_n^2,\beta_n^2}\left[\exp\left(\frac{\nu^2\left(\omega_n^2+\beta_n^2\right)t}{2}\alpha^{2(t-1)}\right)\right]\frac{\mathrm{d}\alpha}{\mathrm{B}\left(g_\alpha,h_\alpha\right)}.
\end{aligned}$$

By

$$\begin{aligned}
\mathcal{E}_\xi\left[\exp\left(b\xi\right)\right] &= \frac{2h_\omega^{g_\omega}}{\Gamma\left(g_\omega\right)}\int_0^\infty \xi^{2g_\omega-1}\exp\left(b\xi-h_\omega\xi^2\right)\mathrm{d}\xi \\
&\simeq \frac{2\sqrt{\pi}}{\Gamma\left(g_\omega\right)}\left(\frac{b}{2\sqrt{h_\omega}}\right)^{2g_\omega-1}\exp\left(\frac{b^2}{4h_\omega}\right)
\end{aligned} \tag{20}$$

for large $b\in\mathcal{R}_+$ [e.g. see Abadir's (1995) fractional Hermite polynomials], we can take the required expectations with respect to $\xi\equiv\omega^2$ and $\zeta\equiv\beta^2$ as

$$\begin{aligned}
H_t \simeq{}& \frac{4\pi}{\Gamma\left(g_\omega\right)\Gamma\left(g_\beta\right)\mathrm{B}\left(g_\alpha,h_\alpha\right)}\left(\frac{\nu^2 t}{4\sqrt{h_\omega}}\right)^{2g_\omega-1}\left(\frac{\nu^2 t}{4\sqrt{h_\beta}}\right)^{2g_\beta-1} \\
&\int_0^1 \alpha^{g_\alpha+4(t-1)\left(g_\omega+g_\beta-1\right)-1}\left(1-\alpha\right)^{h_\alpha-1}\exp\left(\nu\mu t\alpha^{t-1}+\frac{\nu^4t^2}{16}\left(\frac{1}{h_\omega}+\frac{1}{h_\beta}\right)\alpha^{4(t-1)}\right)\mathrm{d}\alpha.
\end{aligned}$$

By a change of variable replacing α by $\alpha^{1/(t-1)}$, then approximating for large t, we get

$$\begin{aligned}
H_t &\simeq \frac{4\pi}{\Gamma(g_\omega)\Gamma(g_\beta)\mathrm{B}(g_\alpha,h_\alpha)(t-1)}\left(\frac{\nu^2 t}{4\sqrt{h_\omega}}\right)^{2g_\omega-1}\left(\frac{\nu^2 t}{4\sqrt{h_\beta}}\right)^{2g_\beta-1} \\
&\quad \int_0^1 \alpha^{4(g_\omega+g_\beta-1)-1+\frac{g_\alpha}{t-1}}\left(1-\alpha^{\frac{1}{t-1}}\right)^{h_\alpha-1}\exp\left(\nu\mu t\alpha+\frac{\nu^4 t^2}{16}\left(\frac{1}{h_\omega}+\frac{1}{h_\beta}\right)\alpha^4\right)\mathrm{d}\alpha \\
&\simeq \frac{4\pi}{\Gamma(g_\omega)\Gamma(g_\beta)\mathrm{B}(g_\alpha,h_\alpha)(t-1)^{h_\alpha}}\left(\frac{\nu^2 t}{4\sqrt{h_\omega}}\right)^{2g_\omega-1}\left(\frac{\nu^2 t}{4\sqrt{h_\beta}}\right)^{2g_\beta-1} \\
&\quad \int_0^1 \alpha^{4(g_\omega+g_\beta-1)-1}(1-\alpha)^{h_\alpha-1}\exp\left(\nu\mu t\alpha+\frac{\nu^4 t^2}{16}\left(\frac{1}{h_\omega}+\frac{1}{h_\beta}\right)\alpha^4\right)\mathrm{d}\alpha \\
&= \frac{4\pi}{\Gamma(g_\omega)\Gamma(g_\beta)\mathrm{B}(g_\alpha,h_\alpha)(t-1)^{h_\alpha}}\left(\frac{\nu^2 t}{4\sqrt{h_\omega}}\right)^{2g_\omega-1}\left(\frac{\nu^2 t}{4\sqrt{h_\beta}}\right)^{2g_\beta-1} \\
&\quad \sum_{j=0}^{\infty}\frac{\left(\frac{\nu^4 t^2}{16}\right)^j\left(\frac{1}{h_\omega}+\frac{1}{h_\beta}\right)^j}{j!}\int_0^1 \alpha^{4(j+g_\omega+g_\beta-1)-1}(1-\alpha)^{h_\alpha-1}\exp(\nu\mu t\alpha)\,\mathrm{d}\alpha,
\end{aligned}$$

the middle step having been obtained by

$$1-\alpha^{\frac{1}{t-1}} = 1-(1-(1-\alpha))^{\frac{1}{t-1}} \simeq \frac{1-\alpha}{t-1}.$$

Noting that

$$\int_0^1 \alpha^{b-1}(1-\alpha)^{h_\alpha-1}\exp(\nu\mu t\alpha)\,\mathrm{d}\alpha \simeq \frac{\Gamma(h_\alpha)}{(\nu\mu t)^{h_\alpha}}\exp(\nu\mu t) \tag{21}$$

for large t [e.g. see Kummer's function in Abadir (1995)] with $\nu\mu \in \mathcal{R}_+$, we can write

$$\begin{aligned}
H_t &\simeq \frac{4\pi}{\Gamma(g_\omega)\Gamma(g_\beta)\mathrm{B}(g_\alpha,h_\alpha)(t-1)^{h_\alpha}}\left(\frac{\nu^2 t}{4\sqrt{h_\omega}}\right)^{2g_\omega-1}\left(\frac{\nu^2 t}{4\sqrt{h_\beta}}\right)^{2g_\beta-1} \\
&\quad \sum_{j=0}^{\infty}\frac{\left(\frac{\nu^4 t^2}{16}\right)^j\left(\frac{1}{h_\omega}+\frac{1}{h_\beta}\right)^j}{j!}\frac{\Gamma(h_\alpha)}{(\nu\mu t)^{h_\alpha}}\exp(\nu\mu t) \\
&= \frac{4\pi\Gamma(g_\alpha+h_\alpha)}{\Gamma(g_\alpha)\Gamma(g_\omega)\Gamma(g_\beta)(\nu\mu t(t-1))^{h_\alpha}}\left(\frac{\nu^2 t}{4\sqrt{h_\omega}}\right)^{2g_\omega-1}\left(\frac{\nu^2 t}{4\sqrt{h_\beta}}\right)^{2g_\beta-1} \\
&\quad \exp\left(\nu\mu t+\frac{\nu^4 t^2}{16}\left(\frac{1}{h_\omega}+\frac{1}{h_\beta}\right)\right).
\end{aligned} \tag{22}$$

As in Granger (1980), and in spite of the different setting here (geometric AR), we find that the Beta parameter b of (21) is relatively unimportant in determining the time-series features. This is no wonder, since AR roots near 0 have little impact on aggregate memory, while roots near unity are more critical in this respect. In terms of the original first Beta parameter, g_α acts just as a 'scaling' for H_t rather than an important parameter (e.g. power of t).

Going back to the required $\widetilde{V}_{t,t+\tau}$ of (19), we have now derived an approximation for H_t, and accordingly $H_{t+\tau}$. We need to do the same for $G_{t,t+\tau}$. We start with the same approximation of the integrals (expectations) near $\alpha = 1$, and

$$\begin{aligned}
&G_{t,t+\tau}\\
\simeq\;& \mathcal{E}_n\left[\exp\left(\nu\mu t\alpha_n^{t-1}+\frac{\nu^2 t\left(\omega_n^2+\beta_n^2\right)}{2}\alpha_n^{2(t-1)}\right)\right.\\
&\left.\mathcal{E}_k\left[\exp\left(\nu\mu\left(t+\tau\right)\alpha_k^{t+\tau-1}+\frac{\nu^2\left(t+\tau\right)\left(\omega_k^2+\beta_k^2\right)}{2}\alpha_k^{2(t+\tau-1)}+\nu^2 t\beta_n\beta_k\alpha_k^{t+\tau-1}\alpha_n^{t-1}\right)\right]\right]\\
=\;& \frac{1}{\left(\mathrm{B}\left(g_\alpha,h_\alpha\right)\right)^2}\mathcal{E}_{\omega_n^2,\beta_n^2}\left[\int_0^1\alpha_n^{g_\alpha-1}\left(1-\alpha_n\right)^{h_\alpha-1}\exp\left(\nu\mu t\alpha_n^{t-1}+\frac{\nu^2 t\left(\omega_n^2+\beta_n^2\right)}{2}\alpha_n^{2(t-1)}\right)\right.\\
&\mathcal{E}_{\omega_k^2,\beta_k^2}\left[\int_0^1\alpha_k^{g_\alpha-1}\left(1-\alpha_k\right)^{h_\alpha-1}\right.\\
&\left.\left.\exp\left(\nu\left(\mu\left(t+\tau\right)+\nu t\beta_n\beta_k\alpha_n^{t-1}\right)\alpha_k^{t+\tau-1}+\frac{\nu^2\left(t+\tau\right)\left(\omega_k^2+\beta_k^2\right)}{2}\alpha_k^{2(t+\tau-1)}\right)\mathrm{d}\alpha_k\right]\mathrm{d}\alpha_n\right].
\end{aligned}$$

By the transformations replacing α_k by $\alpha_k^{1/(t+\tau-1)}$ and α_n by $\alpha_n^{1/(t-1)}$, followed by the same large-t expansion as before,

$$\begin{aligned}
G_{t,t+\tau}\simeq\;& \frac{1}{\left(\mathrm{B}\left(g_\alpha,h_\alpha\right)\right)^2\left(\left(t-1\right)\left(t+\tau-1\right)\right)^{h_\alpha}}\\
&\mathcal{E}_{\omega_n^2,\beta_n^2}\left[\int_0^1\frac{\left(1-\alpha_n\right)^{h_\alpha-1}}{\alpha_n}\exp\left(\nu\mu t\alpha_n+\frac{\nu^2 t\left(\omega_n^2+\beta_n^2\right)}{2}\alpha_n^2\right)\mathcal{E}_{\omega_k^2,\beta_k^2}\left[\int_0^1\frac{\left(1-\alpha_k\right)^{h_\alpha-1}}{\alpha_k}\right.\right.\\
&\left.\left.\exp\left(\nu\left(\mu\left(t+\tau\right)+\nu t\beta_n\beta_k\alpha_n\right)\alpha_k+\frac{\nu^2\left(t+\tau\right)\left(\omega_k^2+\beta_k^2\right)}{2}\alpha_k^2\right)\mathrm{d}\alpha_k\right]\mathrm{d}\alpha_n\right].
\end{aligned}$$

As before, taking expectations with respect to $\xi \equiv \omega^2$ then integrating α_k out,

$$
\begin{aligned}
& G_{t,t+\tau} \\
\simeq\ & \frac{4\pi}{\left(\mathrm{B}\left(g_\alpha, h_\alpha\right)\Gamma\left(g_\omega\right)\right)^2\left(\left(t-1\right)\left(t+\tau-1\right)\right)^{h_\alpha}}\left(\frac{\nu^4 t\left(t+\tau\right)}{16h_\omega}\right)^{2g_\omega-1} \\
& \mathcal{E}_{\beta_n^2}\left[\mathcal{E}_{\beta_k^2}\left[\int_0^1 \alpha_n^{4g_\omega-3}\left(1-\alpha_n\right)^{h_\alpha-1}\exp\left(\nu\mu t\alpha_n+\frac{\nu^2 t\beta_n^2}{2}\alpha_n^2+\frac{\nu^4 t^2}{16h_\omega}\alpha_n^4\right)\int_0^1\left(1-\alpha_k\right)^{h_\alpha-1}\right.\right. \\
& \left.\left.\alpha_k^{4g_\omega-3}\exp\left(\nu\left(\mu\left(t+\tau\right)+\nu t\beta_n\beta_k\alpha_n\right)\alpha_k+\frac{\nu^2\left(t+\tau\right)\beta_k^2}{2}\alpha_k^2+\frac{\nu^4\left(t+\tau\right)^2}{16h_\omega}\alpha_k^4\right)\mathrm{d}\alpha_k\mathrm{d}\alpha_n\right]\right] \\
\simeq\ & \frac{4\pi\Gamma\left(h_\alpha\right)}{\left(\mathrm{B}\left(g_\alpha, h_\alpha\right)\Gamma\left(g_\omega\right)\right)^2\left(\left(t-1\right)\left(t+\tau-1\right)\nu\right)^{h_\alpha}}\left(\frac{\nu^4 t\left(t+\tau\right)}{16h_\omega}\right)^{2g_\omega-1} \\
& \exp\left(\nu\mu\left(t+\tau\right)+\frac{\nu^4\left(t+\tau\right)^2}{16h_\omega}\right)\mathcal{E}_{\beta_n^2}\left[\mathcal{E}_{\beta_k^2}\left[\exp\left(\frac{\nu^2\left(t+\tau\right)\beta_k^2}{2}\right)\right.\right. \\
& \left.\left.\int_0^1\frac{\alpha_n^{4g_\omega-3}\left(1-\alpha_n\right)^{h_\alpha-1}}{\left(\mu\left(t+\tau\right)+\nu t\beta_n\beta_k\alpha_n\right)^{h_\alpha}}\exp\left(\nu t\left(\mu+\nu\beta_n\beta_k\right)\alpha_n+\frac{\nu^2 t\beta_n^2}{2}\alpha_n^2+\frac{\nu^4 t^2}{16h_\omega}\alpha_n^4\right)\mathrm{d}\alpha_n\right]\right] \\
=\ & \frac{4\pi\Gamma\left(h_\alpha\right)}{\left(\mathrm{B}\left(g_\alpha, h_\alpha\right)\Gamma\left(g_\omega\right)\right)^2\left(\left(t-1\right)\left(t+\tau-1\right)\nu\mu\left(t+\tau\right)\right)^{h_\alpha}}\left(\frac{\nu^4 t\left(t+\tau\right)}{16h_\omega}\right)^{2g_\omega-1} \\
& \exp\left(\nu\mu\left(t+\tau\right)+\frac{\nu^4\left(t+\tau\right)^2}{16h_\omega}\right)\mathcal{E}_{\beta_n^2}\left[\mathcal{E}_{\beta_k^2}\left[\exp\left(\frac{\nu^2\left(t+\tau\right)\beta_k^2}{2}\right)\sum_{j=0}^{\infty}\binom{-h_\alpha}{j}\left(\frac{\nu t\beta_n\beta_k}{\mu\left(t+\tau\right)}\right)^j\right.\right. \\
& \left.\left.\int_0^1\alpha_n^{j+4g_\omega-3}\left(1-\alpha_n\right)^{h_\alpha-1}\exp\left(\nu t\left(\mu+\nu\beta_n\beta_k\right)\alpha_n+\frac{\nu^2 t\beta_n^2}{2}\alpha_n^2+\frac{\nu^4 t^2}{16h_\omega}\alpha_n^4\right)\mathrm{d}\alpha_n\right]\right]
\end{aligned}
$$

by a binomial expansion with binomial coefficients $\binom{a}{j} \equiv \Gamma(a+1) / [\Gamma(a-j+1)\, j!]$. Integrating α_n out as before,

(23) $$G_{t,t+\tau}$$

$$\simeq \frac{4\pi}{((t-1)(t+\tau-1)\nu^2\mu t(t+\tau))^{h_\alpha}} \left(\frac{\Gamma(h_\alpha)}{\mathrm{B}(g_\alpha, h_\alpha)\Gamma(g_\omega)}\right)^2 \left(\frac{\nu^4 t(t+\tau)}{16h_\omega}\right)^{2g_\omega-1}$$
$$\exp\left(\nu\mu(t+\tau) + \frac{\nu^4(t+\tau)^2}{16h_\omega}\right) \mathcal{E}_{\beta_n^2}\left[\mathcal{E}_{\beta_k^2}\left[\exp\left(\frac{\nu^2(t+\tau)\beta_k^2}{2}\right)\sum_{j=0}^{\infty}\binom{-h_\alpha}{j}\right.\right.$$
$$\left.\left.\left(\frac{\nu t\beta_n\beta_k}{\mu(t+\tau)}\right)^j \frac{1}{(\mu+\nu\beta_n\beta_k)^{h_\alpha}} \exp\left(\nu t(\mu+\nu\beta_n\beta_k) + \frac{\nu^2 t\beta_n^2}{2} + \frac{\nu^4 t^2}{16h_\omega}\right)\right]\right]$$
$$= \frac{4\pi}{((t-1)(t+\tau-1)\nu^2 t)^{h_\alpha}} \left(\frac{\Gamma(g_\alpha+h_\alpha)}{\Gamma(g_\alpha)\Gamma(g_\omega)}\right)^2 \left(\frac{\nu^4 t(t+\tau)}{16h_\omega}\right)^{2g_\omega-1}$$
$$\exp\left(\nu\mu(2t+\tau) + \frac{\nu^4\left(t^2+(t+\tau)^2\right)}{16h_\omega}\right)$$
$$\mathcal{E}_{\beta_n^2}\left[\exp\left(\frac{\nu^2 t\beta_n^2}{2}\right)\mathcal{E}_{\beta_k^2}\left[\frac{\exp\left(\nu^2 t\beta_n\beta_k + \frac{\nu^2(t+\tau)\beta_k^2}{2}\right)}{(\mu+\nu\beta_n\beta_k)^{h_\alpha}(\mu(t+\tau)+\nu t\beta_n\beta_k)^{h_\alpha}}\right]\right].$$

where the terms of the binomial expansion have been collected. For large t, we can use Watson's lemma to approximate

$$(\mu+\nu\beta_n\beta_k)(\mu(t+\tau)+\nu t\beta_n\beta_k)$$

by $\mu^2(t+\tau)$ when integrating for the expectation with respect to $\zeta \equiv \beta^2$. Then, we need to derive

$$\begin{aligned}
&\mathcal{E}_{\beta_n^2}\left[\exp\left(\frac{\nu^2 t\beta_n^2}{2}\right)\mathcal{E}_{\beta_k^2}\left[\exp\left(\nu^2 t\beta_n\beta_k+\frac{\nu^2(t+\tau)\beta_k^2}{2}\right)\right]\right]\\
&= \frac{4h_\beta^{2g_\beta}}{(\Gamma(g_\beta))^2}\int_0^\infty \zeta_n^{2g_\beta-1}\exp\left(\frac{\nu^2 t\zeta_n}{2}-h_\beta\zeta_n^2\right)\\
&\qquad\int_0^\infty \zeta_k^{2g_\beta-1}\exp\left(\nu^2 t\sqrt{\zeta_n\zeta_k}+\frac{\nu^2(t+\tau)\zeta_k}{2}-h_\beta\zeta_k^2\right)\mathrm{d}\zeta_k\mathrm{d}\zeta_n\\
&= \frac{4h_\beta^{2g_\beta}}{(\Gamma(g_\beta))^2}\sum_{j=0}^\infty\frac{(\nu^2 t)^j}{j!}\int_0^\infty \zeta_n^{2g_\beta-1+\frac{j}{2}}\exp\left(\frac{\nu^2 t\zeta_n}{2}-h_\beta\zeta_n^2\right)\mathrm{d}\zeta_n\\
&\qquad\int_0^\infty \zeta_k^{2g_\beta-1+\frac{j}{2}}\exp\left(\frac{\nu^2(t+\tau)\zeta_k}{2}-h_\beta\zeta_k^2\right)\mathrm{d}\zeta_k\\
&\simeq \frac{4\pi h_\beta^{2g_\beta-1}}{(\Gamma(g_\beta))^2}\exp\left(\frac{\nu^4 t^2}{16h_\beta}+\frac{\nu^4(t+\tau)^2}{16h_\beta}\right)\sum_{j=0}^\infty\frac{(\nu^2 t)^j}{j!}\left(\frac{\nu^2 t}{4h_\beta}\right)^{2g_\beta-1+\frac{j}{2}}\left(\frac{\nu^2(t+\tau)}{4h_\beta}\right)^{2g_\beta-1+\frac{j}{2}}\\
&= \frac{4\pi}{(\Gamma(g_\beta))^2}\left(\frac{\nu^4 t(t+\tau)}{16h_\beta}\right)^{2g_\beta-1}\exp\left(\frac{\nu^4}{16h_\beta}\left(t^2+4t\sqrt{t(t+\tau)}+(t+\tau)^2\right)\right)
\end{aligned}$$

by the same integration used in (20). Substituting into (23),

$$\begin{aligned}
G_{t,t+\tau} \simeq\; & \frac{16\pi^2}{((t-1)(t+\tau-1)\nu^2\mu^2 t(t+\tau))^{h_\alpha}}\left(\frac{\Gamma(g_\alpha+h_\alpha)}{\Gamma(g_\alpha)\Gamma(g_\omega)\Gamma(g_\beta)}\right)^2\\
&\left(\frac{\nu^4 t(t+\tau)}{16h_\omega}\right)^{2g_\omega-1}\left(\frac{\nu^4 t(t+\tau)}{16h_\beta}\right)^{2g_\beta-1}\\
&\exp\left(\nu\mu(2t+\tau)+\frac{\nu^4}{16}\left(t^2+(t+\tau)^2\right)\left(\frac{1}{h_\omega}+\frac{1}{h_\beta}\right)+\frac{\nu^4 t\sqrt{t(t+\tau)}}{4h_\beta}\right).
\end{aligned}$$

Together with (22), this gives

$$\begin{aligned}
\widetilde{V}_{t,t+\tau} \equiv\; & G_{t,t+\tau}-H_t H_{t+\tau}\\
\simeq\; & \frac{16\pi^2}{((t-1)(t+\tau-1)\nu^2\mu^2 t(t+\tau))^{h_\alpha}}\left(\frac{\Gamma(g_\alpha+h_\alpha)}{\Gamma(g_\alpha)\Gamma(g_\omega)\Gamma(g_\beta)}\right)^2\\
&\left(\frac{\nu^4 t(t+\tau)}{16h_\omega}\right)^{2g_\omega-1}\left(\frac{\nu^4 t(t+\tau)}{16h_\beta}\right)^{2g_\beta-1}\\
&\exp\left(\nu\mu(2t+\tau)+\frac{\nu^4}{16}\left(t^2+(t+\tau)^2\right)\left(\frac{1}{h_\omega}+\frac{1}{h_\beta}\right)\right)\left(\exp\left(\frac{\nu^4 t\sqrt{t(t+\tau)}}{4h_\beta}\right)-1\right),
\end{aligned}$$

and the corresponding autocorrelation function is

$$
\begin{aligned}
R_{t,t+\tau} &\equiv \frac{V_{t,t+\tau}}{\sqrt{V_{t,t}V_{t+\tau,t+\tau}}} \simeq \frac{\widetilde{V}_{t,t+\tau}}{\sqrt{\widetilde{V}_{t,t}\widetilde{V}_{t+\tau,t+\tau}}} \\
&\simeq \frac{\exp\left(\frac{\nu^4 t\sqrt{t(t+\tau)}}{4h_\beta}\right)-1}{\sqrt{\exp\left(\frac{\nu^4 t^2}{4h_\beta}\right)-1}\sqrt{\exp\left(\frac{\nu^4 (t+\tau)^2}{4h_\beta}\right)-1}},
\end{aligned}
$$

which completes the proof. Q.E.D.

Proof of Corollary 2. From (12), using the backshift (lag) operator $\mathcal{B}$ and letting $X_t \equiv \theta_t^\nu$,

$$\log y_t = \frac{\gamma}{1-\gamma}\log s^* + \frac{1}{\nu\left(1-\gamma\mathcal{B}\right)}\log X_t,$$

apart from the inconsequential initial conditions (see the discussion following (13) in the text). The constants $(1-\gamma)^{-1}\gamma\log s^*$ and ν have no impact on the relation between the autocorrelation functions of $\{\log X_t\}$ and $\{\log y_t\}$, and $1-\gamma\mathcal{B}$ has an exponentially-decaying effect on the transfer function (when $|\gamma|<1$) linking the long-memory process $\{\log X_t\}$ to $\{\log y_t\}$. Q.E.D.

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