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EULER, LAMBERT, AND THE LAMBERT W-FUNCTION TODAY

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Abstract

The Lambert W-function has found applications in an extraordinary number of scientific fields. In this paper we present a short historical review, a brief description of the function, and a survey of some of its applications.

Keywords: Lambert W-function; delay differential equation

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1. Introduction

To celebrate the 300th anniversary of Leonhard Euler's birth, we offer an historical look at a well-known mathematical function.

The Lambert W-function dates back to Johann Lambert (1728–1777) and Leonhard Euler (1707–1783). These two mathematicians developed a series solution for the trinomial equation, but left it unnamed. The series was christened the Lambert W-function two centuries later, when it was included in the algebraic system MAPLE®.

The Lambert W-function, represented by $W(z)$, is defined as the inverse of the function $f(z) = z\mathrm{e}^z$, satisfying

$$W(z)\mathrm{e}^{W(z)} = z.$$

According to [21], the mathematical history of $W(z)$ began in 1758 when Lambert solved the trinomial equation

$$x = q + x^m,$$

subsequently transformed by Euler into the form

$$x^\alpha - x^\beta = (\alpha - \beta)v x^{\alpha+\beta}.$$

After expansion in series, a new function, $T(z)$, known as the *tree function*, was obtained as

$$T(z) = \sum_{n\geq 1} n^{n-1}\frac{z^n}{n!}.$$

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According to [32], in 1844 Eisenstein noticed that the tree function was a generating function that generates *rooted labelled trees,* and satisfies the functional equation

$$T(z) = z\mathrm{e}^{T(z)}. \tag{1}$$

Thus, the tree function $T(z)$ is related to the Lambert W-function by $T(z) = -W(-z)$.

In [26] and [41], the designation of *Cayley trees* is associated with the function that satisfies (1). The *Cayley function,* $C(z)$, is introduced as the function satisfying the functional equation

$$C(z) = z\mathrm{e}^{C(z)},$$

which suggests that the tree function $T(z)$ is an example of a Cayley function, which is named after the mathematician Arthur Cayley who enumerated rooted labelled trees in the nineteenth century. Consequently, the two functions, W and T, have found relevant applications in combinatorics, particulary in enumerating trees.

It is also interesting to mention the reason for choosing the letter W for the Lambert W-function. The reason for the choice of the letter W when it was included in MAPLE (see [21]) is not entirely clear. However, '...fortuitously, the letter W has additional significance because of the pioneering work on many aspects of W by Wright' [21, p. 330]. Wright [42], [43] made quite significant contributions to delay differential equations (DDEs).

From 1949 (see [42]) to 1959 (see [43]), Wright obtained relevant results in the solution to $z\mathrm{e}^z = a$, which is the characteristic equation associated with the DDE $x' = Bx(t - r)$. Assuming that $x(t) = C\mathrm{e}^{\lambda t}$ is a solution for some value of λ, we obtain after substitution a transcendental equation $z\mathrm{e}^z = a$, where $z = \lambda r$ and $a = rB$. As Wright stated in [43], the contribution of Hayes [30] in 1950 was also undeniably important in the study of that transcendental equation. Today, a more appealing solution involving the operator W can be expressed by $z = W(a)$.

Furthermore, in 1973, Crowley *et al.* [22] presented an algorithm, the 443 algorithm, for solving the transcendental equation $w\mathrm{e}^w = x$. Recently, this algorithm has been superseded by algorithm 743 (see [5]).

As mentioned in [19], the Lambert W-function is sometimes known as the *omega function* because of several contributions where the Greek letter ω is used. Values of the Lambert W-function can be found at http://functions.wolfram.com, and an informative poster is available at http://www.orcca.on.ca/LambertW.

2. The two real branches of the Lambert W-function

The Lambert W-function is a multivalued function displaying two real branches. Figure 1 represents the two real branches of $W(x)$. If x is real in the interval $-1/\mathrm{e} < x < 0$, then there are two real values for $W(x)$. The branch satisfying $W(x) \geq -1$ is defined as the *principal branch* of the Lambert W-function and is denoted by $W(x)$ or $W_0(x)$. The branch satisfying $W(x) \leq -1$ is denoted by $W_{-1}(x)$. An explanation of this notation can be found in [21].

If x is real, then we have the following result on the existence of real values for $W(x)\mathrm{e}^{W(x)} = x$. If $x \geq 0$ then there is an unique real root which is positive except for $W(0) = 0$; if $-1/\mathrm{e} < x < 0$ then there are two negative real roots, $W_0(x)$ and $W_{-1}(x)$; if $x = -1/\mathrm{e}$ then there is only one negative real solution, $W_0(-1/\mathrm{e}) = W_{-1}(-1/\mathrm{e}) = -1$; and if $x < -1/\mathrm{e}$ then there are no real solutions.

The main properties and calculus of W are given in [16] and [17]; in particular, expressions for its derivative, for integrals containing W, and asymptotic expansions of the complex branches of W.

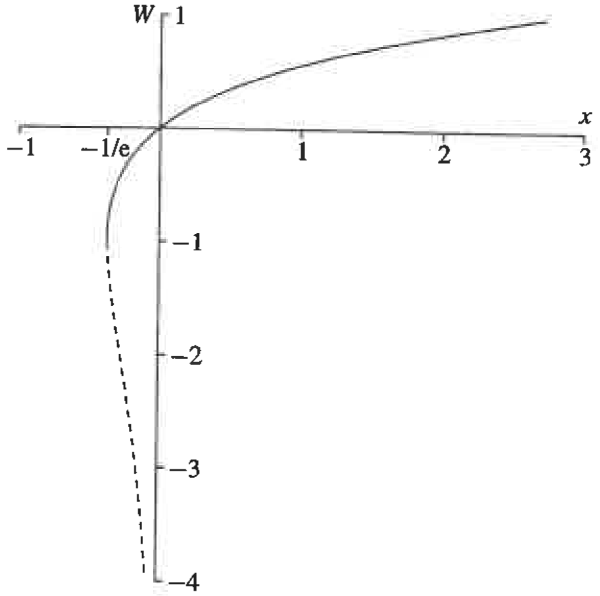

FIGURE 1: The two real branches of $W(x)$; the solid line is $W_0(x)$ and the dashed line is $W_{-1}(x)$.

We conclude this section noting that

$$W(x) = \sum_{n\geq 1} \frac{(-n)^{n-1}}{n!} x^n$$

is a convergent series for $|x| < 1/e$.

3. Applications of the Lambert W-function: a short survey

Knowledge on the Lambert W-function, as a mathematical tool, has allowed the derivation of closed form solutions for models in numerous scientific disciplines, for which explicit or exact solutions were not known, and alternative iterative methods or approximate solutions had been used.

Mathematics, physics, biology, geology, engineering, and even risk theory, are some areas in which the usefulness of the Lambert W-function has been proved. In physics, we point out the applications of the Lambert W-function to optics [17], particle physics [6], general relativity [13], and geophysics [9]. In engineering, we may mention electronics [12], [37] and acoustics [11]. Other applications of the Lambert W-function can be found in biochemistry [36], geology [35], risk theory [2], technological systems [25], and in information theory [40].

In the field of mathematics, the applications of the Lambert W-function are also very rich. Corless *et al.* [19] highlighted the study of real values of $W(x)$ and its manipulation in MAPLE.

Shih [39] used the inverse of the function xe^x to represent periodic orbits of two nonlinear relaxation oscillations systems. The main contribution of [39] consists of studying the singular behaviour of each function $W(-k, x)$ at the branch point $x = -e^{-1}$, where $W(-k, x)$ are denoted as `LambertW(k,x)` in MAPLE.

Corless *et al.* [20] discussed the relationship that exists between the inverse of $y^\alpha e^y$ and the *Stirling numbers of the first and second kind*, $\left[{n \atop m}\right]$ and $\left\{{n \atop m}\right\}$ respectively. On this issue, see [33], where a definition of the Stirling numbers based on the analysis of the transformation from a polynomial expressed in powers of k to a polynomial expressed in terms of binomial coefficients is presented. Stirling numbers are the coefficients involved in this transformation. In particular, Stirling numbers of the first kind are used to convert from binomial coefficients to powers, i.e.

$$n!\binom{x}{n} = \sum_k (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} x^k,$$

while Stirling numbers of the second kind are used to convert from powers to binomial coefficients, i.e.

$$x^n = \sum_k \begin{Bmatrix} n \\ k \end{Bmatrix} \binom{x}{k} k!.$$

Following Knuth's notation [34], the Stirling numbers of the first kind, $\left[{n \atop k}\right]$, should be called *Stirling cycle numbers*, while Stirling numbers of the second kind, $\left\{{n \atop k}\right\}$, should be called *Stirling subset numbers*.

The *unwinding number* definition proposed by Corless *et al.* [14], enables the establishment of a precise distinction between the different branches of the Lambert W-function. This contribution is fundamental in obtaining an asymptotic expansion for all branches of $W(z)$ and for all z (see [16]), and has also led to a better solution for transcendental equations (see [15]).

It is a curious coincidence that, two centuries ago, Euler and Lambert were studying the solution to particular algebraic equations, while, in 1998, the Lambert W-function was again used in solving some transcendental equations; see [15].

More recently, the Lambert W-function has been applied to infinite exponentials [28] and graph theory [27].

However, as noted earlier, it is in the field of time-delay dynamics that the modern Lambert W-function has found a special area of application. Two of the most recent contributions to this field are [1] and [38].

4. Delay differential equations and the Lambert W-function

DDEs were introduced by Condorcet and Laplace in the eighteenth century. *Delay systems* are sometimes called *hereditary systems*, *retard equations*, or *differential-difference equations*. Literature surveys can be found in [4] and [23], and the classics [7], [24], and [29].

During the eighteenth century, Condorcet and Laplace were studying DDEs while Euler and Lambert were considering certain other equations. Why did their results only cross two centuries later? One good reason is that the vehicle that joins the two fields is the characteristic equation associated with $x' = Bx(t-r)$. If we assume a solution of the type $e^{\lambda t}$, then we have

$$\lambda - Be^{-\lambda r} = 0 \quad \Longleftrightarrow \quad \lambda r e^{\lambda r} = rB \quad \Longleftrightarrow \quad \lambda r = W(rB).$$

The solution of the DDE can then be expressed as

$$x(t) = C \exp\left(\frac{W(rB)}{r} t\right),$$

satisfying $x' = Bx(t-r)$.

If we introduce an initial function to obtain a unique solution, then the problem is less simple, as can be seen in [3], where the authors point out the advantage of using the Lambert function to derive a closed form solution to the linear DDE system.

Recently, Corless *et al.* [18] examined the solutions to the matrix equation $S\exp(S) = A$, which is the analogue to the previous equation, $\lambda r e^{\lambda r} = rB$. The first simple equation that Corless *et al.* studied, $y'(t) = Ay(t-1)$, is very useful in mathematical biology models because the delay logistic equation has this particular DDE form, after linearization at the equilibrium point [10]. Though efficient numerical computational routines are available, e.g. in `DynPac` or `dde23`, Corless *et al.* [18] were concerned with finding analytical solution methods by exploring the relationships between the matrix function $W_k(A)$ and the solutions of $S\exp(S) = A$.

Heffernan and Corless [31] instead examined some computer algebra approaches for solving DDEs. The main methods discussed are the method of steps, the Laplace transform method, and, more recently, the least squares method. It is interesting to note that this last method was used in [3], where the Lambert W-function concept was used to generalize the state transition matrix concept from ordinary differential equations to DDEs.

A more specific work which relates the Lambert W-function with the discussion of the roots of the characteristic equation is [8]. In [7] we find the use of expansions of the form

$$u(t) = \sum_r p_r(t)\exp(s_r t)$$

for the solution, $u(t)$, of

$$\begin{aligned} a_0u'(t) + b_0u(t) + b_1u(t-w) &= 0, \qquad t > w,\ a_0 \neq 0,\\ u(t) &= g(t), \qquad 0 \le t \le w, \end{aligned}$$

where the sum is taken over all the characteristic roots s_r. The subject of the asymptotic location of the zeros of the characteristic function

$$h(s) = a_0 s + b_0 + b_1 e^{-ws},$$

has since been extensively studied. In [8] the characteristic roots of $y'(t) = ay(t-1)$ were studied by using the Lambert W-function, to obtain an asymptotic distribution of the roots that provide a stability result on the completeness of exponential systems.

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