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From the classification of quadrilaterals to the classification of prisms: An experiment with prospective teachers

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ABSTRACT

This article reports a research in the context of a K-6 prospective teacher education experiment developed in a geometry course in the 2nd year of their preparation program. This course included the study of the classification of quadrilaterals and prisms. The research is guided by the following question: how does the learning of hierarchical classification of geometric figures evolve from a teacher education experiment that includes the classification of quadrilaterals and prisms and follows an exploratory approach to teaching? Data was collected from audio and video records from the lessons and from the participants' written reports about the classification of quadrilaterals and prisms. The results show that, in the first stage that focused on quadrilaterals, the participants' difficulties in classifying derived mainly from their inexperience with the process of classifying geometrical objects and from their strong conceptualization of some quadrilaterals, very attached to prototypical images. In the second stage, the classification of prisms showed a positive and significant evolution, with a lower influence of prototypical images and a higher understanding about the classification process and the identification of hierarchical relationships among "close" and "distant" figures. However, the final evaluation test showed that the prospective teachers still had misunderstandings, most often related to the interpretation of the discourse and logical reasoning than to limited figural concepts.

1. Introduction

In usual mathematics teaching, the content is taught to students according to a pre-organized structure, in which all the concepts and respective definitions are pre-established by teachers, based on textbook and curricula. This way of presenting concepts to students is not productive from a didactic perspective, if we assume that the student must play a major role in the construction of their knowledge, nor corresponds to the way concepts are developed in mathematics. This system, where the students do not have opportunities to organize their spatial experiences, has been criticised for several decades by educators and mathematicians, notably by [Freudenthal \(1973\)](#) who underlined the importance of activities in which the students learn to organize a subject (and thus learn what organizing is), learn to conceptualize (and what conceptualization is), and learn to define (and what a definition is).

Among the various activities to be valued in geometry teaching, lies the classification of figures, referred by many researchers (e.g., [De Villiers, 1994](#); [Mariotti & Fischbein, 1997](#)) and suggested by the most recent curricular guidelines for mathematics teaching in preschool and elementary and middle school ([National Council of Teachers of Mathematics, 2000](#)). However, research carried out in different countries has showed several difficulties that affect students of all levels of education, as well as in prospective teachers.

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In general, studies on the classification of quadrilaterals show that geometric reasoning is frequently affected by mental images of the figures, lacking flexibility in the majority of times (prototype effect), by difficulties in deductive reasoning and by misunderstanding of the classification process itself (Brunheira & Ponte, 2015; De Villiers, 1994; Fujita, 2012; Fujita & Jones, 2007; Monaghan, 2000; Okazaki & Fujita, 2007; Tempera, 2010). Nonetheless, little is known about the classification of solids, as well as the progression on learning to classify. Thereby, it is necessary that research focuses on the learning of these topics and overcomes the lack of knowledge about teachers and prospective teachers in geometry, referred by some researchers (Chapman, 2013; Clements & Sarama, 2011; Steele, 2013). In addition, research should develop productive strategies to diminish the diagnosed difficulties, supporting prospective teachers in the development of geometric reasoning. Addressing this issue, we conducted an experiment in a geometry course of a degree in basic education¹. Our investigation addresses the following research question: how does the learning of hierarchical classification of geometric figures evolve from a teacher education experiment that includes the classification of quadrilaterals and prisms and follows an exploratory approach to teaching?

2. Theoretical framework and literature review

2.1. The classification process in geometry

According to Markman (1989), the systematic organization of concepts into categories is a major intellectual achievement of human conceptualization. As this psychologist states, from an early age, we are capable of recognizing some individual objects as individuals – babies do not see their family members as any other person or their toys as any other toys. By 18 months to 2 years of age, children become capable of distinguishing stuffed animals from real animals and name them. This type of simple categorization, in which an object belongs to one or another group, is the first problem of this nature that children face. The second problem is the organization of concepts in inclusive classes, establishing a hierarchical relationship among them. Organized like this, a categorized object in a certain hierarchical level has all the properties of the objects belonging to the classes in upper hierarchical levels, and this has multiple advantages for the production and organization of knowledge.

Therefore, we may assume that the classification process is inherent to our human condition and common to several everyday activities, as well as to all fields of knowledge. In the case of mathematics, the classification process is most relevant. Mariotti and Fischbein (1997) state in the following way what means to classify:

A classification task consists of stating an equivalence among similar but figurally different objects, towards a generalisation. That means overcoming the particular case and consider this particular case as an instance of a general class. In other terms, the process of classification consists of identifying pertinent common properties, which determine a category. (p. 244)

In mathematics, we may consider several types of classifications. Regarding quadrilaterals, De Villiers (1994) refers that figures may be organized using partition classifications or hierarchical classifications. In either case, we can distinguish descriptive classifications (*a posteriori*) from constructive classifications (*a priori*). We undertake *a posteriori* classifications when we have already studied the properties of figures and we have organized objects regarding those properties. In *a priori* classifications, we begin with an object and we build all the relations based on generalization or specialization processes. For example, beginning with a particular quadrilateral – the square – we built other more general concepts by “erasing” some of the properties or by replacing some properties with more general ones; otherwise, we can begin with a more general concept, like the parallelogram, and add properties or replace some properties with more specific ones. The main function of *a priori* classification is the construction of new concepts.

Regarding partition and hierarchical classifications, De Villiers (1994) points that the term “hierarchical classification” designates a type of “classification of a set of concepts in such a manner that the more particular concepts form subsets of the more general concepts” (p. 11); on the other hand, in partition classifications “the various subsets of concepts are considered to be disjoint from one another” (p. 11). Both concepts correspond to the two ways of classification aforementioned by Markman (1989) – partition classifications corresponds to those we spontaneously construct in the beginning of our lives and the hierarchical ones those demanding a greater maturity. However, as this author notes, partition classifications frequently have some underlying hierarchical organization.

As De Villiers (1994) states, there is a close relation between the definition and the classification of objects. For instance, the definition “a quadrilateral having at least one pair of opposite sides parallel” for trapezia generates a hierarchical classification in which the parallelogram is the particular case of a trapezium; however, if the definition states that there is only one pair of opposite sides parallel, we have a partition classification, with the parallelogram belonging to a disjoint set of the trapezium. This relationship between classification and definition is the basis of some advantages that the author attributes to the hierarchical classification, such as leading to a greater economy in the formulation of definitions and theorems, and simplifying the deduction of properties to more specific concepts. Despite the advantages of hierarchical classifications, partition classifications are not wrong – many are quite important, as the classification of polygons on convex and concave.

2.2. Learning to classify geometric figures hierarchically

Even though the classification process is inherent to our human condition and is present in many activities, there are several

¹ In Portugal, K-6 prospective teachers take a bachelor's degree in Basic Education for three years and then a master's degree targeted to the level and subject areas in which they will be teaching.

factors that make it considerably complex. According to [Mariotti and Fischbein \(1997\)](#), in geometry, theoretical classifications frequently resort to structural criteria which are not immediately clear and “are far from those perceptual criteria to which we are used to refer in our spontaneous activity of classification” (p. 244). For instance, mathematically it does not make sense to put rectangles and rectangular parallelepipeds into the same category, but it is natural that individuals consider them to have the “same shape”. In addition, by declaring an equivalence among objects of the same category, we attend to the similarities that the objects have among them, but we need to ignore their differences, which contradicts our natural tendency to differentiate objects. Individuals tend to add unneeded or false attributes to figures based on a prototype, a behaviour that [Hershkowitz \(1989\)](#) characterizes as follows: prototypical judgement type 1 – “the prototypical example is used as the frame of reference and visual judgment is applied to other instances” (p. 74); prototypical judgement type 2 – “the prototypical example is used as the frame of reference but subject bases his judgement on the prototype self attributes and tries to impose them on other concept examples” (p. 74). In both cases, the prototypical example is considered “the representative concept” and the other cases are judged by their “distance” from this representative case. Opposed to these judgements, the desirable behaviour is to develop an analytical judgement (type 3) where “the critical attributes are used as a frame of reference in the formation of geometrical concepts” (p. 74).

The difficulty to classify tends to be bigger when we are dealing with hierarchical classifications. In addition to the above-mentioned aspects, the establishment of hierarchical classifications involves the comprehension of an important set of understandings, systematized by [Markman \(1989\)](#), which we illustrate using quadrilaterals:

- We may apply two labels to the same figure;
- Transitivity relation between concepts – for example, if a square is a rhombi and if a rhombi is a parallelogram, then a square is a parallelogram;
- Asymmetry of the relations between quadrilaterals – for instance, all rectangles are parallelograms, but not all parallelograms are rectangles;
- Asymmetry of the properties between quadrilaterals – for example, the properties of the rectangles are still properties of the squares, but not all properties of the squares are properties of the rectangles.

The last aspect corresponds to “the opposing direction inclusion relationship”, an expression used by [Hershkowitz, Bruckheimer, and Vinner \(1987\)](#) to underline a potentially conflicting idea: the set of attributes of a rectangle is a subset of the attributes of a square, but a square is a rectangle (meaning that the set of squares is a subset of rectangles).

Several studies about the classification of figures draw on the Van Hiele levels. According with [Battista \(2009\)](#), at level 1 (visual-holistic reasoning), figures are recognized by their appearance as visual wholes; the language is mainly influenced by prototypes and by the orientation of figures. At level 2 (descriptive-analytic reasoning), students attend to, conceptualize and specify shapes by describing their parts and relations among parts, initially, using informal language and progressively incorporating formal geometric concepts, as they are taught in the curricula. In the end, they only use formal concepts, but cannot relate properties or understand that a subset of properties implies another. At level 3 (relational-inferential reasoning), individuals infer relations among properties of figures and make simple logic deductions; it is at this level that they understand the hierarchical classification of figures, although they may resist accepting some relations in an initial phase at this level. At level 4 (formal deductive proof), students completely understand logic deduction and demonstrate; lastly, at level 5 (rigor), students use, understand and analyse axiomatic systems.

Still in the tradition of the research developed based on the theory of Van Hiele, other researchers adapted the levels to other topics. One of those adaptations was proposed by [Gutiérrez, Jaime, and Fortuny \(1991\)](#) and concerns three-dimensional figures, while keeping coherence with the original formulation to plane figures: at level 1 (recognition), solids are visually identified in a holistic way, without reference to their components or properties; at level 2 (analysis), students identify components and properties of solids, which are described informally, but still do not logically relate nor classify families of solids; at level 3 (informal deduction), students can classify families of solids and understand their definitions, as well as provide informal arguments for their deductions; at level 4 (formal deduction), students understand the role of different elements in an axiomatic system and produce formal proofs. In this adaptation of the Van Hiele levels, the researchers put the classification of solids at level 3, the same level other authors consider for the classification of plane figures ([Battista, 2009; Fujita & Jones, 2007](#)).

The conclusion that only at Van Hiele level 3 people may dominate the hierarchical classification of figures may be explained by using [Hershkowitz \(1989\)](#) categorization of the reasoning that students use to analyse figures. At Van Hiele level 1, students recognise figures by appearance, so they use the prototype example as reference and the judgement is affected by the visual attributes of prototypes, which they try to impose on other figures. For instance, students may argue that a square is not a parallelogram because is not slanted (prototypical judgement type 1). At level 2, students conceptualize figures and relations between their parts, but they do not relate properties, so they may accept that a square is a parallelogram because it has opposite sides parallel (analytical judgment), but they also may argue that a square is not a parallelogram because it does not have two acute and two obtuse angles (prototypical judgement type 2). At level 3, students infer relations among figures and make simple logic deductions. In this sense, students may recognise that if a square has four congruent sides, the opposite sides are congruent, so it is also a parallelogram (analytical judgment).

According to [Hershkowitz \(1989\)](#), prototypical judgment type 2 might represent the transition between the first two Van Hiele levels, which contradicts the idea that these levels are discrete. Moreover, her research shows that behaviour change from one concept to another and that it “differs with experience and knowledge within a given concept” (p. 75). Also [Gutiérrez et al. \(1991\)](#), in their study, point that the most significant aspect highlighted is the possibility to situate the same student at two different levels, which is “probably depending on the difficulty of the problem” (p. 250). Such result does not question the hierarchical structure of

these levels, but reminds that “people do not behave in a simple, linear manner, which the assignment of one single level would lead us to expect” (p. 250).

2.3. The hierarchical classification of figures and the use of DGE

The hierarchical classification of geometric figures requires the mastery of many aspects previously mentioned, which have been identified as problematic in empirical studies involving students, teachers and prospective teachers. Some studies report difficulties related to the use of the prototypical examples (e.g., [Erdogan & Dur, 2014](#); [Fujita & Jones, 2007](#); [Monaghan, 2000](#)), which might reflect in a different manner depending on the individual and the figure ([Okazaki & Fujita, 2007](#)), even when individuals know a correct definition ([Fujita, 2012](#)). Furthermore, [De Villiers \(1994\)](#) refers that many difficulties frequently arise from misunderstandings about the classification process itself and not necessarily from a lack of knowledge of the properties of the figures.

The prototypical effect, most referred in these studies, is often related to static mental images of shapes. For instance, the work of [Monaghan \(2000\)](#) suggests that “students use terms such as diagonal, vertical, and horizontal, as fixed and defining attributes of specific shapes rather than as descriptors of particular representations of individual cases” (p. 190), showing a static image of the concept. This phenomenon relates to the roles that diagrams play in geometric reasoning. As [Battista \(2008a\)](#) states, on one hand, particular instances of diagrams and objects contribute to the formation of geometric conceptualizations because geometric concepts derived from the analysis of those instances, i.e., they are usually abstractions of particular instances. On the other hand, diagrams and objects can be used to represent formal geometric concepts. So, for instance, a set of representations of the concept of rectangle contributes to the formation of this concept, but the concept may be represented by a single case. Therefore, students often “attribute irrelevant characteristics of a diagram to the geometric concept it is intended to represent” ([Battista, 2008a](#), p. 347).

Dynamic geometric environments (DGEs) play an important role in dealing with this problem. As [Battista \(2008a\)](#) claims, research on DGEs presents two major perspectives on “draggable drawings”: (i) they are seen as generating numerous examples, which might remedy the difficulties that students face when they encounter a restricted lack of examples of a concept; (ii) geometrical relationships can be visualized as invariants when the draggable figure moves. Some empirical studies reflect these perspectives, but also unveil issues to consider and questions for further research. Next, we present three studies, involving children of different ages and grades, which constitute examples of these perspectives.

[Sinclair and Moss \(2012\)](#) conducted a study among 4–5 year-old children working with Geometer’s Sketchpad on a 30-min lesson during which the children observed, described, created and transformed triangles of different sizes, proportions, and orientations. The researchers did not intend to classify triangles, only that the children would broaden their restricted range of three-sided polygons recognized as triangles. In the beginning of the experience, the children agreed that, in order to construct a triangle, “you do three sides and connect them”, but some of them did not recognise long and skinny examples as triangles, or suggested that some of them were “upside down”. Throughout the intervention, the diversity of three-sided polygons they were prepared to call “triangle” grew substantially, as the possibility of manipulating the figures encouraged the children to abandon some restrictions that they had formerly imposed. However, considering the short amount of time and experience, the researchers question if, once children find themselves in a static environment, they would maintain their thinking.

[Battista \(2008b\)](#) also developed a study based on an intervention with 5th graders using Shape Makers microworld, a special add-on to the DGE Geometer’s Sketchpad. This DGE allowed dragging quadrilaterals and triangles maintaining their properties, as the students worked on a sequence of instructional tasks. Some of the tasks focused on classification issues. For instance, students were asked to use the seven Shape Makers (square, rectangle, trapezoid, kite, rhombus, parallelogram and quadrilateral) to reproduce a static drawing of a clown consisting of three squares, two oblique parallelograms, a rectangle and a concave kite. To do so, they had to transform the Shape Makers to overlap those quadrilaterals (for example, the Parallelogram Maker had to overlap a rectangle), which promotes the idea that some quadrilaterals are particular cases of others, i.e., the hierarchical classification of quadrilaterals. [Battista \(2008b\)](#) suggests that “investigating shapes through Shaper Maker transformations makes the essence of their properties more psychologically salient to students than simply comparing examples of shapes as is done in traditional instruction” (p. 152). However, the author alerts that the conceptualization of the movement constrains requires much guidance, reflection and experimentation so that “students’ conceptualizations become positively stated, not has what a Shape Maker cannot do, but as what regularities or invariants hold for Shape Makers movements” (p. 149).

[Kaur \(2015\)](#) built on the work of [Battista \(2008b\)](#) and designed a project to work with younger children (ages 7–8, grades 2–3) in order to use the potential of DGEs to develop an understanding of, and reasoning about, the properties and behaviours of different triangles, including the classification of an equilateral triangle as an isosceles triangle. The children dragged the sides and vertices of the triangles, thus generating many non-prototypical examples. At the beginning they used mostly action verbs (“go”, “moves”, “staying”, “paralysed”...) showing the tendency to reason in terms of motion when comparing different types of triangles. During the discussions, “the children’s routines moved from informal dynamic descriptions to formal geometric properties as well as from particular to more general discourse about different types of triangles” (p. 407). From the point of view of classification, the initial colour-coding of the triangles (different colours for isosceles and equilateral), while useful as an initial means of communication, may have worked against the idea of inclusive relations.

Returning to general research on classification of geometric figures, although there is a significant number of studies involving students of different ages, teachers and prospective teachers, the results of research concern, in fact, almost totally, the hierarchical classification of quadrilaterals and, in a few cases, triangles. For example, the recent revision of studies presented within the scope of PME ([Jones & Tzekaki, 2016](#)) from 2006 to 2015, includes a section referring to the hierarchical relations involving solely those polygons. Also, the review of published studies since 2008 in articles, proceedings and books, prepared by [Sinclair et al. \(2016\)](#) in an

ICMI13 survey, includes an item about the definitions of triangles and quadrilaterals and consequent analysis of their classifications, but with no reference to other types of figures.

Therefore, there is a significant body of knowledge about the classification of quadrilaterals that points to common obstacles, namely the influence of prototypical images that tend to subdue the reasoning about geometric figures. This fact is considered problematic as, like Clements and Sarama (2011) refer, “such limited education, from their own early years on, leaves teachers under-prepared for teaching geometry” (2011, p. 136), whether at preschool or at more advanced stages of schooling.

2.4. Prospective elementary teacher education in geometry

For the National Council of Teachers of Mathematics [NCTM] the knowledge necessary for teaching includes “the content and discourse of mathematics, including mathematical concepts and procedures and the connections among them; multiple representations of mathematical concepts and procedures; ways to reason mathematically, solve problems, and communicate mathematics effectively at different levels of formality” (NCTM, 1991, p. 132). This perspective is coherent with the idea advocated by Ma (1999) that teachers need a profound understanding of fundamental mathematics. Regarding geometry, the NCTM (1991) states that all teachers should understand how geometry is used to describe the world we live in and how it is used to solve concrete problems; analyse a diverse set of two and three dimensional figures; use synthetic geometry, coordinates and transformations; improve their skills in producing arguments, justifications and privilege spatial visualization.

The education of teachers concerns also the ways they are taught. Regarding the results of several studies about prospective teachers’ knowledge of mathematics, Watson and Mason (2007) propose that courses should prompt prospective teachers to engage in mathematical thinking through working on suitable mathematical tasks, develop their understanding about the features and power of those tasks, reflect on the experience of doing mathematical tasks individually or with others, challenge approaches dominated by procedures which depend on rote memorization, and observe and listen to learners. These orientations are also consistent with ideas underlined by other researchers, according to whom prospective teachers should learn using the same methods that are recommended they should use with students (Ponte & Chapman, 2008) and also that connecting subject matter knowledge and pedagogy is a promising strategy to develop both kinds of knowledge and their integration, a critical to teach well (Ball, 2000).

The experiment that we present follows this perspective, as we focus on prospective teachers’ learning as they work on exploratory tasks and reflect on their own learning (Ponte, 2005). Exploratory tasks demand learners to engage actively in the construction of their knowledge by solving tasks where there is no clear solving method, as it is the case of problems. Such is also the case of investigations, in which learners are challenged to ask questions or extend the purpose of the task. They need to interpret the given information, develop strategies, represent and communicate their solutions. This promotes their understanding of representations, concepts, and procedures, and also develops the ability to argue about ideas, as they communicate such ideas to others. Work on exploratory tasks develops usually in three phases: (i) presenting and interpreting the task; (ii) carrying out the task individually, in pairs, or in small groups; and (iii) presenting and discussing results in whole class, ending with a final synthesis.

3. Research methodology

This paper addresses an investigation with an intervention, in order to change practices and enhance prospective teachers’ preparation in geometry, particularly the process of classifying geometric figures, where we found difficulties in previous experiences also reported in Portuguese studies (e.g., Brunheira & Ponte, 2015; Tempera, 2010). The research focus is on learning in context, beginning from the conception of strategies and teaching tools, following a design-based research as methodology, in the form of a prospective teacher experiment (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) in which the teacher also plays the role of researcher. The design of the experiment is guided by the following conjecture: the classification process of geometric figures i) promotes knowledge about properties of geometric figures; ii) promotes the understanding of the classification process and of the concept of class; and iii) promotes reasoning.

The experiment took place in a class of 25 prospective teachers that attended a geometry course in the second year of a bachelor’s degree in basic education. This course lasts 15 weeks, with two lessons of 2 h 15 min per week, making a total of 67 h 30 min of classroom work. We designed and implemented two sets of lessons focusing on the hierarchical classification of figures. The first set concerned the classification of quadrilaterals and the second, two months later, the classification of prisms. In both cases, the class produced *a posteriori* classifications, as this work was preceded by the analysis of some of the properties of the figures. Table 1

Table 1
General plan of the intervention.

Week	Task	Time spent
1	Diagnostic test*; four individual interviews*	2 h
2	Investigation of the quadrilaterals’ properties (using GeoGebra)*	4h30
3	Classification of quadrilaterals*	2 h15
3	Definition of quadrilaterals (construction of inclusive definitions)	2 h15
12	Investigation of prisms’ properties (using manipulative solids)	2 h15
12	Classification of prisms*; Definition of parallelepiped	2 h15
15	Final evaluation test*	2 h15

Table 2

Levels of understanding of the inclusive relations in the class of quadrilaterals and prisms (adapted from Fujita, 2012).

Level	Description
Hierarchical classification	Learners can accept squares, rectangles and rhombi [cubes, rectangular parallelepipeds, parallelepipeds and quadrangular prisms] are also parallelograms [prisms]. “The opposing direction inclusion relationship” of definitions and attributes is understood. Their judgement is likely to be analytical (type 3, based on critical attributes).
Partial prototypical classification	Learners have begun to extend their figural concepts, so they accept some relationships but not others. For example, they accept that rhombi are also parallelograms [parallelepipeds are prisms] but not squares and rectangles [cubes]. Their judgement is likely prototypical type 2 (based on attributes).
Prototypical classification	Learners who have limited personal figural concepts. Their judgement is either prototypical type 1 (based on visual appearance) or 2 (based on attributes).
0	Learners do not have basic knowledge of parallelograms [prisms].

presents all the tasks related to the process of classifying geometric figures, indicating those reported in this article (with an asterisk), the week in which they were implemented and the time spent on each. From the methodological point of view, the lessons followed an exploratory approach, beginning with an introduction of the task by the teacher, followed by work in small groups, and finishing with a whole class discussion (Ponte, 2005). Although they generally worked in groups, in several occasions the participants were encouraged initially to find an individual solution, discussing it within the group and, at the end, in a whole class discussion.

We present quantitative data to give a general idea about the class performance, but also qualitative data from participants’ records of tasks or from dialogues among them that illustrate the reasoning underlying the solutions. All dialogues were collected from the same group (with Tita, Helena, Fernanda and Cristina), the same participants we interviewed in the beginning of the course. This group was chosen because they had different backgrounds concerning mathematics, different attitudes concerning their confidence towards the learning of mathematics and different expectations relating their professional career (as kindergarten teachers, primary teachers or middle school mathematics teachers). These data were collected from audio and video records of the lessons, as well as from the participants’ records of tasks.

To analyse the way prospective teachers classify quadrilaterals and prisms, we use a model by Fujita (2012) for the levels of understanding of the classification of quadrilaterals. This researcher assumes that the understanding of the inclusion relations between quadrilaterals develops through certain levels by considering Van Hiele’s visual, analytical, informal deductive and deductive levels. The model also takes into account the theory of figural concepts by Fischbein (1993), who suggests that geometric figures are mental entities controlled by a figural component and a conceptual component and, given the importance of the prototype phenomena, incorporates Hershkowitz (1989) types of prototypical judgment in order to characterize students reasoning when they add unnecessary or untrue attributes to figures. Table 2 illustrates Fujita’s (2012) framework for the parallelograms class but, analogously, we may consider similar frameworks for the families of trapezia or kites. For the classes of rectangles or rhombi, the framework has one less category (partial prototypical classification) since the only type of quadrilateral belonging to those classes is the square. In our study, we follow Fujita’s categories of analysis but we extended it to the classification of prisms, using the analogy to the three-dimensional space.

At the hierarchical level, the analysis an individual does of the figures is based on critical attributes, disregarding attributes such as acute and obtuse angles that only oblique parallelograms present. Individuals also realize that, although the critical attributes of parallelograms are a subset of the critical attributes of rectangles, the set of rectangles is a subset of the parallelograms, which requires logical reasoning. At the partial prototypical classification level and the prototypical classification level, individuals judge figures by their “distance” to the prototypical image. So, if they began to extend their figural concepts, they might accept some quadrilaterals as parallelograms, particularly if they have more common critical attributes. However, they still consider attributes that are noncritical, such as different consecutive sides, which prevent a complete hierarchical classification. In addition, if their figural concepts are very limited, they do not relate parallelograms to other quadrilaterals because they focus on noncritical attributes that may be verbalized or associated with the visual appearance of the prototype. Finally, the last level concerns individuals that do not identify simple attributes, so they do not have the minimal knowledge, even for prototypes.²

4. Results

4.1. Classifying quadrilaterals

In the first lesson, the class did a multiple-choice diagnostic test, which included a question about the inclusive relation between squares and rectangles. The results showed that only 44% consider true the statement “all squares are rectangles” and acknowledged that not all rectangles are squares. Additionally, in an interview to four of the participants that answered correctly to this question (Tita, Helena, Fernanda and Cristina), we realized that only two (Helena and Cristina) understood the reason why a square is a rectangle, answering “because it has four right angles”. These data confirm that there are some participants who know something about this relation, but only as a fact. No other hierarchical relations were acknowledged, which was no surprise because the

² An interrater reliability study done independently by the two authors showed 94% of agreement. The few cases of disagreement concerned responses with contradictory or insufficient information.

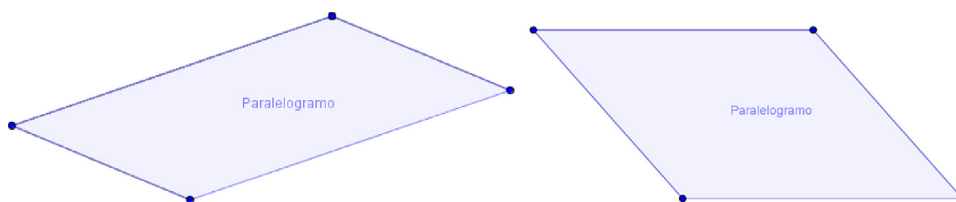


Fig. 1. Parallelogram's representation before and after dragging.

Portuguese curriculum did not include this subject at the time the participants were at school.

The work on the hierarchical classification of quadrilaterals begun with an investigation about their properties, using GeoGebra and dynamic pre-constructed sketches of squares, rectangles, parallelograms, rhombi, trapezia and kites.³ The prospective teachers were supposed to recognise each figure shown in the screen as a representative of its class and to study the invariant properties related to sides, angles, diagonals and axis of symmetry, making it possible to indicate the critical attributes of each class. Although we did not ask for relationships between quadrilaterals, this task intended that the prospective teachers began to expand the figural concepts, thus, understanding that some quadrilaterals are particular cases of others. Simultaneously, this activity gave us more elements about the way participants conceptualized quadrilaterals in the beginning of the course.

The participants had no previous experience using a DGE which strongly influenced the participants' behaviour in manipulating the figures. Instead of dragging the points and analysing the figure through multiple representations, initially some of the prospective teachers avoided the ones that did not match the prototypical image. For instance, the first reaction Tita had when she opened the parallelogram's sketch was to transform it as shown in Fig. 1.

In fact, Tita's group wrote the conclusions about the properties of sides and angles of the parallelogram using a static representation, which resulted in statements such as "two acute angles and two obtuse angles; opposite angles are congruent." However, as they moved to more unfamiliar quadrilaterals and the investigation of invisible elements such as diagonals, they began to drag the dynamic sketches. This action led to cognitive conflicts because they had to deal with non-prototypical examples and particular cases. The following dialogue shows how participants were intrigued by the representation of a kite that appeared on the screen (Fig. 2) that is different from a prototypical image:

Tita: So, this is still a kite?

Teacher: It is still a kite, exactly. It does not correspond to the image we usually have of a kite. Nonetheless, it is still a kite.

Tita: We tried to change it, but, afterwards, we thought that it stopped being a kite.

Teacher: Well, you can change it as you wish, it is constructed in a way that...

Tita: But we can put it a rhombus, but in that case it is not one [kite], is it?

Teacher: If it comes to be a rhombus that means [the rhombus] is a particular case. [Helena looked at Cristina, suspiciously]

This dialogue shows that, just like younger students from previous studies, these participants had limited figural concepts and that the DGE gave them the opportunity to broaden the images associated to the concept of kite. In addition, their language focus much more on the movement and transformation, than on the properties. From the point of view of the levels of understanding of inclusive relations, concerning kites, these prospective teachers were at the *prototypical classification* comprehension level and their judgement about the figures probably was based on their appearance, because they had no measures nor tried to test any property (prototypical judgment type 1). In fact, it was necessary to explicitly stimulate the participants to consider all the representations on the screen as particular cases of a class, instead of disregarding the examples that did not match the prototypical image. After the teacher's intervention, Tita, Cristina and Helena returned to the analysis of the kite:

Tita: So the diagonals... The largest bisect the smallest. No, wait! It may be the other way around!

And it may happen that they have the same sides, with the same length!

Helena: No...

Tita: But that is an exception.

So, although the task did not ask explicitly to consider the relationships among figures, the participants began to show up when each quadrilateral was investigated using GeoGebra. In most cases, like this group of prospective teachers, the teacher insisted on considering many representations of each class, including particular cases. Besides, usually one of the elements of the group understood better this suggestion and prompted the others to consider it, as it happened with Tita in this group. This confirms the strong need for support and reflection suggested by Battista (2008b) and that the link between movement constrains or possibilities to the conceptualization of the properties of figures is far from immediate.

After investigating the properties of the quadrilaterals using GeoGebra, the class presented and discussed collectively their findings. Then, they moved to the next task, in order to systematize the relationships between the quadrilaterals that emerged from the investigation, using a Venn diagram and a flowchart (Fig. 3).

³ As described for the Shape Makers microworld (Battista, 2008b).

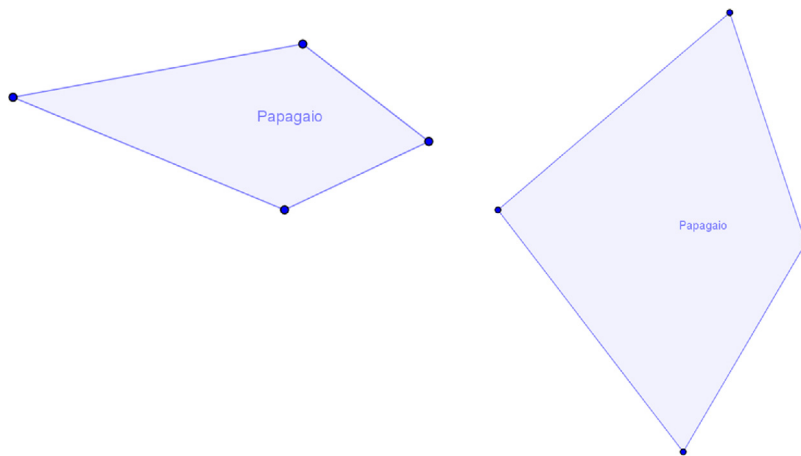


Fig. 2. Kite's representation before and after dragging.

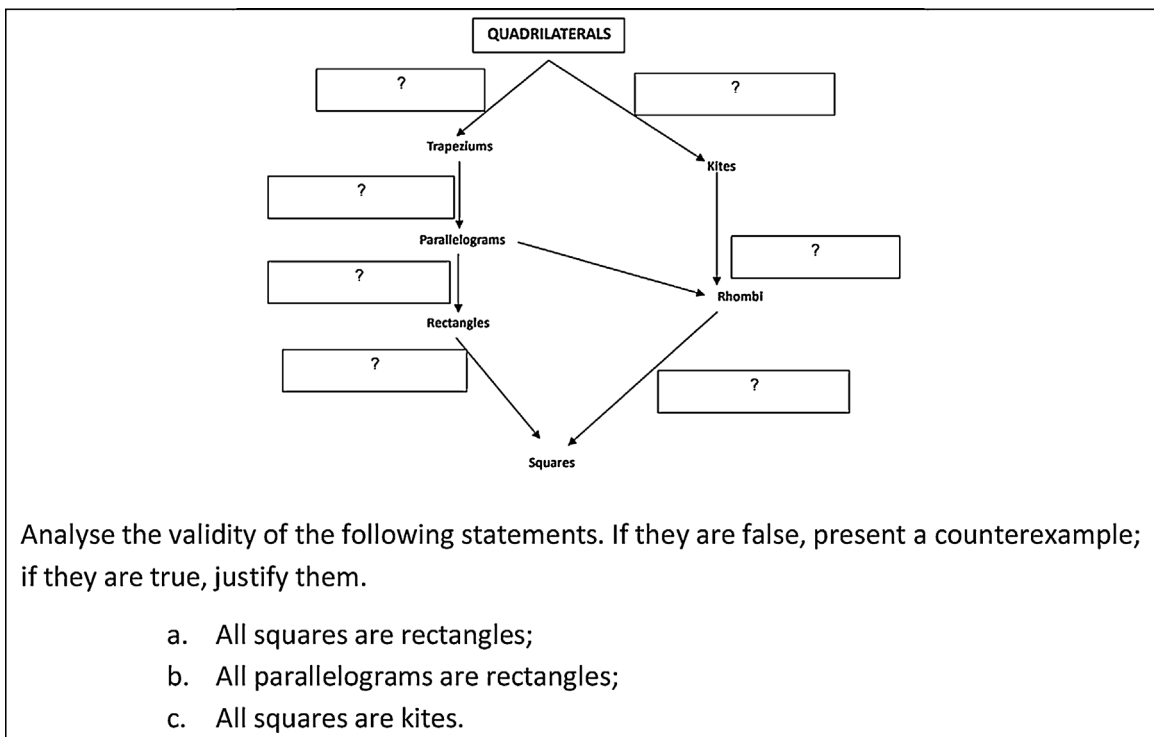


Fig. 3. Excerpt from the task on classification of quadrilaterals.

In the flowchart the participants had to fill in the spaces with the properties that would lead to the next quadrilateral. In some cases, there was more than one possibility, but they had to consider also the quadrilaterals appearing in “lower levels”. This prompted many discussions:

Cristina: It is “Having parallel sides/Without parallel sides” [referring to the first two spaces above]. No, but the rhombus has parallel sides...

Tita: No, we’re talking about the kite!

Cristina: Yes, but here it can’t be “Without parallel sides” because the rhombus has parallel sides.

Tita: So it must be... “Two pairs of consecutive sides equal”... It is also true for rhombi. What about “perpendicular diagonals”?

Cristina: No...

- a. True. All properties of the rectangles are included in the properties of the squares, such as four right angles and two pairs of parallel sides. The square is a particular case of a rectangle.

Fig. 4. Isabel's answer to question a of the quadrilaterals task.

Tita: It fits the kite, rhombus and square.

Helena: Yes, I think we should say “perpendicular diagonals”.

The group ended up writing “two pairs of consecutive sides equal” for the kite. However, even if they did not realize that perpendicular diagonals was not enough to produce a kite, this discussion turned their attention to the critical attributes of a class, which represents a significant step towards the hierarchical classification. This properties were further explored by the definition task, where they had to critique and produce definitions for some quadrilaterals.

We will now discuss how the prospective teachers analysed the validity of the three statements presented in Fig. 3. The first statement had a 100% success, as all answered correctly, presenting an answer in accordance to the *hierarchical classification level* of rectangles. The justifications were very similar to the answer in Fig. 4, and only three participants argued for the statement's veracity solely based on the existence of four right angles. These answers focus on the critical attributes (analytical judgment) and acknowledge “the opposing direction inclusion relationship”.

The second statement (all parallelograms are rectangles) obtained an identical success. The given answers showed different levels of sophistication, potentially indicative of the different ways of thinking. Among these answers, 79% were based on the critical attributes of parallelograms, showing the understanding that a rectangle is a parallelogram, but not all parallelograms are rectangles. It is the case of the answer shown in Fig. 5. The remaining 21% answers were solely based on the asymmetry of relations between the parallelogram and the rectangle, simply stating “false, since the rectangle is a particular case of the parallelogram”, a statement that, although correct, does not provide a proper justification.

Therefore, the first two statements had complete success, putting the prospective teachers at the level of *hierarchical classification* for the figures in question. However, the analysis of the answers to the third statement suggests that the fact that the square and the kite do not have a “direct” relation (considering the flowchart or the Veen diagram previously analysed) or have less common attributes, may alter significantly the degree of difficulty of the question. In fact, slightly more than half of the participants acknowledged that squares belong to the class of kites, making it possible to organize their answers in the following way (Table 3).

Let us analyse some answers corresponding to these categories. In the group of 10 participants corresponding to the *hierarchical classification level* of kites, four of them used the transitivity relation, without reference to any properties, assuming that a square is a rhombus and a rhombus is a kite, therefore a square is a kite. The remaining six gave similar answers to the one presented in Fig. 6, distinguishing among themselves by the quantity of properties mentioned. This answer focus on critical attributes of the kite (analytical judgement) and acknowledges that the square has an attribute that is noncritical to be considered a kite.

The answer in Fig. 7 is representative of the answer of five prospective teachers that only recognises “direct relationships” and corresponds the *partial prototypical classification level* of kites. In this level, individuals identify some inclusive relations, but not others. These participants accept the inclusion of squares in the class of rhombi; others also refer the inclusion of rhombi in the class of kites. However, they seem to have the idea that only one type of quadrilateral can be considered a particular case of another.

Finally, we have another group of four prospective teachers that presented answers that seem to correspond to a level of *partial prototypical classification* of kites because they argue based on critical attributes. However, as the answer shown in Fig. 8 suggests, the underlying reasoning maybe also based on prototypical images (prototypical judgement type 1 and 2). With respect to properties, the participants did not seem to find the relation between the properties “four equal sides” and “consecutive equal sides two by two”, meaning that they did not understand that the first statement implies the second one. An explanatory hypothesis is the incorrect interpretation of the property “consecutive equal sides two by two”, which might have been understood in a strict sense, that is, “solely consecutive equal sides two by two”, which coincides and is reinforced by the image of the prototype figure.

To better access the reasons that might restrain individuals of assuming this relation, we present the following episode that took place in the group we follow, when each one shared their answer. In the individual record, Tita answered correctly based on the relation of transitivity, while Fernanda and Helena gave incorrect answers:

Tita: Kites have no parallel sides... I mean... Here it is written “the square is a particular case of a rhombus”. If a rhombus is a kite, then... then... Did you answered “false”, in the third one?

- b. False, because rectangles are particular cases of parallelograms – both have two pairs of parallel sides – but parallelograms may not have four right angles.



Fig. 5. Anabela's answer to question b of the quadrilaterals task.

Table 3

Performance of the prospective teachers regarding question c, using Fujita's (2012) model.

Level	Hierarchical classification		Partial prototypical	
Frequencies	10 (53%)		9 (47%)	
Type of answer and frequencies	Uses transitivity 4 (21%)	Uses the properties of the figures 6 (32%)	Only recognizes "direct" relationships 5 (26%)	Misinterprets "two pairs of equal consecutive sides" 4 (21%)

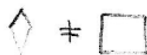
c. It is true. In order to be labelled a "kite", a figure has to have two pairs of consecutive sides of the same length, a condition satisfied by the square, although this is a particular case, having all sides equal.

Fig. 6. Carla's answer to question c of the quadrilaterals task.

c. False because all squares are rhombi. A square is a particular case of a rhombi.

Fig. 7. Sandra's answer to question c of the quadrilaterals task.

c. False. A kite has to have consecutive sides equal 2 by 2, on the contrary, a square has to have 4 sides equal.

**Fig. 8.** Anita's answer to question c of the quadrilaterals task.

Fernanda: Yes... The rhombus is a particular case of a kite.

Tita: And, if a square is a particular case of a rhombus... then that means it is also a kite! Oh, I can't understand a thing... This makes no sense...

Tita: [consults her record sheet of the properties of quadrilaterals] Two pairs of equal consecutive sides, a pair of opposite congruent angles, the diagonals are perpendicular, one of the diagonals bisects the other. Of course, since the square has all the characteristics of the kite... And more. So... it makes sense!

Meanwhile, the teacher was passing by the group and notices Helena's confusing expression:

Teacher: Helena is not convinced... Talk to me.

Helena: I don't know...

Teacher: Is it hard to accept it, is that it?

Helena: Yes! But I also think it is because we see that they are so different... because a rhombus and a square are... ok, now these are so different...

Tita: Yes, for me it makes no sense at all...

The teacher left the group to reflect upon the subject a bit longer. Although they kept on thinking, the prospective teachers seemed yet confused:

Helena: That they are "descendent", I get it, but it frustrates me to see that a kite has no parallel sides, so how is it a square also a kite??? If a square has all parallel sides! This was the first thing I wrote on the worksheet!

Tita: Yes, yes, kites have no parallel sides and squares have them all.

Helena: Those opposite.

Tita: Yes. I do not know what to say, seriously... I am confused.

Our first comment concerns Tita's reasoning. Although she answered correctly, whether using transitive property or deducing the relation from the critical attributes of the kite that does not prevent her from doubting or being disturbed by her own conclusions. In her discourse, she begins by producing a prototypical judgement type 2 because she bases her reasoning on the attributes of kites but

adds unnecessary ones. However, when she restricts her reasoning to critical attributes, she accepted the relationship. Fernanda had a very limited intervention, which prevents us from truly understanding her reasoning, besides the fact that she already acknowledged some relationships. However, Helena's statements give us interesting clues about the obstacles that this prospective teacher found in accepting that a square is a particular case of a kite. On one hand, Helena's representation of the square was very "distant" from the one she associates to the kite, which leads us to the prominent influence of prototypical images (prototypical judgement type 1). On the other hand, the analysis of the properties did not help her, as Helena made the mistake of imposing a noncritical attribute to kites (nonexistence of parallel sides) and judge the square by it (prototypical judgement type 2).

In the collective discussion, the teacher addressed the difficulties that she found while supporting the work of the group. She was surprised by the fact that the participants acknowledge the relationship between squares and rhombi, and between rhombi and kites, but did not recognise the relationship between the squares and kites. She asked Helena to share her thoughts:

Helena: I find it very strange because when I see a kite – this is the one that I think is the most shocking – I never see a square there, that's why I find it hard to see that a square is a particular case of a kite.

Teacher: So let's see. The image that we have of a kite is very different from the image we have of a square and so of course we react against it, it costs us to accept that. This is natural. But this is what we have to see: does the square meet the conditions of the kite? Let's check.

The discussion continued with the prospective teachers stating the properties that they wrote when they made the investigation using GeoGebra and verifying if the square met those properties. This was also discussed using other quadrilaterals, such as the parallelogram.

Therefore, in the beginning of the course, the participants showed very limited figural concepts of quadrilaterals, ignored some properties (especially those related to diagonals or more unfamiliar quadrilaterals) and did not relate quadrilaterals or think about them as elements of a class. The work undertaken favoured the evolution of the participants that extended their figural concepts and begun to consider the organization in hierarchical classes. However, just about half of the class seemed to evidence understanding of all relationships among quadrilaterals and still struggled with it. The teacher addressed those difficulties in the collective discussion of the task, encouraging an analytical judgment of the geometric figures, instead of an analysis based on the prototypical images.

4.2. Classifying prisms

The classification of quadrilaterals begun with the analysis of their properties (using GeoGebra), so the classification of prisms took the same path as quadrilaterals, but, in this case, using manipulative solids. The participants were engaged in a classification game following these rules: the teacher presents a set of solids on a table; she thinks of a criterion for separating the solids into two subsets – one set has the attributes that meet the criterion (set A) and the other does not (set B). A person gives a solid to the teacher who places it in the respective subset. This action is repeated until the criterion thought by the teacher is discovered. In order to do so, the participants have to identify the attributes that are common to the set A but that no element belonging to the set B meets. This reasoning is consistent with the process of classifying, has suggested by [Mariotti and Fischbein \(1997\)](#). This task was developed using the following criteria: polyhedra/other solids; prisms/other solids; parallelepipeds/other solids (see [Fig. 9](#)).

After the game, the class worked on a classification task focusing on prisms (see [Fig. 9](#) for an excerpt of the task). They analysed which properties characterized straight and oblique prisms, and parallelepipeds. Subsequently, they organized a new Venn diagram (question 8) and identified the relations among classes (question 9).



Fig. 9. Organization of the solids during the game using the parallelepipeds criterion.

Prisms

Regular prisms **Parallelepipeds**

2. What distinguishes regular prisms from other prisms?
...
6. Analyse the validity of the following statements:
a. The parallelepipeds are regular prisms; b. There are regular prisms that are parallelepipeds; c. All quadrangular prisms are parallelepipeds.
...
8. Construct a Venn diagram that includes the following classes of solids: quadrangular prisms, cubes, parallelepipeds, prisms, rectangular parallelepipeds.
9. State two relations that result from the previous organization.

Fig. 10. Excerpt from the prisms task.

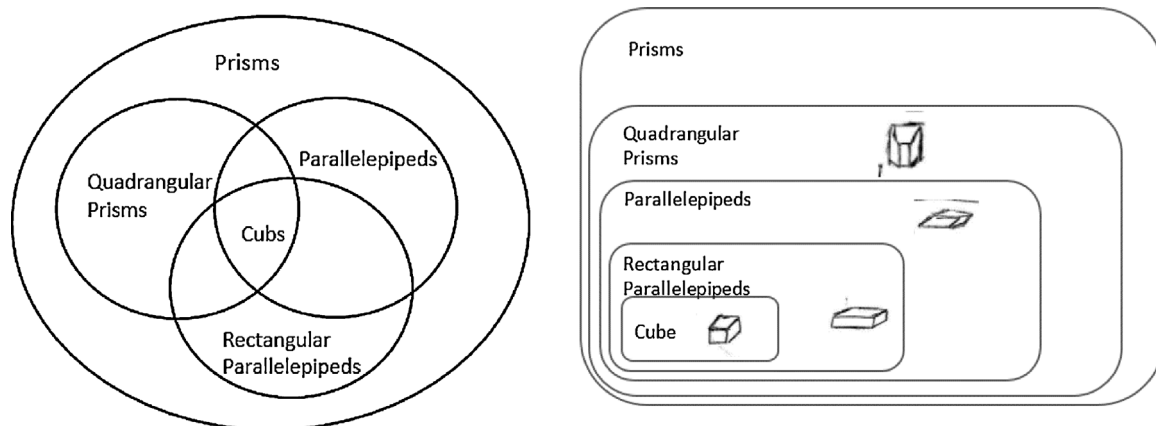


Fig. 11. a and b – Júlia’s first attempt and final answer of question 8 of the task.

The participants showed difficulties in answering question 2 because they were looking for a single attribute that distinguishes regular prisms from other prisms, and did not consider a conjunction of attributes. On the contrary, they answered question 6 easily, although the teacher had to clarify the meaning of “quadrangular”.

We focus now on the results of questions 8 and 9. Up until that moment, all tasks involving Venn diagrams presented the criteria setting each class or examples for each class, like the one presented in Fig. 10. Question 8 asked for the first time to create a Venn diagram without this information, solely knowing which classes to include. Therefore, knowing that the participants had already

shown difficulties in classifying quadrilaterals, we consider surprising that all of them presented a completely accurate diagram. This does not mean that the participants immediately presented the correct answer, as in several cases the diagram originated discussion and reflection among them, as well as attempts that were subsequently abandoned, as shown in Julia's records (Fig. 11a and b).

Tita, Cristina, Fernanda, and Helena presented answers that were globally correct, but needed to sketch provisory diagrams that were used to think and discuss amongst themselves, as the following dialogue shows:

Cristina: Easy... Quadrangular prisms... It may be a paralelepiped or not... But it can also be a rectangular paralelepiped...

Tita: I think that this [class] is wider.

Cristina: It can be a cube...

Helena: This one?! Can it be a cube?

Cristina: Not this one, the set!

Tita: Are you drawing inside?

Cristina: Why, how are you doing it?

Tita: I am drawing separately... [Disjoint sets]

Cristina: This is a quadrangular prism. It has four sides [the base polygon]. But there are also paralelepipeds that are quadrangular prisms.

Helena: All of them. All paralelepipeds are quadrangular prisms!

Cristina: OK. So, this one [paralelepiped] has to be on both.

Helena: It has to be inside!!! If all paralelepipeds are quadrangular prisms... It has to be inside! Here ["outside" the paralelepipeds but "inside" the quadrangular prisms] is this one, for instance. It has four [edges in the base] and it is not a paralelepiped, but it is a quadrangular prism.

Fernanda: The cube can't also be a quadrangular prism?

Tita: I already know it!!!

This dialogue concerns the inclusion of paralelepipeds in the class of quadrangular prisms. Although there is not an explicit discussion on the properties, we realize that the participants were making that analysis focusing on a critical attribute of these prisms – the base has four edges (analytical judgement). The discussion led to a correct organization, however we highlight another aspect: the participants were thinking about the classes of solids and, so, were considering each solid as an element of such class. Tita's first intervention about the inclusiveness of the class she was thinking of is a sign of that, and so is Cristina's comment about the class of solids that includes the cube (referring to quadrangular prisms), Helena's statement about paralelepipeds being quadrangular prisms, and even Fernanda's question at the end of the episode. Another aspect that arises from this discussion concerns the way the participants faced intersections among classes. Although, initially, Tita and Cristina did not consider the hypothesis of putting each class inside another, for Helena that idea was perfectly clear.

This stage of the work shows that the participants acknowledged relations among several solids and were seeking to find an organization that accommodated all relations coherently. Venn diagrams produced by each one are similar to Júlia's final one (Fig. 11b), globally corresponding to the intended organization. Only Fernanda presented a slightly different diagram, as it does not include the class of cubes that, like we saw in the dialog, raised her some questions. This difficulty concerning the cube may derive from the fact that it is more difficult to hierarchically classify figures for which a person has a stronger conceptualization, as it is the case of the square or cube. In addition, the image of a cube is more "distant" from a common quadrangular prism than it is from a rectangular paralelepiped, an aspect that affects the analyses of the figures.

Concerning the establishment of relations (Question 9), with the exception of a mistake, all participants presented correct relations. Besides, contrasting to what happened with relations among quadrilaterals, the participants presented "indirect" relations, based on the Venn diagram, as Helena's answer (Fig. 12) illustrates.

Helena's answer is a correct example among several others, although very significant, taking into account that, during the classification of quadrilaterals, she struggled with the idea that a square is also a kite. In the class, there is only one partially correct answer (Fig. 13).

Although Vânia made a correct Venn diagram and her first statement is consistent with the diagram's organization, she reverses the implication by saying that "all rectangular paralelepipeds are cubes". This happened often in the participants' oral speech, which they self-corrected most of the times.

Therefore, the organization of classes in a Venn diagram and the establishment of hierarchical relations among prisms correspond

9. A cube is a paralelepiped. A rectangular paralelepiped is a quadrangular prism.

Fig. 12. Helena's answer to question 9 of the task.

9. All cubes, rectangular parallelepipeds and parallelepipeds are quadrangular prisms. All rectangular parallelepipeds are cubes.

Fig. 13. Vânia, question 9 of the prisms task.

Table 4

Performance of the prospective teachers regarding the question in the test, using Fujita's (2012) model.

Quadrilateral		Parallelogram		Kite		
Level	Hierarchical classification	Prototypical classification	Not classified ^a	Hierarchical classification	Partial Prototypical classification	Not classified
Frequencies	15 (60%)	5 (20%)	5 (20%)	14 (56%)	3 (12%)	8 (32%)
			Both hierarchical classification 8 (32%)			
			Both prototypical classification 1 (4%)			

^a These cases correspond to responses with insufficient information or that involve logical contradictions or language inadequacies.

to the level of *hierarchical classification* for the whole class, with some reservations regarding Fernanda that did not put the cube in the diagram and Vânia who made a mistake concerning the relation involving cubes and rectangular parallelepipeds.

4.3. Returning to quadrilaterals

The evolution that we recognise in learning the hierarchical classification of prisms leads us to question a general evolution in the classification process and the reasoning itself. In particular, this means that the participants were, at the end of the course, able to classify figures hierarchically, including the quadrilaterals?

In the final and individual evaluation test, the class was asked to evaluate the validity of the following statements:

A: Parallelograms do not have symmetry axis.

B: All squares are kites.

The answers allowed us to situate the participants in the following levels: Hierarchical classification of parallelograms, if they considered sentence A false and justified using particular cases; Prototypical classification of parallelograms, if they considered sentence A true and justified using oblique parallelograms; Hierarchical classification of kites, if they considered sentence B true and justified using critical attributes of kites; Partial prototypical classification of kites, if they considered sentence B false but mentioned that squares are particular cases of other quadrilaterals. In some cases, the justification was inconclusive, so we present those cases separately. Table 4 present the performance of the prospective teachers regarding this question.

As the results show, approximately 1/3 of the prospective teachers gave answers consistent with a hierarchical classification for the kite and the parallelogram, only one participant answered according to a prototypical classification for both quadrilaterals and most of the participants showed a mixed understanding. In order to get a sense of these cases, we present two answers that address the most common mistakes:

A. False. Common parallelograms do not have symmetry axis. However, rectangles are particular cases of parallelograms because all the angles measure 90° and they do have symmetry axis.

B. False. The correct sentence should be: "all kites are squares" because kites are particular cases of squares since they meet the condition consecutive sides are equal 2 by 2.

Fig. 14. Isabel's answer to the question in the final test.

We may find Isabel's answer (Fig. 14) disturbing because she relates correctly a particular case to a more general case for the class of parallelograms (hierarchical classification level), but she inverts the relation of the square and the kite, even though she identifies a critical attribute that supports the correct relation. In fact, in this case she does not apply "the opposing direction inclusion relationship". So, she probably knows the properties of the quadrilaterals involved, she identifies common critical attributes, but still her logical reasoning shows some weaknesses. It is not a case of prototypical judgment because Isabel does not justify based on noncritical attributes of kites. Although this answer has contradictory elements about her thinking, in fact we found these problems in most of the participants in the category "other". Lucia's answer is also inconsistent from the hierarchical classification point of view. She (Fig. 15) also shows understanding some properties and, in sentence B, she justifies correctly why we should consider a figure as a special case of a class, when using the square and the kite. In this case, she makes an analytical judgment and understands "the opposing direction inclusion relationship" (hierarchical classification level). In fact, this answer shows she knows that a square is a parallelogram. However, in sentence A she is clearly thinking about oblique parallelograms and she disregards particular cases (prototypical classification). Because this type of answer was very common, the teacher asked why they made this

- A. True, if we overlap two halves of the figure they do not match.
- B. True, because squares are particular cases of all kinds of quadrilaterals so, in particular they are special kinds of kites. Squares have consecutive sides congruent 2 by 2.

Fig. 15. Lucia's answer to the question in the final test.

mistake and some participants said they assumed we meant oblique parallelograms and “not all parallelograms” as the word “all” was not present.

These data from the final test shows that most of the participants evolved significantly from the beginning of the course, when only some of them knew the relationship between squares and rectangles. At this final stage, 1/3 responded according to the hierarchical classification of both quadrilaterals and almost the entire class showed some knowledge of the hierarchical classification. In fact, most of the wrong answers are related to difficulties that were not caused by prototypical images or ignorance about the properties of quadrilaterals, but by misunderstandings relating the interpretation of the discourse and logical reasoning. This is consistent with [De Villiers \(1994\)](#) on the nature of difficulties that result from the process of classifying itself.

5. Conclusion

In the beginning of the course, the participants showed very limited figural concepts of quadrilaterals and ignored some properties, especially those related to diagonals or more unfamiliar quadrilaterals. From the hierarchical classification point of view, the prospective teachers did not acknowledge the relationships between quadrilaterals and only few of them considered a square as a particular case of a rectangle and could explain why.

As suggested by [Battista \(2008a\)](#), the work undertaken using GeoGebra broaden their figural concepts as the prospective teachers generated numerous examples through dragging, including non-prototypical images and particular cases. In addition, they identified the critical attributes by noticing the invariant relations. However, this process needed much support from the teacher and even from some participants who played an important role in challenging their colleagues' fixed prototypes. Following this activity, in the classification task, we noticed some evolution because all participants acknowledged some relationships and justified it correctly, achieving the *level of hierarchical classification* for rectangles and parallelograms ([Fujita, 2012](#)). However, just about half of them reasoned according to that level for the kites. Such difficulties were largely due to the inexperience in classifying geometric objects (including the notion of class) and to the strong conceptualization of some quadrilaterals. This aspect seemed to influence the participants' reasoning particularly when the relationship between the quadrilaterals was not “direct” (like squares and rhombi), which lead them to make prototypical judgments based on the appearance or the critical attributes of a prototypical image that they take for reference ([Hershkowitz, 1989](#)). In order to change this behaviour, the teacher invited the prospective teachers to share their difficulties, discussed their reasoning and encouraged them to make an analytical judgment of the geometric figures in order to classify them correctly.

Concerning the hierarchical classification of prisms, the construction of the Venn diagram and the indication of the relations among figures showed a remarkable evolution. On one hand, the participants constructed correct diagrams and presented hierarchical relations among “close” figures – like the cube being a parallelepiped – and even more examples of “distant” ones – like the parallelepiped being a quadrangular prism. Thus, the prospective teachers did not seem to analyse the different examples by their “distance” to the prototypical example. Instead, they were more able to produce analytical judgments ([Hershkowitz, 1989](#)). On the other hand, their discourse shows how they thought about the solids as classes of figures, facing diagrams as representations of concepts ([Battista, 2008a](#)), a fundamental aspect of classifying geometric figures. Regarding the levels of understanding of the inclusive relations, in the case of prisms, all prospective teachers were at the level of *hierarchical classification* ([Fujita, 2012](#)), with reservations for two participants. This conclusion seems to be paradoxical if we take into account that prisms are more complex figures than quadrilaterals and that, ultimately, their hierarchical organization is dependent on the organization of quadrilaterals, where the class had more difficulties. However, there are two reasons that might explain this evolution. On one hand, from one task to the other, the prospective teachers learned about the classification process, both regarding its meaning, the analytical judgment underlying the relationships and the ways of representation that we use to highlight relationships among figures. On the other hand, the strong conceptualization of quadrilaterals may conflict with the need of, simultaneously, emphasizing properties that are common to the class while ignoring others ([Mariotti & Fischbein, 1997](#)). Thus, the fact that the hierarchy among prisms was more easily established by the participants than the hierarchy among quadrilaterals, may be due to the lower familiarity that they have with such solids, which may release them from the fixation in noncritical attributes.

Therefore, comparative analysis of the results between the participants' classification of quadrilaterals and the classification of prisms shows a positive and significant evolution. However, when we turned again to the quadrilaterals in the final test, we still encounter misunderstandings, most often related to the interpretation of the discourse and logical reasoning and not so much to the limited figural concepts. This reinforces two ideas. On one hand, the nature of the analysis individuals make of figures (prototypical or analytical) also depends on the type of figure ([Hershkowitz, 1989](#)) which leads to different levels of understanding of the hierarchical classification ([Fujita, 2012](#)). This is probably due to the fact that individuals may be at different Van Hiele levels or in their transition ([Battista, 2009](#); [Gutiérrez et al., 1991](#)). On the other hand, the hierarchical classification of geometric figures is a complex

process, which does not depend only of the recognition of their properties (De Villiers, 1994). This suggests the need for further research that focus on the way individuals classify figures hierarchically taking into account the aspects already considered by Fujita (2012) and also including other factors that influence reasoning, such as language interpretation and logical reasoning.

Finally, this study supports De Villiers (1994) perspective, when he says that the hierarchy among figures must not be imposed by the teacher; rather there must be an effective involvement of the students in the construction and discussion of different classifications, as well as a negotiation of meanings and an appreciation of the different types of classification. This suggestion is validated by the results of our teacher education experiment, based on an exploratory approach. The dialogues that we present show how the interaction between the teacher and the prospective teachers and among them promotes the verbalization of the reasoning and the debate of divergent ideas, leading to argumentation. Through interaction, the participants supported each other, but also challenged each other. This approach gave the prospective teachers the opportunity to engage actively in the construction of their knowledge and challenged their previous conceptions as they were asked to interpret the given information, and to represent and communicate their solutions of the task to each other and to the teacher.

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