Universidade de Lisboa
Faculdade de Ciências
Departamento de Matemática

Cluster-Tilting Theory

Raquel Coelho Guardado Simões

Mestrado em Matemática
(Álgebra, Lógica e Fundamentos)

2008

Universidade de Lisboa
Faculdade de Ciências
Departamento de Matemática

Cluster-Tilting Theory

Raquel Coelho Guardado Simões

Dissertação orientada pelos Professores Doutores
Robert Marsh e Catarina Santa Clara

Mestrado em Matemática
(Álgebra, Lógica e Fundamentos)

2008

Sumário

A noção de álgebra de clusters surgiu em 2001, num artigo de S. Fomin e A. Zelevinsky, com vista a estudar as bases canónicas duais de um grupo quântico de uma álgebra de Lie simples de dimensão finita sobre o corpo dos complexos. A teoria das álgebras de clusters tem bastantes ligações e aplicações a diversas áreas, tais como a álgebra, a combinatória e a física. A interessante conexão entre as álgebras de clusters e a teoria da representação foi estudada nos artigos [8, 11, 13, 22], entre muitos outros.

As categorias de clusters, introduzidas por A. Buan, R. Marsh, M. Reineke, I. Reiten e G. Todorov [11], são certos quocientes da categoria derivada da categoria dos módulos de uma álgebra de caminhos associada a um quiver finito sem ciclos orientados nem laços (aresta cujo vértice de partida e o vértice de chegada coincidem), e as álgebras cluster-tilted, introduzidas por A. Buan, R. Marsh e I. Reiten [8], são definidas como sendo álgebras de endomorfismos de certos objectos na categoria de clusters (conhecidos como objectos cluster-tilting). As categorias de clusters e as álgebras cluster-tilted foram independentemente introduzidas por P. Caldero, F. Chapoton e R. Schiffler [13] o tipo A.

O objectivo principal desta dissertação é o estudo das álgebras cluster-tilted associadas a um quiver de Dynkin de tipo A, usando a definição gráfica, que envolve diagonais e triangulações de polígonos regulares, da categoria de clusters e da álgebra cluster-tilted associada, um assunto desenvolvido por P. Caldero, F. Chapoton e R. Schiffler, em 2006 [13]. Esta abordagem das álgebras cluster-tilted é particularmente interessante porque evita o uso das categorias derivadas e dá-nos uma definição geométrica da categoria de clusters e das álgebras cluster-tilted associadas. Os objectos da categoria de clusters podem ser vistos como diagonais de um polígono regular, e os morfismos da categoria são dados por certas transformações dessas diagonais. A remoção (de uma maneira apropriada) de diagonais de uma triangulação do polígono dá origem a uma categoria quociente equivalente à categoria dos módulos de uma álgebra cluster-tilted.

As noções de álgebras cluster-tilted e de categorias de clusters conduzi-

ram à teoria cluster-tilting, um novo método de comparação de álgebras que generaliza a teoria tilting clássica (ver [3], por exemplo).

O artigo [7] descreve os desenvolvimentos recentes na teoria cluster-tilting e o artigo [23] de C. M. Ringel (a aparecer em *Handbook of Tilting theory*) inclui uma secção extensa dedicada à teoria cluster-tilting. Ringel também incluiu as álgebras cluster-tilted na lista de *Striking new results* na página da fdlist (http://www.mathematik.uni-bielefeld.de/fdlist/).

Como já foi referido, o principal objecto de estudo desta tese é o artigo [13]. Para a sua compreensão, precisamos de ter alguns conhecimentos sobre teoria da representação de álgebras, nomeadamente a representação de quivers e a teoria de Auslander-Reiten, e sobre a teoria das álgebras de clusters e as suas principais propriedades, com particular referência para a descrição das álgebras de clusters de tipo A em termos de diagonais e triangulações de um polígono regular e do conjunto de raízes quase-positivas de um sistema de raízes de tipo A. Dividimos assim esta dissertação em quatro capítulos.

O primeiro capítulo introduz as noções e resultados gerais da teoria das categorias, da teoria dos módulos e dos sistemas de raízes, necessários para o que se segue.

O capítulo 2 é dedicado ao estudo da teoria da representação de quivers. Na primeira secção vemos que cada álgebra A de dimensão finita sobre um corpo algebricamente fechado corresponde a um quiver com certas relações, e que esta conexão permite-nos visualizar qualquer A-módulo de dimensão finita em termos da representação de quivers. Na secção 2.2, apresentamos uma computação explícita dos A-módulos simples, projectivos indecomponíveis e injectivos indecomponíveis, considerados como representações do quiver associado à álgebra A. As noções e resultados importantes sobre morfismos irredutíveis e morfismos minimais quase-cindíveis (duas classes de morfismos intimamente ligadas) são dadas na Secção 2.3. As noções de morfismos minimais quase-cindíveis dão origem a um tipo especial de sequências exactas de módulos, as sequências quase-cindíveis. A existência destas sequências (começando, ou acabando, num módulo indecomponível arbitrário) é provada na Secção 2.4. O importante conceito de translacções de Auslander-Reiten é usado para a prova deste resultado. A descrição de alguns aspectos da teoria de Auslander-Reiten feita na Secção 2.4 permite-nos definir e descrever o quiver de Auslander-Reiten (Secção 2.5). De uma maneira informal, os vértices deste quiver correspondem aos módulos indecomponíveis e as arestas correspondem aos morfismos irredutíveis. Este quiver é importante para o estudo da categoria dos A-módulos de dimensão finita, na medida em que 'armazena' toda a sua informação. De entre as propriedades deste quiver que são enunciadas, destacamos uma que vai ser directamente usada na demonstração de um dos resultados do artigo [13], e que diz que se o quiver de

Auslander-Reiten, denotado por $\Gamma(mod\, A)$ contém uma componente conexa $\mathcal{C}$ tal que o comprimento dos módulos associados aos seus vértices é menor ou igual que um certo número natural, então $\Gamma(mod\, A) = \mathcal{C}$ e é finito.

No capítulo 3 damos uma introdução básica da teoria das álgebras de clusters. Exibimos as diferentes versões da definição desta classe de álgebras (baseando-nos nos artigos [16, 17]) e vemos as relações entre elas. De uma maneira informal, a álgebra de clusters é um certo tipo de álgebra comutativa, definida combinatorialmente, gerada por um conjunto de elementos designados por variáveis de cluster, que é dividido por certos subconjuntos de igual cardinalidade (os clusters). Destacamos algumas propriedades desta classe de álgebras, como por exemplo o fenómeno de Laurent, que diz que qualquer variável de cluster pode ser escrita como um polinómio de Laurent nas variáveis de um cluster arbitrário fixo, e a classificação das álgebras de clusters de tipo finito (as que têm um número finito de variáveis de clusters), um dos principais resultados na teoria das álgebras de clusters, que foi provado por S. Fomin e A. Zelevinsky em [17]. Esta classificação é, surpreendentemente, paralela à classificação de Cartan-Killing dos sistemas de raízes. Para o caso finito, enunciamos um teorema, conhecido por 'teorema dos denominadores', que é um caso particular do fenómeno de Laurent para as álgebras de clusters de tipo finito, e que estabelece uma bijecção entre as variáveis de clusters e as raízes quase-positivas do sistema de raízes associado à álgebra de clusters. O exemplo de álgebras de clusters que apresentamos (que é o que precisamos de conhecer para a compreensão do artigo [13]) é do tipo A_n e é descrito em termos de diagonais e triangulações de um polígono regular. Referimos também que este exemplo está relacionado com o Grassmanniano $Gr_{2,n+3}$.

O último capítulo é então dedicado ao estudo de [13]. Este artigo dá uma descrição geométrica da categoria dos módulos sobre uma álgebra cluster-tilted de tipo A em termos de diagonais de polígonos regulares, cujos morfismos advêm da rotação das diagonais em torno dos seus vértices. Apresentamos a demonstração do resultado chave que nos diz que esta categoria geométrica é equivalente à categoria dos módulos de um quiver com relações (designadas por relações triangulares) oriundas da álgebra de clusters de tipo A, e do resultado principal deste artigo, que dá uma generalização do teorema dos denominadores para qualquer cluster (do tipo A).

Palavras-Chave: Álgebras de Cluster, Teoria Cluster-Tilting, Teoria da Representação de Quivers, Teoria Auslander-Reiten

Abstract

The notion of Cluster Algebra first appeared in 2001, in a paper by S. Fomin and A. Zelevinsky, studying the dual canonical basis of the quantum group of a finite dimensional simple Lie algebra over the complex numbers, and also total positivity for algebraic groups.

Cluster categories, introduced by A. Buan, R. Marsh, M. Reineke, I. Reiten and G.Todorov [11] in 2004, are certain quotients of the derived category of the module category of a finite dimensional path algebra, and the cluster-tilted algebras, defined by A. Buan, R. Marsh and I. Reiten [8], are the endomorphism algebras of certain objects in a cluster category.

Our aim is to study the cluster-tilted algebras associated to a Dynkin quiver of type A, using the graphical definition, involving diagonals of polygons, of the cluster category and its associated cluster-tilted algebras, introduced by P. Caldero, F. Chapoton and R. Schiffler, in 2006.

In order to understand the cluster-tilted algebras, we need some background on the theory of representations of algebras (with emphasis on representations of quivers and the Auslander-Reiten theory) and on the theory of cluster algebras (with emphasis on the description of cluster algebras of type A in terms of diagonals and triangulations of a regular polygon and the set of almost positive roots in the root system of type A).

Key words: Cluster Algebras, Cluster-Tilting Theory, Representation Theory of Quivers, Auslander-Reiten Theory

Acknowledgements

My deepest thanks go to my supervisors, Dr. Robert J. Marsh and Dr. Catarina Santa Clara, for their guidance, crucial help and encouragement. For additional assistance, I also wish to thank Professor José A. Perdigão Dias da Silva, Dr. Luís F. R. Sequeira and Dr. Pedro J. S. Freitas.

My thanks go also to Centro de Álgebra da Universidade de Lisboa and to Departamento de Matemática, Faculdade de Ciências, Universidade de Lisboa, for the magnificent work conditions.

I am indebted to Fundação para a Ciência e Tecnologia, for their financial support through Grant SFRH/BM/29573/2006.

Last but not least I would like to thank my family for all their support.

Contents

Introduction

The notion of Cluster Algebra first appeared in 2001, in a paper by S. Fomin and A. Zelevinsky, studying the dual canonical basis of the quantum group of a finite dimensional simple Lie algebra over the complex numbers, and also total positivity for algebraic groups.

The theory of cluster algebras has many connections and applications in many different areas, such as algebra, geometry, combinatorics and physics. The interesting connection between cluster algebras and representation theory was studied in the papers [8, 11, 13, 22] and many other articles.

Cluster categories, introduced by A. Buan, R. Marsh, M. Reineke and G. Todorov [11], are certain quotients of the derived category of the module category of a finite dimensional path algebra, and the cluster-tilted algebras, defined by A. Buan, R. Marsh and I. Reiten [8], are the endomorphism algebras of certain objects in a cluster category (known as cluster-tilting objects). Cluster categories and cluster-tilted algebras were introduced independently by P. Caldero, F. Chapoton and R. Schiffler [13] for type A.

Our aim is to study the cluster-tilted algebras associated to a Dynkin quiver of type A, using the graphical definition, involving diagonals of polygons, of the cluster category and its associated cluster-tilted algebras, introduced by P. Caldero, F. Chapoton and R. Schiffler [13]. The objects of the cluster category can be thought of as diagonals in a regular polygon, with morphisms given by certain transformations of the diagonals. Removing the diagonals in a triangulation of the polygon, in an appropriate way, provides a quotient category equivalent to the module category of a cluster-tilted algebra.

The notions of cluster-tilted algebras and cluster categories led to cluster-tilting theory, which is a new method of comparing algebras, generalising the classical tilting theory (see, for example, [3]).

The survey article [7] gives a description of recent developments in cluster-tilting theory and C. M. Ringel's article to appear in the Handbook of Tilting Theory [23] includes a large section on cluster-tilting theory. Ringel has also listed cluster-tilted algebras on the fdlist page of striking new results (http://www.mathematik.uni-bielefeld.de/ fdlist/).

As was already mentioned, the core of this dissertation is the study of [13]. Some knowledge of the theory of representations of algebras and the theory of cluster algebras is required for the understanding of this paper. With this purpose in mind, we divide this dissertation into four chapters.

Chapter 1 covers the background necessary for what follows, namely category theory, module theory and root systems.

Chapter 2 is devoted to the study of representation of quivers. In the first section we see that each finite dimensional algebra A over an algebraically closed field is associated to a quiver, and that this connection allow us to visualize any finite dimensional A-module as a representation of a quiver. In Section 2.2, we exhibit an explicit computation of the simple, the indecomposable projective and the indecomposable injective A-modules, viewed as representations of the quiver associated to A. The concepts of and main results on irreducible morphisms and minimal almost split morphisms are given in Section 2.3. The notions of minimal almost split morphisms give rise to a special type of exact sequences, called the almost split sequences, which play an important role in representation theory. The existence of these sequences is proved in Section 2.4. The important concept of Auslander-Reiten translation, studied also in this section, is used to the proof of this result. The description of some aspects of Auslander-Reiten theory, which were introduced in Section 2.4, enables us to define and describe the Auslander-Reiten quiver (Section 2.5), which can be thought of as a combinatorial picture of the module category of A.

In Chapter 3 we give an introduction to the theory of cluster algebras. We exhibit the variations of the definition of cluster algebras (given in [16, 17]) and see the connection between them. Informally speaking, a cluster algebra is a certain kind of combinatorially defined commutative algebra generated by a set of elements known as cluster variables, divided into overlapping subsets of equal cardinality, called clusters. We present some important results, such as the Laurent phenomenon, which asserts that any cluster variable can be written as a Laurent polynomial in the cluster variables of any given cluster. We mention the classification of the cluster algebras of finite type (the ones that have a finite number of cluster variables), which is parallel to the Cartan-Killing classification of root systems. The theorem known as 'the denominator theorem' is stated. This theorem establishes a bijection between the cluster variables and the almost positive roots of the root system associated to the cluster algebra of finite type. We focus our attention on the cluster algebras of type A, which can be described in terms of diagonals and triangulations of a regular polygon.

Finally, Chapter 4 is dedicated to the study of the paper [13], which, as we have already stated, gives a geometric description of the module category of a cluster-tilted algebra of type A in terms of diagonals in polygons, and whose morphisms come from rotating diagonals about their end-points. We give the proof of the key result that this geometric category is equivalent to the module category of a quiver with relations arising from a cluster algebra of type A, and we finish with the proof of the main result of this paper, which is a generalization of the 'denominator theorem' by Fomin and Zelevinsky [17] to any cluster (in type A).

Chapter 1

Background

In this preliminary chapter, we will fix some notation and state a few results about algebras and modules, categories and functors, and root systems, that will be used in the sequel. For more details and proofs, the reader is referred to the following textbooks [1, 2, 4, 19, 20, 21, 24].

1.1 Category Theory

Categories

Definition 1.1.1. A *category* $\mathcal{C}$ consists of:

- A class $Obj\,\mathcal{C}$, whose elements are called *objects* of $\mathcal{C}$.
- For each pair of objects (X, Y) of $\mathcal{C}$, there is a set $Hom_{\mathcal{C}}(X, Y)$, or for short, $Hom(X, Y)$, whose elements are called *morphisms* from X to Y, such that for different pairs of objects $(X, Y) \neq (W, Z)$, $Hom_{\mathcal{C}}(X, Y) \cap Hom_{\mathcal{C}}(W, Z) = \emptyset$. We denote a morphism $f \in Hom_{\mathcal{C}}(X, Y)$ by
$$X \xrightarrow{f} Y \qquad or \qquad f : X \to Y.$$
- For every triple X, Y, Z of objects of $\mathcal{C}$, there is an operation $\circ$, called the *composition*, defined by
$$\begin{array}{rccc} \circ: & Hom_{\mathcal{C}}(Y, Z) \times Hom_{\mathcal{C}}(X, Y) & \to & Hom_{\mathcal{C}}(X, Z) \\ & (g, f) & \mapsto & g \circ f, \end{array}$$
that satisfies the conditions:
  - For every object X of $\mathcal{C}$, there exists a morphism $1_X \in Hom_{\mathcal{C}}(X, X)$, such that for all $X \xrightarrow{f} Y$ and $Z \xrightarrow{g} X$, $f \circ 1_X = f$ and $1_X \circ g = g$,

– $h \circ (g \circ f) = (h \circ g) \circ f$, for all $X \xrightarrow{f} Y$, $Y \xrightarrow{g} Z$, $Z \xrightarrow{h} W$.

The composition of two morphisms f and g will be denoted by gf or fg, for short.

Definition 1.1.2. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Their product $\mathcal{C} \times \mathcal{D}$ is the category whose objects are of the form (X, Y), with $X \in Obj\, \mathcal{C}, Y \in Obj\, \mathcal{D}$, the morphisms are of the form $f = (f_1, f_2) : (X, Y) \to (X', Y')$, where $f_1 \in Hom_{\mathcal{C}}(X, X')$, $f_2 \in Hom_{\mathcal{D}}(Y, Y')$, and the composition $\circ$ is defined by $(f_1, f_2) \circ (g_1, g_2) = (f_1 \circ g_1, f_2 \circ g_2)$, for all $g_1 \in Hom_{\mathcal{C}}(X, Y)$, $f_1 \in Hom_{\mathcal{C}}(Y, Z)$, $g_2 \in Hom_{\mathcal{D}}(X', Y')$, $f_2 \in Hom_{\mathcal{D}}(Y', Z')$.

In order to define an additive category, we need the definition of a direct sum of objects. The symbol $[n]$ ($n \in \mathbb{N}$), which is introduced in the following definition and will be used throughout this dissertation, denotes the set $\{1, \ldots, n\}$.

Definition 1.1.3. Let $X_1, \ldots, X_n$ be objects of $\mathcal{C}$. A *direct sum* of these objects is a pair $(X_1 \oplus \ldots \oplus X_n, (u_j)_{j \in [n]})$, where $X_1 \oplus \ldots \oplus X_n \in Obj\, \mathcal{C}$, and $u_j : X_j \to X_1 \oplus \ldots \oplus X_n$, $j \in [n]$, are morphisms, denominated *summand embeddings*, that verifies the following condition:
If Z is an object of $\mathcal{C}$ and $f_1 : X_1 \to Z, \ldots, f_n : X_n \to Z$ are morphisms in $\mathcal{C}$, then there is one and only one morphism $f : X_1 \oplus \ldots \oplus X_n \to Z$ such that $f_j = f u_j$, for all $j \in [n]$.

Note that if such object $X_1 \oplus \ldots \oplus X_n$ exists, it is unique up to isomorphism.

Definition 1.1.4. Let $\mathcal{C}$ be a category.

1. The category $\mathcal{C}$ is said to be *additive* if it satisfies the following axioms:

   - For every finite set $X_1, \ldots, X_n$ of objects, there exists a direct sum $X_1 \oplus \ldots \oplus X_n$ in $\mathcal{C}$.
   - The set $Hom_{\mathcal{C}}(X, Y)$ has an abelian group structure, for each pair (X, Y) of objects of $\mathcal{C}$.
   - Let X, Y, Z be arbitrary objects of $\mathcal{C}$. The composition of morphisms in $\mathcal{C}$ is bilinear, i.e.,

     $$(f + f') \circ g = f \circ g + f' \circ g,$$

     $$f \circ (g + g') = f \circ g + f \circ g',$$

     for all morphisms $f, f' \in Hom_{\mathcal{C}}(Y, Z)$, $g, g' \in Hom_{\mathcal{C}}(X, Y)$.

- There is an object $0 \in Obj\,\mathcal{C}$ such that the identity morphism 1_0 is the element zero of the abelian group $Hom_{\mathcal{C}}(0,0)$.

2. Let $\mathcal{C}$ be additive. The *opposite category* of $\mathcal{C}$, denoted by $\mathcal{C}^{op}$, is the additive category such that $Obj\,\mathcal{C}^{op} = Obj\,\mathcal{C}$, $Hom_{\mathcal{C}^{op}}(X,Y) = Hom_{\mathcal{C}}(Y,X)$, for all $X, Y \in Obj\,\mathcal{C}$, the addition in $Hom_{\mathcal{C}^{op}}(X,Y)$ is the addition in $Hom_{\mathcal{C}}(Y,X)$, and the composition $*$ in $Hom_{\mathcal{C}^{op}}$ is given by $f * g = g \circ f$, where $\circ$ is the composition in $Hom_{\mathcal{C}}$. Note that $(\mathcal{C}^{op})^{op} = \mathcal{C}$.

3. Let K be a field. The category $\mathcal{C}$ is called a *K-category* if, for every pair X, Y of objects of $\mathcal{C}$, $Hom_{\mathcal{C}}(X,Y)$ has a K-vector space structure such that the composition $\circ$ of morphisms in $\mathcal{C}$ is a K-bilinear map.

Remark 1.1.5. Let $\mathcal{C}$ be an additive category and let $X_1 \oplus \ldots \oplus X_n$ be the direct sum of the objects $X_1, \ldots, X_n$ of $\mathcal{C}$. By the direct sum property, for each $j \in [n]$, there exists a morphism $p_j : X_1 \oplus \ldots \oplus X_n \to X_j$ such that $p_j \circ u_j = 1_{X_j}$, $p_j \circ u_i = 0$, for $i \neq j$, and $u_1 \circ p_1 + \ldots + u_n \circ p_n = 1_{X_1 \oplus \ldots \oplus X_n}$. This morphism p_j is called the *jth projection.* Furthermore, if $(g_i : X \to X_i)_{i \in [n]}$ is a set of morphisms, there exists one and only one morphism $g : X \to X_1 \oplus \ldots \oplus X_n$ such that $p_j \circ g = g_j$, for $j \in [n]$.

Thus, in an additive category, for any two direct sums $X = X_1 \oplus \ldots \oplus X_n$, $Y = Y_1 \oplus \ldots \oplus Y_m$, any morphism $f : X \to Y$ can be written in the form

$$f = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1} & f_{m2} & \cdots & f_{mn} \end{bmatrix},$$

where $f_{ij} = p_i \circ f \circ u_j \in Hom_{\mathcal{C}}(X_j, Y_i)$.

Let $\mathcal{C}$ be an additive category and $f : X \to Y$ be a morphism in $\mathcal{C}$.

A *kernel* of f is a pair $(Ker\,f, u : Ker\,f \to X)$, where $Ker\,f \in Obj\,\mathcal{C}$ and u is a morphism in $\mathcal{C}$, such that

- $f \circ u = 0$,
- If $Z \in Obj\,\mathcal{C}$ and $h : Z \to X$ is a morphism in $\mathcal{C}$ such that $f \circ h = 0$, then there exists a unique morphism $h' : Z \to Ker\,f$ such that $h = u \circ h'$.

A *cokernel* of f is a pair $(Coker\,f, p : Y \to Coker\,f)$, where $Coker\,f$ is an object of $\mathcal{C}$ and p is a morphism in $\mathcal{C}$ such that

- $p \circ f = 0$,

- If $Z \in Obj\,\mathcal{C}$, and $g : Y \to Z$ is a morphism in $\mathcal{C}$ such that $g \circ f = 0$, then there exists a unique morphism $g' : Coker\,f \to Z$ such that $g = g' \circ p$.

Definition 1.1.6. A category $\mathcal{C}$ is said to be *abelian* if:

- $\mathcal{C}$ is additive,
- Every morphism $f : X \to Y$ in $\mathcal{C}$ admits a kernel $u : Ker\,f \to X$ and a cokernel $p : Y \to Coker\,f$,
- For each morphism $f : X \to Y$ in $\mathcal{C}$, the unique morphism $\bar{f} : Coker\,u \to Ker\,p$, where $u : Ker\,f \to X$ is a kernel of f and $p : Y \to Coker\,f$ is a cokernel of f, is an isomorphism, i.e., there exists a morphism $\bar{f}' : Ker\,p \to Coker\,u$ such that $\bar{f}\bar{f}' = 1_{Ker\,p}$ and $\bar{f}'\bar{f} = 1_{Coker\,u}$.

Remark 1.1.7. In an additive category $\mathcal{C}$ where every morphism $f : X \to Y$ admits a kernel u and a cokernel p, there exists such a morphism $\bar{f}$ and it is unique. For more details, see [2, Appendix A.1].

Functors

The functors can be viewed as 'morphisms between categories'.

Definition 1.1.8. Let $\mathcal{C}, \mathcal{D}, \mathcal{C}'$ be arbitrary categories.

1. We say that F is a *covariant functor* from $\mathcal{C}$ to $\mathcal{D}$, and we write $F : \mathcal{C} \to \mathcal{D}$ if:
   - For each object X of $\mathcal{C}$, $F(X) \in Obj\,\mathcal{D}$,
   - For each morphism $f : X \to Y$ in $\mathcal{C}$, a morphism $F(f) : F(X) \to F(Y)$ is defined in $\mathcal{D}$,
   - For every object X in $\mathcal{C}$, $F(1_X) = 1_{F(X)}$,
   - For each pair of morphisms $f : X \to Y, g : Y \to Z$, $F(g \circ f) = F(g) \circ F(f)$.
2. We say that F is a *contravariant functor* from $\mathcal{C}$ to $\mathcal{D}$, and we write $F : \mathcal{C} \to \mathcal{D}$ if:
   - For each object X of $\mathcal{C}$, $F(X) \in Obj\,\mathcal{D}$,
   - For each morphism $f : X \to Y$ in $\mathcal{C}$, a morphism $F(f) : F(Y) \to F(X)$ is defined in $\mathcal{D}$,

- For every object X in $\mathcal{C}$, $F(1_X) = 1_{F(X)}$,
- For each pair of morphisms $f : X \to Y, g : Y \to Z$, $F(g \circ f) = F(f) \circ F(g)$.

3. Any functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{C}'$, is said to be a *bifunctor*.

Remark 1.1.9. A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ can be considered as a covariant functor $F : \mathcal{C}^{op} \to \mathcal{D}$ or $F : \mathcal{C} \to \mathcal{D}^{op}$.

Definition 1.1.10. The composition of two functors $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C}' \to \mathcal{C}''$ is the functor defined by: $GF(X) = G(F(X))$, for $X \in Obj\,\mathcal{C}$, and $GF(f) = G(F(f))$, for each morphism $f : X \to Y$ in $\mathcal{C}$.

Now, we give the notion of functorial morphism, that compares two functors between the same categories.

Definition 1.1.11. Let $F, F' : \mathcal{C} \to \mathcal{D}$ be two functors. A *functorial morphism* $\varphi = (\varphi_X)_{X \in Obj\,\mathcal{C}} : F \to F'$ is a family of morphisms $\varphi_X : F(X) \to F'(X)$, $X \in Obj\,\mathcal{C}$, such that, for any morphism $f : X \to Y$ in $\mathcal{C}$, $\varphi_Y\, F(f) = F'(f)\varphi_X$, i.e., such that the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\varphi_X} & F'(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle F'(f)} \\ F(Y) & \xrightarrow{\varphi_Y} & F'(Y) \end{array}$$

in $\mathcal{D}$ commutes. If each morphism $\varphi_X, X \in Obj\,\mathcal{C}$, is an isomorphism in $\mathcal{D}$, then φ is said to be a *functorial isomorphism*. In this case, we write $F \overset{\varphi}{\cong} F'$, or briefly, $F \cong F'$.

Definition 1.1.12. Let $\mathcal{C}$ and $\mathcal{D}$ be arbitrary categories.

1. A covariant functor $F : \mathcal{C} \to \mathcal{D}$ is a *category equivalence* provided there is a functor $F' : \mathcal{D} \to \mathcal{C}$ such that $FF' \cong 1_{\mathcal{D}}$ and $F'F \cong 1_{\mathcal{C}}$, where $1_{\mathcal{C}}$ (resp. $1_{\mathcal{D}}$) is the identity functor of $\mathcal{C}$ (resp. $\mathcal{D}$). The functor F' is called an *inverse equivalence* of F.

2. A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories, and is called a *duality*, if the covariant functor $F : \mathcal{C}^{op} \to \mathcal{D}$ is an equivalence of categories.

3. We say that $\mathcal{C}$ and $\mathcal{D}$ are *equivalent*, and we write $\mathcal{C} \cong \mathcal{D}$, if there is a category equivalence between them.

Definition 1.1.13. Let $\mathcal{C}, \mathcal{D}, \mathcal{C}'$ be categories and $F : \mathcal{C} \to \mathcal{D}$ be a covariant functor.

1. F is called *dense* if, for any object Y of $\mathcal{D}$, there exists an object X in $\mathcal{C}$ such that $F(X) \cong Y$.

2. F is said to be *full* if, for all objects X, Y of $\mathcal{C}$ and for any morphism $g : F(X) \to F(Y)$ in $\mathcal{D}$, there is a morphism $f : X \to Y$ in $\mathcal{C}$ such that $F(f) = g$. In other words, F is full if, for any pair X, Y of objects of $\mathcal{C}$, the map

$$\begin{array}{rccc} F_{X,Y}: & Hom_{\mathcal{C}}(X,Y) & \to & Hom_{\mathcal{D}}(F(X), F(Y)) \\ & f & \mapsto & F(f) \end{array}$$

   is surjective.

3. If the map $F_{X,Y}$ is injective, for all $X, Y \in Obj\,\mathcal{C}$, the functor F is called *faithful*.

4. Suppose $\mathcal{C}$ and $\mathcal{C}'$ are additive. Then F is called *additive* if, for any objects X, Y in $\mathcal{C}$, $F(X) \oplus F(Y) \cong F(X \oplus Y)$ and the map $F_{X,Y}$ verifies the condition: $F(f+g) = F(f) + F(g)$, for all $f, g : X \to Y$ in $\mathcal{C}$.

5. If $\mathcal{C}$ and $\mathcal{D}$ are K-categories, then F is *K-linear* if F is additive and $F_{X,Y}$ is a K-linear map for all $X, Y \in Obj\,\mathcal{C}$.

6. If $\mathcal{C}$ and $\mathcal{D}$ are K-categories, then F is called a *fully faithful embedding* functor if the map $F_{X,Y}$ is an isomorphism of K-vector spaces, i.e., if F is full, faithful and K-linear.

The following theorem gives a useful characterization of equivalence of categories.

Theorem 1.1.14. [2, Theorem A.2.5] *Let $\mathcal{C}$ and $\mathcal{D}$ be two arbitrary categories and $F : \mathcal{C} \to \mathcal{D}$ be a covariant functor. Then F is an equivalence of categories if and only if F is full, faithful, and dense.*

The radical of a category

Definition 1.1.15. Let $\mathcal{C}$ be an additive K-category and let I be a class of morphisms of $\mathcal{C}$. The class I is called a *two-sided ideal* in $\mathcal{C}$ if the following conditions hold:

- If $X \in Obj\,\mathcal{C}$, then $0_X \in I$,

- If $f : X \to Y \in I$ and $g : Y \to Z$ is a morphism of $\mathcal{C}$, then $g \circ f \in I$,
- If $f : X \to Y \in I$ and $g : Z \to X$ is a morphism of $\mathcal{C}$, then $f \circ g \in I$.

The *quotient category* $\mathcal{C}/I$ is the category such that $Obj\,\mathcal{C}/I = Obj\,\mathcal{C}$ and, for a pair of objects X, Y in $\mathcal{C}$, $Hom_{\mathcal{C}/I}(X, Y) = Hom_{\mathcal{C}}(X, Y)/I(X, Y)$, where $I(X, Y)$ is the set of morphisms of I from X to Y.

Definition 1.1.16. Let $\mathcal{C}$ be an additive K-category. The *radical* of $\mathcal{C}$, denoted by $rad_{\mathcal{C}}$, is a class of morphisms in $\mathcal{C}$ whose set of morphisms from X to Y, $rad_{\mathcal{C}}(X, Y)$, where $X, Y \in Obj\,\mathcal{C}$, consists of the morphisms $h \in Hom_{\mathcal{C}}(X, Y)$ for which $1_X - g \circ h$ is invertible, for any $g \in Hom_{\mathcal{C}}(Y, X)$.

Let $n \in \mathbb{N}$. The nth power of $rad_{\mathcal{C}}$, denoted by $rad_{\mathcal{C}}^n$, is the class of morphisms of the form $f = f_n \circ f_{n-1} \circ \ldots \circ f_1 : X \to Y$, where $f_i \in rad_{\mathcal{C}}(X_{i-1}, X_i)$, for each $i \in [n]$, considering $X_0 = X$ and $X_n = Y$.

It is easy to check that the radical of an additive K-category is a two-sided ideal, as well as its nth powers, for $n \in \mathbb{N}$.

Proposition 1.1.17. [2, A.3.4] *Let $\mathcal{C}$ be an additive K-category and $X = X_1 \oplus \ldots \oplus X_n, Y = Y_1 \oplus \ldots \oplus Y_m$ be two direct sums in $\mathcal{C}$. Let*

$$f = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1} & f_{m2} & \cdots & f_{mn} \end{bmatrix} : X \to Y,$$

be a morphism in $\mathcal{C}$. Then $f \in rad_{\mathcal{C}}(X, Y)$ if and only if $f_{ji} : X_i \to Y_j \in rad_{\mathcal{C}}(X_i, Y_j)$, for all $i \in [n], j \in [m]$.

Proposition 1.1.18. [2, A.3.5] *Let $X, Y \in Obj\,\mathcal{C}$ such that $Hom_{\mathcal{C}}(X, X)$ and $Hom_{\mathcal{C}}(Y, Y)$ are local, i.e., $Hom_{\mathcal{C}}(X, X)$ and $Hom_{\mathcal{C}}(Y, Y)$ have a unique maximal ideal. Then $rad_{\mathcal{C}}(X, Y)$ is the vector space of all nonisomorphisms from X to Y in $\mathcal{C}$.*

1.2 Module Theory

Throughout, K will denote an algebraically closed field, and A a finite dimensional algebra over K. All A-modules are, unless otherwise specified, right finite dimensional A-modules. The category whose objects are the right A-modules and the morphisms the A-module morphisms is called the category of modules and is denoted by $Mod\,A$. If all the objects are the finite-dimensional A-modules, we use the notation $mod\,A$ instead.

The radical, the socle and the top

Definition 1.2.1.

1. The *radical of* A, denoted by $rad\, A$, is the intersection of all the maximal right (or equivalently, left) ideals in A.

2. The *radical of* M, denoted by $rad\, M$, is the intersection of all the maximal submodules of M.

Observe that the radical $rad\, A_A$ of the right A-module A_A is the radical $rad\, A$ of the algebra A. The following proposition collects the main properties of the radical of an algebra and of a module.

Proposition 1.2.2. [2, I.1.3, I.1.4, I.3.7]

1. $rad\, A = \{a \in A \mid 1 - ba$ *has an inverse,* $\forall b \in A\}$.

2. *The radical of* A *is the intersection of all the maximal left ideals of* A, *and so it is a two-sided ideal.*

3. *All two-sided nilpotent ideals of* A *are contained in* $rad\, A$. *Furthermore, if* I *is a two sided nilpotent ideal of* A *such that* A/I *is isomorphic to a product* $K \times \cdots \times K$ *of copies of* K, *then* $I = rad\, A$.

4. *For all right* A*-modules* M *and* N, $rad(M \oplus N) = rad\, M \oplus rad\, N$.

5. $rad\, M = M\, rad\, A$.

Definition 1.2.3.

1. A module is said to be *simple* if it is nonzero and the only submodules of M are zero and itself. If M is a direct sum of simple modules, then M is called *semisimple*.

2. The submodule of M generated by all simple submodules of M is called the *socle* of M, and is denoted by $soc\, M$.

3. The right $A/rad\, A$-module $M/rad\, M$, whose action of $A/rad\, A$ is defined by $(m + rad\, M).(a + rad\, A) = ma + rad\, M$, where $m \in M, a \in A$, is called the *top of* M and is denoted by $top\, M$.

Note that $top\, M$ is indeed a right $A/rad\, A$-module, by 1.2.2 (5).

Definition 1.2.4. An algebra is called *local* if it has a unique maximal right (or equivalently, left) ideal.

Proposition 1.2.5. [2, I.4.6, I.4.8] *Let A be a K-algebra and M be a right A-module.*

1. *If A is finite dimensional, then A is local if and only if the K-algebra $A/rad\, A$ is isomorphic to K.*

2. *Let M be an arbitrary right A-module. If $End\, M$ is local, then M is indecomposable.*

3. *If M is finite dimensional and indecomposable, then $End\, M$ is local and any endomorphism f of M is either an isomorphism or is nilpotent, i.e., $f^m = 0$, for some natural m.*

The next corollary follows directly from 1.2.5 (3) and 1.1.18 with $\mathcal{C} = mod\, A$.

Corollary 1.2.6. *If X and Y are indecomposable modules in $mod\, A$, then $rad_A(X, Y)$ is the K-vector space of all noninvertible homomorphisms from X to Y.*

Definition 1.2.7. Let A be a finite dimensional K-algebra. We say that A is *representation-finite* if there is only a finite number of isomorphism classes of indecomposable finite dimensional right A-modules, and that A is *representation-infinite* otherwise.

The length of a module

Definition 1.2.8. A *composition series* of M is a chain

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M$$

of submodules of M such that M_{i+1}/M_i is simple for $i = 0, \ldots, n-1$.

Since the algebra A and the A-module M are finite dimensional, there exists a composition series of M (cf. [1]). Moreover, it follows from the Jordan-Hölder theorem (cf. [1, 11.3]), that the number n of submodules of M in a composition series of M is well determined. This number is called the *length* of M and is denoted by $l(M)$.

The following proposition gives us some properties of the length, which are easy consequences of the Jordan-Hölder theorem.

Proposition 1.2.9.

1. *Let $f : M \to N$ be a morphim between A-modules. If f is an epimorphism, then $l(N) \leq l(M)$. If f is a monomorphism, then $l(M) \leq l(N)$.*

2. *For any submodule N of M, $l(M/N) = l(M) - l(N)$.*

3. *For any pair L, N of submodules of M, $l(L+N) = l(L)+l(N)-l(L\cap N)$.*

The idempotents and the decomposition into indecomposables

Definition 1.2.10. An element e of A is an *idempotent* if $e^2 = e$.

1. Two idempotents e_1, e_2 of A are said to be *orthogonal* if $e_1e_2 = e_2e_1 = 0$.
2. An idempotent e is called *primitive* if there are no orthogonal idempotents $e_1, e_2 \in A$ such that $e = e_1 + e_2$.
3. An idempotent e is *central* if $ea = ae$, for all $a \in A$. If 0 an 1 are the only central idempotents of A, A is said to be *connected*.

Remark 1.2.11. Any finite dimensional algebra A admits a decomposition $A = e_1A \oplus \ldots \oplus e_nA$, where $e_1, \ldots, e_n$ are primitive pairwise orthogonal idempotents of A such that $1 = e_1 + \cdots + e_n$. Note that, since each e_i is primitive, each e_iA is an indecomposable right ideal of A. Such a decomposition is called an *indecomposable decomposition* of A.

Definition 1.2.12. A set $\{e_1, \ldots, e_n\}$ of idempotents of A satisfying the conditions:

- $e_1, \ldots, e_n$ are pairwise orthogonal and primitive;
- $e_1 + \ldots + e_n = 1$,

is said to be *complete set of primitive orthogonal idempotents* of A.

Proposition 1.2.13. [2, Proposition I.4.5] *Let $e \in A$ be a primitive idempotent.*

1. *The $(A/rad\, A)$-module $top\, e_A$ is simple.*
2. *$rad\, eA = e\, rad\, A$ is the unique maximal proper submodule of eA.*

Proposition 1.2.14. [2, I.6.2] *Let $\{e_1, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents of the finite dimensional K-algebra A. The following conditions are equivalent:*

1. *$e_iA \ncong e_jA$, for all $i \neq j$,*
2. *The algebra $A/rad\, A$ is isomorphic to a product of copies of K.*

Definition 1.2.15. An algebra that satisfies one of the equivalent conditions of proposition 1.2.14 is said to be *basic*.

Proposition 1.2.16. [2, I.4.2] *Let e be an idempotent of A and M be a right A-module. The right eAe-modules $Hom_A(eA, M)$ and Me are isomorphic.*

Proposition 1.2.17. [2, I.5.17] *Let $\{e_1, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents of A.*

1. *If S is a simple right A-module, then S is isomorphic to $top\, e_iA$, for some $i \in [n]$.*

2. *If P is an indecomposable projective right A-module, then P is isomorphic to e_iA, for some $i \in [n]$.*

3. *If I is an indecomposable injective right A-module, then I is isomorphic to $D(Ae_i)$, for some $i \in [n]$.*

Exact sequences

Definition 1.2.18. Let $f : M \to N$ be a morphism of right A-modules.

1. We say that f is a *section* if it admits a left inverse, i.e., if there exists $g : N \to M$ such that $gf = 1_M$.

2. We say that f is a *retraction* if it admits a right inverse, i.e., if there exists $h : N \to M$ such that $fh = 1_N$.

Remark 1.2.19. Let $f : M \to N$ be a section and $g : N \to M$ be a left inverse of f. Then f is injective and g is surjective. Moreover, we have $N = Im\, f \oplus Ker\, g \cong M \oplus Ker\, g$. Indeed, if $y \in Im\, f \cap Ker\, g$, then $y = f(x)$ for some $x \in M$, and $0 = g(y) = gf(x) = x$. So $x = 0$ and consequently, $y = 0$. Any element n of N can be written in the form $n = n - fg(n) + fg(n)$, where $n - fg(n) \in Ker\, g$ and $fg(n) \in Im\, f$.

Analogously, if $f : M \to N$ is a retraction and h is a right inverse of f, then f is surjective, h is injective and $M = Ker\, f \oplus Im\, h$.

Lemma 1.2.20. [2, IV.1.9] *(a) Let $f : L \to M$ be a nonzero A-module homomorphism, with L indecomposable. Then f is not a section if and only if*

$$Im\, Hom_A(f, L) \subseteq rad\, End\, L.$$

(b) Let $g : M \to N$ be a nonzero A-module homomorphism, with N indecomposable. Then g is not a retraction if and only if $Im\, Hom_A(N, g) \subseteq rad\, End\, N$.

Definition 1.2.21.

1. A sequence

$$\dots \to M_{n-1} \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \xrightarrow{f_{n+1}} M_{n+2} \to \dots,$$

   where each M_i is a right A-module and each f_i is a morphism, is called an *exact sequence* if $Ker\, f_n = Im\, h_{n-1}$ for all n. In particular, the sequence

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

   is said to be a *short exact sequence* if f is a monomorphism, g is an epimorphism and $Im\, f = Ker\, g$.

2. A short exact sequence

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

   splits if f is a section.

Remark 1.2.22. In a short exact sequence

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0,$$

f is a section if and only if g is a retraction. Indeed, suppose that f is a section. Then, by 1.2.19, $Im\, f \cong Ker\, g$ is a direct summand of M, i.e., there exists a submodule K of M such that $K \oplus Ker\, g = M$. The morphism $g|_K : K \to N$ is an isomorphism, for $N = g(M) = g(Ker\, g + K) = g(K) = g|_K(K)$ and $Ker\, g|_K = Ker\, g \cap K = 0$. Consider $h = g|_K^{-1} : N \to K$ as a monomorphism from N to M. For $n \in N$, we have $gh(n) = g|_K h(n) = n$, so g is a retraction. The converse is proved similarly.

Projective and Injective modules

Definition 1.2.23. Let U be a right A-module.

1. U is said to be *projective* if, for any A-modules M, N, any morphism $f : U \to N$ and any epimorphism $g : M \to N$, there exists a morphism $f' : U \to M$ such that $gf' = f$, i.e., such that the next diagram is commutative.

$$\begin{tikzcd} & U \arrow[ld, dotted, "f'"'] \arrow[d, "f"] & \\ M \arrow[r, "g"'] & N \arrow[r] & 0 \end{tikzcd}$$

2. U is said to be *injective* if, for any A-modules M, N, any morphism $f : N \to U$ and any monomorphism $g : N \to M$, there exists a morphism $f' : M \to U$ such that $f'g = f$, i.e., such that the next diagram is commutative.

$$\begin{tikzcd} & U & \\ 0 \arrow[r] & N \arrow[u, "f"] \arrow[r, "g"'] & M \arrow[ul, dotted, "f'"'] \end{tikzcd}$$

Lemma 1.2.24. *Let P be a projective A-module and $M \xrightarrow{\alpha} N \xrightarrow{\beta} L$ be an exact sequence. If f is a morphism from P to N such that $\beta \circ f = 0$, then there exists a morphism $g : P \to M$ such that $\alpha \circ g = f$.*

Proof. Write $\alpha = \iota \circ \bar{\alpha}$, where $\iota : Im\,\alpha \hookrightarrow N$ is the inclusion map and $\bar{\alpha} : M \to Im\,\alpha$ is the epimorphism induced by α. Since $\beta \circ f = 0$, we have $Im\,f \subseteq Ker\,\beta = Im\,\alpha$. Let then $\bar{f} : P \to Im\,\alpha$ be the morphism induced by f. Because $\bar{\alpha}$ is an epimorphism and P is projective, there exists a morphism $g : P \to M$ such that $\bar{\alpha} \circ g = \bar{f}$. Hence $\alpha \circ g = \iota \circ \bar{\alpha} \circ g = \iota \circ \bar{f} = f$, and we are done. $\square$

The following lemma will be frequently used in the sequel.

Lemma 1.2.25.

1. *If P is a projective A-module and $v \in Hom_A(V, P)$ is an epimorphism, then v is a retraction.*

2. *If I is an injective A-module and $u \in Hom_A(I, U)$ is a monomorphism, then u is section.*

Proof. (1) Suppose that v is an epimorphism. Then, by definition of projective module, there exists $v' : P \to V$ such that $vv' = 1_P$, which means that v is a retraction. The second statement is similar. $\square$

Definition 1.2.26. Let M be an A-module.

1. A *projective resolution* of M is an exact sequence

$$\ldots \to P_n \xrightarrow{f_n} P_{n-1} \to \ldots \to P_1 \xrightarrow{h_1} P_0 \xrightarrow{h_0} M \to 0, \qquad (1.1)$$

where the P_i are projective A-modules.

2. An *injective resolution* of M is an exact sequence

$$0 \to M \xrightarrow{g_0} I^0 \xrightarrow{g_1} I^1 \to \ldots \to I^m \xrightarrow{g_{m+1}} I^{m+1} \to \ldots, \tag{1.2}$$

where the I_i are injective A-modules.

Definition 1.2.27. Let M, N be two arbitrary A-modules.

1. A morphism $f : M \to N$ is called a *minimal epimorphism* if it is an epimorphism and $Ker\, f$ is *superfluous* in M, i.e., if $Ker\, f + X = M$ for some submodule X of M, then $X = M$.

2. A morphism $f : M \to N$ is said to be a *projective cover* of N if it is a minimal epimorphism and M is projective.

3. A *minimal projective presentation* of M is an exact sequence

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0,$$

such that $p_0 : P_0 \to M$ and the morphism $\bar{p}_1 : P_1 \to Ker\, p_0 = Im\, p_1$ induced by p_1 are projective covers.

4. A projective resolution (1.1) (definition 1.2.26) is said to be *minimal* provided $f_j : P_j \to Im\, P_j$ is a projective cover for all j.

5. A morphism $f : M \to N$ is called a *minimal monomorphism* provided it is a monomorphism and $X \cap Im\, f \neq 0$, for every nonzero submodule X of N.

6. A morphism $f : M \to N$ is said to be a *injective envelope* of M if it is a minimal monomorphism and N is injective.

7. A *minimal injective presentation* of M is an exact sequence

$$0 \to M \xrightarrow{u_0} I_0 \xrightarrow{u_1} I_1,$$

such that the morphisms $u_0 : N \to I_0$ and $Im\, u_1 \hookrightarrow I_1$ are injective envelopes.

8. An injective resolution (1.2) (definition 1.2.26) is said to be *minimal* provided the morphisms $Im\, g_m \hookrightarrow I^m$, for $m \geq 1$, and $g_0 : M \to I^0$ are injective envelopes.

Proposition 1.2.28. [2, I.5.8, I.5.14] *Every module M in $\mathrm{mod}\, A$ admits a projective cover and an injective envelope and they are uniquely determined up to isomorphism.*

Corollary 1.2.29. [2, I.5.10, I.5.16] *Any finite dimensional A-module M admits a minimal projective (resp. injective) presentation and a minimal projective (resp. injective) resolution.*

Lemma 1.2.30. [4, I.3.7] *Let M be an A-module, $f : P \to M$ be a projective cover of M and $g \in End\, P$. If $fg = f$, then g is an automorphism.*

Proposition 1.2.31. *Let M be an A-module. Let*

$$S : P_1 \xrightarrow{s} P_0 \xrightarrow{b} M \to 0$$

be a projective presentation of M, and

$$Q_1 \xrightarrow{r} Q_0 \xrightarrow{a} M \to 0$$

be a minimal projective presentation of M. Then there exist direct sum decompositions $P_1 = E_1 \oplus E_1' \oplus E_1''$, $P_0 = E_0 \oplus E_0' \oplus E_0''$, a morphism $u : E_1 \to E_0$ and an isomorphism $v : E_1' \to E_0'$ such that the sequence S is isomorphic to the sequence

$$E_1 \oplus E_1' \oplus E_1'' \to E_0 \oplus E_0' \oplus E_0'' \to M \to 0,$$

where the first map is given by

$$\begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which we denote by $diag(u, v, 0)$.

Proof. Since Q_0 is projective and $b : P_0 \to M$ is surjective, there exists a morphism $f : Q_0 \to P_0$ such that $bf = 1_M a = a$. Analogously, there exists a morphism $g : P_0 \to Q_0$ such that $ag = b$. Since $a : Q_0 \to M$ is a projective cover and $agf = bf = a$, it follows that gf is an automorphism (cf. 1.2.30). Now, let $x \in Ker\, a$. Then $f(x) \in Ker\, b$, as $bf(x) = a(x) = 0$. So f induces a map f' from $Ker\, a$ to $Ker\, b$. Similarly, g induces a map g' from $Ker\, b$ to $Ker\, a$. Since gf is an automorphism and $g'f'$ is a restriction of gf, it is also an automorphism. Because $s : P_1 \to Ker\, b$ is surjective and Q_1 is projective, there exists a morphism $p : Q_1 \to P_1$ such that $f'r = sp$. Analogously, there exists a morphism $q : P_1 \to Q_1$ such that $g's = rq$. Let us see now that pq is an automorphism. Since $r : Q_1 \to Ker\, a$ is surjective and Q_1 is projective, there exists $t : Q_1 \to Q_1$ such that $(g'f')^{-1}r = rt$. Then we have $rtqp = (g'f')^{-1}g'f'r = r$, and since $r : Q_1 \to Ker\, a$ is a projective cover, it follows from 1.2.30 that tqp is an automorphism. Hence qp is injective and because

Q_1 is finite dimensional, qp is an automorphism. Since $(gf)^{-1}gf = 1_{Q_0}$, we have $P_0 = Im\, f \oplus Ker\,(gf)^{-1}g = Im\, f \oplus Ker\, g$. Note that f is injective, so $Im\, f \cong Q_0$. Thus, we have $P_0 \cong Q_0 \oplus Ker\, g$. Analogously, since $(qp)^{-1}qp = 1_{Q_1}$, $P_1 = Im\, p \oplus Ker\, q \cong Q_1 \oplus Ker\, q$. Consider now the map s restricted to $Ker\, q$. Because $rq = gs$, we have that $s(Ker\, q) \subseteq Ker\, g$. Thus, s induces a map from $Ker\, q$ to $Ker\, g$. Let α be this map, i.e., $\alpha = s\,|_{Ker\, q}$. We will check now that this map is surjective. Let $y \in Ker\, g$. Then $0 = ag(y) = b(y)$, so $y \in Ker\, b = Im\, s$. Hence, there is $x \in P_1$ such that $y = s(x)$. Consider the element $x - p(qp)^{-1}q(x)$ of P_1. It is clear that this element lies in $Ker\, q$, and $s(x - p(qp)^{-1}q(x)) = s(x) - fr(qp)^{-1}q(x) = s(x) - f(gf)^{-1}rq(x) = s(x) - f(gf)^{-1}gs(x) = s(x) - f(gf)^{-1}g(y) = s(x)$, as $y \in Ker\, g$. So α is surjective. Note that $Ker\, g$ is projective as it is a direct summand of P_0. Hence α is a retraction (cf. 1.2.25). Let $\beta : Ker\, g \to Ker\, q$ be a right inverse of g. By 1.2.19, $Ker\, q = Ker\, \alpha \oplus Im\, \beta$. So $\alpha : Ker\, \alpha \oplus Im\, \beta \to Ker\, g$ is an isomorphism on $Im\, \beta$ and zero on $Ker\, \alpha$. It is easy to check that the diagram,

$$\begin{tikzcd}
Q_1 \oplus Im\,\beta \oplus Ker\,\alpha \arrow[r, "diag(r,v,0)"] \arrow[d, "{[p\,\iota]}"'] & Q_0 \oplus Ker\,g \oplus 0 \arrow[r, "{[a\,0]}"] \arrow[d, "{[f\,\iota]}"] & M \arrow[r] \arrow[d, "1_M"] & 0 \\
P_1 \arrow[r, "s"] & P_0 \arrow[r, "b"] & M \arrow[r] & 0
\end{tikzcd}$$

where ι denotes the inclusion map and $v := \alpha\,|_{Im\,\beta}$, is commutative and the vertical maps are isomorphisms. This proves the proposition. ☐

Definition 1.2.32. The *projective dimension* of an A-module M is the length of a minimal projective resolution of M. Similarly, the *injective dimension* of an A-module M is the length of a minimal injective resolution of M.

Remark 1.2.33. The projective dimension (resp. injective dimension) of M is zero if and only if M is projective (resp. injective).

Examples of functors

We now present a number of functors that will be frequently used in the sequel, particularly in the next chapter.

Firstly, we need the definition of opposite algebra. Let $A = (A, +, \cdot)$ be a K-algebra. The *opposite algebra* of A is the K-algebra $(A, +, *)$, with the same K-vector space structure as A and with the multiplication $*$ defined by $a * b = b \cdot a$, for all $a, b \in A$.

We begin with the definition of the *Hom-functors*. Let A, B be K-algebras and let M be an A-B-bimodule, i.e., M is a left A-module, a right B-module and satisfies the condition $(ax)b = a(xb)$, for all $x \in M, a \in A$ and $b \in B$. The contravariant Hom-functor

$$Hom_B(-, M) : Mod\, B \to Mod\, A^{op}$$

is defined as follows:

- Each right A-module N is associated to the K-vector space $Hom_B(N, M)$ of all B-module morphisms from N_B to M_B endowed with the left A-module structure given by $(af)(x) = a(f(x))$, for all $a \in A, f \in Hom_B(N, M), x \in N$.

- For each morphism of right B-modules $f : N \to L$, $Hom_B(f, M)$ is the morphism of left A-modules from $Hom_B(L, M)$ to $Hom_B(N, M)$ defined by the formula $Hom_B(f, M)(g) = gf$.

The covariant Hom-functor

$$Hom_B(M, -) : Mod\, B \to Mod\, A$$

is defined as follows:

- Each right B-module N is associated to the right A-module $Hom_B(M, N)$, whose multiplication $Hom_B(M, N) \times A \to Hom_B(M, N)$ is defined by the formula $(fa)(m) := f(am)$, for $f \in Hom_B(M, N), a \in A$ and $m \in M$.

- For each morphism of B-modules $f : N \to L$, $Hom_B(M, f)$ is the morphism of right A-modules from $Hom_B(M, N)$ to $Hom_B(M, L)$ defined by the formula $Hom_B(M, f)(g) = fg$, for $g \in Hom_B(M, N)$.

It is easy to check that the Hom-functors are additive and left exact functors, i.e., for any exact sequence

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$$

in $Mod\, B$, then the sequences

$$0 \to Hom_B(M, X) \xrightarrow{Hom_B(M,f)} Hom_B(M, Y) \xrightarrow{Hom_B(M,g)} Hom_B(M, Z),$$

and

$$0 \to Hom_B(Z, M) \xrightarrow{Hom_B(g,M)} Hom_B(Y, M) \xrightarrow{Hom_B(f,M)} Hom_B(X, M)$$

are left exact. For more details, see [1, 16.6], for example.

We will pay special attention to a specific case of the contravariant Hom-functors, where $B = A$ and $M = A_A$:

$$(-)^t = Hom_A(-, A) : mod\, A \to mod\, A^{op},$$

which will be called A-dual functor.

This functor has the following important property.

Proposition 1.2.34. *If P_A is a projective right A-module, then P^t is a projective left A-module.*

Proof. If $P_A \cong eA$, with $e \in A$ a primitive idempotent, then

$$P^t = Hom_A(eA, A) \cong Ae,$$

and this result thus follows from the additivity of $(-)^t$. □

This functor induces a duality, also denoted by $(-)^t$, between the category *proj* A of projective right A-modules, and the category *proj* A^{op} of projective left A-modules (this is due to the homomorphism $\epsilon_M : M \to M^{tt}$ given by $\epsilon_M(z)(f) = f(z)$ (for $z \in M$ and $f \in M^t$), which is functorial in M and it is an isomorphism whenever M is projective.

Another important functor is the *K-duality functor* $D : mod\, A \to mod\, A^{op}$, also known as *standard duality*, is defined as follows:

- For each right A-module M, $D(M)$, also denoted by M^*, is the K-vector space $Hom_K(M, K)$ with a structure of left A-module given by the formula $(af)(m) = f(ma)$, for $f \in Hom_K(M, K), a \in A$ and $m \in M$.

- For each A-module morphism $f : M \to N$, the K-morphism $D(h) : D(N) \to D(M)$ of left A-modules is defined by $\varphi \mapsto \varphi h$.

It is easy to show that D is a duality. The following proposition provides some important properties of this functor.

Proposition 1.2.35. [2, I.5.13] *Let L, M, N be A-modules. The standard duality D verifies the following assertions.*

1. *M is a projective A-module if and only if $D(M)$ is an injective A^{op}-module.*

2. *M is an injective A-module if and only if $D(M)$ is a projective A^{op}-module.*

3. *A morphism* $f : M \to N$ *is a minimal monomorphism if and only if* $D(f) : D(N) \to D(M)$ *is a minimal epimorphism.*

4. *A morphism* $f : M \to N$ *is a minimal epimorphism if and only if* $D(f) : D(N) \to D(M)$ *is a minimal monomorphism.*

5. *A sequence* $0 \to L \xrightarrow{f} N \xrightarrow{g} M \to 0$ *in* $mod\, A$ *is exact if and only if* $0 \to D(M) \xrightarrow{D(g)} D(N) \xrightarrow{D(f)} D(L) \to 0$ *is an exact sequence in* $mod\, A^{op}$.

Now we define the *tensor product functors*. Let A, B be K-algebras and M be an $A-B$-bimodule. The covariant functor

$$(-) \otimes M : Mod\, A \to Mod\, B$$

is defined as follows:

- Each right A-module N is associated to the right B-module $N \otimes M$, whose multiplication $(N \otimes M) \times B \to N \otimes M$ is defined by the formula $(m \otimes n)b := mb \otimes n$, for $m \in M, n \in N, b \in B$.

- For each morphism of right A-modules $f : N \to L$, $f \otimes M : N \otimes M \to L \otimes M$ is defined by the formula $(f \otimes M)(n \otimes m) = f(n) \otimes m$, for $n \in N, m \in M$.

Analogously, we define the covariant tensor product $M \otimes (-) : Mod\, B^{op} \to Mod\, A^{op}$.

Let M be an A-module. It is easy to see that

$$\varphi^M : (-) \otimes_A M^t \to Hom_A(M, -),$$

defined as follows: for an A-module N,

$$\begin{array}{rccc} \varphi^M_N : & N \otimes_A M^t & \to & Hom_A(M, N) \\ & n \otimes f & \mapsto & \varphi^M_N(n \otimes f) : m \mapsto nf(m), \end{array}$$

where $m \in M, n \in N$ and $f \in M^t$, is a functorial homomorphism. Moreover, if M is projective, then φ^M is a functorial isomorphism, i.e., φ^M_N is an isomorphism for any A-module N, and if N is projective, then φ^M_N is an isomorphism as well.

The mth extension bifunctor $Ext_A^m(-,-)$

Let A be a K-algebra and let M, N be two A-modules. Take a projective resolution of M

$$\ldots \to P_m \stackrel{h_m}{\to} P_{m-1} \to \ldots \to P_1 \stackrel{h_1}{\to} P_0 \stackrel{h_0}{\to} M \to 0.$$

Then $Ext_A^m(M, N)$ is defined to be $Ker\, Hom_A(h_{m+1}, N)/Im\, Hom_A(h_m, N)$, for $m \geq 1$. For $m = 0$,

$$Ext_A^0(M, N) := Ker\, Hom_A(h_1, N).$$

Note that $Hom_A(h_{m+1}, N)Hom_A(h_m, N) = 0$ since $h_{m+1}h_m = 0$. Thus $Im\, Hom_A(h_m, N) \subseteq Ker\, Hom_A(h_{m+1}, N)$. Moreover, this definition doesn't depend on the choice of the projective representation of M, so $Ext_A^m(-, N)$ is well defined on the objects.

The following proposition gives another definition of $Ext_A^m(M, N)$ via any injective resolution of N.

Proposition 1.2.36. [24, 6.2.5] *Let M, N be A-modules, and let*

$$0 \to N \stackrel{p_0}{\to} I_0 \stackrel{p_1}{\to} I_1 \stackrel{p_2}{\to} I_2 \to \ldots$$

be an injective resolution of N. Then

$$Ext_A^m(M, N) \cong Ker\, Hom_A(M, p_{n+1})/Im\, Hom_A(M, p_n).$$

Consider now an A-module morphism $f : M \to M'$ and a projective resolution of M':

$$\ldots \to P'_m \stackrel{h'_m}{\to} P'_{m-1} \to \ldots \to P'_1 \stackrel{h'_1}{\to} P'_0 \stackrel{h'_0}{\to} M' \to 0.$$

There exists a family of morphisms $\{f_m\}_{m \in \mathbb{N}_0}$ satisfying the condition $h'_m f_m = f_{m-1}h_m$, for $m \geq 1$ and $h'_0 f_0 = f h_0$. The existence of f_0 is due to the fact that $h' : P'_0 \to M'$ is an epimorphism and P_0 is projective, and the existence of f_m, for $m \geq 1$ follows from 1.2.24.

Define $Ext_A^m(f, N) : Ext_A^m(M', N) \to Ext_A^m(M, N)$ to be $Hom_A(f_m, N)$.

Note that

$$Hom_A(h_{m+1}, N)Hom_A(f_m, N) = Hom_A(f_{m+1}, N)Hom_A(h'_{m+1}, N),$$

with $m \in \mathbb{N}$. Hence, if $g \in Ker\, Hom_A(h'_{m+1}, N)$, then $Hom_A(f_m, N)(g) \in Ker\, Hom_A(h_{m+1}, N)$, and if $g \in Im\, Hom_A(h'_m, N)$, then $Hom_A(f_m, N)(g) \in$

$Im\, Hom_A(h_m, N)$. Moreover, this definition does not depend on the choice of $\{f_m\}_{m\in\mathbb{N}_0}$, so $Ext^m_A(-, N)$ is well defined on the morphisms.

The functor

$$Ext^m_A(-, N) : Mod\, A \to Mod\, K$$

is contravariant and additive. Analogously, we define the covariant additive functor

$$Ext^m_A(M, -) : Mod\, A \to Mod\, K.$$

Proposition 1.2.37. [2, A.4.6] *Let M be an A-module.*

1. *The projective dimension of M is m if and only if $Ext^m_A(M, -) \neq 0$ and $Ext^{m+1}_A(M, -) = 0$.*
2. *The injective dimension of M is m if and only if $Ext^m_A(-, M) \neq 0$ and $Ext^{m+1}_A(-, M) = 0$.*

We will now see another characterization of the elements of $Ext^1_A(M, N)$, for two arbitrary A-modules M, N.

Definition 1.2.38. Let M, N be two A-modules. An *extension of M by N* is an exact sequence

$$0 \to M \xrightarrow{f} L \xrightarrow{g} N \to 0.$$

Two extensions of M by N:

$$0 \to M \xrightarrow{f} L \xrightarrow{g} N \to 0,$$

and

$$0 \to M \xrightarrow{f'} L' \xrightarrow{g'} N \to 0,$$

are *equivalent* if there exists a morphism $\alpha : L \to L'$ such that $\alpha f = f'$ and $g'\alpha = g$. The morphism α is in fact an isomorphism.

This defines an equivalence relation in the set of the extensions of M by N. We denote by $ext(N, M)$ the set of all equivalence classes of extensions of M by N.

Proposition 1.2.39. [24, 7.1.5] *There is a one-to-one correspondence between $ext(N, M)$ and $Ext^1_A(N, M)$.*

Proposition 1.2.40. [24, 7.1.8, 7.1.9] *Let $S : 0 \to M \xrightarrow{f} L \xrightarrow{g} N \to 0$ be an extension of M by N, v a morphism from N' to N and u a morphism from M to M'.*

1. *There is a unique (up to equivalence) extension* $S' : 0 \to M \xrightarrow{f'} L' \xrightarrow{g'} N' \to 0$ *of* M *by* N'*, and a morphism* $v' : L' \to L$ *such that the diagram*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f'"] \arrow[d, "1"] & L' \arrow[r, "g'"] \arrow[d, "v'"] & N' \arrow[r] \arrow[d, "v"] & 0 \\
0 \arrow[r] & M \arrow[r, "f"'] & L \arrow[r, "g"'] & N \arrow[r] & 0
\end{tikzcd}$$

is commutative.

2. *There is a unique (up to equivalence) extension* $S'' : 0 \to M' \xrightarrow{f''} L'' \xrightarrow{g''} N \to 0$ *and a morphism* $u' : L \to L''$ *such that the diagram*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "u"] & L \arrow[r, "g"] \arrow[d, "u'"] & N \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & M' \arrow[r, "f''"'] & L'' \arrow[r, "g''"'] & N \arrow[r] & 0
\end{tikzcd}$$

is commutative.

We denote by $ext(v, M)([S])$, or briefly $ext(M, v)(S)$, the equivalence class in $ext(N', M)$ represented by S', and $ext(N, u)([S])$, or $ext(N, u)(S)$ for short, the equivalence class in $ext(N, M')$ represented by S''.

Remark 1.2.41. It can be proved (see proof of [19, 3.2.4], for example) that the bijection ψ between $ext(N, M)$ and $Ext^1_A(N, M)$ given in 1.2.39 is natural in N, that is, for any morphism $v : N' \to N$, the square

$$\begin{tikzcd}
ext(N, M) \arrow[r, "\psi"] \arrow[d, "ext(v,M)"] & Ext^1_A(N, M) \arrow[d, "Ext^1_A(v,M)"] \\
ext(N', M) \arrow[r, "\psi"] & Ext^1_A(N', M)
\end{tikzcd}$$

commutes, and it is also natural in M, in the sense that, for any morphism $u : M \to M''$, the square

$$\begin{tikzcd}
ext(N, M) \arrow[r, "\psi"] \arrow[d, "ext(N,u)"] & Ext^1_A(N, M) \arrow[d, "Ext^1_A(N,u)"] \\
ext(N, M'') \arrow[r, "\psi"] & Ext^1_A(N, M'')
\end{tikzcd}$$

commutes.

Definition 1.2.42. An extension $0 \to M \xrightarrow{f} L \xrightarrow{g} N \to 0$ of M by N is said to be a *split extension* if it splits.

It can be proved ([24, 7.1.3, 7.1.4]) that two split extensions of M by N are equivalent and that an extension equivalent to a split extension also splits. Hence the split extensions of M by N form an equivalence class.

Proposition 1.2.43. [24, 7.1.6] *Under the correspondence between the equivalence classes of extensions of M by N and $Ext^1_A(N, M)$ given in 1.2.39, the equivalence class of split extensions of M by N corresponds to the zero element of $Ext^1_A(N, M)$.*

1.3 Root Systems

We will now give the basic notions on root systems, necessary to understand what follows. For more details on this subject, we refer the reader to [6, 20].

Throughout this section, V will denote an Euclidean space of finite dimension $n + 1$.

Let H be a hyperplane of V orthogonal to $\alpha \in V$. Then, there exists one and only one reflection such that $s(v) = v$, for all $v \in H$. This reflection is given, for every $v \in V$, by

$$s(v) = v - \frac{2(v, \alpha)}{(\alpha, \alpha)}\alpha.$$

Definition 1.3.1. A *(crystallographic) root system* in V is a finite subset Φ of generators of V that satisfies the following conditions:

- $0 \notin \Phi$,
- $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$, for all $\alpha \in \Phi$,
- For each $\alpha \in \Phi$, $s_\alpha(\Phi) = \Phi$, where s_α is the reflection in $H_\alpha = \{x \in V \mid (x, \alpha) = 0\}$.
- $2(\alpha, \beta)/(\alpha, \alpha) \in \mathbb{Z}$, for all $\alpha, \beta \in \Phi$.

A root system Φ is said to be *reducible* if it can be decomposed into two disjoint root systems Φ' and Φ'' such that every $\beta \in \Phi'$ is normal to every $\gamma \in \Phi''$. A root system is *irreducible* if such a decomposition doesn't exist.

Given a root system $\Phi \subseteq V$, we define the *Weyl group* associated to Φ to be the group generated by the reflections s_α, for $\alpha \in \Phi$.

A *simple system*, $\Pi \subseteq \Phi$, is a basis for V and such that each $\alpha \in \Phi$ can be written as a linear combination of Π with all coefficients of the same sign. The elements of Π are called the *simple roots*, the elements of Φ for which the coefficients are all positive, are called the *positive roots* and the others (whose coefficients are all negative) are called the *negative roots*.

Definition 1.3.2. Let Φ be a root system and $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subseteq \Phi$ be a simple system. The *Cartan matrix* A of Φ is the $n \times n$ integer matrix with

$$a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}.$$

Observe that the diagonal entries of the Cartan matrix are 2, $a_{ij} \leq 0$, for all $i \neq j$, and $a_{ij} = 0$ if and only if $a_{ji} = 0$.

We can associate a diagram, called the *Dynkin diagram*, to the Cartan matrix A as follows: the vertices $1, \ldots, n$ correspond to the simple roots $\alpha_1, \ldots, \alpha_n$, and the vertices i, j, with $i \neq j$, are linked by $a_{ij}a_{ji}$ edges. Note that this number is indeed a non-negative integer. Moreover, $a_{ij}a_{ji} = 0, 1, 2$ or 3.

Proposition 1.3.3. *A root system is indecomposable if and only if the Dynkin diagram of the corresponding Cartan matrix is connected.*

The following proposition provides the completed classification of irreducible root systems, obtained by E. Cartan and W. Killing.

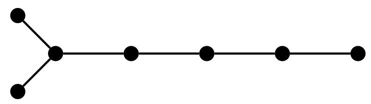
Proposition 1.3.4. *The complete list of Dynkin diagrams of irreducible root systems is given by:*

A_n $(n \geq 1)$ $\quad \bullet - \bullet - \bullet - \bullet - \bullet - \bullet$

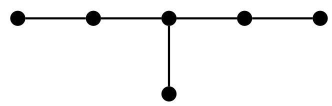
B_n $(n \geq 2)$ $\quad \bullet \Longleftarrow \bullet - \bullet - \bullet - \bullet - \bullet$

C_n $(n \geq 3)$ $\quad \bullet \Longrightarrow \bullet - \bullet - \bullet - \bullet - \bullet$

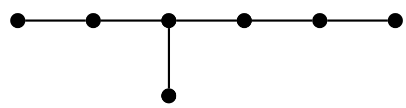
D_n $(n \geq 4)$



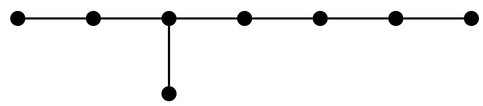
E_6



E_7



E_8



F_4 $\quad \bullet - \bullet \Longrightarrow \bullet - \bullet$

G_2 $\quad \bullet \Lleftarrow \bullet$

The meaning of the arrows on the double and triple edges is as follows. If $a_{ij}a_{ji} > 1$, we place an arrow on the edge joining i to j pointing from the longer root toward the shorter root, i.e., if $|\alpha_i| > |\alpha_j|$, then we have an arrow pointing from j to i.

We will be mainly interested in the root system A_n. Let $(e_1, \ldots, e_{n+1})$ be the standard basis of V. Write $\alpha_{ij} = e_i - e_{j+1}$, with $1 \leq i < j \leq n+1$. Then $\Phi := \{\pm\alpha_{ij} \mid 1 \leq i < j \leq n+1\}$ is a root system of type A_n. A simple root system is given by $\alpha_1 = \alpha_{11}, \ldots, \alpha_n = \alpha_{nn}$, and the corresponding Cartan matrix is given by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i = j-1, j+1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $i < j$, $\alpha_{ij} = \alpha_i + \ldots + \alpha_j$.

The following definition was introduced by Fomin and Zelevinsky.

Definition 1.3.5. Let Φ be a root system, and $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots. A root α of Φ is said to be *almost positive* if $\alpha = -\alpha_i$, with $\alpha_i \in \Pi$ or the coefficients are all positive in the expression of α as a linear combination of the simple roots. The set of almost positive roots is denoted by $\Phi_{\geq -1}$.

For each $i \in [n]$, let $\sigma_i : \Phi_{\geq -1} \to \Phi_{\geq -1}$ be the map defined by

$$\sigma_i(\alpha) = \begin{cases} s_i(\alpha) & \text{if } s_i(\alpha) \in \Phi_{\geq -1}, \\ \alpha & \text{otherwise.} \end{cases}$$

The map σ_i is an involution, i.e., $\sigma_i^2 = 1$. Indeed, if $\alpha \in \Phi_{\geq -1}$ is such that $s_i(\alpha) \in \Phi_{\geq -1}$, then $\sigma_i^2(\alpha) = \sigma(s_i(\alpha)) = s_i(s_i(\alpha)) = \alpha$, and if $s_i(\alpha) \notin \Phi_{\geq -1}$, then $\sigma_i^2(\alpha) = \sigma_i(\alpha) = \alpha$.

Note also that $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $a_{ij} = 0$.

Let $\triangle$ be a Dynkin diagram. We can divide the set of vertices into two disjoint subsets I_+ and I_-, such that $a_{ij} = 0$, for all pairs i, j of different vertices, belonging both to either I_+ or $I-$.

Let

$$\tau_+ : \Phi_{\geq -1} \to \Phi_{\geq -1}, \tau_+ = \prod_{i \in I_+} \sigma_i,$$

and

$$\tau_- : \Phi_{\geq -1} \to \Phi_{\geq -1}, \tau_- = \prod_{i \in I_-} \sigma_i.$$

Since the σ_i commute with one another, we have $\tau_+^2 = 1$. For the same reason, we have $\tau_-^2 = 1$, as well. Therefore, τ_+ and τ_- are permutations of $\Phi_{\geq -1}$ which are both involutions.

We end this section with a proposition that will be useful for the sequel, namely for chapter 3 and chapter 4.

Proposition 1.3.6. [18, Theorem 2.6] *Every $\langle \tau_-, \tau_+ \rangle$-orbit in $\Phi_{\geq -1}$ contains a negative simple root.*

Chapter 2

Representation theory of quivers

This chapter is dedicated to the study of representation theory, specifically the notions and the results necessary to the understanding of [13], the core of this thesis. Quivers give a way to visualize finite dimensional algebras and modules. For each basic, connected and finite algebra A, we associate a finite and connected quiver Q. The modules over A can be viewed as representations of Q, connecting each vertex of Q to a K-vector space and each arrow to a K-linear map. We prove that the category of the modules over a path algebra associated to a quiver is equivalent to the category of representations of that quiver. This approach provides an explicit computation of some special modules as the simple and the indecomposable projective and injective modules. Section 2 is devoted to this subject.

Irreducible and minimal almost split morphisms are of great importance in the area of representation theory. The notion of minimal almost split morphisms gives rise to a special type of exact sequences, the almost split sequences. The main purpose of section 4 is to prove the existence of these sequences, which requires the notion of Auslander-Reiten translations.

A special quiver that contains all the information known about $mod\, A$, called the Auslander-Reiten quiver, whose vertices are in relation with the indecomposable modules and the arrows are in relation with the irreducible morphisms, is defined in the last section of this chapter.

There are some results with minor relevance for the objective of this dissertation whose proof is ommited. In these cases we refer the reader to [2] and [4].

2.1 Basic notions: quivers, path algebras and representations

Firstly, let us enumerate the basic notions and terminology that we need to know about quivers.

A *quiver* $Q = (Q_0, Q_1)$ is an oriented graph, where Q_0 is the set of vertices and Q_1 the set of arrows between vertices. Throughout this chapter we will only consider *finite* and *connected* quivers, that is, quivers with a finite number of vertices and arrows, whose vertices are all connected with one another.

Two vertices a, b of Q are *neighbours* if they are connected by an arrow α. Suppose α is oriented from a to b. In this case we say that a is the source of α and b is its target. Then a is said to be a *predecessor* of b and b is said to be a *successor* of a. A vertex a is a *source* (resp. *sink*) if it is not a target (resp. source) of any arrow in Q.

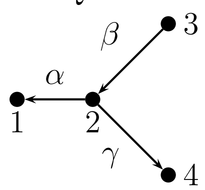
A *path of length* n, ω, in Q is a sequence of n arrows $\alpha_1 \alpha_2 \dots \alpha_n$ such that the source of α_i is the target of α_{i-1}, for $i = 2, \dots, n$. The source of ω is the source of α_1 and the target of ω is the target of α_n. If the source and the target of ω coincide, the path ω is called a *cycle*. If the quiver Q has no cycles, it is called *acyclic*. To each vertex a of Q, we associate a *trivial path*, a path of length zero, and it will be denoted by ϵ_a.

A *subquiver* of Q is a quiver $Q' = (Q'_0, Q'_1)$ such that $Q'_0 \subseteq Q_0$ and $Q'_1 \subseteq \{\, \alpha \in Q_1 \mid s(\alpha), t(\alpha) \in Q'_0 \,\}$. If $Q'_1 = \{\, \alpha \in Q_1 \mid s(\alpha), t(\alpha) \in Q'_0 \,\}$, then the subquiver Q' is said to be *full*.

We will now associate a K-algebra, where K denotes an algebraically closed field, to a quiver.

Definition 2.1.1. Let Q be a quiver. The *path algebra* KQ of Q is the K-algebra with basis of the underlying K-vector space the paths in Q, and with the product of two paths ω, ω', given by the obvious composition $\omega\omega'$, if it makes sense, i.e., if the target of ω is the source of ω', and zero otherwise. This product is extended to arbitrary elements of KQ, by distributivity.

Example 2.1.2. Consider the quiver Q



The basis of the path algebra KQ is the set:

$$\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \alpha, \beta, \gamma, \beta\gamma, \beta\alpha\},$$

where ϵ_i is the trivial path associated to the vertex i ($i \in [4]$), and the multiplication table for the basis is:

	ϵ_1	ϵ_2	ϵ_3	ϵ_4	α	β	γ	$\beta\gamma$	$\beta\alpha$
ϵ_1	ϵ_1	0	0	0	0	0	0	0	0
ϵ_2	0	ϵ_2	0	0	α	0	γ	0	0
ϵ_3	0	0	ϵ_3	0	0	β	0	$\beta\gamma$	$\beta\alpha$
ϵ_4	0	0	0	ϵ_4	0	0	0	0	0
α	α	0	0	0	0	0	0	0	0
β	0	β	0	0	$\beta\alpha$	0	$\beta\gamma$	0	0
γ	0	0	0	γ	0	0	0	0	0
$\beta\gamma$	0	0	0	$\beta\gamma$	0	0	0	0	0
$\beta\alpha$	$\beta\alpha$	0	0	0	0	0	0	0	0

The following proposition gathers some basic properties of the path algebra.

Proposition 2.1.3. [2, II.1.4, II.1.5, II.1.7] *Let A be the path algebra KQ of the finite and connected quiver Q.*

1. *A is associative.*

2. *The trivial paths ϵ_i, with $i \in Q_0$, are orthogonal idempotents and $\sum_{i\in Q_0} \epsilon_i$ is the identity of A.*

3. *The trivial paths ϵ_i, with $i \in Q_0$, are primitive. The set $\{\epsilon_i \mid i \in Q_0\}$ is, therefore, a complete set of primitive orthogonal idempotents.*

4. *A is connected*

5. *A is finite dimensional if and only if Q is acyclic.*

The next proposition tells us what is the radical of the path algebra of a finite, connected and acyclic quiver.

Proposition 2.1.4. *Let Q be a finite, connected and acyclic quiver. The two-sided ideal R_Q of the path algebra KQ generated (as an ideal) by the arrows of Q is the radical of KQ.*

Proof. Since Q is acyclic, there is a path in Q with maximal length l. So the two-sided ideal of KQ, R_Q^{l+1}, which is generated by the paths of length $\geq l+1$, is zero. Thus R_Q is nilpotent.

Consider the elements $\bar{\epsilon}_i = \epsilon_i + R_Q$ $(i \in Q_0)$ of KQ/R_Q. These elements form a basis for KQ/R_Q, since R_Q contains all the paths of length ≥ 1, and we have the decomposition

$$KQ/R_Q = \bigoplus_{i \in Q_0} \bar{\epsilon}_i (KQ/R) \bar{\epsilon}_i$$

as a K-vector space. Because the underlying vector space of the K-algebra $\bar{\epsilon}_i(KQ/R_Q)\bar{\epsilon}_i$ is generated by $\bar{\epsilon}_i$, $\bar{\epsilon}_i(KQ/R_Q)\bar{\epsilon}_i$ is isomorphic to K. Therefore, KQ/R is isomorphic to a product of $|Q_0|$ copies of K. By 1.2.2 (3), $rad\, KQ = R_Q$. □

Remark 2.1.5. Since $KQ/R_Q = KQ/rad\, KQ$ is isomorphic to a product of copies of K, it follows from 1.2.14 that $\epsilon_i KQ \not\cong \epsilon_j KQ$, for all $i, j \in Q_0, i \neq j$, i.e., KQ is basic.

We will now define special quotients of the path algebra of a finite and connected quiver Q, which are finite dimensional even when KQ isn't, that is, even when Q is not acyclic. We will see later that any basic, connected and finite dimensional K-algebra is isomorphic to one of these quotients of the path algebra of a finite and connected quiver.

Definition 2.1.6.

1. A *relation* ρ on a quiver Q over K is a linear combination

$$\rho = \sum_{i=1}^{m} \lambda_i \omega_i,$$

   where $\lambda_i \in K$ (all not zero) and ω_i are paths of Q with length ≥ 2, all with the same target and source.

2. Let $(\rho_i)_{i \in J}$ be a set of relations on Q over K. The pair $(Q, (\rho_i)_{i \in J})$ is called a *quiver bound by the relations* ρ_i, or by the relations $\rho_i = 0, i \in I$.

We can associate with $(Q, (\rho_i)_{i \in J})$ the K-algebra $KQ/ \langle \rho_i \mid i \in J \rangle$, where $\langle \rho_i \mid i \in J \rangle$ is the ideal of KQ generated by the relations ρ_i. Since the paths involved in each ρ_i have length at least 2, $\langle \rho_i \mid i \in J \rangle \subseteq R_Q^2$. If this ideal contains R_Q^m for some $m \geq 2$, it is called *admissible*.

Example 2.1.7. Consider the quiver Q

$$
\begin{tikzcd}
& \bullet \arrow[dl, "\beta"'] & \\
\bullet \arrow[loop left, "\lambda"] & & \bullet \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\
& \bullet \arrow[ul, "\delta"] &
\end{tikzcd}
$$

The ideal $I = \langle \alpha\beta - \gamma\delta, \beta\lambda, \lambda^3 \rangle$ is admissible, because $I \subseteq R_Q^2$ and $R_Q^4 \subseteq I$. Indeed, every path of length ≥ 4 in Q must contain the path λ^3 (when its source is 1, 2 or 3) or the path $\alpha\beta\lambda^2$ or $\gamma\delta\lambda^2$ (when the source is 4). Either way, such path lies in I. In the first case because λ^3 belongs to I, in the second case because $\beta\lambda$ belongs to I and in the third case because $\gamma\delta\lambda^2 = (\lambda\delta - \alpha\beta)\lambda^2 + \alpha\beta\lambda^2 \in I$. An example of an ideal that is not admissible is $J = \langle \beta\lambda, \alpha\beta - \gamma\delta \rangle$. In fact, $R^l \nsubseteq J$ for all $l \geq 2$, since $\lambda^l \in R^l \setminus J$.

Suppose $I = \langle \rho_i \mid i \in J \rangle$ is admissible, and let $m \geq 2$ be such that $R_Q^m \subseteq I$. Then KQ/R_Q^m is clearly finite dimensional, since it is generated by the finite set $\{\omega + R_Q^m \mid \omega \text{ is a path of length} < \text{m}\}$. Because KQ/I is the image of KQ/R_Q^m by the canonical epimorphism, KQ/I is finite dimensional. Also, if I is admissible, the radical of KQ/I is easily determined.

Proposition 2.1.8. [2, 2.2.5, 2.2.10] *Let Q be a finite and connected quiver and $I = (\rho_i)_{i \in J}$ be an admissible ideal of KQ.*

1. *KQ/I is basic and connected.*
2. *The radical of KQ/I is the ideal R_Q/I.*

Let now A be a basic, connected and finite dimensional K-algebra. Let $\{e_1, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents of A. Note that there exists such a set (cf. 1.2.11). The *quiver of A*, Q_A is defined as follows:

- The vertices of Q_A correspond to the idempotents $e_1, \ldots, e_n$.
- The arrows $\alpha : i \to j$, for $i, j \in (Q_A)_0$, correspond to the vectors in a basis of the K-vector space $e_i(rad\, A/rad^2\, A)e_j$.

The quiver is well defined, i.e., it doesn't depend on the choice of the complete set of primitive orthogonal set (cf. [2, II.3.2]).

Note that Q_A is finite since A is finite dimensional and so is the K-vector space $e_i(rad\, A/rad^2\, A)e_j$, for all $i, j \in (Q_A)_0$. Moreover, the fact that A is connected implies that Q_A is connected (cf. [2, II.3.4]).

At first sight, the definition for the arrows in Q_A may seem laboured, but considering a finite and connected quiver Q and an admissible ideal I of the path algebra KQ, the set $\{\bar{\alpha} + rad^2(KQ/I) \mid \bar{\alpha} = \alpha + I,\, \alpha \in Q_1\}$ is a basis of

$$rad\,(KQ/I)/rad^2\,(KQ/I) = (R_Q/I)/(R_Q/I)^2 \cong R_Q/R_Q^2.$$

Therefore, the arrows from i to j in Q are in bijective correspondence with the vectors in a basis of the K-vector space $e_i(rad\,A/rad^2\,A)e_j$.

So, in the case where $A = KQ/I$ for some finite and connected quiver Q and some admissible ideal I, the quiver of A is Q.

Proposition 2.1.9. [2, II.3.7] *Let A be a basic and connected finite dimensional K-algebra. There exists an admissible ideal I of KQ_A such that $A \cong KQ_A/I$.*

The following remark will be useful for the last chapter.

Remark 2.1.10. Let A be a basic, connected and finite dimensional algebra. There is an admissible ideal $I = \langle \rho_i \mid i \in J\rangle$ such that $A = KQ_A/I$. Write $Q = Q_A$, for short. The quiver of the opposite algebra A^{op} is the quiver Q^{op}, obtained from Q by changing the order of the arrows, i.e., if $Q_1^{op} = \{\alpha^{op} : b \to a \mid \alpha : a \to b \in Q_1\}$. Moreover, $A^{op} = (KQ^{op})/I^{op}$, where $I^{op} = \langle \rho_i^{op} \mid i \in J\rangle$.

Let A be a finite dimensional algebra. We will now relate the quivers with finite dimensional A-modules. The category of the finite dimensional modules over a finite dimensional K-algebra is equivalent to the category of the finite dimensional modules over a basic, connected and finite dimensional K-algebra (cf. [2, I.6.19]). Hence we will assume that the modules considered from now on are over an algebra satisfying such conditions. We know that A is isomorphic to KQ/I, for some finite and connected quiver Q and an admissible ideal I. An A-module can be viewed as a representation of (Q, I), which assigns to each vertex of Q a K-vector space, and to each arrow of Q a K-linear map and satisfies some relations induced by I.

Definition 2.1.11. Let Q be a finite quiver. A *representation* M, (M_a, φ_α) $(a \in Q_0, \alpha \in Q_1)$, of Q is a set of K-vector spaces $\{M_a \mid a \in Q_0\}$ together with a set of K-linear maps $\{\varphi_\alpha : M_a \to M_b \mid \alpha : a \to b \in Q_1\}$. If all the vector spaces M_a, $a \in Q_0$ are finite dimensional, then (M_a, φ_α) is said to be a *finite dimensional* representation of Q.

For a finite quiver Q, we define a category of representations of Q, denoted by $Rep(Q)$, where:

- The objects are the representations of Q,
- Given two representations of Q, $M =: (M_a, \varphi_\alpha)$ and $M' =: (M'_a, \varphi'_\alpha)$, a *morphism* (of representations) $f : M \to M'$ is defined to be a family $f = (f_a)_{a \in Q_0}$ of K-linear maps $(f_a : M_a \to M'_a)_{a \in Q_0}$ for which the following diagram is commutative:

$$\begin{tikzcd} M_a \arrow[r, "\varphi_\alpha"] \arrow[d, "f_a"] & M_b \arrow[d, "f_b"] \\ M'_a \arrow[r, "\varphi'_\alpha"] & M'_b \end{tikzcd}$$

  for all $\alpha : a \to b \in Q$,
- The composition of two morphisms of representations of Q, $f = (f_a)_{a \in Q_0} : M \to M'$ and $g = (g_a)_{a \in Q_0} : M' \to M''$ is the family $gf = (g_a f_a)_{a \in Q_0}$. Clearly gf is a morphism from M to M''.

We are only interested in finite dimensional representations of Q, and in this case the category is denoted by $rep(Q)$.

Example 2.1.12. Consider the quiver

$$\begin{tikzcd} 1 & 2 \arrow[l, "\beta"'] & 3 \arrow[l, shift right, "\gamma"'] \arrow[l, shift left, "\alpha"] \end{tikzcd}$$

and the finite dimensional representations M:

$$\begin{tikzcd} K & K^2 \arrow[l, "{[0\,1]}"'] & K \arrow[l, shift right, "{[0\,1]^t}"'] \arrow[l, shift left, "{[1\,0]^t}"] \end{tikzcd}$$

and M':

$$\begin{tikzcd} K & K \arrow[l, "1"] & K^2. \arrow[l, shift right, "{[0\,1]}"'] \arrow[l, shift left, "{[1\,0]}"] \end{tikzcd}$$

We have a morphism $f = (f_a)_{a \in Q_0} : M \to M'$, where $f_1 = 1, f_2 = [0\,1]$ and $f_3 = [0\,1]^t$.

$$\begin{tikzcd} K \arrow[d, "1"'] & K^2 \arrow[l, "{[0\,1]}"'] \arrow[d, "{[0\,1]}"'] & K \arrow[l, shift right, "{[0\,1]^t}"'] \arrow[l, shift left, "{[1\,0]^t}"] \arrow[d, "{[0\,1]^t}"] \\ K & K \arrow[l, "1"] & K^2. \arrow[l, shift right, "{[0\,1]}"'] \arrow[l, shift left, "{[1\,0]}"] \end{tikzcd}$$

Let $I = \langle \rho_i \mid i = 1, \ldots, m\rangle$ be an admissible ideal. Suppose Q is bound by the relations in Q. We say that a representation $M = (M_a, \varphi_\alpha)$ is *bound by* I or *satisfies the relations* in I if

$$\varphi_\rho = 0, \text{ for all relations } \rho \in I,$$

where φ_ρ is the *evaluation* of M on ρ, which is defined as follows:

$$\varphi_\omega = \varphi_{\alpha_k}\varphi_{\alpha_{k-1}} \cdots \varphi_{\alpha_1},$$

for a path $\omega = \alpha_1 \ldots \alpha_k$ in Q, and

$$\varphi_\rho = \sum_{i=1}^{l} \lambda_i \varphi_{\omega_i},$$

where $\rho = \sum_{i=1}^{l} \lambda_i \omega_i$ is an arbitrary element of KQ.

Example 2.1.13. Consider the quiver Q and the representation M given in the example 2.1.12. This representation is bound by the relation $\alpha\beta = 0$ but it doesn't satisfy the relation $\gamma\beta = 0$.

The full subcategory $Rep_K(Q, I)$ of $Rep_K(Q)$ is composed by the representations of Q bound by I. If the representations are finite dimensional, then the subcategory is denoted by $rep_K(Q, I)$.

Theorem 2.1.14. *Let Q be a finite and connected quiver, I an admissible ideal of KQ and $A = KQ/I$. The categories $rep_K(Q, I)$ and $mod\, A$ are equivalent.*

Proof. We only present the equivalence $F : mod\, A \to rep_K(Q, I)$ and its equivalence inverse. Denote by $e_a = \epsilon_a + I$, with $a \in Q_0$, the primitive idempotents of A. Consider the functor

$$F : mod\, A \to rep_K(Q, I),$$

defined as follows:

1. For $M \in mod\, A$, $F(M) = (M_a, \varphi_\alpha)$, where
   - $M_a := Me_a$, for each $a \in Q_0$,
   - For each $\alpha : a \to b \in Q_1$, $\varphi_\alpha(x) := x\bar{\alpha} \in Me_b$, where $x = xe_a \in M_a$, and $\bar{\alpha} = \alpha + I$.
2. For an A-module morphism $f : M \to M'$, $F(f) = (f_a)_{a \in Q_0}$, where $f_a = f|_{M_a}$. Note that, for $x = xe_a \in Me_a$, we have $f_a(xe_a) = f(xe_a) = f(x)e_a \in M'e_a = M'_a$.

Let us check that (M_a, φ_α) is bound by the relations in I. Let $\rho = \sum_{i=1}^m \lambda_i \omega_i$, where $\lambda_i \in K$ and $\omega_i = \alpha_{i,1} \ldots \alpha_{i,l_i}$ is a path in KQ, be a relation in I. Then, we have:

$$\begin{aligned}
\varphi_\rho(x) &= \sum_{i=1}^m \lambda_i \varphi_{\omega_i}(x) \\
&= \sum_{i=1}^m \lambda_i \varphi_{\alpha_{i,l_i} \ldots \alpha_{i,1}}(x) \\
&= \sum_{i=1}^m \lambda_i (x \bar{\alpha}_{i,1} \ldots \bar{\alpha}_{i,l_i}) \\
&= x \cdot \sum_{i=1}^m \lambda_i (\bar{\alpha}_{i,1} \ldots \bar{\alpha}_{i,l_i}) \\
&= x \cdot \bar{\rho} = x \cdot 0 = 0.
\end{aligned}$$

We also have to check that $F(f)$ is a morphism of representations, i.e., $\varphi'_\alpha f_a = f_b \varphi_\alpha$, for each arrow $\alpha : a \to b$. Indeed, if $x \in M_a$, we have

$$\varphi'_\alpha f_a(x) = f_a(x)\bar{\alpha} = f(x)\bar{\alpha} = f(x\bar{\alpha}) = f_b(x\bar{\alpha}) = f_b \varphi_\alpha(x).$$

It is clear that F sends finite A-modules to finite bounded representations. The inverse equivalence of F is $G : rep_K(Q, I) \to mod\, A$, such that:

1. For $M = (M_a, \varphi_\alpha) \in rep_K(Q, I)$, $G(M) = \oplus_{a \in Q_0} M_a$ with a KQ-module structure given by:

$$x\omega = \begin{cases} x_a & \text{if } \omega = e_a, \\ (\delta_{bc} \varphi_\omega(x_a))_c & \text{if } \omega \text{ is a path from } a \text{ to } b, \end{cases}$$

   where $x = (x_a)_{a \in Q_0} \in G(M)$.

2. $f = \oplus_{a \in Q_0} f_a : G(M) \to G(M')$, for each morphism $(f_a)_{a \in Q_0} : M = (M_a, \varphi_\alpha) \to M' = (M'_a, \varphi'_\alpha)$ in $rep_K(Q, I)$.

Note that if $\rho \in I$ and $x \in G(M)$, then $x\rho = 0$, since $\varphi_\rho = 0$. So $G(M)$ is, indeed, a KQ/I-module. On the other hand, $f = \oplus_{a \in Q_0} f_a$, which is clearly a K-linear map, is in fact an A-module morphism. Indeed, for $x = (x_a)_{a \in Q_0}$ and $\bar{\omega} = (\sum_{i=1}^m \lambda_i \omega_i) + I \in KQ/I$, where ω_i is a path from a_i to b_i in Q, we have

$$\begin{aligned}
f(x\bar{\omega}) &= f(\sum_{i=1}^{m} \lambda_i x\omega_i) = \sum_{i=1}^{m} \lambda_i f(x\omega_i) \\
&= \sum_{i=1}^{m} \lambda_i f(0, \ldots, 0, \varphi_{\omega_i}(x_{a_i}), 0, \ldots, 0) \\
&= \sum_{i=1}^{m} \lambda_i (0, \ldots, 0, f_{b_i}\varphi_{\omega_i}(x_{a_i}), 0, \ldots, 0) \\
&= \sum_{i=1}^{m} \lambda_i (0, \ldots, 0, \varphi'_{\omega_i} f_{a_i}(x_{a_i}), 0, \ldots, 0) \\
&= \sum_{i=1}^{m} \lambda_i (f_a(x_a))_a \omega_i = f(x)\bar{\omega}.
\end{aligned}$$

If $M = (M_a, \varphi_\alpha)$ is a finite dimensional representation of (Q, I), then $dim_K\, M_a < \infty$, for all $a \in Q_0$. Because Q is finite, it follows that the dimension of $G(M)$, which is equal to $dim_K \oplus_{a \in Q_0} M_a$, is also finite. Therefore, G sends finite representations of (Q, I) to finite dimensional A-modules. □

From now on, we identify the A-modules with the corresponding representations given in the proof of theorem 2.1.14, and we use the same notation for both.

2.2 Indecomposable projectives and injectives

This interpretation of modules over $A \cong KQ/I$ in terms of representations of (Q, I) is very useful to their description. In this section, we will compute the simple modules and the indecomposable projectives and injectives.

Firstly, let us fix some notation. We denote by (Q, I) a finite and connected quiver bound by the admissible ideal I, and by A the quotient K-algebra KQ/I. Recall that the radical of A is R_Q/I, where R_Q is the ideal of KQ generated by the arrows of Q, and that the set $\{e_1 = \epsilon_1 + I, \ldots, e_n = \epsilon_n + I\}$, where n is the number of vertices of Q, is a complete set of primitive orthogonal idempotents of A.

Let a be a vertex of Q. Consider the A-module $S(a) = (S(a)_b, \varphi_\alpha)$ given by $S(a)_b = K$, for $b \neq a$, $S(a)_a = K$ and $\varphi_\alpha = 0$, for every arrow of Q. Note that, because all the K-linear maps φ_α are zero, $S(a)$ is bound by I. It is clear that each $S(a)$ is a simple module. Indeed, any submodule of $S(a)$ is of the form $N(a) = (N(a)_b, \psi_\alpha)$, where $N(a)_b$ is a K-vector subspace of $S(a)_b$

and $\psi_\alpha = \varphi_\alpha|_{N(a)_b}$, for each arrow $\alpha : b \to c$. So $N(a)_b = 0$, for $b \neq a$, $\psi_\alpha = 0$, for all $\alpha \in Q_1$ and $N(a)_a$ is either K or 0, which implies that $N(a) = S(a)$ or $N(a)$ is zero, as required.

Proposition 2.2.1. *The set $\{S(a) \mid a \in Q_0\}$ is a complete set of representatives of the isomorphism classes of the simple A-modules.*

Proof. Since $Hom_A(S(a), S(b)) = 0$, for $a \neq b$, the A-modules $S(a)$, with $a \in Q_0$, are pairwise nonisomorphic. If we prove that $S(a) \cong top\, e_aA$, the proposition follows from 1.2.17. By 1.2.16, $Hom_A(e_aA, S(a)) \cong S(a)e_a = S(a)_a \neq 0$. So, there is a nonzero morphism ϕ from $e_aA \to S(a)$, which is surjective, since $S(a)$ is simple. Since $e_aA/Ker\,\phi \cong Im\,\phi = S(a)$, and $S(a)$ is simple, $Ker\,\phi$ is a maximal proper submodule of e_aA. Hence, $Ker\,\phi = rad\,(e_aA)$, by 1.2.13 (2), and so $S(a) \cong e_aA/rad\,(e_aA) = top\,(e_aA)$, as required. □

The following proposition provides a computation of the socle, the radical and the top of an A-module M.

Proposition 2.2.2. *Let $M = (M_a, \varphi_\alpha)$ be a representation of (Q, I).*

1. *$soc\, M = N$, where $N = (N_a, \psi_\alpha)$ with*

$$N_a = \begin{cases} M_a & \text{if } a \text{ is a sink} \\ \bigcap_{\alpha:a\to b} Ker(\varphi_\alpha : M_a \to M_b) & \text{if } a \text{ is not a sink,} \end{cases}$$

and $\psi_\alpha = \varphi_\alpha|_{N_a} = 0$ for every arrow α of source a.

2. *$rad\, M = J$, where $J = (J_a, \gamma_\alpha)$ with $J_a = \sum_{\alpha:b\to a} Im(\varphi_\alpha : M_b \to M_a)$ and $\gamma_\alpha = \varphi_\alpha|_{J_a}$ for every arrow α of source a.*

3. *$top\, M = L$, where $L = (L_a, \psi_\alpha)$ with*

$$L_a = \begin{cases} M_a & \text{if } a \text{ is a source} \\ \sum_{\alpha:b\to a} Coker(\psi_\alpha : M_b \to M_a) & \text{otherwise.} \end{cases}$$

and $\psi_\alpha = 0$ for every arrow α of source a.

Proof. (1) If we prove that N is a semisimple submodule of M that contains every simple submodule of M, we are done. Since $N_a \subseteq M_a$, for all $a \in Q_0$, and $\psi_\alpha = \varphi_\alpha|_{N_a}$, N is a submodule of M. Note that a module $X = (X_a, \tau_\alpha)$ is semisimple if and only if $\tau_\alpha = 0$, for all $\alpha \in Q_1$. Indeed, if X is semisimple, then it is a direct sum of simple modules. Write $X = \oplus_{a\ inQ_0} S(a)^{d_a}$. By definition of a direct sum of representations, it follows that $\tau_\alpha = 0$, for all $\alpha \in Q_1$. Conversely, suppose that for each $\alpha \in Q_1$, $\tau_\alpha = 0$. For every

$a \in Q_0$, $X_a = K^{d_a}$, for some d_a. Then, again by definition of a direct sum of representations, $M = \oplus_{a \in Q_0} S(a)^{d_a}$, since $S(a)$ is the representation with K at vertex a and 0 elsewhere. Therefore, N is semisimple, since $\psi_\alpha = 0$ for every α. Let S be a simple submodule of M. Then, by 2.2.1, $S \cong S(a)$ for some $a \in Q_0$. Because S embeds into M, the square

$$\begin{tikzcd}
K = S(a)_a \arrow[r] \arrow[d] & S(a)_b = 0 \arrow[d] \\
M_a \arrow[r, "\varphi_\alpha"'] & M_b
\end{tikzcd}$$

is commmutative, for each $\alpha : a \to b \in Q_1$. Thus, $S(a)_a \subseteq Ker\, \varphi_\alpha$ for all $\alpha : a \to b$, which implies that $S(a)_a \subseteq N_a$. Hence $S(a) \subseteq N$, as required.

(2) By 1.2.2 (4) and 2.1.8, $rad\, M = M \cdot rad\, A = M \cdot R_Q/I$. Since R_Q/I is generated by $\{\bar{\alpha} = \alpha + I \mid \alpha \in Q_1\}$, we have $rad\, M = \sum_{\alpha \in Q_1} M\bar{\alpha}$. Hence, for each $a \in Q_0$,

$$J_a = \left(\sum_{\alpha \in Q_1} M\bar{\alpha} \right) e_a = \sum_{\alpha \in Q_1} M(\bar{\alpha} e_a) = \sum_{\alpha : b \to a} M\bar{\alpha},$$

and, for each arrow $\alpha : b \to a$ of target a, we have $M\bar{\alpha} = Me_b\bar{\alpha} = M_b\bar{\alpha} = \varphi_\alpha(M_b) = Im\, \varphi_\alpha$, according to the definition of the functor F in 2.1.14. For every arrow α of source a, $\gamma_\alpha = \varphi_\alpha|_{J_a}$, because J is a submodule of M.

(3) is a direct consequence of (2). $\square$

Example 2.2.3. Consider the quiver Q

$$\begin{tikzcd}
 & 3 \\
1 \arrow[ur, "\gamma"] \arrow[r, "\alpha"'] & 2 \arrow[u, "\beta"']
\end{tikzcd}$$

and the representation M of Q given by

$$\begin{tikzcd}
 & K \\
K \arrow[ur, "1"] \arrow[r, "{[1\,0]^t}"'] & K^2 \arrow[u, "{[0\,1]}"']
\end{tikzcd}$$

Let us compute $soc\, M$. We have

- $(soc\, M)_1 = Ker\, \varphi_\gamma \cap Ker\, \varphi_\alpha = 0$,

- $(soc\, M)_2 = Ker\, \varphi_\beta = \langle(1,0)\rangle_K \cong K$,
- $(soc\, M)_3 = M_3 = K$, because the vertex 3 is a sink.

Thus

$$soc\, M \quad = \quad \begin{tikzcd} & K \\ 0 \arrow[ur] \arrow[r] & K \arrow[u, "0"'] \end{tikzcd} \quad \cong \quad S(2) \oplus S(3)$$

In this case $rad\, M = soc\, M$, since

- $(rad\, M)_1 = \sum_{\{\mu | t(\mu)=1\}} Im\, \varphi_\mu = 0$,
- $(rad\, M)_2 = Im\, \varphi_\alpha = \langle(1,0)\rangle_K \cong K$,
- $(rad\, M)_3 = Im\, \varphi_\gamma + Im\, \varphi_\beta = K + K = K$,
- $\gamma_\beta = \varphi_\beta|_{\langle(1,0)\rangle_K} = 0$.

Finally the $top\, M$ is given by

$$\begin{tikzcd} & 0 \\ K \arrow[ur] \arrow[r, "0"'] & K \arrow[u] \end{tikzcd} \quad \cong \quad S(1) \oplus S(2)$$

Observe that M is indecomposable. Indeed an endomorphism of M is given by a triple (f_1, f_2, f_3), where

$$\begin{aligned} f_1 &= [f_1(1)] : M_1 \to M_1, \\ f_2 &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} : M_2 \to M_2, \\ f_3 &= [f_3(1)] : M_3 \to M_3 \end{aligned}$$

are such that $f_3 \circ 1 = 1 \circ f_1$, $[1\ 0]^t \circ f_1 = f_2 \circ [1\ 0]^t$ and $f_3 \circ [0\ 1] = [0\ 1] \circ f_2$. This means that $f_1 = f_3, f_1(1) = f_3(1) = a = d$ and $c = 0$. Thus

$$End\, M \cong \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in K \right\}.$$

Consider the ideal

$$\mathcal{I} = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \mid b \in K \right\}$$

of $End\,M$. We have $\mathcal{I}^2 = 0$, which is to say $\mathcal{I}$ is nilpotent, and $End\,M/\mathcal{I} \cong K$. Hence $\mathcal{I} = rad\,(End\,M)$ (1.2.2(3)) and so $End\,M$ is local. By 1.2.5 (3), M is indecomposable.

The next proposition describes the indecomposable projective and injective A-modules. Recall that every indecomposable projective right A-module is isomorphic to e_iA, for some $i \in [n]$, and every indecomposable injective right A-module is isomorphic to $D(Ae_i)$, for some $i \in [n]$ (see 1.2.17).

Proposition 2.2.4. *Let (Q, I) be a bound quiver, $A = KQ/I$, $P(a) = e_aA$ and $I(a) = D(Ae_a)$, with $a \in Q_0$.*

1. *The indecomposable projective A-module $P(a)$ is given by the representation $(P(a)_b, \varphi_\beta)$, where:*
   - *The K-vector space $P(a)_b$ is generated by $\bar{\omega} = \omega + I$, with ω a path from a to b,*
   - *For an arrow $\beta : b \to c$, the K-linear map $\varphi_\beta : P(a)_b \to P(a)_c$ is given by the right multiplication by $\bar{\beta} = \beta + I$.*
2. *The indecomposable injective A-module $I(a)$ is given by the representation $(I(a)_b, \varphi_\beta)$, where:*
   - *The K-vector space $I(a)_b$ is the dual of the K-vector space with basis the set of all $\bar{\omega} = \omega + I$, with ω a path from b to a,*
   - *For an arrow $\beta : b \to c$, the K-linear map $\varphi_\beta : I(a)_b \to I(a)_c$ is given by the dual of the left multiplication by $\bar{\beta} = \beta + I$.*

Proof. (1) According to the functor F of the proof of the Theorem 2.1.14, we have

$$P(a)_b = P(a)e_b = e_aAe_b = e_a(KQ/I)e_b = (\epsilon_a(KQ)\epsilon_b)/(\epsilon_aI\epsilon_b),$$

which implies that $P(a)_b$ is generated by the paths from a to b, and for an arrow $\beta : b \to c$ in Q, φ_β is given by the right multiplication by $\bar{\beta} = \beta + I$.

(2) We have

$$I(a)_b = I(a)e_b = D(Ae_a)e_b \cong D(e_bAe_a) \cong D(\epsilon_b(KQ)\epsilon_a/\epsilon_bI\epsilon_a),$$

which proves the first part. Let now $\beta : b \to c$ be an arrow of Q. The K-linear map $\varphi_\beta : I(a)_b \to I(a)_c$ is such that $\varphi_\beta(f) = f\bar{\beta}$ for $f \in I(a)_b = Hom_K(\epsilon_b(KQ)\epsilon_a/\epsilon_bI\epsilon_a, K)$. According to the KQ/I-module structure of $D(\epsilon_c(KQ)\epsilon_a/\epsilon_cI\epsilon_a)$, we have $\varphi_\beta(f)(\bar{(\omega)}) = f\bar{\beta}(\bar{\omega}) = f(\bar{\beta}\bar{\omega})$. Thus, $\varphi_\beta = D(\mu_\beta)$, where

$$\mu_\beta : \epsilon_c(KQ)\epsilon_a/\epsilon_cI\epsilon_a \to \epsilon_b(KQ)\epsilon_a/\epsilon_bI\epsilon_a$$

is the left multiplication $\bar{\omega} \mapsto \bar{\beta}\bar{\omega}$. $\square$

Remark 2.2.5.

- It follows from 2.2.2 (2) and from 2.2.4 (1) that the radical of $P(a)$ is given by the representation $(P'(a)_b, \varphi'_\beta)$, where $P'(a)_b = P(a)_b$ for $b \neq a$, $P'(a)_a$ is the K-vector space generated by $\bar{\omega} = \omega + I$, with ω a nonstationary path from a to a, $\varphi'_\beta = \varphi_\beta$ for any arrow β of source $b \neq a$ and $\varphi'_\alpha = \varphi_\alpha|_{P'(a)_a}$ for any arrow α of source a.

- For every vertex a of Q, the simple module $S(a)$ is isomorphic to $soc\, I(a)$. We use 2.2.2 (1) and 2.2.4 (2) to prove this. Let $soc\, I(a) = (N(a)_b, \psi_\alpha)$ and $I(a) = (I(a)_b, \varphi_\alpha)$. Let $b \in Q_0 \setminus \{a\}$. If b is a sink, then $N(a)_b = I(a)_b = D(e_b A e_a) = 0$, since it can't be a path from b to a. Suppose now that b is not a sink. Then

$$N(a)_b = \bigcap_{\alpha : b \to c} Ker\,(\varphi_\alpha : I(a)_b \to I(a)_c)\,.$$

By 2.2.4 (2), $I(a)_b$ is spanned by $\bar{\omega}^\star$, where ω is a path from b to a, $\bar{\omega} = \omega + I$ and $\bar{\omega}^\star$ is the dual linear map, given by

$$\bar{\omega}^\star(\bar{x}) = \begin{cases} 1 & \text{if } \bar{x} = \bar{\omega} \\ 0 & \text{otherwise,} \end{cases}$$

for $\bar{x} \in e_b A e_a$. Similarly, $I(a)_c$ is spanned by $\bar{v}^\star$, where v is a path from c to a, $\bar{v} = v + I$ and $\bar{v}^\star$ is the dual linear map. Let $\alpha : b \to c$ be an arrow of Q and let $\sum_\omega = \lambda_\omega \bar{\omega}^\star$ be an element of $I(a)_b$. We have,

$$\varphi_\alpha \left(\sum_\omega = \lambda_\omega \bar{\omega}^\star \right) (\bar{v}) = \sum_\omega \lambda_\omega \bar{\omega}^\star(\bar{\alpha}\bar{v}) = \lambda_{\alpha v},$$

according to the definition of the dual linear map $\bar{\omega}^\star$. Hence, if $\sum_\omega = \lambda_\omega \bar{\omega}^\star \in \cap_{\alpha : b \to c} \varphi_\alpha$, then $\lambda_{\alpha v} = 0$, for any path αv from b to a via some vertex c, i.e., $\lambda_\omega = 0$, for all ω spanning $I(a)_b$. Therefore, $N(a)_b = \cap_{\alpha : b \to c} Ker\, \varphi_\alpha = 0$, for $b \neq a$. Since $soc\, I(a)$ is simple (see [4, II.4.1], for example), we have $N(a)_a = K$. Furthermore, the K-linear maps ψ_α are all zero. Hence $S(a) = soc\, I(a)$, as required.

- For each $a \in Q_0$, $I(a)/soc\, I(a) = I(a)/S(a)$ is given by the representation (L_b, ψ_β), where L_b is the quotient space of $I(a)_b$ which is dual to the space spanned by the residual classes of paths from b to a of length at least one, and ψ_β is the induced map.

Example 2.2.6. Let Q be the quiver

$$\begin{tikzcd}
1 & & 4 \arrow[dl, "\beta"'] & \\
& 3 \arrow[ul, "\lambda"'] \arrow[dl, "\mu"'] & & 6 \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\
2 & & 5 \arrow[ul, "\delta"] &
\end{tikzcd}$$

bound by $\alpha\beta = \gamma\delta, \delta\mu = 0$ and $\beta\lambda = 0$.

Then $P(1) = S(1)$, $P(2) = S(2)$ (in fact, $P(a)$ is simple if and only if a is a sink), $rad\, P(1) = 0 = rad\, P(2)$,

$$P(3) = \begin{tikzcd}
K & & 0 \arrow[dl] & \\
& K \arrow[ul, "1"] \arrow[dl, "1"'] & & 0 \arrow[ul] \arrow[dl] \\
K & & 0 \arrow[ul] &
\end{tikzcd} \qquad rad\, P(3) = S(1) \oplus S(2)$$

$$P(4) = \begin{tikzcd}
0 & & K \arrow[dl, "1"] & \\
& K \arrow[ul] \arrow[dl, "1"'] & & 0 \arrow[ul] \arrow[dl] \\
K & & 0 \arrow[ul] &
\end{tikzcd} \qquad rad\, P(4) = \begin{tikzcd}
0 & & 0 \arrow[dl] & \\
& K \arrow[ul] \arrow[dl, "1"'] & & 0 \arrow[ul] \arrow[dl] \\
K & & 0 \arrow[ul] &
\end{tikzcd}$$

$$P(5) = \begin{tikzcd}
K & & 0 \arrow[dl] & \\
& K \arrow[ul, "1"'] \arrow[dl] & & 0 \arrow[ul] \arrow[dl] \\
0 & & K \arrow[ul, "1"] &
\end{tikzcd} \qquad rad\, P(5) = \begin{tikzcd}
K & & 0 \arrow[dl] & \\
& K \arrow[ul, "1"] \arrow[dl] & & 0 \arrow[ul] \arrow[dl] \\
0 & & 0 \arrow[ul] &
\end{tikzcd}$$

$$P(6) = \begin{tikzcd}
0 & & K \arrow[dl, "1"'] & \\
& K \arrow[ul] \arrow[dl] & & K \arrow[ul, "1"'] \arrow[dl, "1"] \\
0 & & K \arrow[ul, "1"] &
\end{tikzcd} \qquad rad\, P(6) = \begin{tikzcd}
0 & & K \arrow[dl, "1"'] & \\
& K \arrow[ul] \arrow[dl] & & 0 \arrow[ul] \arrow[dl] \\
0 & & K \arrow[ul, "1"] &
\end{tikzcd}$$

$$I(4) = \begin{tikzcd}
0 & & K \arrow[dl] & \\
& 0 \arrow[ul] \arrow[dl] & & K \arrow[ul, "1"'] \arrow[dl] \\
0 & & 0 \arrow[ul] &
\end{tikzcd} \qquad I(5) = \begin{tikzcd}
0 & & 0 \arrow[dl] & \\
& 0 \arrow[ul] \arrow[dl] & & K \arrow[ul] \arrow[dl, "1"] \\
0 & & K \arrow[ul] &
\end{tikzcd}$$

$I(1) = P(5), I(2) = P(4), I(3) = P(6)$ and $I(6) = S(6)$ (note that $I(a)$ is simple if and only if a is a source).

2.3 Irreducible morphisms and almost split sequences

This section is devoted to the study of special types of morphisms which play an important role in the representation theory, the minimal almost split morphisms and the irreducible morphisms. The inclusion $rad\,P \hookrightarrow P$, for any indecomposable projective module P, and the canonical epimorphism $I \to I/soc\,I$, for any indecomposable injective module I are examples of minimal almost split morphisms. The irreducible morphisms can be viewed as components of minimal almost split morphisms and they have an important characterization via the radical of the category $mod\,A$. We also give the notion and some properties of almost split sequences.

Throughout, K denotes an algebraically closed field, A denotes a finite dimensional K-algebra, and all A-modules are, unless otherwise specified, right finite dimensional A-modules.

Definition 2.3.1.

1. A morphism $f : X \to Y$ is *left minimal* if every $h \in End\,Y$ such that $hf = f$ is an automorphism.
2. A morphism $f : X \to Y$ is *left almost split* if
   - f is not a section,
   - Every morphism $u : X \to U$ that is not a section factors through f, i.e., there exists $u' : Y \to U$ such that $u'f = u$.
3. A morphism $f : X \to Y$ that is left minimal and left almost split is said to be *left minimal almost split*.
4. A morphism $g : Y \to Z$ is *right minimal* if every $k \in End Y$ such that $gk = g$ is an automorphism.
5. A morphism $g : Y \to Z$ is *right almost split* if
   - g is not a retraction,
   - Every morphism $v : V \to Z$ that is not a retraction factors through g, i.e., there exists $v' : V \to Y$ such that $gv' = v$.
6. A morphism $g : Y \to Z$ that is right minimal and right almost split is said to be *right minimal almost split*.

Proposition 2.3.2.

1. *Any two left minimal almost split morphisms with the same domain, $f : M \to N$ and $g : M \to L$, are equivalent, i.e., there exists an isomorphism $h : N \to L$ such that $g = hf$.*

2. *Any two right minimal almost split morphisms with the same codomain, $f : N \to M$ and $g : L \to M$, are equivalent, i.e., there exists an isomorphism $h : N \to L$ such that $f = gh$.*

Proof. (1) Since g is left almost split, g is not a section. Hence there exists a morphism $h : N \to L$ such that $g = hf$, because f is left almost split. Analogously, there is a morphism $h' : L \to N$ such that $h'g = f$. Therefore we have $hh'g = hf = g$ and $h'hf = h'g = f$, which implies that both hh' and $h'h$ are automorphisms, as g and f are left minimal. In particular $h'h$ is injective, so h is a monomorphism, and hh' is surjective, so h is an epimorphism. Thus h is an isomorphism. The proof of (2) is similar to (1). □

Example 2.3.3. Let P be an indecomposable projective A-module. The inclusion $\iota : rad\,P \hookrightarrow P$ is right minimal almost split. The right minimality follows directly from the fact that ι is a monomorphism. Because $rad\,P \neq P$, g is not an epimorphism, and so g cannot be a retraction. Let now $v : V \to P$ be a homomorphism that is not a retraction. We want to prove the existence of a morphism $v' : V \to rad\,P$ such that $gv' = v$, which means that $Im\,v \subseteq rad\,P$. By lemma 1.2.25 v is not an epimorphism, i.e., $Im\,v \subsetneqq P$. On the other hand, it follows from 1.2.13 (2) that $rad\,P$ is the unique maximal submodule of P, since P is projective. Thus $Im\,v \subseteq rad\,P$.

Morever, it follows from 2.3.2 that any right minimal almost split morphism with P as codomain is a monomorphism and its image is $rad\,P$.

Analogously, if I is an indecomposable injective A-module, the canonical projection $f : I \to I/soc\,I$ is left minimal almost split, and any left minimal almost split with I as domain is an epimorphism and its kernel is $soc\,I$.

Definition 2.3.4. A short exact sequence in $mod\,A$

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is said to be an *almost split sequence* provided f is left minimal almost split and g is right minimal almost split.

Remark 2.3.5.

1. In such a sequence, L and N are indecomposable. This follows from the fact that the domain of a left almost split morphism and the target of a

right almost split morphism are indecomposable modules. Indeed, let $f : L \to M$ be left almost split and suppose, for a contradiction, that we have a nontrivial decomposition $L = L_1 \oplus L_2$. Consider the projections $p_i : L \to L_i$ (with $i = 1, 2$). Each morphism p_i is not a section, by lemma 1.2.25. Thus, by definition of left almost split morphism, there exists a morphism $u_i : M \to L_i$ such that $u_i f = p_i$, for $i = 1, 2$, and so $u = [u_1\, u_2]^t : M \to L$ is such that $uf = 1_L$. Therefore, f is a section, a contradiction. The proof that the target of a right almost split morphism is indecomposable is analogous.

2. L is not injective and N is not projective. Otherwise f would be a section and g would be a retraction.

The following theorem gives several characterizations of almost split sequences.

Theorem 2.3.6. [2, IV.1.13] *The following are equivalent for an exact sequence*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0 .$$

1. *The sequence is almost split.*
2. *L is indecomposable, and g is right almost split.*
3. *N is indecomposable, and f is left almost split.*
4. *The homomorphism f is left minimal almost split.*
5. *The homomorphism g is right minimal almost split.*
6. *L and N are indecomposable, and f and g are irreducible.*

We will go back to the almost split sequences, proving their existence and 'uniqueness' in the next section.

Now we give the definition of an irreducible morphism and study the connection with the minimal almost split morphisms.

Definition 2.3.7. A morphism $f : X \to Y$ in $mod\, A$ is called *irreducible* if:

1. f is neither a section nor a retraction, and
2. if $f = f_1 f_2$ either f_1 is a retraction or f_2 is a section.

Remark 2.3.8. If $f : M \to N$ is an irreducible morphism, then f is either a epimorphism or a monomorphism. Indeed, suppose f is not an epimorphism, and consider the canonical factorization $M \xrightarrow{u} Im\, f \xhookrightarrow{v} N$ of f. Since f is not an epimorphism, v is not surjective, so it cannot be a retraction. Because f is irreducible, u must be a section, which implies that u is injective, and so f is a monomorphism, as required.

An important characterization of the irreducible morphisms, in connection with the radical of $mod\, A$, is given below, without proof.

Proposition 2.3.9. [2, IV.1.6] *A morphism $f : X \to Y$ between two indecomposable modules X, Y is irreducible if and only if $f \in rad_A(X,Y) \setminus rad_A^2(X,Y)$.*

This proposition induces the following definition.

Definition 2.3.10. Let M, N be two indecomposable A-modules. We call the quotient

$$Irr(M,N) = rad_A(M,N)/rad_A^2(M,N)$$

of the K-vector spaces $rad_A(M,N)$ and $rad_A^2(M,N)$, the *space of irreducible morphisms*.

Proposition 2.3.11. *Every left (or right) minimal almost split morphism is irreducible.*

Proof. Suppose $f : L \to M$ is a left minimal almost split morphism. By definition, f is not a section. Suppose, for a contradiction, that f is a retraction. Then, f is surjective and $Ker\, f$ is a direct summand of L. Since f is left almost split, L is indecomposable (2.3.5 (1)). Thus, $Ker\, f = 0$ or $Ker\, f = L$. If the first case holds, then f is an isomorphism, which is impossible because f is not a section. The second case implies that $f = 0$, and because f is surjective, $M = 0$, a contradiction. Therefore, f is not a retraction. Let $f = f_1 f_2$, where $f_2 \in Hom_A(L,X), f_1 \in Hom_A(X,M)$ and suppose f_2 is not a section. Because f is left almost split, there exists $f_2' : M \to X$ such that $f_2' f = f_2$. Thus, $f = f_1 f_2 = f_1 f_2' f$. Since f is left minimal, $f_1 f_2'$ is an automorphism, so, in particular, f_1 is a retraction. The proof that a right minimal almost split morphism is irreducible is analogous. □

Example 2.3.12. It follows from this proposition and by 2.3.3 that the inclusion $radP \hookrightarrow P$ for an indecomposable projective A-module P, and the projection $I \to I/soc\, I$, for an indecomposable injective A-module, are irreducible morphisms.

The following theorem establishes the connection between irreducible morphisms and minimal almost split morphisms.

Theorem 2.3.13.

1. *Let $f : L \to M$ be left minimal almost split in $\operatorname{mod} A$. A morphism $f' : L \to M'$ of A-modules is irreducible if and only if*
   - $M' \neq 0$,
   - *There exists a direct sum decomposition $M \cong M' \oplus M''$ and a morphism $f'' : L \to M''$ such that $[f' \ f'']^t : L \to M' \oplus M''$ is left minimal almost split.*
2. *Let $g : M \to N$ be right minimal almost split in $\operatorname{mod} A$. A morphism $g' : M' \to N$ of A-modules is irreducible if and only if*
   - $M' \neq 0$,
   - *There exists a direct sum decomposition $M \cong M' \oplus M''$ and a morphism $g'' : M'' \to N$ such that $[g' \ g''] : M' \oplus M'' \to N$ is right minimal almost split.*

Proof. (1) Suppose $f' : L \to M'$ is irreducible. Then clearly $M' \neq 0$. Since f' is not a section and f is left almost split, there exists $h : M \to M'$ such that $f' = hf$. Because f' is irreducible and f is not a section, h is a retraction. Let g be an A-module homomorphism from M' onto M such that $hg = 1_{M'}$ and let $M'' = Ker\, h$. Consider the morphism $q : M \to M''$ defined by $q(x) = x - gh(x)$ for $x \in M$. Then $[h \ q]^t : M \to M' \oplus M''$ is an isomorphism, being $[g \ \iota] : M' \oplus M'' \to M$, where ι is the inclusion $\iota : M'' \hookrightarrow M$, its inverse. This permits to conclude that $[h \ q]^t f = [f' \ qf]^t : L \to M' \oplus M''$ is left minimal almost split. Indeed, for $u \in End(M' \oplus M'')$ such that $u[h \ q]^t f = [h \ q]^t f$, we have $[g \ \iota]u[h \ q]^t f = f$. Hence $[g \ \iota]u[h \ q]^t$ is an automorphism, and so is u. Clearly $[h \ q]^t f$ is not a section, otherwise f would be a section as well. Let now $u : L \to U$ be an A-homomorphism that is not a section. Since f is left almost split, there is $u' : M \to U$ such that $u' f = u$. So $u'' = u'[g \ \iota] : M' \oplus M''$ is such that $u''[h \ q]^t f = u$.

Conversely, let $f' \in Hom_A(L, M')$, where $M' \neq 0$, and suppose $M \cong M' \oplus M''$, for some module M'', and there exists a morphism $f'' : L \to M''$ such that $[f' \ f'']^t : L \to M' \oplus M''$ is left minimal almost split. Because $[f' \ f'']^t$ is not a section, f' is not an isomorphism. Moreover, L is indecomposable (2.3.5 (1)), so f' is not a retraction. Suppose that f' is a section, and let $h : M' \to L$ be its left inverse. Then $[h \ 0][f' \ f'']^t = 1_L$, which means that$[f' \ f'']^t$ is a section, a contradiction. Hence, f' is not a section. Suppose that $f' = f_1 f_2$, with $f_2 \in Hom_A(L, X), f_1 \in Hom_A(X, M')$ and f_2 is not a section. We have

$$[f' \ f'']^t = \begin{bmatrix} f_1 & 0 \\ 0 & 1 \end{bmatrix} [f_2 \ f'']^t,$$

where

$$[f_2 \ f'']^t \in Hom_A(L, X \oplus M''),$$

and

$$\begin{bmatrix} f_1 & 0 \\ 0 & 1 \end{bmatrix} \in Hom_A(X \oplus M'', M' \oplus M'').$$

By 1.2.20, $Im\, Hom_A(f_2, L) \subseteq rad\, End\, L$, since f_2 is not a section. Since f'' is not a section either, we have $Im\, Hom_A(f'', L) \subseteq rad\, End\, L$. Therefore, $Im\, Hom_A([f_2 \ f'']^t, L) \subseteq rad\, End\, L$. So $[f_2 \ f'']^t$ is not a section, by 1.2.20. On the other hand, $[f' \ f'']^t$ is irreducible since it is left minimal almost split. So $\begin{bmatrix} f_1 & 0 \\ 0 & 1 \end{bmatrix}$ is a retraction, which implies that f_1 is a retraction. The proof of (2) is similar. □

Corollary 2.3.14. *If $u' : U \to V'$ is irreducible and $v : V' \to W$ is a retraction, then $vu' : U \to W$ is irreducible.*

Proof. Since u' is irreducible, there exist a module V'' and a morphism $u'' : U \to V''$ such that $[u' \ u'']^t : U \to V' \oplus V''$ is left minimal almost split. Because v is a retraction, it admits a right inverse v' and we have $V' = Ker\, v \oplus Im\, v' \cong Ker\, v \oplus W$. The isomorphism between V' and $Ker\, v \oplus W$ is given by $x \mapsto (x - v'v(x)) + v(x)$. Let $\psi : V' \oplus V'' \to Ker\, v \oplus W \oplus V''$ be the isomorphism given by $(x, y) \mapsto (x - v'v(x)) + v(x) + y$. We have that vu' is a component of $\psi[u' \ u'']^t$. Since $\psi[u' \ u'']^t$ is left minimal almost split (because $[u' \ u'']^t$ is left minimal almost split and ψ is an isomorphism), it follows from 2.3.13 that vu' is irreducible, as required. □

2.4 The Nakayama functor, the transposition and the Auslander-Reiten translations

In this section we introduce the notion of the Nakayama functor and the Auslander-Reiten translations, which are very useful to prove the existence of the almost split sequences.

Recall that the A-dual functor

$$(-)^t : mod\, A \to mod\, A^{op}$$

is exact and it is a duality between $proj\, A$ and $proj\, A^{op}$ (see Chapter 1).

Definition 2.4.1. The *Nakayama functor* of $mod\,A$ is defined to be the functor $\nu = D(-)^t : mod\,A \to mod\,A$.

The following proposition states that this functor is an equivalence between $proj\ A$ and $inj\ A$.

Proposition 2.4.2. *The restriction of the Nakayama functor ν to $proj\ A$ induces an equivalence between $proj\ A$ and $inj\ A$, with quasi-inverse $\nu^{-1} = Hom_A(D(A_A), -) : inj\ A \to proj\ A$.*

Proof. We want to prove that

$$\nu\, Hom_A(D(A), -)|_{inj\ A} = 1_{inj\ A}$$

and

$$Hom_A(D(A), -)\, \nu|_{proj\ A} = 1_{proj\ A}.$$

By additivity, it is sufficient to prove this for indecomposable (projective and injective) modules. Let $P(i) = e_i A$, for some idempotent e_i, be an indecomposable projective module (recall 1.2.17), and denote by $I(i)$ the indecomposable injective module $D(Ae_i)$ associated to e_i. By 1.2.16, we have that

$$\nu\, P(i) = \nu\, e_i A = D\, Hom_A(e_i A, A) \cong D(Ae_i) = I(i).$$

Moreover,

$$\begin{aligned} Hom_A(D(A), I(i)) &= Hom_A(D(A), D(Ae_i)) \\ &\cong Hom_{A^{op}}(Ae_i, A) \cong e_i A = P(i). \end{aligned}$$

So $\nu\, Hom_A(D(A), I(j)) = I(j)$ for any indecomposable injective module $I(j)$, and $Hom_A(D(A), \nu\, P(i)) = P(i)$ for any indecomposable projective module $P(i)$, as required. □

Definition 2.4.3. Let $M \in mod\,A$ and

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0$$

be a minimal projective presentation of M. Then we have $M \cong Coker\, p_1$. Applying the functor $(-)^t$ to the morphism p_1, we get a morphism $p_1^t : P_0^t \to P_1^t$. The cokernel of this morphism is the *transpose* of M, and will be denoted by $Tr\, M$.

Remark 2.4.4. The left A-module $Tr\, M$ is uniquely determined up to isomorphism, because the minimal projective presentations are uniquely determined up to isomorphism.

The main properties of $Tr\,M$ are stated in the following proposition.

Proposition 2.4.5. *Let M, N be a indecomposable right A-modules.*

1. *M is projective if and only if $Tr\,M = 0$.*
2. *If M is not projective and*
$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$$
is a minimal projective presentation of M, then
$$P_0^t \xrightarrow{p_1^t} P_1^t \to Tr\,M \to 0$$
is a minimal projective presentation of $Tr\,M$.
3. *If M is not projective, then $Tr\,M$ is indecomposable and $Tr(Tr\,M) \cong M$.*
4. *If M and N are not projective, then $M \cong N$ if and only if $Tr\,M \cong Tr\,N$.*

Proof. (1) Suppose M is projective. Then the sequence $0 \to M \xrightarrow{1_M} M \to 0$ is a minimal projective presentation of M, which clearly implies that $Tr\,M = 0$. Conversely, suppose $Tr\,M = 0$ and let $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$ be a minimal projective presentation of M. The fact that $Tr\,M = 0$ means that p_1^t is an epimorphism. Because P_1^t is projective, it follows from 1.2.25 (1) that p_1^t is a retraction. Hence p_1 is a section, and so p_0 is a retraction (cf. 1.2.22). Let $p_0' : M \to P_0$ be a right inverse of p_0. We will prove that M is projective by definition. Let $f : N \to L$ be an epimorphism and $g \in Hom_A(M, L)$. Because P_0 is projective, there exists $f' : P_0 \to N$ such that $gp_0 = ff'$. Therefore, the morphism $f'p_0' : M \to N$ is such that $f(f'p_0') = gp_0p_0' = g$, and we are done.

(2) Suppose M is not projective, and let $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$ be a minimal projective presentation of M. It is clear that the sequence

$$S : P_0^t \xrightarrow{p_1^t} P_1^t \to Tr\,M \to 0$$

is a projective presentation of $Tr\,M$. Suppose, for a contradiction that it is not minimal. By 1.2.31, the sequence S is isomorphic to

$$S' : E_0 \oplus E_0' \oplus E_0'' \to E_1 \oplus E_1' \oplus E_1'' \to Tr\,M \to 0,$$

where the first map is given by the matrix $diag(u, v, 0)$, where u is a morphism from E_0 to E_1 and v is an isomorphism from E'_0 to E'_1. Consider the exact sequence

$$0 \to M^t \xrightarrow{p_0^t} P_0^t \xrightarrow{p_1^t} P_1^t \to Tr\, M \to 0.$$

Applying $(-)^t$ to it, we get

$$P_1^{tt} \xrightarrow{p_1^{tt}} P_0^{tt} \xrightarrow{p_0^{tt}} M \to 0.$$

So, applying $(-)^t$ to the isomorphic sequence S', we obtain

$$E_1^t \oplus E_1'^t \oplus E_1''^t \to E_0^t \oplus E_0'^t \oplus E_0''^t \to M \to 0,$$

where the first map is $diag(u^t, v^t, 0^t = 0)$. The cokernel of this map is $E_0^t/Im\, u^t \oplus E_0'^t/Im\, v^t \oplus E_0''^t/0$ and it is isomorphic to M which is indecomposable. Since v is an isomorphism, so is v^t, as $(-)^t$ is a duality on projectives. Hence $E_0'^t/Im\, v^t = 0$. Since the cokernel is indecomposable, either $E_0^t/Im\, u^t$ or $E_0''^t$ is zero. If $E_0^t/Im\, u^t = 0$, then $M \cong E_0''^t$, a contradiction since M is not projective and $E_0''^t$ is projective. Hence $M \cong E_0^t/Im\, u^t$. We thus have a projective presentation

$$E_1^t \xrightarrow{u^t} E_0^t \to M \to 0,$$

contradicting the minimality of the presentation $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$.

(3) Suppose M is not projective. Applying $(-)^t$ to the sequence S' of (2), we get the exact sequence

$$P_1^{tt} \xrightarrow{p_1^{tt}} P_0^{tt} \xrightarrow{p_0^{tt}} Tr\, Tr\, M \to 0.$$

Because P_1 and P_0 are projective modules, we have $P_1 \overset{\epsilon_{P_1}}{\cong} P_1^{tt}$ and $P_0 \overset{\epsilon_{P_2}}{\cong} P_0^{tt}$. Consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
P_1 \arrow[r, "p_1"] \arrow[d, "\epsilon_{P_1}"', "\cong"] & P_0 \arrow[r, "P_0"] \arrow[d, "\epsilon_{P_0}"', "\cong"] & M \arrow[r] & 0 \\
P_1^{tt} \arrow[r, "p_1^{tt}"'] & P_0^{tt} \arrow[r, "p_0^{tt}"'] & Tr\, Tr\, M \arrow[r] & 0.
\end{tikzcd}$$

Let $m \in M$. There exists $x \in P_0$ such that $p_0(x) = m$, for p_0 is surjective. It is easy to see that the map $m \mapsto p_0^{tt}\epsilon_{P_0}(x)$ is well defined and it is an isomorphism between M and $Tr\, Tr\, M$.

The last statement is an easy consequence of (3). $\square$

Since $Tr\,N = 0$ for an indecomposable projective right A-module N, there can't be a duality from $mod\,A$ to $mod\,A^{op}$ that sends M to $Tr\,M$. But there is a duality that maps M to $Tr\,M$ between the categories $mod\,A$ and $mod\,A^{op}$ modulo projectives, defined as follows.

Let $\mathcal{P}$ be the class of A-module morphisms that factors through a projective A-module, and denote by $\mathcal{P}(M,N)$ the set of these morphisms that goes from M to N.

Remark 2.4.6. Let $f, g \in \mathcal{P}(M,N)$ and $\lambda \in K$. Write $f = f_1 f_2$, with $f_2 : M \to P, f_1 : P \to N$, where P is a projective A-module, and $g = g_1, g_2$, with $g_2 : M \to P', g_1 : P' \to N$, where P' is a projective A-module. Then clearly, $\lambda f \in \mathcal{P}(M,N)$ and $f + g = [f_1\ g_1][f_2\ g_2]^t : M \to P \oplus P' \to N$, and so $f+g$ factors through the projective A-module $P \oplus P'$, which means that $f + g \in \mathcal{P}(M,N)$. Let now $f = f_1 f_2 \in \mathcal{P}(M,N)$ as before and g be an arbitrary A-module morphism from N to L. Then, clearly $gf = (gf_1)f_2$ factors through the projective P, and so $gf \in \mathcal{P}(M,L)$. Analogously, $gf \in \mathcal{P}(M,L)$, for $f \in Hom_A(M,N), g \in \mathcal{P}(N,L)$. This proves that $\mathcal{P}$ is an ideal in $mod\,A$.

Dually, we have the ideal $\mathcal{I}$ in $mod\,A$ such that, for each pair M, N of A-modules, $\mathcal{I}(M,N)$ is the subset of $Hom_A(M,N)$ of the modules that factors through an injective A-module.

Definition 2.4.7.

1. The quotient category $\underline{mod}\,A = mod\,A/\mathcal{P}$ such that:

   - $Obj\,\underline{mod}\,A = Obj\,mod\,A$,
   - $\underline{Hom}_A(M,N) = Hom_A(M,N)/\mathcal{P}(M,N)$, and
   - The composition of morphisms is induced from the composition in $mod\,A$,

   is called the *projectively stable category*.

2. The quotient category $\overline{mod}\,A = mod\,A/\mathcal{I}$ such that:

   - $Obj\,\overline{mod}\,A = Obj\,mod\,A$,
   - $\overline{Hom}_A(M,N) = Hom_A(M,N)/\mathcal{I}(M,N)$, and
   - The composition of morphisms is induced from the composition in $mod\,A$,

   is called the *injectively stable category*.

The category $\underline{mod}\,A$ can be identified with another quotient category, denoted by $\vec{proj}\,A/\vec{proj}_1\,A$, which we will define now.

The category $\vec{proj}\,A$ is such that the objects are triples (P_1, P_0, f), where P_1, P_0 are projective A-modules and $f \in Hom_A(P_1, P_0)$, and for a pair of objects (P_1, P_0, f), (P_1', P_0', f'), a morphism from (P_1, P_0, f) to (P_1', P_0', f') is of the form (u_1, u_0), where $u_1 \in Hom_A(P_1, P_1')$, $u_0 \in Hom_A(P_0, P_0')$ are such that the following diagram is commutative.

$$\begin{tikzcd} P_1 \arrow[r, "f"] \arrow[d, "u_1"'] & P_0 \arrow[d, "u_0"] \\ P_1' \arrow[r, "f'"'] & P_0' \end{tikzcd}$$

The class $\vec{proj}_1\,A((P_1, P_0, f), (P_1', P_0', f'))$ is the set of morphisms (u_1, u_0) in $\vec{proj}\,A$ from (P_1, P_0, f) to (P_1', P_0', f') such that the A-module morphism

$$\begin{array}{rrcl} u: & Coker\,f & \to & Coker\,f' \\ & x + Im\,f & \mapsto & u_0(x) + Im\,f, \end{array} \tag{2.1}$$

which is well defined, is zero, i.e., u factors through a projective module.

The class $\vec{proj}_1\,A$ is indeed an ideal in $\vec{proj}\,A$.

The quotient category $\vec{proj}\,A/\vec{proj}_1\,A$ is then the category whose objects are the objects of $\vec{proj}\,A$ and, for each pair (P_1, P_0, f), (P_1', P_0', f') of objects, a morphism from A morphism from (P_1, P_0, f) to (P_1', P_0', f'), is of the form $(u_1, u_0) + \vec{proj}_1\,A((P_1, P_0, f), (P_1', P_0', f'))$, where (u_1, u_0) is a morphism between these two objects in $\vec{proj}\,A$.

Consider the functor $F : \vec{proj}\,A/\vec{proj}_1\,A \to \underline{mod}\,A$ defined as follows:

- For each object (P_1, P_0, f), $F(P_1, P_0, f) := Coker\,f$,
- For each morphism $(u_1, u_0) + \vec{proj}_1\,((P_1, P_0, f), (P_1', P_0', f'))$, $F((u_1, u_0) + \vec{proj}_1\,((P_1, P_0, f), (P_1', P_0', f')) = u$, where u is defined by 2.1.

Write $F(u_1, u_0)$ instead of $F((u_1, u_0) + \vec{proj}_1\,((P_1, P_0, f), (P_1', P_0', f'))$, for short. Note that if $(u_1, u_0) \in \vec{proj}_1\,A$, then $F(u_1, u_0)$ belongs to $\mathcal{P}(Coker\,f, Coker\,f')$, according to the definition of $\vec{proj}_1\,A$, so F is well defined.

Let M be an object in $\underline{mod}\,A$ and let $P_1 \xrightarrow{f} P_0 \to M \to 0$ be a minimal projective presentation of M. Then $M = F(P_1, P_0, f)$. Let $u : M \to M'$ be a morphismin $\underline{mod}\,A$, and write $M = F(P_1, P_0, f)$ and $M' = F(P_1', P_0', f')$. Denote by g (resp. g') the canonical epimorphism $P_0 \to M = P_0/Im\,f$ (resp. $P_0' \to M' = P_0'/Im\,f'$). Since g' is an epimorphism and P_0 is projective, there exists $u_0 : P_0 \to P_0'$ such that $ug = g'u_0$. Because $g'(u_0 f) = ugf = 0$, by the exactness of the sequence $P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \to 0$, we have that $Im\, u_0 f \subseteq Ker\, g'$. As the morphism $f' : P_1' \to Ker\, g' = Im\, f'$ is surjective and P_1 is projective, there exists a morphism $u_1 : P_1 \to P_1'$ such that $u_0 f = f' u_1$. We have that (u_1, u_0) is a morphism in $\overrightarrow{proj}\, A / \overrightarrow{proj}_1\, A$ and $F(u_1, u_0) = u$. This proves that the category $\underline{mod}\, A$ is equivalent to the quotient of $\overrightarrow{proj}\, A$ modulo $\overrightarrow{proj}_1\, A$, as required.

Let us construct now a duality $Tr : \underline{mod}\, A \to \underline{mod}\, A^{op}$, that maps M to its transpose.

- $Tr\, M = F(P_0^t, P_1^t, f^t)$, for $M = F(P_1, P_0, f) \in \underline{mod}\, A$,
- Let $u : M \to M'$ be a morphism in $\underline{mod}\, A$, where $M = F(P_1, P_0, f)$ and $M = F(P_1', P_1', f')$. Then $u = F(u_1, u_0)$, such that the following diagram, with exact rows, is commutative.

$$\begin{tikzcd}
P_1 \arrow[r, "f"] \arrow[d, "u_1"] & P_0 \arrow[r] \arrow[d, "u_0"] & M \arrow[r] \arrow[d, "u"] & 0 \\
P_1' \arrow[r, "f'"'] & P_0' \arrow[r] & M' \arrow[r] & 0
\end{tikzcd}$$

If we apply the functor $(-)^t$, we get a diagram

$$\begin{tikzcd}
P_0^t \arrow[r, "f^t"] & P_1^t \arrow[r] & Tr\, M \arrow[r] & 0 \\
P_0'^t \arrow[r, "f'^t"'] \arrow[u, "u_0^t"] & P_1'^t \arrow[r] \arrow[u, "u_1^t"] & Tr\, M' \arrow[r] & 0
\end{tikzcd}$$

with exact rows and a commutative left square. We define $Tr\, u : Tr\, M' \to Tr\, M$ to be the unique morphism that makes the right square commutative.

Note that (u_0^t, u_1^t) is a morphism in $\overrightarrow{proj}\, A^{op} / \overrightarrow{proj}_1\, A^{op}$. Indeed, it can be proved that if $(u_1, u_0) : (P_1, P_0, f) \to (P_1', P_0', f')$ belongs to $\overrightarrow{proj}_1\, A$, then $(u_0^t, u_1^t) : (P_0'^t, P_1'^t, f'^t) \to (P_0^t, P_1^t, f^t)$ belongs to $\overrightarrow{proj}_1\, A^{op}$.

The functor $Tr : \underline{mod}\, A \to \underline{mod}\, A^{op}$ is well defined and it is indeed a duality, which is called the *transposition*.

For the missing details, see [2, IV.2.2].

Definition 2.4.8. The composition of D with Tr, $\tau = D\, Tr$ and its inverse equivalence, $\tau^{-1} = Tr\, D$ are called the *Auslander-Reiten translations*.

The next proposition records some basic properties of Auslander-Reiten translations.

Proposition 2.4.9. *Let M and N be indecomposable modules in $\operatorname{mod} A$.*

1. *The module $\tau\, M$ is zero if and only if M is projective.*
2. *The module $\tau^{-1}\, N$ is zero if and only if N is injective.*
3. *If M is a nonprojective module, then $\tau\, M$ is indecomposable noninjective and $\tau^{-1}\,\tau\, M \cong M$.*
4. *If N is a noninjective module, then $\tau^{-1}\, N$ is indecomposable nonprojective and $\tau\,\tau^{-1}\, N \cong N$.*
5. *If M and N are nonprojective, then $M \cong N$ if and only if there is an isomorphism $\tau M \cong \tau N$.*
6. *If M and N are noninjective, then $M \cong N$ if and only if there is an isomorphism $\tau^{-1} M \cong \tau^{-1} N$.*

Proof. This proposition is an easy consequence of 2.4.5 and 1.2.35. □

Proposition 2.4.10. *Let M be an indecomposable nonprojective module in $Mod\, A$ and let*

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0$$

be a minimal projective presentation of M. Then the sequence

$$0 \longrightarrow \tau\, M \longrightarrow \nu\, P_1 \xrightarrow{\nu\, p_1} \nu\, P_0$$

is a minimal injective presentation of $\tau\, M$.

Proof. The sequence $P_0^t \xrightarrow{p_1^t} P_1^t \longrightarrow Tr\, M \longrightarrow 0$ is a minimal projective presentation of $Tr\, M$ (2.4.5 (2)) and the result follows by 1.2.35. □

Recall the functorial morphism φ^M, where M is an A-module, defined in the first chapter. For an arbitrary A-module N, we have that $Coker\, \varphi_N^M \cong \underline{Hom}_A(M, N)$. The proof of this fact can be found in [2, IV.2.12].

The following proposition is useful for the proof of 2.4.13.

Proposition 2.4.11. *For two A-modules M, N, we have that*

$$Ext_A^1(M, N) \cong D\underline{Hom}_A(\tau^{-1}N, M) \cong D\overline{Hom}_A(N, \tau M),$$

and these isomorphisms are functorial in both variables.

Proof. If N is injective, then $Ext^1_A(-,N) = 0$ by 1.2.37, and $\tau^{-1}N = 0$, by 2.4.9 (3). Hence, $\underline{Hom}_A(\tau^{-1}N,-) = 0$. Moreover, since every morphism from N to $\tau\,M$ factors through the injective module N, we have $\overline{Hom}_A(N,\tau\,M) = 0$. So the result holds if N is injective. Because the Hom-functor, D and $Ext^1_A(-,N)$ are additive functors, it is enough to prove the theorem in the case where N has no injective direct summand. Let then N be an A-module that doesn't have any injective direct summand. Since N is not injective, by 2.4.9 (4), we can write $N = \tau\,L$, where $L = \tau^{-1}\,N$. Let $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} L \to 0$ be a minimal projective presentation of L. We get the exact sequence

$$0 \to \tau\,L \to I_0 \xrightarrow{Dp_1^t} I_1 \xrightarrow{Dp_0^t} DL^t \to 0,$$

where $I_0 = DP_1^t$ and $I_1 = DP_0^t$ are injective, by applying the Nakayama functor. Write $\bar{p}_1 = Hom_A(M,Dp_1^t)$ and $\bar{p}_0 = Hom_A(M,Dp_0^t)$, for short. Let

$$0 \to DL^t \xrightarrow{p_2} I_2 \xrightarrow{p_3} I_3 \to \dots$$

be an injective resolution of DL^t. Then we obtain the following injective resolution of $\tau\,L$:

$$0 \to \tau\,L \to I_0 \xrightarrow{Dp_1^t} I_1 \xrightarrow{p_2Dp_0^t} I_2 \xrightarrow{p_3} I_3 \to \dots$$

Note that this is indeed an injective resolution of $\tau\,L$ because $Ker\,(p_2\,Dp_0^t) = Ker\,Dp_0^t = Im\,Dp_1^t$ and $Im\,(p_2\,Dp_0^t) = p_2(Im\,Dp_0^t) = p_2(DL^t) = Ker\,p_3$.

Applying the left exact functor $Hom_A(M,-)$, we get:

$$0 \to (\tau\,L)' \to (I_0)' \xrightarrow{\bar{p}_1} (I_1)' \xrightarrow{\bar{p}_2\bar{p}_0} (I_2)' \xrightarrow{\bar{p}_3} (I_3)' \to \dots$$

where $\bar{p}_i = Hom_A(M,p_i)$ $(i \geq 2)$, $(\tau\,L)' = Hom_A(M,\tau\,L)$, and $(I_j)' = Hom_A(M,I_j)$, for $j \geq 0$. By 1.2.36, we have

$$Ext^1_A(M,N) \cong Ker\,(\bar{p}_2\bar{p}_0)/Im\,\bar{p}_1 = Ker\,\bar{p}_0/Im\,\bar{p}_1.$$

Now, for an A-module X, let

$$\omega^X : DHom_A(X,-) \to Hom_A(-,DX^t)$$

be the composition of $D\varphi^X : DHom_A(X,-) \to D((-)\otimes_A X^t)$ with

$$\eta^X : D((-)\otimes_A X^t) \xrightarrow{\cong} Hom_A(-,DX^t),$$

defined as follows: for an A-module Y,

$$\eta^X_M(f)(y)(g) := f(y \otimes g),$$

where $f \in D(Y \otimes X^t), y \in Y$ and $g \in X^t$. It is easy to check that η^X is a functorial isomorphism. Hence ω^X is a functorial morphism and, if X is projective, φ^X is an isomorphism and so is ω^X.

Write $\tilde{p}_1 = DHom_A(p_1, M)$ and $\tilde{p}_0 = DHom_A(p_0, M)$, where p_1 and p_0 are the morphisms involved in the minimal projective presentation of L. Consider the diagram

$$\begin{tikzcd}
Hom_A(M, DP_1^t) \arrow[r, "\bar{p}_1"] & Hom_A(M, DP_0^t) \arrow[r, "\bar{p}_0"] & Hom_A(M, DL^t) & \\
DHom_A(P_1, M) \arrow[r, "\tilde{p}_1"] \arrow[u, "\omega_M^{P_1}"] & DHom_A(P_0, M) \arrow[r, "\tilde{p}_0"] \arrow[u, "\omega_M^{P_0}"] & DHom_A(L, M) \arrow[r] \arrow[u, "\omega_M^{L}"] & 0.
\end{tikzcd}$$

The lower row, which is obtained from the minimal projective presentation of L by applying the functor $DHom_A(-, M)$, is exact since $DHom_A(-, M)$ is right exact. Furthermore, the diagram is commutative because ω^X is functorial, and the morphisms $\omega_M^{P_1}, \omega_M^{P_0}$ are isomorphisms as P_1 and P_0 are projective modules.

Let $f \in Ker\, \bar{p}_0$. We have that $\omega_M^L \left(\tilde{p}_0 \left(\omega_M^{P_0}\right)^{-1} (f) \right) = \bar{p}_0 \omega_M^{P_0} \left(\omega_M^{P_0}\right)^{-1} (f) = \bar{p}_0(f) = 0$. Thus $Im\, \tilde{p}_0 \left(\omega_M^{P_0}\right)^{-1} \subseteq Ker\, \omega_M^L$. Consider then the morphism

$$\psi = \tilde{p}_0 \left(\omega_M^{P_0}\right)^{-1} : Ker\, \bar{p}_0 \to Ker\, \omega_M^L.$$

This morphism is surjective, since $\omega_M^{P_0}$ is an isomorphism and $\tilde{p}_0$ is an epimorphism. Therefore, by the isomorphism theorem, we have $Im\, \psi = Ker\, \omega_M^L \cong Ker\, \bar{p}_0 / Ker\, \psi$. Moreover, because $\omega_M^{P_0}, \omega_M^{P_1}$ are isomorphisms and the lower row of the previous diagram is exact,

$$Ker\, \psi \cong Ker\, \tilde{p}_0 = Im\, \tilde{p}_1 \cong Im\, \bar{p}_1.$$

It follows that

$$\begin{aligned}
Ext_A^1(M, N) = Ker\, \bar{p}_0 / Im\, \bar{p}_1 &\cong Ker\, \bar{p}_0 / Ker\, \psi \\
&\cong Ker\, \omega_M^L \\
&= Ker\, D\varphi_M^L (\eta^X \text{ is an isomorphism}) \\
&\cong DCoker\, \varphi_M^L
\end{aligned}$$

Because $Coker\, \varphi_M^L \cong \underline{Hom}_A(L, M) = \underline{Hom}_A(\tau^{-1} N, M)$, the proof of the first isomorphism is complete. The proof of the other isomorphism is similar. $\square$

The next lemma is necessary to the proof of 2.4.13.

Lemma 2.4.12. *Let M, L be a indecomposable A-modules. If L is nonprojective, then $\mathcal{P}(L, M) \subseteq rad_A(L, M)$.*

Proof. Let $f \in \mathcal{P}(L, M) \setminus rad_A(L, M)$. The morphism f can be written in the form $f = f_1 f_2$, where $f_2 : L \to P$ and $f_1 : P \to M$, with P projective. Because $f \notin rad_A(L, M)$, f admits an inverse $h : M \to L$. In particular, hf_1 is a left inverse of f_2. Consequently, L is projective. Indeed, if $u : L \to N$ is a morphism and $v : V \to N$ is an epimorphism, then there is a morphism $r : P \to U$ such that $vr = uhf_1$ because P is projective, and so the morphism $rf_2 : L \to U$ satisfies $vrf_2 = uhf_1f_2 = u$, and we are done. □

Now we are able to prove the very important theorem of Auslander and Reiten about the existence of the almost split sequences. But first we introduce a new functor that will be used in the proof of this theorem. Let M be a right A-module. The covariant functor $S(M, -) : mod\,A \to mod\,A$ is defined as follows:

- For an A-module N, $S(M, N) := Hom_A(M, N)/rad_A(M, N)$,
- For two A-modules N, L, and $f \in Hom_A(N, L)$, $S(M, f) : S(M, N) \to S(M, L)$ is given by the formula

$$S(M, f)(g + rad_A(M, N)) = fg + rad_A(M, L),$$

  where $g \in Hom_A(M, N)$.

Note that if $g \in rad_A(M, N)$, then $fg \in rad_A(M, N)$, because rad_A is an ideal of $mod\,A$. So $S(M, f)$ is well-defined. It is easy to check that $S(M, -)$ is indeed a covariant functor.

Theorem 2.4.13. *Let M be an indecomposable nonprojective A-module and N be an indecomposable noninjective A-module.*

1. *There exists an almost split sequence $0 \to \tau M \to E \to M \to 0$ in $mod\,A$.*
2. *There exists an almost split sequence $0 \to N \to F \to \tau^{-1}N \to 0$ in $mod\,A$.*

Proof. Let M, X be two indecomposable nonprojective A-modules. It follows from 2.4.11 that

$$Ext_A^1(M, \tau X) \cong D\underline{Hom}_A(X, M),$$

as $\tau^{-1}\tau X \cong X$, by 2.4.9 (3).

Denote by $p_{X,M} : \underline{Hom}_A(X,M) \to S(X,M)$ the canonical epimorphism. This morphism is well defined by 2.4.12. By applying the contravariant functor D, we get the canonical monomorphism $Dp_{X,M} : DS(X,M) \to D\underline{Hom}_A(X,M)$.

Set $X = M$ and let ξ' be a nonzero element of $DS(M,M)$, and $\xi = Dp_{M,M}(\xi')$. Since $\xi' \neq 0$ and $Dp_{M,M}$ is a monomorphism, ξ is also nonzero. Hence $\xi \in \underline{Hom}_A(M,M) \cong Ext^1_A(M,\tau M)$ is represented by a short exact sequence

$$S : 0 \to \tau M \xrightarrow{f} E \xrightarrow{g} M \to 0 \tag{2.2}$$

that doesn't split (see 1.2.39 and 1.2.43).

Because M is not projective, τM is indecomposable, by 2.4.9 (3). Let us show now that g is right almost split. This morphism cannot be a retraction as 2.2 doesn't split. Let $v \in Hom_A(V,M)$ that is not a retraction. We want to prove the existence of a morphism $v' : V \to E$ such that $gv' = v$. Write

$$v = [v_1 \ldots v_n] : V = V_1 \oplus \ldots \oplus V_n \to M,$$

where $V_1, \ldots, V_n$ are indecomposable modules. Suppose $v_i : V_i \to M$ is a retraction, for some $i \in [n]$, and let v_i' be its right inverse. Then $v' = [0 \ldots 0\, v_i'\, 0 \ldots 0] : M \to V$ is a right inverse of v, contradicting the fact that v is not a retraction. Hence none of the components v_i of v is a retraction, in particular, none of them is an isomorphism. It follows from 1.2.5 (3) and 1.1.18 that $v_i \in rad_A(V_i,M)$, for all $i \in [n]$. By 1.1.17, $v \in rad_A(V,M)$. Consider the diagram

$$\begin{tikzcd}
DS(M,M) \arrow[r, "Dp_{M,M}"] \arrow[d, "DS(M,v)"] & D\underline{Hom}_A(M,M) \arrow[r, "\cong"] \arrow[d, "D\underline{Hom}_A(M,v)"] & Ext^1_A(M,\tau M) \arrow[d, "Ext^1_A(v,\tau M)"] \\
DS(M,V) \arrow[r, "Dp_{M,V}"] & D\underline{Hom}_A(M,V) \arrow[r, "\cong"] & Ext^1_A(V,\tau M)
\end{tikzcd}$$

which is commutative because $(Dp_{M,Y})_{Y \in mod\,A}$ is functorial and so are the isomorphisms given in 2.4.11. Because $v \in rad_A(V,M)$, $DS(M,v)(\xi') = 0$, which implies that $Ext^1_A(v,\tau M)(\xi) = 0$, due to the commutativity of the diagram. This means that the sequence

$$S' : 0 \to \tau M \xrightarrow{f'} E' \xrightarrow{g'} V \to 0,$$

that represents $Ext^1_A(v,\tau M)(\xi)$, splits. Note that $[S'] = ext(v,\tau M)([S])$ (cf. 1.2.41), and so we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \tau M \arrow[r, "f'"] \arrow[d, "1_{\tau M}"] & E' \arrow[r, "g'"] \arrow[d, "w"] & V \arrow[r] \arrow[d, "v"] & 0 \\
0 \arrow[r] & \tau M \arrow[r, "f"'] & E \arrow[r, "g"'] & M \arrow[r] & 0.
\end{tikzcd}$$

Since the first row splits, we can consider a right inverse $g'' : V \to E'$ of g'. Then the morphism $v' = wg'' : V \to E$ satisfies the required condition, i.e., $gv' = gwg'' = vg'g'' = v$. Hence, g is right almost split. By 2.3.6 (b), the sequence 2.2 is almost split and the first statement of the theorem is proved. The second statement has a similar proof. □

The almost split sequences are also unique in the following sense.

Proposition 2.4.14. *An almost split sequence is uniquely determined (up to isomorphism) by each of its end terms.*

Proof. Suppose that

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

and

$$0 \to L' \xrightarrow{f'} M' \xrightarrow{g'} N' \to 0$$

are two almost split sequences in $mod\, A$ and $L \cong L'$. Let $r : L \to L'$ be an isomorphism. Since f is left almost split and $f'r : L \to M'$ is not a section, there is a morphism $h : M \to M'$ such that $hf = f'r$. Analogously, since f' is left almost split and $fr^{-1} : L' \to M$ is not a section, there exists a morphism $h' : M' \to M$ such that $h'f' = fr^{-1}$. Thus, $f' = hfr^{-1}$, and $f = h'f'r$. It follows that $f' = hh'f'$ and $f = h'hf$. Because both f and f' are left minimal, hh' and $h'h$ are automorphisms, and so h is an isomorphism. It is easy to check that $h' : N \to N'$ defined by $h'(n) = g'h(m)$, for $n \in N$ and where $m \in M$ is such that $g(m) = n$ is well defined, is an isomorphism and $h'g = g'h$. Consequently, the two sequences are isomorphic, i.e., the terms of the sequences are isomorphic and the following diagram is commutative.

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "r", "\cong"'] & M \arrow[r, "g"] \arrow[d, "h", "\cong"'] & N \arrow[r] \arrow[d, "h'", "\cong"'] & 0 \\
0 \arrow[r] & L' \arrow[r, "f'"'] & M' \arrow[r, "g'"'] & N' \arrow[r] & 0
\end{tikzcd}$$

The proof that the two sequences are isomorphic in the case when $N \cong N'$ is similar. □

The following proposition, which is a consequence of 2.4.13, gives an important characterization of the minimal almost split morphisms in connection with the space of irreducible morphisms.

Proposition 2.4.15. *Let M be an A-module and write it as a direct sum $\oplus_{i=1}^{m} M_i^{n_i}$, where the modules $M_1, \ldots, M_m$ are indecomposable and such that $M_i \ncong M_j$, with $i \neq j$.*

1. *Let L be an indecomposable A-module and $f \in Hom_A(L, M)$. Write $f = [f_1 \ \dots \ f_m]^t$, where*

$$f_i = [f_{i1} \ \dots \ f_{in_i}]^t \in Hom_A(L, M_i^{n_i}), i \in [m].$$

Then f is left minimal almost split if and only if the following conditions hold:

- $f_{ij} \in rad_A(L, M_i)$, *for* $i \in [m], j \in [n_i]$,
- *For each* $i \in [m]$, $\{\bar{f}_{ij} = f_{ij} + rad_A^2(L, M_i) \mid j \in [n_i]\}$ *is a K-basis of* $Irr(L, M_i)$,
- *If M' is an indecomposable A-module such that $Irr(L, M') \neq 0$, then $M' \cong M_i$ for some i.*

2. *Let N be an indecomposable A-module and $g \in Hom_A(M, N)$. Write $g = [g_1 \ \dots \ g_m]$, where*

$$g_i = [g_{i1} \ \dots \ g_{in_i}] \in Hom_A(M_i^{n_i}, N), i \in [m].$$

Then g is right minimal almost split if and only if the following conditions hold:

- $g_{ij} \in rad_A(M_i, N)$, *for* $i \in [m], j \in [n_i]$,
- *For each* $i \in [m]$, $\{\bar{g}_{ij} = g_{ij} + rad_A^2(M_i, N) \mid j \in [n_i]\}$ *is a K-basis of* $Irr(M_i, N)$,
- *If M' is an indecomposable A-module such that $Irr(M', N) \neq 0$, then $M' \cong M_i$ for some i.*

Proof. Let $f \in Hom_A(L, M)$ be left minimal almost split. Then f is irreducible, by 2.3.11. Note that $f_{ij} = v_j f$, where $v_j : \oplus_{k=1}^m M_k^{n_k}$ is the projection, and that v_j is a retraction. Hence by 2.3.14, $f_{ij} : L \to M_i$ is irreducible, for $i \in [m], j \in [n_i]$. So $f_{ij} \in rad_A(L, M_i)$ for all $i \in [m], j \in [n_i]$, by 2.3.9. If M' is an indecomposable module for which there is an irreducible morphism $f' : L \to M'$, then there exists $i \in [m]$ such that $M' \cong M_i$, by 2.3.13.

Fix an $i \in [m]$ and let $\bar{h} = h + rad_A^2(L, M_i)$ be a nonzero element of $Irr(L, M_i)$. We will prove that it can be written as a linear combination of $\bar{f}_{i1}, \dots, \bar{f}_{in_i}$. Since $\bar{h} \neq 0$, $h \in rad_A(L, M_i) \setminus rad_A^2(L, M_i)$, i.e., h is irreducible (2.3.9) and so it is not a section. Since f is left almost split, there exists a homomorphism $h' = [h'_1 \ \dots \ h'_t] : \oplus_{k=1}^t M_k^{n_k} \to M_i$, with $h'_k = [h'_{k1} \ \dots \ h'_{kn_k}] : M_k^{n_k} \to M_i$ such that

$$h = h'f = \sum_{k=1}^{t} \sum_{j=1}^{n_k} h'_{kj} f_{kj}.$$

Because $M_k \not\cong M_i$, for $k \neq i$, none of $h'_{kj} : M_k \to M_i$ is an isomorphism. So $h'_{kj} \in rad_A(M_k, M_i)$, for $k \neq i$, as M_k, M_i are indecomposable modules. It follows from the definition of the ideal of a category and from the fact that $f_{kj} \in rad_A(L, M_k)$ that $h'_{kj} f_{kj} \in rad^2_A(L, M_i)$, for $k \neq i$. Therefore

$$\bar{h} = \sum_{j=1}^{n_i} \bar{h'}_{ij} \bar{f}_{ij}.$$

By 1.2.5 (1), (3), we have that $(End\, M_i / rad\, End\, M_i) \cong K$. So we can write $h'_{ij} = \lambda_j \cdot 1_{M_i} + u_{ij}$, with $\lambda_j \in K$ and $u_{ij} \in rad_A\, End\, M_i = rad_A(M_i, M_i)$ (this follows from 1.2.2 (1), from the definition of the radical of a category and from 1.1.18). Hence $u_{ij} f_{ij} \in rad^2_A(L, M_i)$ and so we have that

$$\bar{h} = \sum_{j=1}^{n_i} \lambda_j \cdot \bar{f}_{ij},$$

as required. Let us prove now that $\{\bar{f}_{i1}, \ldots, \bar{f}_{in_i}\}$ is K-linear independent. Suppose, for a contradiction, that there is a zero linear combination $\sum_j \lambda_j \bar{f}_{ij} = 0$, with $\lambda_j \in K$ and $\lambda_j \neq 0$ for some j. The morphism $l = [\lambda_1 \ \ldots \ \lambda_{n_i}] : M_i^{n_i} \to M_i$ defined by $(x_1, \ldots, x_{n_i}) \mapsto \lambda x_1 + \cdots + \lambda_{n_i} x_{n_i}$ is a retraction (the morphism

$$\begin{array}{rccc} \beta = [\beta_1 \cdots \beta_{n_i}]^t : & M_i & \to & M_i^{n_i} \\ & x & \mapsto & (0, \ldots, 0, \lambda_j^{-1} x, 0, \ldots, 0) \end{array}$$

is a right inverse of l). By 2.3.14 and because f_i is irreducible, we conclude that $v := \sum_{j=1}^{n_i} \lambda_j f_{ij} = l f_i$ is irreducible, which contradicts the fact that $v \in rad^2_A(L, M_i)$, as $\sum_j \lambda_j \bar{f}_{ij} = 0$.

Conversely, suppose that the three conditions hold. Our objective is to prove that f is left minimal almost split. Consider a left minimal almost split morphism $f' : L \to U$. Note that 2.4.13 and 2.3.3 guarantee the existence of such a morphism. Write $U = \oplus_{k=1}^s U_k^{m_k}$, where $U_1, \ldots, U_s$ are indecomposable modules and $U_1, \ldots, U_s$ are pairwise nonisomorphic. Since for each k, j, $Irr(L, U_k) \neq 0$ it follows from the third condition that $U_k \cong M_j$ for some j, and so $dim_K\, Irr(L, U_k) = dim_K\, Irr(L, M_j)$. Thus $m_k = n_j$ by the second condition. Consequently, $U = \oplus_{k=1}^s U_k^{m_k} \cong \oplus_{j=1}^t M_j^{n_j} = M$ and so we may then assume that $U = M$. Observe that f is not a section. Indeed, if f was a section, then there would exist $g : M \to L$ such that $gf = 1_L$. Write $g = [g_1 \ \ldots \ g_m]$, with $g_i = [g_{i1} \ \ldots \ g_{in_i}]$. So we have

$$\sum_{i=1}^{m} \sum_{j=1}^{n_i} g_{ij} f_{ij} = 1_L.$$

But each f_{ij} lies in $rad_A(L, M_j)$, and so $\sum_{i=1}^{m}\sum_{j=1}^{n_i} f_{ij}g_{ij} \in rad_A(L, L)$, which contradicts the fact that $1_L \notin rad_A(L, L)$, as L is indecomposable and 1_L is an isomorphism. Thus, because f' is left almost split and f is not a section, there exists $h \in End(M)$ such that $f = hf'$. Let us prove that h is an isomorphism. Let $h_{ir,js} : M_j \to M_i$, with $i, j \in [m], r \in [n_i], s \in [n_j]$ be the components of h. We have that, for each $i \in [m]$ and $r \in [n_i]$, $f_{ir} = \sum_{j=1}^{m}\sum_{s=1}^{n_j} h_{ir,js}f'_{js}$, and so

$$\bar{f}_{ir} = \sum_{j=1}^{m}\sum_{s=1}^{n_j} \bar{h}_{ir,js}\bar{f'}_{js},$$

where $\bar{f}_{ir} = f_{ir} + rad^2(L, M_i)$, $\bar{h}_{ir,js} = h_{ir,js} + rad(M_j, M_i)$ and $\bar{f'}_{js} = f'_{js} + rad(L, M_j)$. It follows from the fact that $M_i \ncong M_j$, if $i \neq j$, and from 1.1.18 that $h_{ir,js} \in rad(M_j, M_i)$, and so $\bar{h}_{ir,js} = 0$, for $i \neq j$. Hence $\bar{f}_{ir} = \sum_{s=1}^{n_i} \bar{h}_{ir,is}\bar{f'}_{is}$. Because both $\{\bar{f}_{ir} \mid r \in [n_i]\}$ and $\{\bar{f'}_{is} \mid s \in [n_i]\}$ are K-basis of $Irr(L, M_i)$, the matrix $(\bar{h}_{ir,is} : r, s \in [n_i])$ is invertible. Let $h_i : M_i^{n_i} \to M_i^{n_i}$ be the map whose components are $h_{ir,is}$, with $r, s \in [n_i]$, and let $\bar{h}_i = h_i + rad(M_i^{n_i}, M_i^{n_i})$. By 1.1.17, $\bar{h}_i$ is given by the matrix $(\bar{h}_{ir,is}; r, s \in [n_i])$, and so $\bar{h}_i$ is invertible. So there is $k_i \in End(M_i^{n_i})$ such that $\bar{h}_i\bar{k}_i = \bar{1}_{M_i^{n_i}}$. Thus, $1_{M_i^{n_i}} - h_ik_i \in rad(M_i^{n_i}, M_i^{n_i})$. It follows by definition that $h_ik_i = 1_{M_i^{n_i}} - 1_{M_i^{n_i}}(1_{M_i^{n_i}} - h_ik_i)$ is invertible, and so is h_i. Let $g : M \to M$ be the morphism such that $g_{ir,is} = 0$, for all $i \in [m]$ and $r, s \in [n_i]$, and $g_{ir,js} = h_{ir,js}$, for $i \neq j$, and let $\sigma : M \to M$ be the morphism $\sigma = \sum_{i=1}^{m} h_i$. Note that $h = \sigma + g$. Because $M_i \ncong M_j$, for $i \neq j$, we have $g_{ir,js} \in rad(M_j, M_i)$, if $i \neq j$. By definition $g_{ir,is} = 0 \in rad(M_i, M_i)$. Therefore, $g \in rad(M, M)$, by 1.1.17. On the other hand, σ is invertible since each h_i is invertible, as we have seen. Let σ^{-1} be the inverse of σ. Then $\sigma^{-1}h = \sigma^{-1}(\sigma + g) = 1_M - (-\sigma^{-1})g$. Because g lies in $rad(M, M)$, $\sigma^{-1}h$ is invertible, and so is h, as required. We can now conclude that f is left minimal almost split. Indeed, if $k \in End(M)$ is such that $kf = f$, then $khf' = hf'$, and so $(h^{-1}kh)f' = f'$, which implies that $h^{-1}kh$ is an automorphism. Therefore, k is an automorphism and f is left minimal. To prove that f is left almost split it only remains to check that for any morphism $u : L \to U$ that is not a section, there is a morphism $u'' : M \to U$ such that $u''f = u$. Since f' is left almost split, there is $u' : M \to U$ such that $u'f' = u$. Set $u'' = u'h^{-1}$, and we are done. The proof of (2) is similar. □

Remark 2.4.16. Let $P(a) = e_aA$ be an indecomposable projective A-module and $I(a) = D(Ae_a)$ an indecomposable injective A-module. We have seen that the inclusion $rad\,P(a) \hookrightarrow P(a)$ is right minimal almost split and the canonical epimorphism $I(a) \to I(a)/soc\,I(a)$ is left minimal almost split (cf. 2.3.3). It follows from 2.4.15 that every indecomposable A-module X such that $Irr(X, P(a)) \neq 0$ is isomorphic to a direct summand of $rad\, P(a)$, and every indecomposable A-module Y such that $Irr(I(a), Y) \neq 0$ is isomorphic to a direct summand of $I(a)/soc\, I(a)$. Moreover the dimension of $Irr(X, P(a))$ is the multiplicity of X as a direct summand of $rad(P(a))$, and the dimension of $Irr(I(a), Y)$ is the multiplicity of Y as a direct summand of $I(a)/soc\, I(a)$.

Corollary 2.4.17. *Let $0 \to L \xrightarrow{f} \bigoplus_{i=1}^{t} M_i^{n_i} \xrightarrow{g} N \to 0$ be an almost split sequence, with L, N indecomposable and $M_1, \ldots, M_t$ indecomposable and pairwise nonisomorphic. Then, for each $i \in [t]$, we have*

$$dim_K\, Irr(L, M_i) = dim_K\, Irr(M_i, N).$$

2.5 The Auslander-Reiten quiver

This section is devoted to the concept of the Auslander-Reiten quiver of a basic and connected finite dimensional K-algebra A. Some properties of this quiver will be given, with particular reference to 2.5.10, which claims that if the Auslander-Reiten quiver of A, denoted by $\Gamma(mod\, A)$, has a finite connected component $\mathcal{C}$, then $\Gamma(mod\, A) = \mathcal{C}$ and it is finite. We will also give a simple example just to illustrate the notion of this quiver.

The Auslander-Reiten quiver is a quiver that collects the information of the category $mod\, A$. Since any module decomposes as a direct sum of indecomposable modules, and such decomposition is unique up to isomorphism, it is natural to consider the vertices of the quiver as isomorphism classes of indecomposable modules. Moreover, the morphisms in $mod\, A$ that admit no nontrivial factorization are precisely the irreducible morphisms (cf. 2.3.8), so it is also natural that the arrows of the AR-quiver correspond to the irreducible morphisms. The formal definition is given below.

Definition 2.5.1. Let A be a basic and connected finite dimensional K-algebra. The quiver $\Gamma(mod\, A)$, of $mod\, A$ is defined as follows:

1. The vertices of $\Gamma(mod\, A)$ are the isomorphism classes $[X]$ of indecomposable modules X in $mod\, A$.

2. Let $[M], [N]$ be the vertices in $\Gamma(mod\, A)$ corresponding to the indecomposable modules M, N in $mod\, A$. The arrows $[M] \to [N]$ are in bijective correspondence with the vectors of a basis of the K-vector space $Irr(M, N)$.

The quiver $\Gamma(mod\, A)$ of the module category $mod\, A$ is called the *Auslander-Reiten quiver* (or *AR-quiver*, for short) of A.

We gather some characteristics of the AR-quiver in the following remark.

Remark 2.5.2.

1. Let M be an indecomposable module. It follows from 2.4.15, 2.4.13 and 2.3.3 that if M is projective, then the predecessors of $[M]$ are the points $[L]$ with L an indecomposable direct summand of $rad\, M$, and if M is not projective, the predecessors of $[M]$ are the points $[L]$ where L is an indecomposable direct summand of the middle term of the almost split sequence ending with M. Similarly, if M is injective, the successors of $[M]$ are the points $[N]$ such that N is an indecomposable summand of $M/soc\, M$, and if M is not injective, the successors of $[M]$ are the indecomposable direct summands of the middle term of the almost split sequence starting with M. In particular, for every indecomposable module M, the set of the predecessors and the set of the successors of $[M]$ are finite, which means that each vertex of $\Gamma(mod\, A)$ has a finite number of neighbours. A quiver that satisfies such property is said to be *locally finite*. Thus, $\Gamma(mod\, A)$ is locally finite.

2. The quiver $\Gamma(mod\, A)$ is finite if and only if A is representation-finite (recall the definition 1.2.7).

3. The AR-quiver has no loops. Indeed, let $M \in mod\, A$ and suppose $f : M \to M$ is an irreducible morphism. Then by 2.3.8, f is a epimorphism or a monomorphism. Since M is finite dimensional, it follows that f is an isomorphism, contradicting the fact that f is neither a section nor a retraction. So the source and the target of the arrows of the AR-quiver must be different from each other.

4. If A is a representation-finite algebra, then $\Gamma(mod\, A)$ has no multiple arrows (see [2, IV.4.9]).

5. Recall the Auslander-Reiten translation τ (cf. 2.4.8). For each indecomposable nonprojective module M, $\tau\, M$ is indecomposable noninjective, and for each indecomposable noninjective module N, $\tau^{-1}\, N$ is indecomposable nonprojective, by 2.4.9. Moreover, we have $\tau^{-1}\tau\, M \cong M$, and $\tau\tau^{-1}N \cong N$. This permits to define a bijection (also denoted by τ):
$$\begin{array}{rccc} \tau : & \Gamma'_0 & \to & \Gamma''_0 \\ & [M] & \mapsto & [\tau\, M], \end{array}$$
where $\Gamma'_0 = \{[M] \in \Gamma\,(mod\, A) \mid \text{M is not projective}\}$ and $\Gamma''_0 = \{[N] \mid$

N is not injective}. The inverse bijection, denoted by τ^{-1} is given by

$$\begin{array}{rccc} \tau^{-1}: & \Gamma_0'' & \to & \Gamma_0' \\ & [N] & \mapsto & [\tau^{-1} N]. \end{array}$$

The bijection τ is called the *translation* of the AR-quiver $\Gamma\,(mod\,A)$. Let M be an indecomposable nonprojective module and let

$$0 \to \tau\, M \to \oplus_{i\in[t]} N_i^{n_i} \to M \to 0$$

be an almost split sequence ending with M (cf. 2.4.13). By 2.4.15, the isomorphism classes $[N_i]$ of the direct summands of the middle term are precisely the successors of $\tau\, M$ in $\Gamma\,(mod\,A)$ and the predecessors of M. Moreover, we have $dim_K\, Irr(N_i, M) = dim_K\, Irr(\tau\, M, N_i) = n_i$ (cf. 2.4.17) and so there is a bijection between the set of arrows from $[\tau\, M]$ to $[N_i]$ and the set of arrows from $[N_i]$ to $[M]$.

A quiver with these properties is known as a *translation quiver* (see definition in [2, IV.4.7]).

We will now give a simple example of Auslander-Reiten quiver, but before that we state the following useful proposition that gives us examples of almost split sequences.

Proposition 2.5.3. [2, IV.3.9, IV.3.11]

1. *Let P be a nonsimple indecomposable projective and injective module, $S = soc\,P$ and $R = rad\,P$. The sequence*

$$0 \to R \overset{[q\,i]^t}{\to} R/S \oplus P \overset{[-j\,p]}{\to} P/S \to 0,$$

where q, p are the projections and i, j are the inclusions, is almost split.

2. *The target of any irreducible morphism whose domain is simple, projective and noninjective is projective.*

3. *The domain of any irreducible morphism whose target is simple, injective and nonprojective is injective.*

Remark 2.5.4. Let S be a simple projective noninjective module, and $f : S \to P$ a left minimal almost split morphism. Any component of f, $f' : S \to P'$ corresponding to a direct summand P' of P, is irreducible. By 2.5.3 (2), each direct summand P' of P must be projective. It follows from 2.4.16 that the dimension of the space of irreducible maps from S to P' equals the multiplicity of S in $rad\,P'$. So $P = \oplus P_i^{d_i}$, where d_i is the multiplicity of S in $rad\, P_i$ for each indecomposable projective module P_i. Similarly, if $g : I \to S'$ is a right minimal almost split morphism, for a simple, injective nonprojective module S', then $I = \oplus I_i^{d'_i}$, where I_i are the indecomposable injective modules and d'_i is the multiplicity of S' in $I_i / soc\, I_i$.

Example 2.5.5. Consider the path K-algebra A of the quiver

$$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet}$$

The indecomposable injective modules and projective modules are:

$$P(1) \;=\; K \xrightarrow{1} K \xrightarrow{1} K \;=\; I(3)$$

$$P(2) \;=\; 0 \to K \xrightarrow{1} K$$

$$P(3) \;=\; 0 \to 0 \xrightarrow{1} K \;=\; S(3)$$

$$I(1) \;=\; K \to 0 \to 0 \;=\; S(1)$$

$$I(2) \;=\; K \xrightarrow{1} K \to 0.$$

It is easy to see, according to 2.2.2, that the radicals of the projective modules $P(1), P(2)$, and the quotients $I(i)/soc\, I(i)$ $(i = 2, 3)$ are given by:

$$rad\, P(1) = P(2),\; rad\, P(2) = P(3)$$

$$soc\, I(2) = S(2),\; soc\, I(3) = S(3)$$

$$\frac{I(2)}{soc\, I(2)} = I(1),\; \frac{I(3)}{soc\, I(3)} = I(2).$$

Note that $P(3)$ is simple, projective and noninjective. By 2.5.4, in a left minimal almost split morphism $P(3) \to P$, $P = P(1)^{d_1} \oplus P(2)^{d_2} \oplus P(3)^{d_3}$, where d_i is the multiplicity of $P(3)$ in $rad\, P(i)$, for $i = 1, 2, 3$. In this case, since $P(3)$ is not a summand of $rad\, P(1)$ and $P(3) = rad\, P(2)$, we have $P = P(2)$, and so the only left minimal almost split morphism starting at $P(3)$ is the embedding $P(3) \xrightarrow{i} P(2)$. Therefore, the sequence $0 \to P(3) \xrightarrow{i} P(2) \to Coker\, i \to 0$ is almost split. The cokernel of i is $P(2)/P(3) = S(2)$. So we have an irreducible morphism from $P(3)$ to $P(2)$ and an irreducible morphism from $P(2)$ to $S(2)$. Moreover, by 2.4.15, if M' is an indecomposable module such that $Irr(P(3), M') \neq 0$ then $M' \cong P(2)$ and if N' is an indecomposable module such that $Irr(N', S(2)) \neq 0$, then $N' \cong P(2)$.

Since $P(2) = rad\, P(1)$, we have an irreducible morphism from $P(2)$ to $P(1)$ (the inclusion, to be specific). Consider now $P(1)$. Because $P(1)$ is nonsimple, injective and projective, we have an almost split sequence $0 \to$

$P(2) \to S(2) \oplus P(1) \to I(2) \to 0$, by 2.5.3 (1). In particular, we get irreducible morphisms from $P(2)$ to $S(2)$ and from $P(2)$ to $P(1)$ (cf. 2.4.15). Furthermore, there is no indecomposable module M' nonisomorphic to $S(2)$ and to $P(1)$ such that $Irr(P(2), M') \neq 0$.

Consider now the injective $I(2)$. The projection $I(2) \to I(2)/S(2) = I(1) = S(1)$ is irreducible and left minimal almost split. Again by 2.4.15, there is no other arrow in the Auslander-Reiten quiver of A with source $[I(2)]$. Similarly, we have a left minimal almost split morphism from $P(1) = I(3)$ to $I(3)/soc\, I(3) = I(2)$, and so there is an arrow from $[I(3)]$ to $[I(2)]$, which is the only arrow with source $[I(3)]$.

Now consider the simple, injective and nonprojective module $S(1)$. Since $S(1) = I(2)/soc\, I(2)$ and $S(1)$ is not a summand of $I(3)/soc\, I(3)$, we have a right minimal almost split morphism from $I(2) = S(1) = I(3)$ to $S(1)$, and so by 2.4.15, there is no other arrow in $\Gamma(mod\, A)$ with target $[S(1)]$. Moreover, we have an almost split sequence

$$0 \to Ker\,(I(2) \to S(1)) \to I(2) \to S(1) \to 0,$$

where $Ker\,(I(2) \to S(1)) = S(2)$. Thus, $[I(2)]$ is the only successor of $[S(2)]$. All these observations give rise to the AR-quiver:

$$\begin{tikzcd}
{[P(3)]} \arrow[dr] & & {[S(2)]} \arrow[dr] & & {[S(1)]} \\
& {[P(2)]} \arrow[ur] \arrow[dr] & & {[I(2)]} \arrow[ur] & \\
& & {[P(1)]} \arrow[ur] & &
\end{tikzcd}$$

Note that, when drawing the AR-quiver, we put $[\tau M]$, where M is indecomposable nonprojective, on the same horizontal line as $[M]$, by convention.

We end this chapter with the proof that if $\Gamma(mod\, A)$ has a finite connected component $\mathcal{C}$ then $\Gamma(mod\, A) = \mathcal{C}$, an important fact which is necessary for the last chapter. In order to prove this result we need the following definition and lemmas.

Definition 2.5.6. Let $M_0, M_1, \dots, M_t$ be indecomposable A-modules. A *chain of irreducible morphisms* from M_0 to M_t is a sequence of irreducible morphisms of the form

$$M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \dots \xrightarrow{f_t} M_t.$$

Lemma 2.5.7. *Let $t \in \mathbb{N}$ and let M and N be indecomposable A-modules. Suppose that $Hom_A(M,N) \neq 0$ and that there exists no chain of irreducible morphisms from M to N of length $< t$. Then there is a chain of irreducible morphisms*

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \rightarrow \dots \xrightarrow{f_t} M_t$$

and a morphism $g : M_t \rightarrow N$ such that $gf_t \dots f_1 \neq 0$, and there is a chain of irreducible morphisms

$$N_t \xrightarrow{g_t} N_{t-1} \xrightarrow{g_{t-1}} \dots \rightarrow N_1 \xrightarrow{g_1} N_0 = N$$

and a morphism $f : M \rightarrow N_t$ such that $g_1 \dots g_t f \neq 0$.

Proof. We just prove the existence of the first chain and the morphism g, the argument to the proof of the second chain and morphism f is similar. We prove by induction on t. For $t = 0$ there is nothing to show. Suppose now that M and N are such that $Hom_A(M,N) \neq 0$ and that there is no chain of irreducible morphisms from M to N of length $< t+1$. By induction, there is a chain of irreducible morphisms

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \rightarrow \dots \xrightarrow{f_t} M_t$$

and a morphism $g : M_t \rightarrow N$ with $gf_t \dots f_1 \neq 0$. Suppose, for a contradiction, that g is an isomorphism. Then gf_t is irreducible and so the sequence

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \rightarrow \dots M_{t-1} \xrightarrow{gf_t} N$$

is a chain of irreducible morphisms from M to N of length t, which contradicts the hypothesis. Thus g cannot be an isomorphism. If g is a section, then g is injective and $N = Ker\, g' \oplus Im\, g$, where $g' : N \rightarrow M_t$ is a left inverse of g. Because N is indecomposable, either $Ker\, g' = 0$ or $Im\, g = 0$. If $Ker\, g' = 0$, then g is surjective, which contradicts the fact that g is not an isomorphism, and if $Im\, g = 0$, then $gf_t \dots f_1 = 0$, also a contradiction. Therefore, g is not a section. Consider the left minimal almost split morphism

$$h = [h_1 \ \dots \ h_s]^t : M_t \rightarrow \oplus_{j=1}^{s} L_j,$$

where each L_j is indecomposable (there exists such a morphism by 2.4.13 and 2.3.3). Hence, by definition of left almost split morphism and because g is not a section, there exists

$$u = [u_1 \ \dots \ u_s] : \oplus_{j=1}^{s} L_j \rightarrow N$$

such that $g = uh = \sum_{j=1}^{s} u_j h_j$. Therefore, $g f_t \dots f_1 = \sum_{j=1}^{s} u_j h_j f_t \dots f_1 \neq 0$, and so $u_j h_j f_t \dots f_1 \neq 0$ for some j. Let $M_{t+1} := L_j, f_{t+1} := h_j$ and $g' := u_j$. Note that h_j is irreducible and then

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \dots \xrightarrow{f_t} M_t \xrightarrow{f_{t+1}} M_{t+1}$$

is a chain of irreducible morphisms of length $t+1$, and we are done. $\square$

Lemma 2.5.8. *Let $k, n \in \mathbb{N}$ and*

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to \dots \to M_{2^n-1} \xrightarrow{f_{2^n-1}} M_{2^n}$$

be a sequence of morphisms in $mod\, A$ satisfying the conditions:

1. *Each module M_i is indecomposable, and $l(M_i) \leq k$, $\forall i \in [2^k]$,*
2. *Each morphism is nonzero and is not an isomorphism.*

Then $l(Im\, f_{2^n-1} \dots f_2 f_1) \leq k - n$.

Proof. We proceed by induction on n. Let $n = 1$. Suppose, for a contradiction, that $l(Im\, f_1) = k$. Then, since $Im\, f_1 \subseteq M_2$ and $l(M_2) \leq k$, we have $l(Im\, f_1) = k = l(M_2)$, which implies that f_1 is an epimorphism. Moreover, $k = l(M_2) \leq l(M_1) \leq k$, so $l(M_1) = k$ and $l(Ker\, f_1) = l(M_1) - l(Im\, f_1) = 0$ (cf. 1.2.9). Thus $Ker\, f_1 = 0$ and f_1 is an isomorphism, which contradicts the hypothesis. So $l(Im\, f_1) \leq k - 1$. Suppose now that the statement is valid for n, and let

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \dots \to M_{2^n-1} \xrightarrow{f_{2^n-1}} M_{2^n} \xrightarrow{f_{2^n}} M_{2^n+1} \xrightarrow{f_{2^n+1}} \dots \xrightarrow{f_{2^{n+1}-1}} M_{2^{n+1}}$$

be a sequence of nonzero nonisomorphisms between indecomposable modules of length $\leq k$. Write $f = f_{2^n-1} \dots f_2 f_1$, $h = f_{2^{n+1}-1} \dots f_{2^n+1}$ and $g = f_{2^n}$ to simplify the notation. Since the chains

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \dots \to M_{2^n-1} \xrightarrow{f_{2^n-1}} M_{2^n}$$

and

$$M_{2^n+1} \xrightarrow{f_{2^n+1}} \dots \xrightarrow{f_{2^{n+1}-1}} M_{2^{n+1}}$$

have length $2^n - 1$, we have by induction that $l(Im\, f) \leq k - n$ and $l(Im\, h) \leq k - n$. We have to show that $l(Im\, hgf) \leq k - n - 1$. If $l(Im\, h) < k - n$, then we are done because $l(Im\, hgf) \leq l(Im\, h)$. Also if $l(Im\, f) < k - n$, we have the required inequality since $Im\, hgf \cong Im\, f/(Im\, f \cap Ker\, hg)$ and so $l(Im\, hgf) = l(Im\, f) - l(Im\, f \cap Ker\, hg) \leq l(Im\, f)$. Suppose then that

$l(Im\, f) = l(Im\, h) = k - n > 0$, and assume, for a contradiction, that $l(Im\, hgf) > k - n - 1$. Hence $l(Im\, hgf) = k - n$ since $k - n - 1 < l(Im\, hgf) \leq l(Im\, h) = k - n$. From the equality $l(Im\, hgf) = l(Im\, f) - l(Im\, f \cap Ker\, hg)$ given before, it follows that $l(Im\, f \cap Kerhg) = 0$ and so $Im\, f \cap Ker\, hg = 0$. On the other hand, $Im\, hgf \subseteq Im\, hg \subseteq Im\, h$ and since $l(Im\, hgf) = l(Im\, h) = k - n$, this yields $l(Im\, hg) = k - n$. Therefore

$$l(Ker\, hg) = l(M_{2^n}) - l(Im\, hg) = l(M_{2^n}) - (b - n) = l(M_{2^n}) - l(Im\, f).$$

Consequently, $M_{2^n} = Im\, f \oplus Ker\, hg$. But M_{2^n} is indecomposable and $f \neq 0$, so $Ker\, hg = 0$, which implies that hg and hence g are monomorphisms. Analogously, $M_{2^n+1} = Im\, gf \oplus Ker\, h$. Because M_{2^n+1} is indecomposable and $gf \neq 0$, we have $M_{2^n+1} = Im\, gf$, that is, gf is an epimorphism. Therefore, g is an epimorphism. The contradiction was reached for g is assumed to be a nonisomorphism. □

The following corollary follows directly from 2.5.8.

Corollary 2.5.9. *Let $k \in \mathbb{N}$ and*

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to \ldots \to M_{2^k-1} \xrightarrow{f_{2^k-1}} M_{2^k}$$

be a sequence of nonzero nonisomorphisms in $mod\, A$, where each module M_i is indecomposable and has length $\leq k$. Then $f_{2^k-1} \ldots f_2 f_1 = 0$.

Theorem 2.5.10. *Suppose $\Gamma(mod\, A)$ admits a connected component $\mathcal{C}$ such that its modules have length bounded by a certain natural number k. Then $\mathcal{C}$ is finite and $\mathcal{C} = \Gamma(mod\, A)$. In particular A is representation-finite.*

Proof. Firstly we claim that if M, N are two indecomposable modules such that $[M] \in \mathcal{C}$ and $Hom_A(M, N) \neq 0$, then there exists a chain of irreducible morphisms from M to N of length $< 2^k - 1 = t$. Indeed, if such a chain didn't exist, there would exist a chain of irreducible morphisms

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \ldots \to M_{t-1} \xrightarrow{f_t} M_t$$

and a morphism $g : M_t \to N$ with $gf_t \ldots f_1 \neq 0$, by 2.5.7. But each module M_i is such that $[M_i] \in \mathcal{C}$, so $l(M_i) \leq k$. Hence, by 2.5.9, $f_t \ldots f_1 = 0$, contradicting the fact that $gf_t \ldots f_1 \neq 0$. This implies that $[N] \in \mathcal{C}$. Similarly, if $[N] \in \mathcal{C}$, then $[M] \in \mathcal{C}$.

Let now M be an indecomposable module whose isomorphism class $[M]$ lies in $\mathcal{C}$. Since every module admits a projective cover, there exists an indecomposable projective module P such that $Hom_A(P, M) \neq 0$. Hence $[P] \in \mathcal{C}$, according to the previous claim. Let N be an arbitrary A-module. There exists an indecomposable projective module P' such that $Hom_A(P', N) \neq 0$. Let $P \cong e_a A$ and $P' \cong e_b A$. Because the quiver Q_A is connected, there exists an unoriented path $a \to 1 \to 2 \to \cdots \to r \to b$ from a to b. Consider the indecomposable projective modules $P_1, \ldots, P_r$ associated do the vertices $1, \ldots, r$ of such path. Write $P_0 = P$ and $P_{r+1} = P'$. Since $Hom_A(P_i, P_{i+1}) \cong e_{i+1}Ae_i$, we have that $Hom_A(P_i, P_{i+1}) \neq 0$ or $Hom_A(P_{i+1}, P_i) \neq 0$, for $i = 0, \ldots, r$. Consequently, $[P'] \in \mathcal{C}$ because $[P] \in \mathcal{C}$, and so $[N]$ lies in $\mathcal{C}$ as well, which proves that $\Gamma(mod\, A) = \mathcal{C}$. It remains to prove that $\mathcal{C}$ is finite. As we have seen, for each indecomposable projective A-module P and each indecomposable A-module M such that $Hom_A(P, M) \neq 0$, there exists a chain of irreducible morphisms from P to M of length smaller than $t = 2^k - 1$. Because there are only finitely many nonisomorphic indecomposable projectives, there are only finitely many nonisomorphic indecomposable modules corresponding to points in $\mathcal{C}$. Hence $\mathcal{C}$ is finite. $\square$

Chapter 3

Cluster Algebras

The cluster algebras were introduced by Fomin and Zelevinsky in [16]. Fix $n, m \in \mathbb{N}$, $n \leq m$, and let $\mathbb{F} = \mathbb{Q}(x_1, \ldots, x_m)$ be the field of rational functions in the m variables $x_1, \ldots, x_m$ over $\mathbb{Q}$. Informally speaking, a cluster algebra $\mathcal{A}$ is a commutative ring contained in $\mathbb{F}$ generated by a (possibly infinite) set of generators. These generators are organised into distinguished subsets of cardinality n, known as *clusters*, together with some *coefficients*, and are obtained by an initial transcendence basis of $\mathbb{F}$, changing it according to certain rules to produce new transcendence basis. The process of mutating the transcendence basis are governed by matrices associated to the clusters.

In this chapter we present the variations of the definition of cluster algebras given in [16] and [17], and state some important results, namely the Laurent phenomenon, which says that all the clusters variables are a Laurent polynomial in the cluster variables of a given initial cluster. We focus our attention on the cluster algebras of finite type, the ones with finitely many distinct clusters variables. Fomin and Zelevinsky proved that the classification of the cluster algebras of finite type is completely parallel to the Cartan-Killing classification of root systems. Hence, there is a cluster algebra of finite type for each Cartan matrix of finite type. An important result (cf. 3.4.5) gives the correspondence between cluster variables and the almost positive roots of the root system associated to the finite cluster algebra in question. We will see too that the cluster algebras of type A have a nice description in terms of the diagonals and triangulations of a regular polygon.

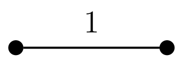
Most results presented in this chapter will not be proved as their proofs are very extensive and are not relevant for the purpose of this thesis, which is the study of [13]. For more details and the omitted proofs, we refer the reader to [16, 17, 18].

3.1 Definitions

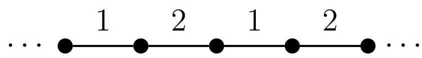
The first part of this section is devoted to the definition of cluster algebras given in [16], and the second part is devoted to the definition given in [17] and the connection between these two notions. The definition of cluster algebras requires the concept of exchange patterns, which involves (exchange) matrices and coefficients. There are different classes of cluster algebras, such as symmetrizable cluster algebras, normalized cluster algebras and cluster algebras of geometric type, according to the variations of those (exchange) matrices and coefficients. The paper [17] is concerned to the normalized cluster algebras, which is the class we will be interested in.

Let I be a set of cardinality n, write $I = \{1, \dots, n\}$ and let T_n be a graph, called *n-regular tree*, where each vertex has valency n and the edges incident in each vertex are labeled by the elements of I. We illustrate the examples for $n = 1, 2, 3$.

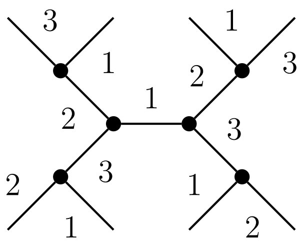
T_1



T_2



T_3



Let $\mathbb{P}$ be a torsion-free multiplicative group, which is called the *coefficient group*. For each vertex t of T_n we associate n commuting indeterminates $x_1(t), \dots, x_n(t)$, called *cluster variables*. The set $\underline{x}(t) = \{x_i(t) \mid i \in I\}$ of the n cluster variables associated to the vertex t is called a *cluster*.

For each $t \in T_n$ and each $j \in I$, we associate a monomial

$$M_j(t) = p_j(t) \prod_{i \in I} x_i^{b_i(t)},$$

where $b_i(t) \in \mathbb{Z}$ and $b_i(t) \geq 0$ for all $i \in I$, and $p_j(t) \in \mathbb{P}$. The clusters of two vertices t, t' joined by an edge labeled by j are related by the *exchange relations*:

$$\begin{aligned} x_i(t) &= x_i(t'), \text{if } i \neq j, \\ x_j(t)x_j(t') &= M_j(t) + M_j(t'). \end{aligned} \tag{3.1}$$

The monomials $M_j(t)$, $t \in T_n$, $j \in I$ must satisfy the following axioms:

1. $x_j(t) \nmid M_j(t)$, $\forall t \in T_n, j \in I$, i.e., $M_j(t)$ doesn't contain $x_j(t)$,

2. If $t \overset{j}{\text{———}} t'$, then $M_j(t)$ and $M_j(t')$ cannot both contain x_i, for any $i \in I$, i.e., if $x_i(t) \mid M_j(t)$ then $x_i(t') \nmid M_j(t')$,

3. If $t \overset{i}{\text{———}} t' \overset{j}{\text{———}} t''$, then $M_i(t)$ contains $x_j(t)$ if and only if $M_j(t')$ contains $x_i(t')$, i.e., $x_j(t) \mid M_i(t) \Leftrightarrow x_i(t') \mid M_j(t')$,

4. If $u \overset{i}{\text{———}} t \overset{j}{\text{———}} t' \overset{i}{\text{———}} u'$, then

   $$\frac{M_i(t')}{M_i(u')} = \frac{M_i(t)}{M_i(u)} \mid_{x_j \leftarrow M_0/x_j},$$

   where $M_0 = M_j(t) + M_j(t') \mid_{x_i=0}$, i.e., $M_i(t')/M_i(u')$ is obtained from $M_i(t)/M_i(u)$ by replacing x_j by M_0/x_j, where M_0 is obtained from $M_j(t) + M_j(t')$ by replacing x_i by 0.

In this case, we say that the family of monomials $\mathcal{M} = (M_j(t))_{j \in I, t \in T_n}$ is an *exchange pattern*.

Remark 3.1.1.

1. The substitution $x_j \leftarrow M_0/x_j$ in the last axiom is monomial. Indeed, the monomials $M_j(t), M_j(t')$ cannot both involve x_i and if one of them, say $M_j(t)$, contains x_i then x_j is replaced by $M_j(t')/x_j$. If $x_i \nmid M_j(t)$ and $x_i \nmid M_j(t')$ then, by axiom 3, $x_j \nmid M_i(t)$ and $x_j \nmid M_i(u)$. So the substitution of $x_j \leftarrow M_0/x_j$ in $M_i(t)/M_i(u)$ is irrelevant.

2. The axiom 4 is invariant under the flip $t \leftrightarrow t'$, $u \leftrightarrow u'$. For if we apply this axiom to the reverse sequence, we get:

$$\frac{M_i(t)}{M_i(u)} = \frac{M_i(t')}{M_i(u')} \mid_{x_j \leftarrow M_0/x_j},$$

where $M_0 = (M_j(t') + M_j(t)) \mid_{x_i=0}$. Making the substitution $x_j \leftarrow M_0/x_j$, we obtain:

$$\frac{M_i(t)}{M_i(u)} \mid_{x_j \leftarrow M_0/x_j} = \frac{M_i(t')}{M_i(u')} \mid_{x_j \leftarrow M_0/(M_0/x_j)} = \frac{M_i(t')}{M_i(u')}.$$

3. The information associated with one vertex of T_n can be propagated around the graph T_n, by axiom 4. In other words, fix a vertex t and suppose we are given its $2n$ associated monomials $M_j(t)$, with $j \in I$ and $M_j(t')$, with $j \in I$ and where t' is the vertex joined to t by an edge with label j. Let t' be a vertex adjacent to t, being the edge that links them labeled by j. The monomial $M_j(t')$ is given. Let $i \in I$ such that $i \neq j$, t'' be the vertex linked to t' by the edge i and t_0 be the vertex joined to t by the edge i. The axiom 4 says that the ratio $\frac{M_i(t')}{M_i(t'')}$ is obtained from $\frac{M_i(t)}{M_i(t_0)}$ using $M_j(t)$ and $M_j(t')$. Note that, by the exchange relations, $x_k(t') = x_k(t'')$ for $k \in I \setminus \{i\}$ and by axiom 1, $x_i(t') \nmid M_i(t')$ and $x_i(t'') \nmid M_i(t'')$. Thus by axiom 2, we have:

$$\frac{M_i(t')}{M_i(t'')} = \frac{p_i(t')}{p_i(t'')} \prod_{k \in I} x_k^{c_k(t')}(t'),$$

where

$$c_k(t') = \begin{cases} b_k(t') & \text{if } x_k(t') \mid M_i(t') \\ -b_k(t'') & \text{if } x_k(t'') \mid M_i(t'') \\ 0 & \text{if } k = i. \end{cases}$$

So the exponents of x_k are uniquely determined as well as $p_i(t')/p_i(t'')$.

The exchange matrices

Given an exchange pattern $\mathcal{M} = (M_j(t))_{j \in I, t \in T_n}$, we can associate a family of matrices over $\mathbb{Z}$ that encodes all the exponents in all of its monomials.

For each $t \in T_n$, let $B(t) = (b_{ij}(t))_{i,j \in I}$ be an $n \times n$ matrix over $\mathbb{Z}$ where the entries $b_{ij}(t) \in \mathbb{Z}$ are such that:

$$\frac{M_j(t)}{M_j(t')} = \frac{p_j(t)}{p_j(t')} \prod_{i \in I} x_i^{b_{ij}(t)}(t),$$

where t' is the vertex joined to t by the edge j.

As it was observed in 3.1.1 (3), we have

$$\begin{aligned} M_j(t) &= p_j(t) \prod_{i:\ b_{ij}>0} x_i^{b_{ij}(t)}, \\ M_j(t') &= p_j(t') \prod_{i:\ b_{ij}<0} x_i^{-b_{ij}(t)}. \end{aligned} \tag{3.2}$$

Note that each matrix $B(t) = (b_{ij}(t))$ satisfies these conditions:

1. $b_{ii}(t) = 0$, for all $i \in I$,
2. $b_{ij}(t) > 0$ if and only if $b_{ji}(t) < 0$.

The first one follows from the axiom 1 of the exchange patterns. To prove the second one, suppose $b_{ij} > 0$. Then $x_i(t) \mid M_i(t)$, which implies that $x_j(t') \mid M_j(t')$, by the third axiom of exchange patterns. Hence $b_{ji} < 0$. The converse is similar.

Definition 3.1.2. A square matrix $B = (b_{ij})$ over $\mathbb{Z}$ is said to be *sign-skew-symmetric* if $b_{ij} = b_{ji} = 0$ or $b_{ij}b_{ji} < 0$, for every i, j, i.e., if it satisfies the two conditions above.

We have just seen that the family of matrices $B(t)$, with $t \in T_n$, corresponding to an exchange pattern is a family of sign-skew-symmetric matrices.

Definition 3.1.3. Let $B = (b_{ij})$ and $B' = (b'_{ij})$ be square matrices of the same size over $\mathbb{Z}$. We say that B' is obtained from B by the *matrix mutation* in direction k, which will be denoted by $B' = \mu_k(B)$, if

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik}|\, b_{kj} + b_{ik}\, |b_{kj}|}{2} & \text{otherwise.} \end{cases}$$

Remark 3.1.4. The matrix mutation μ_k is an involution, i.e., $\mu_k^2 = 1$. Write $B' = \mu_k(B)$ and $B'' = \mu_k(\mu_k(B))$. Then we have

$$b''_{ij} = \begin{cases} -b'_{ij} & \text{if } i = k \text{ or } j = k, \\ b'_{ij} + \frac{|b'_{ik}| b'_{kj} + b'_{ik} |b'_{kj}|}{2} & \text{otherwise.} \end{cases}$$

If $i = k$ or $j = k$, $b'_{ij} = -b_{ij}$ and so $b''_{ij} = b_{ij}$. If not, we have

$$\begin{aligned} b''_{ij} &= b'_{ij} + \frac{|b'_{ik}| b'_{kj} + b'_{ik} |b'_{kj}|}{2} \\ &= b_{ij} + \frac{|b_{ik}| b_{kj} + b_{ik} |b_{kj}|}{2} - \frac{|b_{ik}| b_{kj} + b_{ik} |b_{kj}|}{2} \\ &= b_{ij}. \end{aligned}$$

Therefore, $B'' = B$, as required.

Definition 3.1.5. Two matrices are *mutation equivalent* if they can be obtained from each other by a sequence of matrix mutations followed by a simultaneous permutation of rows and columns.

We will now prove that for a pair of vertices t, t' connected by an edge k, the matrices $B = B(t)$ and $B' = B(t')$ are obtained from each other by a matrix mutation in direction k, that is, $B' = \mu_k(B)$.

We have
$$\frac{M_k(t')}{M_k(t)} = \frac{p_k(t')}{p_k(t)} \prod_{i\in I} x_i^{b'_{ik}(t')}$$
Therefore, $b'_{ik}(t') = -b_{ik}(t)$.

For $j \in I \setminus \{k\}$, let u (resp. u') be the vertex connected to t (resp. t') by the edge labeled by j.

We have
$$\frac{M_j(t')}{M_j(u')} = \frac{p_j(t')}{p_j(u')} \prod_{i\in I} x_i^{b'_{ij}} \tag{3.3}$$
$$\frac{M_j(t)}{M_j(u)} = \frac{p_j(t)}{p_j(u)} \prod_{i\in I} x_i^{b_{ij}}.$$

By axiom 4 of the exchange pattern,
$$\frac{M_j(t')}{M_j(u')} = \frac{M_j(t)}{M_j(u)} \;|_{x_k \leftarrow M_0/x_k}, \tag{3.4}$$
where
$$\begin{aligned} M_0 &= M_k(t) + M_k(t') \;|_{x_j=0} \\ &= \left(p_k(t) \prod_{i:\ b_{ik}>0} x_i^{b_{ik}} + p_k(t') \prod_{i:\ b_{ik}<0} x_i^{-b_{ik}} \right) |_{x_j=0} \;. \end{aligned}$$

Assume $b_{jk} \neq 0$. If $b_{jk} < 0$, then $M_0 = p_k(t) \prod_{i:\ b_{ik}>0} x_i^{b_{ik}}$. If $b_{jk} > 0$, then $M_0 = p_k(t') \prod_{i:\ b_{ik}<0} x_i^{-b_{ik}}$.

In either case we have
$$M_0 = p \prod_{i:\ b_{ik}b_{jk}<0} x_i^{|b_{ik}|},$$
where p is either $p_k(t)$ or $p_k(t')$.

Hence, from 3.4 and 3.3, making the substitution $x_k \leftarrow M_0/x_k$, we get:
$$\prod_{i\in I} x_i^{b'_{ij}} = \prod_{i\in I\setminus k} x_i^{b_{ij}} \cdot \frac{\prod_{i:\ b_{ik}b_{jk}<0} x_i^{|b_{ik}|_{kj}}}{x_k^{b_{kj}}}. \tag{3.5}$$

As $b_{kk} = 0$, we deduce from 3.5 that $b'_{kj} = -b_{kj}$.

Let $i \neq k$. Let us compare the exponents of x_i in both sides of 3.5. If $b_{ik}b_{jk} \geq 0$, then $b'_{ij} = b_{ij}$, as M_0 doesn't contain x_i. If $b_{ik}b_{jk} < 0$, then $b'_{ij} = b_{ij} + |b_{ik}|b_{kj}$.

Thus

$$b'_{ij} = \begin{cases} b_{ij} & \text{if } b_{ik}b_{jk} \geq 0 \\ b_{ij} + |b_{ik}|b_{kj} & \text{if } b_{ik}b_{jk} < 0. \end{cases}$$

Suppose $b_{ik}b_{jk} \geq 0$. Then $b_{ik}b_{kj} \leq 0$, as B is sign-skew-symmetric. Suppose $b_{ik} \leq 0$ and $b_{kj} \geq 0$. We have

$$\frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} = \frac{-b_{ik}b_{kj} + b_{ik}b_{kj}}{2} = 0.$$

The other case, when $b_{ik} \geq 0$ and $b_{kj} \leq 0$, is similar.

If $b_{ik}b_{jk} < 0$, then, because B is sign-skew-symmetric, b_{ik}, b_{kj} have the same sign. If they are both positive,

$$\frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} = b_{ik}b_{kj} = |b_{ik}|b_{kj},$$

and if they are both negative, we have

$$\frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} = -b_{ik}b_{kj} = |b_{ik}|b_{kj}.$$

Hence, $b'_{ij} = b_{ij} + \frac{|b_{ik}|\, b_{kj} + b_{ik}\, |b_{kj}|}{2}$.

We have just proved that

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik}|\, b_{kj} + b_{ik}\, |b_{kj}|}{2} & \text{otherwise,} \end{cases} \tag{3.6}$$

assuming that $b_{jk} \neq 0$. But this equation also holds when $b_{jk} = 0$. In this case, we have $b_{kj} = 0$ and $b_{kl} = 0$, for all $k \in I$, since neither $M_j(t)$ nor $M_j(u)$ contains x_k. So 3.6 reduces to the equation $b'_{ij} = b_{ij}$. On the other hand, the axiom 4 of the exchange pattern becomes

$$\frac{M_j(t')}{M_j(u')} = \frac{M_j(t)}{M_j(u)}.$$

And so we have indeed $b'_{ij} = b_{ij}$.

Conversely, given a family of $n \times n$ matrices $(B(t))_{t \in T_n}$ over $\mathbb{Z}$ such that each $B(t)$ is sign-skew-symmetric and $B(t') = \mu_k(B(t))$ if $t \overset{k}{\text{——}} t'$, it is clear that the family of monomials

$$M_j(t) := \prod_{i:\ b_{ij}(t) > 0} x_i^{b_{ij}(t)},$$

with $j \in I$ and $t \in T_n$, is an exchange pattern.

The following proposition summarizes what we have just proved.

Proposition 3.1.6. *A family of $n \times n$ integer matrices $(B(t))_{t \in T_n}$ corresponds to an exchange pattern if and only if the following conditions are satisfied:*

1. *$B(t)$ is sign-skew-symmetric for all $t \in T_n$,*
2. *If t, t' are two vertices of T_n joined by an edge labeled by k, then $B(t') = \mu_k(B(t))$.*

Given an initial vertex $t_0 \in T_n$ and a $n \times n$ integer matrix B, we get a family $(B(t))_{t \in T_n}$, with $B(t_0) = B$, uniquely determined by matrix mutations, as all matrix mutations are involutions. Thus we can get an exchange pattern from just one matrix B, provided B and the matrices obtained from it by a sequence of mutations are sign-skew-symmetric.

There is a large subclass of the class of sign-skew-symmetric matrices for which this is true, i.e., is closed for matrix mutations.

Definition 3.1.7. A square matrix $B = (b_{ij})$ over $\mathbb{Z}$ is said to be *skew-symmetrizable* if there exists a diagonal matrix $D = (d_i)$ with positive integer diagonal entries such that DB is skew-symmetric, i.e., $-(DB) = (DB)^t$. In other words, $d_i b_{ij} = -d_j b_{ji}$ for all i and j.

Proposition 3.1.8. *Let t_0 be an arbitrary vertex of T_n and B be a skew-symmetrizable matrix. There exists a unique family of matrices $(B(t))_{t \in T_n}$ associated with an exchange pattern on T_n and such that $B(t_0) = B$. Moreover, all the matrices $B(t)$ are skew-symmetrizable.*

Proof. It is clear that any skew-symmetrizable matrix is sign-skew-symmetric. Furthermore, if B is skew-symmetrizable so is $B' := \mu_k(B)$. Indeed let D be a diagonal matrix with positive integer entries d_i such that $d_i b_{ij} = -d_j b_{ji}$ for all i and j. For $i = k$ or $j = k$, we have $d_i b'_{ij} = d_i(-b_{ij}) = d_j b_{ji} = -d_j b'_{ji}$. For the other cases, we have

$$\begin{aligned}
d_i b'_{ij} &= d_i \left(b_{ij} + \frac{|b_{ik}| b_{kj} + b_{ik} |b_{kj}|}{2} \right) \\
&= d_i b_{ij} + \frac{d_i \frac{d_k}{d_i} |b_{ki}| b_{kj} - d_k b_{ki} |b_{kj}|}{2} \\
&= -d_j b_{ji} + \frac{|b_{ki} d_k b_{kj} - d_k \frac{d_j}{d_k} |b_{jk}| b_{ki}}{2} \\
&= -d_j \left(b_{ji} + \frac{|b_{ki}| b_{jk} + b_{ki} |b_{jk}|}{2} \right) \\
&= -d_j b'_{ji}.
\end{aligned}$$

Hence $\mu_k(B)$ is skew-symmetrizable. The proposition follows from 3.1.6. $\square$

The coefficients

Recall that given the $2n$ monomials of an exchange pattern corresponding to the edges emanating from a given vertex, axiom 4 of the exchange patterns only prescribes the ratios $p_i(t')/p_i(t'')$ of coefficients. We will now define a special type of exchange patterns for which all the coefficients are uniquely determined.

Definition 3.1.9. A *semifield* is a triple $(\mathbb{P}, \oplus, \cdot)$, where $\mathbb{P}$ is a set and $\cdot$ and $\oplus$ are two binary operations in $\mathbb{P}$ such that $(\mathbb{P}, \cdot)$ is an abelian multiplicative group and $\oplus$, which is called auxiliary addition, is commutative, associative and distributive with respect to the multiplication $\cdot$ in $\mathbb{P}$.

Note that $(\mathbb{P}, \cdot)$ is torsion free. Indeed, if $p \in \mathbb{P}$ is such that $p^m = 1$ for some $m \geq 2$, then

$$p = \frac{p^m \oplus p^{m-1} \oplus \cdots \oplus p}{p^{m-1} \oplus p^{m-2} \oplus \cdots \oplus 1} = \frac{p^{m-1} \oplus \cdots \oplus 1}{p^{m-1} \oplus \cdots \oplus 1} = 1.$$

Definition 3.1.10. An exchange pattern is said to be *normalized* if $\mathbb{P}$ is a semifield, and $p_k(t) \oplus p_k(t') = 1$ for any edge $t \overset{k}{\text{———}} t'$.

All the coefficients of a normalized exchange pattern can be obtained from the $2n$ coefficients $p_i(t)$ (with $i \in I$) and $p_j(t_j)$ (with $j \in [n]$ and where t_j is the vertex linked to t by the edge j) associated to a given vertex t. The following lemma will be useful to prove this.

Lemma 3.1.11. *Let $(B(t))_{t \in T_n}$ be a family of matrices satisfying the conditions in 3.1.6 and $(p_j(t))$, with $j \in I, t \in T_n$, a family of elements of a coefficient group $\mathbb{P}$. Then the monomials $M_j(t)$, with $j \in I, t \in T_n$ given by*

$$M_j(t) = p_j(t) \prod_{i: b_{ij}(t) > 0} x_i^{b_{ij}(t)},$$

form an exchange pattern if and only if the condition

$$p_i(u)p_i(t')p_j(t')^{max(b_{ji}(t'),0)} = p_i(t)p_i(u')p_j(t)^{max(b_{ji}(t),0)}. \tag{3.7}$$

holds for vertices u, t, t', u' and edges i, j such that $u \overset{i}{\text{———}} t \overset{j}{\text{———}} t' \overset{i}{\text{———}} u'$.

Note that $b_{ji}(t_2) = -b_{ji}(t_3)$ by the definition of matrix mutation, so at most one of the $p_j(t_2)$ and $p_j(t_3)$ enters 3.7.

Proof. The first three axioms of exchange patterns are valid by 3.1.6. Hence the family $(M_j(t))$ is an exchange pattern if and only if it satisfies the axiom 4, that is, whenever $u \overset{i}{\text{———}} t \overset{j}{\text{———}} t' \overset{i}{\text{———}} u'$, we have

$$\frac{M_i(t')}{M_i(u')} = \frac{M_i(t)}{M_i(u)} \mid_{x_j \leftarrow M_0/x_j}, \tag{3.8}$$

where $M_0 = (M_j(t) + M_j(t')) \mid_{x_i=0}$. We have

$$M_0 = \begin{cases} p_j(t) \prod_{k:b_{kj}(t)>0} x_k^{b_{kj}(t)} & \text{if } b_{ij}(t') > 0, \\ p_j(t') \prod_{k:b_{kj}(t')>0} x_k^{b_{kj}(t')} & \text{if } b_{ij}(t') < 0, \\ p_j(t) \prod_{k:b_{kj}(t)>0} x_k^{b_{kj}(t)} + p_j(t') \prod_{k:b_{kj}(t')>0} x_k^{b_{kj}(t')} & \text{if } b_{ij}(t') = 0. \end{cases}$$

In the case when $b_{ij}(t') = 0$, then $b_{ji}(t) = 0$, because $b_{ij}(t') = b_{ji}(t')$ as $B(t')$ is sign-skew-symmetric, and $b_{ji}(t) = -b_{ji}(t')$ as $B(t) = \mu_j(B(t'))$. Thus, in terms of coefficients, we get

$$\frac{p_i(t')}{p_i(u')} = \frac{p_i(t)}{p_i(u)}$$

from 3.8. If $b_{ij}(t') > 0$, then from 3.8 we have:

$$\frac{p_i(t')}{p_i(u')} = \frac{p_i(t)}{p_i(u)} p_j(t)^{b_{ji}(t)}.$$

Finally, if $b_{ij}(t') < 0$, we get:

$$\frac{p_i(t')}{p_i(u')} = \frac{p_i(t)}{p_i(u)} p_j(t')^{b_{ji}(t)}.$$

In all cases we have the relation

$$\frac{p_i(t')}{p_i(u')} = \frac{p_i(t)}{p_i(u)} p_j(t)^{max(b_{ji}(t),0)} p_j(t')^{min(b_{ji}(t'),0)},$$

which is 3.7. □

Proposition 3.1.12. *Every family of matrices $(B(t))_{t \in T_n}$ satisfying the conditions in 3.1.6 together with $2n$ elements:*

$$p_i(t_0), \text{ with } i \in I, \text{ and}$$

$$p_j(t) \text{ for every edge } t_0 \overset{j}{\text{———}} t,$$

of a semifield $\mathbb{P}$ associated to a given vertex T_0 and such that the normalization condition is satisfied, i.e., $p_j(t) \oplus p_j(t_0) = 1$, determines a unique normalized exchange pattern.

Proof. Let $t \overset{j}{\text{———}} t'$ be an edge and $p_j(t), p_j(t')$ be its corresponding coefficients in a normalized exchange pattern. The ratio

$$u_j(t) = \frac{p_j(t)}{p_j(t')}$$

determines the coefficients since we have

$$p_j(t) = \frac{u_j(t)}{1 \oplus u_j(t)},\ p_j(t') = \frac{1}{1 \oplus u_j(t)}.$$

Obviously we have

$$u_j(t)u_j(t') = 1. \tag{3.9}$$

We are given the ratios $u_j(t_0) = p_j(t_0)/p_j(t)$, for every edge $t_0 \overset{j}{\text{———}} t$. All the other ratios can be obtained from these via the relations:

$$u_i(t') = u_i(t)u_j(t)^{max(b_{ji}(t),0)}(1 \oplus u_j(t))^{-b_{ji}(t)} \tag{3.10}$$

for any edge $t \overset{j}{\text{———}} t'$ and $i \neq j$. The coefficients determined by these relations give rise to a unique normalized exchange pattern because 3.10 is the same as 3.7:

$$u_i(t') = u_i(t)u_j(t)^{max(b_{ji}(t),0)}(1 \oplus u_j(t))^{-b_{ji}(t)} \Leftrightarrow$$

$$\frac{p_i(t')}{p_i(t_1)} = \frac{p_i(t)}{p_i(t_0)} \left(\frac{p_j(t)}{p_j(t')}\right)^{max(b_{ji}(t),0)} p_j(t')^{b_{ji}(t)} \Leftrightarrow$$

$$p_i(t')p_i(t_0)p_j(t')^{max(b_{ji}(t),0)-b_{ji}(t)} = p_i(t_1)p_i(t)p_j(t)^{max(b_{ji}(t),0)} \Leftrightarrow$$

$$p_i(t')p_i(t_0)p_j(t')^{max(b_{ji}(t'),0)} = p_i(t_1)p_i(t)p_j(t)^{max(b_{ji}(t),0)},$$

where t_0 (resp. t_1) is the vertex joined to t (resp. t') by the edge i. □

An important example of a semifield is the *tropical semifield*, denoted by $Trop\,(p_j : j \in J)$, where $\mathbb{P}$ is a free abelian multiplicative group, with a finite set of generators p_j $(j \in J)$, and with auxiliary addition $\oplus$ given by

$$\prod_j p_j^{a_j} \oplus \prod_j p_j^{b_j} = \prod_j p_j^{min\,(a_j,b_j)}.$$

Definition 3.1.13. A normalized exchange pattern is of *geometric type* if $\mathbb{P} = Trop(p_j : j \in J)$, and each coefficient $p_j(t)$ is a monomial in the generators p_i with all exponents nonnegative.

Remark 3.1.14. For an exchange pattern of geometric type, the normalization condition $p_j(t) \oplus p_j(t') = 1$ for each edge $t \overset{j}{\text{———}} t'$, means that the monomials $p_j(t)$ and $p_j(t')$ have no variable in common.

The following proposition gives a characterization of the coefficients associated to an exchange pattern of geometric type.

Proposition 3.1.15. *Let $\mathbb{P} = Trop(p_i : i \in J)$, $(B(t))_{t \in T_n}$ be a family of matrices satisfying the conditions in 3.1.6, and $(p_j(t) : j \in I, t \in T_n)$ be a family of elements of $\mathbb{P}$. Then $\mathcal{M} = (M_j(t) : j \in I, t \in T_n)$ where each monomial is given by*

$$M_j(t) = p_j(t) \prod_{i : b_{ij}(t) > 0} x_i^{b_{ij}(t)}$$

is an exchange pattern if and only if , for each $j \in I$ and $t \in T_n$, there exists a family of integers $(c_{ij}(t) : t \in T_n, i \in J, j \in I)$ such that, for every edge $t \overset{k}{\text{———}} t'$,

$$c_{ij}(t') = \begin{cases} -c_{ij}(t) & \text{if } j = k, \\ c_{ij}(t) + \frac{|c_{ik}(t)| b_{kj}(t) + c_{ik}(t) |b_{kj}(t)|}{2} & \text{otherwise,} \end{cases} \tag{3.11}$$

and for which

$$p_j(t) = \prod_{i \in J} p_i^{max(c_{ij}(t), 0)}.$$

Proof. For each edge $t \overset{k}{\text{———}} t'$ consider $u_j(t) = p_j(t)/p_j(t')$. Let $C(t)$ be the matrix whose entries are the exponents of p_i in $u_j(t)$, i.e., such that

$$u_j(t) = \prod_{i \in J} p_i^{c_{ij}(t)}.$$

Then we have

$$\begin{aligned} p_j(t) = \frac{u_j(t)}{1 \oplus u_j(t)} &= \prod_i p_i^{c_ij(t) - min(0, c_{ij}(t))} \\ &= \prod_i p_i^{max(c_{ij}(t), 0)}. \end{aligned}$$

Let $j \in T_n \setminus \{k\}$. By 3.10, we have

$$\begin{gathered} u_j(t') = u_j(t) u_k(t)^{max(b_{kj}(t), 0)} (1 \oplus u_k(t))^{-b_{kj}(t)} \Leftrightarrow \\ \prod_{i \in J} p_i^{c_ij(t')} = \prod_i p_i^{c_{ij}(t)} \prod_i p_i^{c_{ik}(t) max(b_{kj}(t), 0)} \prod_i p_i^{-b_{kj}(t) min(0, c_{ik}(t))}. \end{gathered}$$

Picking up the exponent of p_i we get

$$c_{ij}(t') = c_{ij}(t) + c_{ik}(t)max(b_{kj}(t), 0) - b_{kj}(t)min(0, c_{ik}(t))$$

which is equal to 3.11, as required. $\square$

Remark 3.1.16. For each $t \in T_n$, let $\tilde{B}(t) = (b_{ij})_{i\in I\sqcup J, j\in I}$ be a rectangular integer matrix where $B(t) = (b_{ij})_{i,j\in I}$, which is called the *principal part* of $\tilde{B}(t)$, satisfies the conditions in 3.1.6, and $(b_{ij})_{i\in J, j\in I} = C(t)$, where $C(t)$ is as in 3.1.15. Then we conclude by 3.11 that the matrices $\tilde{B}(t)$ for $t \in T_n$ are related to each other by the matrix mutation rule applied to any $i \in I \sqcup J$ and $j \in I$, i.e., $\tilde{B}(t') = \mu_k(\tilde{B}(t))$, for every edge $t \overset{k}{\text{——}} t'$ in T_n. In the case when J is empty, that is, all the coefficients $p_j(t)$ are equal to 1, we have that $\tilde{B}(t) = B(t)$, for all $t \in T_n$.

Corollary 3.1.17. *Let $\tilde{B}$ be an integer matrix whose principal part is skew-symmetrizable. There exists a unique exchange pattern $\mathcal{M} = \mathcal{M}(\tilde{B})$ of geometric type such that $\tilde{B}(t_0) = \tilde{B}$ at a given vertex $t_0 \in T_n$.*

Proof. It follows directly from 3.1.8 and 3.1.15. $\square$

Cluster Algebras

Now we are able to give the definition of cluster algebras.

Let $\mathbb{ZP}$ be the group ring of the coefficient group $\mathbb{P}$ with integer coefficients. Note that $\mathbb{ZP}$ has no zero divisors since $\mathbb{P}$ is torsion-free. For each $t \in T_n$, let $\mathbb{F}(t)$ be the field of rational functions in the cluster variables $x_i(t)$, $i \in I$, with coefficients in $\mathbb{ZP}$. For every edge $t \overset{k}{\text{——}} t'$, consider the $\mathbb{ZP}$-linear field morphism $R_{tt'} : \mathbb{F}(t') \to \mathbb{F}(t)$ defined by

$$\begin{aligned} R_{tt'}(x_i(t')) &= x_i(t) \text{ for } i \neq k, \\ R_{tt'}(x_k(t')) &= \frac{M_k(t) + M_k(t')}{x_k(t)}. \end{aligned}$$

Note that $M_k(t') \in \mathbb{F}(t)$ as $x_k(t') \nmid M_k(t')$ and $x_i(t') = x_i(t)$, for $i \neq k$. Hence $R_{tt'}(x_k(t'))$ belongs to $\mathbb{F}(t)$. It follows from axiom 2 of exchange patterns that $R_{tt'}R_{t't} = id_{\mathbb{F}(t)}$ and $R_{t't}R_{tt'} = id_{\mathbb{F}(t')}$, so $R_{tt'}$ is an isomorphism. Thus we can view all the fields $\mathbb{F}(t)$ as a single field $\mathbb{F}$ that contains all the elements $x_i(t)$, for all $t \in T_n$ and $i \in I$, that satisfy the exchange relations.

Definition 3.1.18.

1. Let $\mathbb{A}$ be a subring with unit in $\mathbb{ZP}$ containing all coefficients $p_i(t)$ for $i \in I$ and $t \in T_n$. The *cluster algebra* $\mathcal{A} = \mathcal{A}_{\mathbb{A}}(\mathcal{M})$ of rank n over $\mathbb{A}$ associated with an exchange pattern $\mathcal{M}$ is the $\mathbb{A}$-subalgebra with unit in $\mathbb{F}$ generated by the union of all clusters $\underline{x}(t)$, for $t \in T_n$.

2. If all the matrices (equivalently, one of them) of the family $(B(t))_{t\in T_n}$ corresponding to the exchange pattern $\mathcal{M}$ is skew-symmetrizable, then $\mathcal{A}(\mathcal{M})$ is said to be *skew-symmetrizable*.

3. The cluster algebra corresponding to a normalized exchange pattern is called a *normalized cluster algebra*.

4. If the exchange pattern is of geometric type, take $\mathbb{A}$ to be the polynomial ring $\mathbb{Z}[p_i : i \in J]$ instead. Then the corresponding cluster algebra, which can be denoted by $\mathcal{A}(\tilde{B})$ in the sense of 3.1.17, is the subring of $\mathbb{F}$ generated by the clusters variables $x_j(t)$ for all $j \in I$ and $t \in T_n$ together with the generators p_i $(i \in J)$ of $\mathbb{P}$.

The paper [17] defines the normalized cluster algebras in a different way, instead of exchange patterns they take seeds. We will now exhibit this anew definition and see the connection with the first version.

Throughout, $\mathbb{P}$ is a semifield and $\mathbb{F}$ is a field isomorphic to the field of rational functions in n independent variables with coefficients in $\mathbb{ZP}$.

Definition 3.1.19. A *seed* in $\mathbb{F}$ is a triple $\Sigma = (\underline{x}, \underline{p}, B)$, where

- $\underline{x} = \{x_1, \ldots, x_n\}$ is a transcendence basis of $\mathbb{F}$ over the field of fractions of $\mathbb{ZP}$.
- $\underline{p} = (p_x^{\pm})_{x\in\underline{x}}$ is a $2n$-tuple of elements of $\mathbb{P}$ that satisfy the normalization condition $p_x^+ \oplus p_x^- = 1$ for all $x \in \underline{x}$.
- $B = (b_{xy})_{x,y\in\underline{x}}$ is a sign-skew-symmetric matrix, with rows and columns indexed by $\underline{x}$.

The matrix B is called the *exchange matrix* of the seed Σ. Note that B is only determined up to a simultaneous permutation of rows and columns.

We now define the mutation of a seed in direction $z \in \underline{x}$.

Definition 3.1.20. Let $\Sigma = (\underline{x}, \underline{p}, B)$ be a seed in $\mathcal{F}$ and let $z \in \underline{x}$.

- Define $z' \in \mathbb{F}$ via the *exchange relation*:

$$zz' = p_z^+ \prod_{\substack{x\in\underline{x}\\ b_{xz}>0}} x^{b_{xz}} + p_z^- \prod_{\substack{x\in\underline{x}\\ b_{xz}<0}} x^{-b_{xz}}. \tag{3.12}$$

  Set $\underline{x}' = (\underline{x} \cup \{z'\}) \setminus \{z\}$.

- Let $\underline{p}' = ({p'_x}^{\pm})_{x\in\underline{x}'}$ be the $2n$-tuple of elements of $\mathbb{P}$ determined by the normalization conditions ${p'_x}^{+} \oplus {p'_x}^{-} = 1$ together with

$$
{p'_x}^{+}/{p'_x}^{-} = \begin{cases} p_z^-/p_z^+ & \text{if } x = z'; \\ (p_z^+)^{b_{zx}} p_x^+/p_x^- & \text{if } b_{zx} \geq 0; \\ (p_z^-)^{b_{zx}} p_x^+/p_x^- & \text{if } b_{zx} \leq 0. \end{cases} \tag{3.13}
$$

- Let B' be the $n\times n$ matrix obtained from B via the matrix mutation in direction z, indexed by $\underline{x}'$, relabeling one row and one column by replacing z by z'.

If the triple $\Sigma' = (\underline{x}', {p'_x}^{\pm}, B')$ is again a seed in $\mathcal{F}$, then we say that Σ admits a *mutation* in the direction z that results in Σ'. We denote also by $\mu_z(\Sigma)$ the triple Σ'.

Remark 3.1.21.

1. As it was observed in the proof of 3.1.12, the normalization condition together with 3.13 uniquely determines the elements ${p'_x}^{\pm}$, as we have ${p'_x}^{+} = u/1\oplus u$ and ${p'_x}^{-} = 1/1\oplus u$, where $u = {p'_x}^{+}/{p'_x}^{-}$ is given by 3.13. In particular, for $x = z'$, ${p'_{z'}}^{+} = p_z^-$ and ${p'_{z'}}^{-} = p_z^+$.

2. It is easy to check that the set $\underline{x}'$ defined above is a transcendence basis over the field of fractions of $\mathbb{ZP}$. So the triple Σ' is a seed if and only if B' is sign-skew symmetric.

3. The mutation of seeds is involutive, i.e., $\mu_{z'}(\mu_z(\Sigma)) = \Sigma$. Indeed, it was already seen that the matrix mutation is an involution (cf. 3.1.4). Let $\underline{x}' = (\underline{x}\setminus\{z\})\cap\{z'\}$ be the cluster of $\mu_z(\Sigma)$. If we apply the mutation in direction z', the new cluster will be $\underline{x}'' = (\underline{x}'\setminus\{z'\})\cap\{z''\}$, where

$$
z'' = \frac{p_{z'}^+ \prod_{\substack{x\in\underline{x}' \\ b_{xz'}>0}} x^{b'_{xz'}} + p_{z'}^- \prod_{\substack{x\in\underline{x}' \\ b_{xz'}<0}} x^{-b'_{xz'}}}{z'}.
$$

But $p_{z'}^{\pm} = p_z^{\mp}$ and $b'_{xz'} = -b_{xz}$, and so

$$
p_{z'}^+ \prod_{\substack{x\in\underline{x}' \\ b_{xz'}>0}} x^{b'_{xz'}} + p_{z'}^- \prod_{\substack{x\in\underline{x}' \\ b_{xz'}<0}} x^{-b'_{xz'}} = p_z^- \prod_{\substack{x\in\underline{x}' \\ b_{xz}<0}} x^{-b_{xz}} + p_z^+ \prod_{\substack{x\in\underline{x}' \\ b_{xz}>0}} x^{b_{xz}}.
$$

Therefore, $z'' = z$, and consequently $\underline{x}'' = \underline{x}$. Finally, the elements $(p''^{\pm}_x)_{x\in\underline{x}''}$ are given by the normalization conditions together with

$$
\frac{p''^{+}_x}{p''^{-}_x} = \begin{cases} p'^{-}_{z'}/p'^{+}_{z'} & \text{if } x = z'' = z, \\ \left(p'^{+}_{z'}\right)^{b'_{z'x}} p'^{+}_x/p'^{-}_x & \text{if } b'_{z'x} \geq 0, \\ \left(p'^{-}_{z'}\right)^{b'_{z'x}} p'^{+}_x/p'^{-}_x & \text{if } b'_{z'x} \leq 0. \end{cases}
$$

Thus we have that $p''^{\pm}_z = p'^{\mp}_{z'} = p^{\pm}_z$, for x such that $b'_{z'x} \geq 0$, $b_{zx} \leq 0$, and so

$$p''^{+}_x/p''^{-}_x = \left(p^{-}_z\right)^{-b_{zx}} \left(p^{-}_z\right)^{b_{zx}} p^{+}_x/p^{-}_x,$$

which implies that $p''^{\pm}_x = p^{\pm}_x$, and analogously $p''^{\pm}_x = p^{\pm}_x$ for x such that $b'_{z'x} \leq 0$. Therefore $\underline{p}'' = \underline{p}$.

Let $\mathcal{S}$ be a set of seeds in $\mathbb{F}$ with the following properties:

- every seed $\Sigma \in \mathcal{S}$ admits mutations in all n conceivable directions;
- for each seed $\Sigma = (\underline{x}, \underline{p}, B)$ in $\mathcal{S}$ and for each $z \in \underline{x}$, $\mu_z(\Sigma)$ belongs to $\mathcal{S}$ as well;
- any two seeds in $\mathcal{S}$ are obtained from each other by a sequence of mutations.

Definition 3.1.22.

1. The sets $\underline{x}$, for $\Sigma = (\underline{x}, \underline{p}, B) \in \mathcal{S}$, are called *clusters* and their elements are known as the *cluster variables*.

2. Denote by $\mathcal{X}$ the set of all cluster variables and by $\mathcal{P}$ the set of all elements $p^{\pm}_x \in \underline{p}$ and let $\mathbb{Z}[\mathcal{P}]$ be the subring of $\mathbb{F}$ generated by $\mathcal{P}$. The normalized *cluster algebra* $\mathcal{A} = \mathcal{A}(\mathcal{S})$ is the $\mathbb{Z}[\mathcal{P}]$-subalgebra of $\mathbb{F}$ generated by $\mathcal{X}$.

3. The *exchange graph* of $\mathcal{A}(S)$ is the graph whose vertices are labeled by the clusters of the cluster algebra, and whose edges correspond to mutations.

A $n \times n$ integer matrix B such that any matrix obtained from B via a sequence of mutations is sign-skew-symmetric and a set $\underline{p}$ of $2n$ elements of $\mathbb{P}$ that satisfy the normalization condition uniquely determines a cluster algebra. This cluster algebra is denoted by $\mathcal{A}(B, \underline{p})$.

We will now see the connection with the first version. Let $\Sigma = (\underline{x} = \{x_1, \dots, x_n\}, \underline{p}, B)$ be a seed. Let $x_1, \dots, x_n$ be the cluster variables associated to a given vertex t (in the notation given before, $x_i = x_i(t)$, for $i \in I = [n]$) and $B = B(t)$. Denote by $b_ij(t)$ the entry $b_{x_i x_j}(t)$ of B, for short. The $2n$-elements $p^{\pm}_x$, with $x \in \underline{x}$, are the the elements of $\mathbb{P}$ associated to vertex t, i.e., for each $x_i \in \underline{x}$, $p_i(t) = p^{+}_{x_i}$ and $p_i(t_i) = p^{-}_{x_i}$, where t_i is the vertex linked to t by the edge i. Then we can associate the seed Σ to the $2n$ monomials corresponding to t (cf. 3.2):

$$M_j(t) = p_j(t) \prod_{k: b_{kj}(t) > 0} x_k^{b_{kj}(t)},$$

$$M_j(t_j) = p_j(t_j) \prod_{k:b_{kj}(t)<0} x_k^{-b_{kj}(t)},$$

where, for each $j \in I$, t_j is the vertex connected to t by the edge j.

The mutation of Σ in direction of $z = x_j = x_j(t)$ propagates the information to the vertex t_j. Note that the right-hand side of 3.12 is $M_j(t) + M_j(t_j)$. Hence the exchange relation 3.1 is the same as 3.12, where $z' = x_j(t_j)$. So the new cluster $\underline{x}'$ is the cluster $\{x_1(t) = x_1(t_j), \dots, x_j(t_j), \dots, x_n(t) = x_n(t_j)\}$ associated to vertex t_j. Furthermore, the right-hand side of 3.13, for $x = x_i$, is equal to

$$p_i(t)/p_i(t_i)p_j(t)^{max(b_{ji}(t),0)}p_j(t_j)^{min(b_{ji}(t),0)},$$

which is a reformulation of 3.9 and 3.10. So $\underline{p}$ is composed of the elements $p_i(t_j), p_i(t_{ji})$, for $i \in I$, where t_{ji} is the vertex linked to t_j by the edge i.

We may conclude that a set of seeds $\mathcal{S}$ satisfying the three properties given above is related to a normalized exchange pattern, by 3.1.6 and 3.1.12. Consequently, the definition 3.1.22 is the same as 3.1.18 (3), for the particular case when $\mathbb{A}$ is the subring with unit in $\mathbb{ZP}$ generated by the elements $p_i(t)$, with $i \in I, t \in T_n$.

In the sequel we only consider normalized cluster algebras and use the definition of [17], since most results of this chapter comes from this paper.

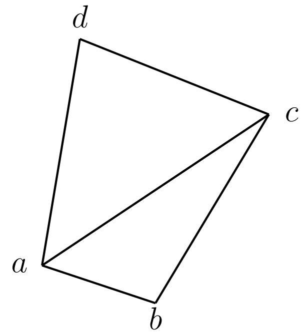
We end this section with a remark that will be useful to the proof of 4.4.4.

Remark 3.1.23. Let $\mathcal{A}(\mathcal{S})$ be a cluster algebra of geometric type, that is, $\mathbb{P} = Trop(p_j : j \in J)$ and each element of $\mathcal{P}$ is a monomial in the generators p_j with all exponents nonnegative. For each seed $(\underline{x}, \underline{p}, B)$, let $\tilde{B}$ be the $(n + |J|) \times n$ matrix defined in 3.1.16. It is clear that the exchange relations 3.12 are not changed if the sign of every element of $\tilde{B}$ is changed.

3.2 Examples - Triangulations and the Grassmannian

Let P_{n+3} be a regular polygon with $n + 3$ vertices. Label the vertices by letters $a, b, c \dots$. We use the notation ab to refer to the diagonal or side of P_{n+3} that links the vertices a and b. Fix a triangulation T in P_{n+3}, i.e., a set of diagonals that don't cross each other and divide the polygon into triangles. Let $\mathbb{P} = Trop(p_{ab} : ab$ is a border edge of $P_{n+3})$ and $\mathbb{A} = \mathbb{Z}$ $[p_{ab} : ab$ is a border edge of $P_{n+3}]$. Associate a variable x_{ab} for each diagonal of P_{n+3}, and use the convention $x_{ab} = 1$ for each side ab.

Each diagonal ac in T belongs to two triangles of the triangulation, which together form a quadrilateral. Let b and d be the other two vertices of the quadrilateral, as the following picture shows.



Define the coefficients by

$$p_{ac}^{+} = q_{ab}q_{cd},\ p_{ac}^{-} = q_{ad}q_{bc},$$

where

$$q_{ab} = \begin{cases} 1 & \text{if } ab \text{ is a diagonal,} \\ p_{ab} & \text{if } ab \text{ is a side.} \end{cases}$$

Note that p_{ac}^{+} and p_{ac}^{-} are coprime, so they satisfy the normalization condition.

Define a $(2n+3)\times n$ matrix $\tilde{B}$, called *edge-adjacency matrix* of T, indexed by the sides of the polygon together with the diagonals of T, such that the first n lines and the n columns (that form the principal part B of $\tilde{B}$) are indexed by the diagonals of T with the same order and the rest of the lines are indexed by the sides, as follows:

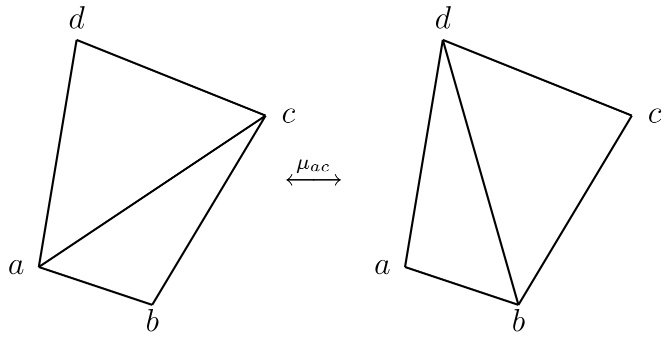
$$b_{ab,cd} = \begin{cases} 1 & \text{if } ab \text{ and } cd \text{ are two sides of a triangle of } T \text{ so that } ab \text{ rotated} \\ & \text{anticlockwise about their common vertex gives } cd, \\ -1 & \text{if } ab \text{ and } cd \text{ are two sides of a triangle of } T \text{ so that } ab \text{ rotated} \\ & \text{clockwise about their common vertex gives } cd, \\ 0 & \text{otherwise} \end{cases}$$

The principal part of $\tilde{B}$ is clearly skew-symmetrizable, since $b_{ab,cd} = -b_{cd,ab}$.

Observe that if ab is a side, and cd is a diagonal, we have

$$b_{ab,cd} = \begin{cases} 1 & \text{if } p_{ab} \text{ is a factor of } p_{cd}^{+}, \\ -1 & \text{if } p_{ab} \text{ is a factor of } p_{cd}^{-}, \\ 0 & \text{otherwise.} \end{cases}$$

If we remove a diagonal ac of the triangulation T, we leave a quadrilateral $[abcd]$ whose other diagonal is bd. Replacing ac with bd we obtain a new triangulation T'.



Denote by $\tilde{B}'$ the edge-adjacency matrix corresponding to T'. The matrix $\tilde{B}'$ is obtained from $\tilde{B}$ by matrix mutation in direction ac. Indeed, according to the definition of the edge-adjacency matrices, we have: $b_{ac,ad} = 1 = b_{ac,bc}$, $b_{ac,cd} = -1 = b_{ac,ab}$, $b_{ad,ab} = 0 = b_{cd,bc}$, $b_{ad,cd} = 1$ and $b_{ab,bc} = -1$. Similarly, we have $b'_{bd,ad} = -1 = b'_{bd,bc}$, $b'_{bd,cd} = 1 = b'_{bd,ab}$, $b'_{ad,ab} = -1$, $b'_{cd,bc} = 1$ and $b'_{ad,cd} = 0 = b'_{ab,bc}$. As we can see, the entries indexed by ac change the sign and it is easy to check that the rest of these entries satisfy the rule for matrix mutation in direction ac. The other entries $b_{xy,zw}$ of B remain unchanged, according to the definition of edge-adjacency matrices, since the rest of the triangles of T remains the same. And in this case, we have $b_{ac,zw} = 0$ or $b_{xy,ac} = 0$, and so by the rule of matrix mutation $b'_{xy,zw} = b_{xy,zw}$.

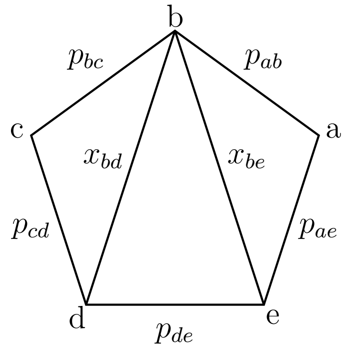
The exchange relations have the form:

$$x_{ac}x_{bd} = p^{+}_{ac}x_{ab}x_{cd} + p^{-}_{ac}x_{ad}x_{bc}.$$

The mutation of seeds correspond to combining these exchange relations with the matrix mutations associated with diagonal flips.

The set of all cluster variables is $\mathcal{X} = \{x_{ab} \mid ab \text{ is a diagonal of } P_{n+3}\}$. The cluster algebra $\mathcal{A}$ is generated by the diagonals and sides of P_{n+3}.

Let us illustrate this example for $n = 2$. We begin with the following triangulation of the pentagon.



So the initial seed is

$$\left(\{x_{be}, x_{bd}\}, \{p^+_{x_{be}}, p^-_{x_{be}}, p^+_{x_{bd}}, p^-_{x_{bd}}\}, \begin{array}{cc} & \begin{array}{cc} \text{be} & \text{bd} \end{array} \\ \begin{array}{c} be \\ bd \end{array} & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{array} \right) =$$

$$\left(\{x_{be}, x_{bd}\}, \{p_{ae}, p_{ab}p_{de}, p_{de}p_{bc}, p_{cd}\}, \begin{array}{cc} & \begin{array}{cc} \text{be} & \text{bd} \end{array} \\ \begin{array}{c} be \\ bd \end{array} & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{array} \right),$$

and the subsequent seeds are

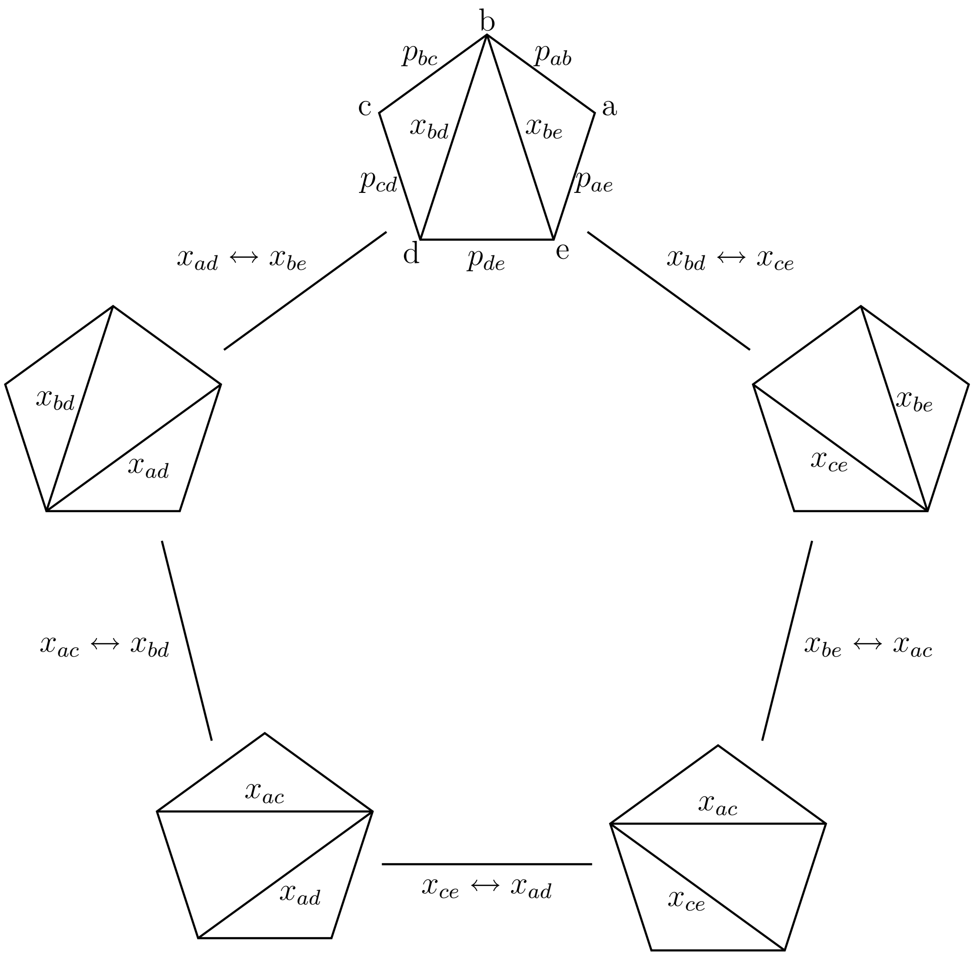
$$\left(\{x_{be}, x_{ce}\}, \{p_{ae}p_{bc}, p_{ab}, p_{cd}, p_{bc}p_{de}\}, \begin{array}{cc} & \begin{array}{cc} \text{be} & \text{ce} \end{array} \\ \begin{array}{c} be \\ ce \end{array} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{array} \right)$$

$$\left(\{x_{ac}, x_{ce}\}, \{p_{ab}, p_{bc}p_{ae}, p_{cd}p_{ae}, p_{de}\}, \begin{array}{cc} & \begin{array}{cc} \text{ac} & \text{ce} \end{array} \\ \begin{array}{c} ac \\ ce \end{array} & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{array} \right)$$

$$\left(\{x_{ac}, x_{ad}\}, \{p_{ab}p_{cd}, p_{bc}, p_{de}, p_{cd}p_{ae}\}, \begin{array}{cc} & \begin{array}{cc} \text{ac} & \text{ad} \end{array} \\ \begin{array}{c} ac \\ ad \end{array} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{array} \right)$$

$$\left(\{x_{bd}, x_{ad}\}, \{p_{bc}, p_{ab}p_{cd}, p_{ab}p_{de}, p_{ae}\}, \begin{array}{cc} & \begin{array}{cc} \text{be} & \text{bd} \end{array} \\ \begin{array}{c} be \\ bd \end{array} & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{array} \right).$$

The exchange graph is given below.



According to the exchange relations, the cluster variables can be written as follows:

$$\begin{aligned}
x_{be} &= 1/x_{be}^{-1}, x_{bd} = 1/x_{bd}^{-1}, \\
x_{ce} &= \frac{p_{de}p_{bc} + p_{cd}x_{be}}{x_{bd}}, \\
x_{ac} &= \frac{p_{ae}p_{bc} + p_{ab}x_{ce}}{x_{be}} \\
&= \frac{p_{ae}p_{bc}x_{bd} + p_{ab}p_{cd}x_{be} + p_{ab}p_{de}p_{bc}}{x_{be}x_{bd}}, \\
x_{ad} &= \frac{p_{cd}p_{ae} + p_{de}x_{ac}}{x_{ce}} \\
&= \frac{x_{bd}(p_{cd}p_{ae}x_{be}xbd + p_{de}p_{ae}p_{bc}x_{bd} + p_{de}^2p_{ab}p_{bc} + p_{de}p_{ab}p_{cd}x_{be})}{x_{be}x_{bd}(p_{de}p_{bc} + p_{cd}x_{be})} \\
&= \frac{x_{bd}p_{ae} + p_{ab}p_{de}}{x_{be}}.
\end{aligned} \tag{3.14}$$

Note that each cluster variable is a Laurent polynomial in the initial cluster $\underline{x_0} = \{x_{be}, x_{bd}\}$, i.e., each cluster can be written as a rational function of $\underline{x_0}$ where the denominator is a monomial in $\underline{x_0}$. Also note that the denominators, $x_{be}^{-1}, x_{bd}^{-1}, x_{be}, x_{bd}$ and $x_{be}x_{bd}$ can be interpreted as the negative simple roots $-\alpha_1, -\alpha_2$ and the positive roots α_1, α_2 and $\alpha_1 + \alpha_2$ in the root system of type A_2. This will be discussed later.

This example is related to the Grassmannian $Gr_{2,5}$. Each 2-dimensional subspace of a 5-dimensional vector space over $\mathbb{C}$ may be described by its two spanning vectors, considered as rows, i.e., a 2×5 matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{15} \\ a_{21} & \cdots & a_{2i} & \cdots & a_{25} \end{pmatrix}.$$

The 2×2 minors

$$\triangle_{ij} = \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix},$$

with $1 \leq i < j \leq 5$, lie in the homogeneous coordinate ring $\mathbb{C}[Gr_{2,5}]$ of the Grassmann variety, and satisfy the relations (called the Plücker relations):

$$\triangle_{ik}\triangle_{jl} = \triangle_{ij}\triangle_{kl} + \triangle_{il}\triangle_{jk},$$

for $i < j < k < l$.

Suppose we write

$$\triangle_{12} = p_{ab},\ \triangle_{23} = p_{bc},\ \triangle_{34} = p_{cd},\ \triangle_{45} = p_{de},\ \triangle_{15} = p_{ae},$$

$$\triangle_{13} = x_{ac},\ \triangle_{14} = x_{ad},\ \triangle_{24} = x_{bd},\ \triangle_{25} = x_{be},\ \triangle_{35} = x_{ce}.$$

Then the Plücker relations coincide with the exchange relations. In fact, the cluster algebra considered above is isomorphic to the coordinate ring $\mathbb{C}[Gr_{2,5}]$ of the Grassmannian $Gr_{2,5}$.

In general, $\mathbb{C}[Gr_{2,n+3}]$ is isomorphic to the cluster algebra related to the triangulations of a regular polygon with $n+3$ sides, defined above. For more details see [17, 12.7].

3.3 The Laurent phenomenon

The exchange relations show that every cluster variable can be expressed as a rational function of $\underline{x}$, where $\underline{x}$ is the cluster of a given initial seed. When we compute the cluster variable z' from z by the exchange relation, the denominator, which is the numerator for z, may contain a large number of monomials. However, the expression obtained can always be simplified, getting just one monomial in the denominator. This means that each cluster variable is a Laurent polynomial in $\underline{x}$. This fact, called the *Laurent phenomenon*, was proved by Zomin and Zelevinsky.

Theorem 3.3.1. [16, Theorem 3.1] *In an arbitrary cluster algebra, any cluster variable can be written as a Laurent polynomial in the variables of an arbitrary fixed cluster and the elements of* $\mathbb{P}$.

Fomin and Zelevinsky conjectured that the coefficients appearing in these polynomials are always nonnegative integer linear combinations of elements in $\mathbb{P}$. This conjecture was proved for some special cases, namely for cluster algebras of finite type, which will be defined in the following section.

3.4 Finite type classification

Cluster algebras with only finitely many distinct seeds are said to be of *finite type*.

Fomin and Zelevinsky proved that the classification of the cluster algebras of finite type is the same as the Cartan-Killing classification of (finite crystallographic) root systems.

Definition 3.4.1. Let $B = (b_{ij})$ be an exchange matrix. Define the *Cartan counterpart* of B to be the generalized Cartan matrix $A = A(B) = (a_{ij})$ with entries:

$$a_{ij} = \begin{cases} 2 & \text{if } i = j; \\ -|b_{ij}| & \text{if } i \neq j. \end{cases}$$

Theorem 3.4.2. [17, Theorem 1.4] *A cluster algebra $\mathcal{A}$ is of finite type if and only if the exchange matrix B of some seed of $\mathcal{A}$ has Cartan counterpart A which is a Cartan matrix of finite type.*

By this theorem, each cluster algebra of finite type has a well-defined *type*, mirroring the Cartan-Killing classification.

The next theorem, obtained by Fomin and Zelevinsky while they were proving 3.4.2, gives a characterization of the cluster algebras of finite type.

Theorem 3.4.3. [17, Theorem 1.8] *Let $\mathcal{A}$ be a cluster algebra. Then the following conditions are equivalent.*

1. *The cluster algebra $\mathcal{A}$ is of finite type.*

2. *The set $\mathcal{X}$ of all cluster variables is finite.*

3. *for every seed $(\underline{x}, \underline{p}, B)$ in $\mathcal{A}$, the entries of the matrix $B = (b_{xy})$ satisfy the inequalities $|b_{xy}b_{yx}| \leq 3$, for all $x, y \in \underline{x}$.*

4. *$\mathcal{A} = \mathcal{A}(B_0, \underline{p}_0)$, for some sign-skew-symmetric matrix $B_0 = (b_{ij})$ such that $b_{ij}b_{ik} \geq 0$ for all i, j, k and $A(B_0)$ is a Cartan matrix of finite type, and $\underline{p}_0$ is a $2n$-tuple of elements in $\mathbb{P}$ satisfying the normalization conditions.*

Example 3.4.4. The cluster algebra $\mathcal{A}$ (which is isomorphic to the coordinate ring $\mathbb{C}[Gr_{2,n+3}]$) generated by the diagonals and sides of the polygon P_{n+3} is of type A_n.

Let Φ be an irreducible finite root system with Cartan matrix $A = (a_{ij})_{i,j \in I}$, and let $\mathcal{A} = \mathcal{A}(B_0, \underline{p}_0)$ be a cluster algebra related to A as in 3.4.3. Let $\Pi = \{\alpha_i : i \in I\}$ be the set of simple roots of Φ and denote by $\Phi_{\geq -1} = \Phi^+ \cup (-\Pi)$, where Φ^+ is the set of positive roots, the set of almost positive roots. Let $\underline{x}_0 = \{x_i : i \in I\}$ be the cluster for the initial seed $(\underline{x}_0, \underline{p}_0, B_0)$.

Theorem 3.4.5 provides a description of the finite set $\mathcal{X}$ of cluster algebras of $\mathcal{A}$ in terms of the roots in $\Phi_{\geq -1}$.

Theorem 3.4.5. [17, Theorem 1.9] *For any root $\alpha = \sum_{i \in I} c_i \alpha_i \in \Phi_{\geq -1}$, there is a unique cluster variable*

$$x[\alpha] = \frac{P_\alpha(x_0)}{\prod_{i \in I} x_i^{c_i}}, \tag{3.15}$$

where P_α is a polynomial over $\mathbb{ZP}$ with nonzero constant term. The map $\alpha \mapsto x[\alpha]$ is a bijection between $\Phi_{\geq -1}$ and $\mathcal{X}$. Under this bijection, $x[-\alpha_i] = x_i$.

Note that the right-hand side of 3.15 is a Laurent polynomial, which agrees with 3.3.1.

Example 3.4.6. Consider the cluster algebra $\mathcal{A}(B_0, \underline{p}_0)$, where

$$B_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \underline{p}_0 = \{p_{ae}, p_{ab}p_{de}, p_{de}p_{bc}, p_{cd}\},$$

presented in Section 3.2 (for $n = 2$). The Cartan counterpart of B_0 is the Cartan matrix for the root system Φ of type A_2. Let α_1, α_2 be the simple roots of Φ and $\Phi_{\leq -1} = \{-\alpha_1, -\alpha_2, \alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. The cluster corresponding to the initial seed $(\underline{x}_0, \underline{p}_0, B_0)$ is $\underline{x}_0 = \{x_{be}, x_{bd}\}$. According to 3.14, the bijection between $\Phi_{\leq -1}$ and $\mathcal{X}$ is given by: $-\alpha_1 \mapsto x_{be}$, $-\alpha_2 \mapsto x_{bd}$, $\alpha_1 \mapsto x_{ad}$, $\alpha_2 \mapsto x_{ce}$, $\alpha_1 + \alpha_2 \mapsto x_{ae}$.

3.5 The compatibility degree and root clusters

Given the connection between root systems and clusters estabilished by 3.4.5, we will now study some properties of clusters in terms of almost positive roots, and give a geometrical description of the clusters of type A_n.

Let $\mathbb{Z}\Pi$ denote the root lattice, i.e., the set of linear combinations $\sum_{i\in I} c_i\alpha_i$ with $c_i \in \mathbb{Z}$. For $\alpha \in \mathbb{Z}\Pi$, denote by $[\alpha : \alpha_i]$ the coefficient of α_i in the expansion of α in the basis Π.

Definition 3.5.1. The *compatibility degree* is the unique map from $\Phi_{\geq -1} \times \Phi_{\geq -1}$ to $\mathbb{Z}_{\geq 0}$, denoted by $(\ \|\ )$, satisfying the following rules:

- $(-\alpha_i\|\beta) = max([\beta : \alpha_i], 0)$,
- $(\tau_\pm\alpha\|\tau_\pm\beta) = (\alpha\|\beta)$, for all $\alpha, \beta \in \Phi_{\geq -1}$.

The proof that this function is well defined can be found in [18, Section 3.1].

We gather some properties of the compatibility degree in the following proposition.

Proposition 3.5.2. [18, Proposition 3.3]

1. *If Φ is simply-laced, then $(\ \|\ )$ is symmetric, that is, $(\alpha\|\beta) = (\beta\|\alpha)$, for all $\alpha, \beta \in \Phi_{\geq -1}$.*
2. *If $(\alpha\|\beta) = 0$, then $(\beta\|\alpha) = 0$.*
3. *If α and β belong to $\Phi(J)_{\geq -1}$ for some proper subset $J \subsetneq I$, where $\Phi(J)$ denotes the root subsystem in Φ spanned by the set of simple roots $\{\alpha_i\,;\, i \in J\}$, then their compatibility degree with respect to the root subsystem $\Phi(J)$ is equal to $(\alpha\|\beta)$.*

Definition 3.5.3.

1. Two roots $\alpha, \beta \in \Phi_{\geq -1}$ are *compatible* if $(\alpha\|\beta) = 0$.
2. A subset of $\Phi_{\geq -1}$ is said to be *compatible* if any pair α, β of its elements is compatible.
3. A maximal (by inclusion) compatible subset of $\Phi_{\geq -1}$ is called a *root cluster*.

Note that by 3.5.2 (2), the compatibility relation between almost positive roots is symmetric.

A root cluster $C = \{\beta_1, \ldots, \beta_n\}$ corresponds to the cluster $\{x[\beta_1], \ldots, x[\beta_n]\}$, where $x[\beta_i]$ is as in 3.4.5. This follows from Lemma 2.4 and Remark 2.5 of [17]. By Lemma 2.4 (2), for each root cluster C there is a seed $\Sigma(C)$. So we also get a cluster corresponding to C. By Lemma 2.4 (3), the cluster in $\Sigma(C)$ is $\{x[\alpha] \mid \alpha \in C\}$. Since the map $\alpha \mapsto x[\alpha]$ of 3.4.5 is a bijection, $C \mapsto \Sigma(C)$ is a bijection between root clusters and seeds, by Remark 2.5.

It follows directly from the next proposition that in cluster algebras of finite type of rank n, two clusters are adjacent in the exchange graph if and only if they agree on $n-1$ elements.

Proposition 3.5.4. [17, Theorem 1.12] *Let $\mathcal{A}$ be a cluster algebra of finite type. Every seed $(\underline{x}, \underline{p}, B)$ in $\mathcal{A}$ is uniquely determined by its cluster $\underline{x}$. For any cluster $\underline{x}$ and any $x \in \underline{x}$, there is a unique cluster $\underline{x}'$ with $\underline{x} \cap \underline{x}' = \underline{x} \setminus \{x\}$.*

The first part of this proposition follows from Lemma 2.4 and Remark 2.5 of [17]. Let $\underline{x}$ be a cluster. By 3.4.5, $\underline{x} = \{x[\beta_1], \ldots, x[\beta_n]\}$, for some almost positive roots $\beta_1, \ldots, \beta_n$. Let Σ be a seed whose cluster is $\underline{x}$. By Remark 2.5, $\Sigma = \Sigma(C)$ for some root cluster C. By Lemma 2.4 (3), $\underline{x} = \{x[\alpha] \mid \alpha \in C\}$. So C must be the set $\{\beta_1, \ldots, \beta_n\}$. Thus, $\Sigma = \Sigma(C)$ is determined, as required. Suppose now that $\underline{x}$ is a cluster and x is a cluster variable that belongs to $\underline{x}$. Let $\underline{x}'$ be a cluster with $\underline{x} \cap \underline{x}' = \underline{x} \setminus \{x\}$. We know that $\underline{x} = \{x[\alpha] \mid \alpha \in C\}$ and $\underline{x}' = \{x[\alpha] \mid \alpha \in C'\}$, for some root clusters C, C'. As the map $\alpha \mapsto x[\alpha]$ is a bijection, $C \cap C' = C \setminus \{\alpha\}$ for some $\alpha \in C$. Hence the uniqueness of C' follows from condition (2.1) at the beginnig of Section 2 of [17]. Note that the condition (2.1) holds, as it is remarked at the beginning of Section 4 of this paper.

The next proposition states some properties of root clusters, that will be needed for the last chapter.

Proposition 3.5.5.

1. *If C is a root cluster for $\Phi_{\geq -1}$, then so are $\tau_+(C)$ and $\tau_-(C)$.*

2. *For every $i \in I$, there is a bijective correspondence between clusters for $\Phi_{\geq -1}$ containing $-\alpha_i$ and clusters for $\Phi_{\geq -1}(I \setminus \{i\})$. This bijection maps C to $C \setminus \{-\alpha_i\}$.*

3. *If Φ is reducible and $\Psi_1, \ldots, \Psi_k$ are the connected components of Φ, then the clusters of Φ are given by the disjoint unions of the clusters for $\Psi_1, \ldots, \Psi_k$.*

Proof. (1) By the second property of the compatibility degree, both τ_+ and τ_- preserve compatibility, which proves the required.

(2) The set $\Phi(I \setminus \{i\})$ is the root subsystem of Φ of all roots in Φ with no α_i in their expansion in terms of the simple roots. It is clear that $\{\alpha_j \mid j \neq i\}$ is a simple system for $\Phi(I \setminus \{i\})$ and $\Phi^+(I \setminus \{i\}) = \Phi^+ \cap \Phi(I \setminus \{i\})$. Let C be a cluster in Φ that contains $-\alpha_i$. Then the almost positive roots of $C \setminus \{-\alpha_i\}$ are compatible with $-\alpha_i$, which means that $C \setminus \{-\alpha_i\} \subseteq \Phi(I \setminus \{i\})$, by definition of compatibility degree. Furthermore, any two almost positive roots of C, other than $-\alpha_i$, are compatible in Φ and so they are also compatible in $\Phi(I \setminus \{i\})$, by 3.5.2 (3). Since C is a maximal compatible set in Φ, $C \setminus \{-\alpha_i\}$ is also a maximal compatible set in $\Phi(I \setminus \{i\})$, i.e., $C \setminus \{-\alpha_i\}$ is a cluster in $\Phi(I \setminus \{i\})$. It follows that removing $-\alpha_i$ from a cluster of Φ containing $-\alpha_i$ gives a bijection from clusters of Φ that contain $-\alpha_i$ to clusters of $\Phi(I \setminus \{i\})$, being the map $C' \mapsto C' \cup \{-\alpha_i\}$, where C' is a cluster in $\Phi(I \setminus \{i\})$, its inverse.

(3) Let C be a cluster in Φ, and write $C = \{\beta_{11}, \ldots, \beta_{1n_1}\} \cup \ldots \cup \{\beta_{k1}, \ldots, \beta_{kn_k}\}$, where the set of roots $\{\beta_{l1}, \ldots, \beta_{ln_l}\}$, which will be denoted by C_l for short, lies in Ψ_l, for $l \in [k]$. We want to prove that C_l is a cluster in Ψ_l, for all $l \in [k]$. Clearly, C_l is a compatible set in Ψ_l for all $l \in [k]$ (cf. 3.5.2 (3)). Suppose one of these sets, say C_j, is not a cluster, i.e., there exists $\beta \in \Psi_j \setminus$ such that $(\beta \| \beta_{jm}) = 0$, with $m \in [n_j]$. If we prove that $(\beta \| \gamma) = 0$, for $\gamma \in \Psi_{j'}$, with $j' \neq j$, we are done because it implies that $C \cup \{\beta\}$ is compatible, contradicting the fact that C is a cluster. Let then γ be a root in $\Psi_{j'}$, with $j' \in [k] \setminus \{j\}$. Let

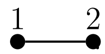
$$\tau_{\pm} = \prod_{k \in I_{\pm}} \sigma_k \qquad (\tau_m)_{\pm} = \prod_{k \in (I_m)_{\pm}} \sigma_k,$$

where I parametrizes the set of simple roots of Φ and I_m ($m \in [k]$) parametrizes the set of simple roots of Ψ_m. For $k \in I_+ \setminus I_j$, we have $\sigma_k(\beta) = \beta$, because $(\beta, \alpha_k) = 0$. Therefore $\tau_{\pm}(\beta) = (\tau_j)_{\pm}(\beta)$. Analogously, $\tau_{\pm}(\gamma) = (\tau_{j'})_{\pm}(\gamma)$. By 1.3.6, there exists $g \in \langle (\tau_j)_+, (\tau_j)_- \rangle$ such that $g(\beta) = -\alpha_l$, for some $l \in I_j$. Let h be the product obtained by replacing each $(\tau_j)_+$ in g with τ_+ and each $(\tau_j)_-$ in g with τ_-, and h' be the product obtained by replacing each $(\tau_j)_+$ in g with $(\tau_{j'})_+$ and each $(\tau_j)_-$ with $(\tau_{j'})_-$. Then we have $h(\beta) = g(\beta) = -\alpha_l$ and $h'(\gamma) = h(\gamma)$. According to the definition of the compatibility degree, we have that $(\beta \| \gamma) = (h(\beta) \| h(\gamma)) = (-\alpha_l \| h'(\gamma))$, and $(-\alpha_l \| h'(\gamma)) = 0$ since $-\alpha_l$ is a negative root that lies in Ψ_j and $h'(\gamma) \in \Psi_{j'}$. $\square$

Proposition 3.5.6. [17, Proposition 3.5, Corollary 4.4] *Let C and $C' = C \setminus \{\beta\} \cup \{\beta'\}$ be two adjacent clusters. Then the roots β and β' are such that $(\beta \| \beta') = (\beta' \| \beta) = 1$. The converse is also true, i.e., for a pair of almost* *positive roots satisfying* $(\beta\|\beta') = (\beta'\|\beta) = 1$, *there exist two adjacent clusters* C *and* C' *such that* $C' = C \setminus \{\beta\} \cup \{\beta'\}$.

Example 3.5.7. We illustrate these results for a root system of type A_2. The Cartan matrix of type A_2 and the Dynkin diagram are given by:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad \text{and}$$



Write $\triangle = I_+ \cup I_-$, where $I_+ = \{1\}$ and $I_- = \{2\}$. The permutations τ_+, τ_- of $\Phi_{\geq -1} = \{-\alpha_1, -\alpha_2, \alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ are given by

$$\begin{aligned} \tau_+ &= (-\alpha_1\, \alpha_1)(-\alpha_2)(\alpha_2\, \alpha_1 + \alpha_2) \\ \tau_- &= (-\alpha_1)(-\alpha_2\, \alpha_2)(\alpha_1\, \alpha_1 + \alpha_2). \end{aligned}$$

For any $\beta \in \Phi_{\geq -1}$, we have $(\beta\|\beta) = 0$, since there exists $g \in \langle \tau_+, \tau_- \rangle$ such that $g(\beta) = -\alpha_i$ for some i, by 1.3.6, and so $(\beta\|\beta) = (g(\beta)\|g(\beta)) = (-\alpha_i\| - \alpha_i) = 0$. Furthermore, we have

$$\begin{aligned} (\alpha_2\|\alpha_1) &= (\tau_+\tau_-(\alpha_2)\|\tau_+\tau_-(\alpha_1)) = (-\alpha_2\|\alpha_2) = 1 \\ (\alpha_1\|\alpha_1 + \alpha_2) &= (\tau_+(\alpha_1)\|\tau_+(\alpha_1 + \alpha_2)) = (-\alpha_1\|\alpha_2) = 0 \\ (\alpha_2\|\alpha_1 + \alpha_2) &= (\tau_-(\alpha_2)\|\tau_-(\alpha_1 + \alpha_2)) = (-\alpha_2\|\alpha_1) = 0. \end{aligned}$$

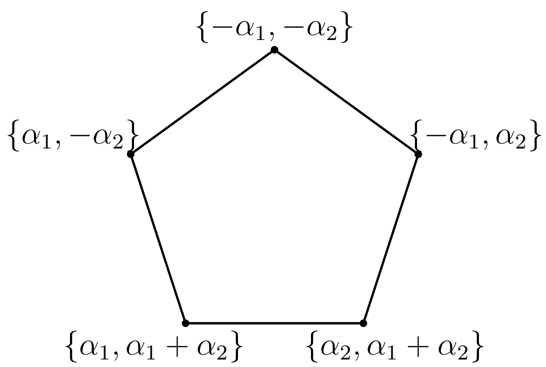
Therefore, by 3.5.2 (1), the compatibility degree is given by the following matrix.

$$\begin{array}{c} \\ -\alpha_1 \\ -\alpha_2 \\ \alpha_1 \\ \alpha_2 \\ \alpha_1+\alpha_2 \end{array} \begin{array}{c} \begin{array}{ccccc} -\alpha_1 & -\alpha_2 & \alpha_1 & \alpha_2 & \alpha_1+\alpha_2 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

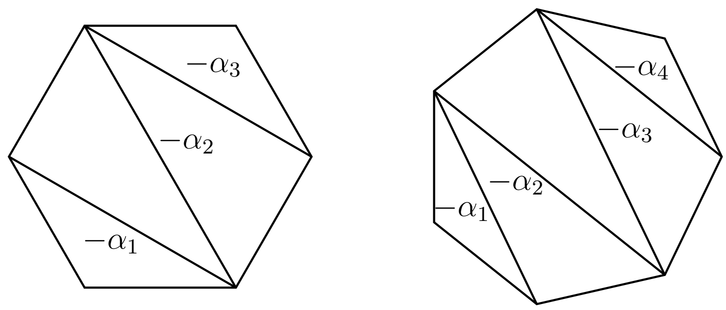
The maximal compatible subsets of $\Phi_{\geq -1}$, i.e., the clusters are

$$\{-\alpha_1, -\alpha_2\}\ \{-\alpha_1, \alpha_2\}\ \{\alpha_2, \alpha_1 + \alpha_2\}\ \{\alpha_1 + \alpha_2, \alpha_1\}\ \{\alpha_1, -\alpha_2\}.$$

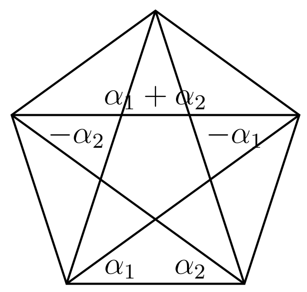
Observe that $(-\alpha_2\|\alpha_2) = (-\alpha_1\|\alpha_1 + \alpha_2) = (\alpha_2\|\alpha_1) = (\alpha_1 + \alpha_2\| - \alpha_2) = 1$, in agreement with 3.5.6. The exchange graph is presented below.



We finish this section with the geometrical description of the clusters of type A_n, given in [18, Section 3.5]. Let P_{n+3} be a regular n+3-gon. Consider a diagonal connecting two vertices which have a common neighbouring vertex, and label it by the root $-\alpha_1$. Then draw a succession of diagonals labelled by $-\alpha_2, \cdots, -\alpha_n$, such that consecutive diagonals have a common vertex. We illustrate for $n = 3$ and $n = 4$.



This set of diagonals of the polygon is called the *snake*. The remaining diagonals of P_{n+3} can be identified with the positive roots by associating each $\alpha_{ij} := \alpha_i + \alpha_{i+1} + \ldots + \alpha_j$, where $i \leq j$, the unique diagonal that crosses the diagonals $-\alpha_i, -\alpha_{i+1}, \ldots, -\alpha_j$ and doesn't cross any other diagonal$-\alpha_k$ of the snake. For example, the diagonals of a regular pentagon can be parametrised by the elements of $\Phi_{\geq -1}$ of type A_2, as the next picture shows.



The following proposition gives a geometric point of view of the compatible sets and clusters of type A_n. The first statement, which is only necessary to the proof of the second one, won't be proved but it can be found in [18, 3.1.4].

Proposition 3.5.8. *Let Φ be a root system of type A_n.*

1. *The involutions τ_+ and τ_- are both symmetries of the polygon P_{n+3}.*
2. *Let $\alpha, \beta \in \Phi_{\geq -1}$. The compatibility degree $(\alpha \| \beta)$ is equal to 1 if the diagonals α and β cross each other, and 0 otherwise.*
3. *Compatible sets are collections of mutually non-crossing diagonals and the clusters correspond to the triangulations.*

Proof. (2) Let β be an almost positive root and let $\alpha = -\alpha_i$ for some i. By definition, $(\alpha \| \beta) = max([\beta : \alpha_i], 0)$, i.e., $(\alpha \| \beta)$ is the coefficient of α_i in the expansion of β in terms of the simple roots if $\beta \neq -\alpha_i$, and 0 otherwise. By definition of the corresponding diagonal, such coefficient is 1 if the diagonal β crosses the diagonal $-\alpha_i$ and zero if not. Hence, the result holds if $\alpha = -\alpha_i$ for some $i \in [n]$. Let now α be an arbitrary almost positive root. By 1.3.6, there exists $g \in \langle \tau_+, \tau_- \rangle$ such that $g(\alpha) = -\alpha_i$, for some i. By definition, $(\alpha \| \beta) = (g(\alpha) \| g(\beta)) = (-\alpha_i \| g(\beta))$. By the above, this is 1 if the diagonals $-\alpha_i$ and $g(\beta)$ cross, and 0 if not. Since g is a symmetry of the polygon (by (1)), it follows that the diagonals α and β cross each other if and only if $g(\beta)$ and $-\alpha_i = g(\alpha)$ cross each other, which proves what we wanted.

The last statement follows from (2). □

3.6 The cluster quiver

For the understanding of the paper [13], we need the definition of a quiver associated to a cluster of cluster algebra of simply-laced case. In this section we present this definition and exhibit a nice rule for quiver mutations associated to the matrix mutations, for the case when the matrices are 2-finite. This subject is studied in sections 7 and 8 of [17].

We begin with a more general definition of a weighted and oriented graph associated to a sign-skew-symmetric matrix. This definition is parallel to the definition of Dynkin diagram for a generalized Cartan matrix.

Definition 3.6.1. Let $B = (b_{ij})_{i,j\in I}$ be a sign-skew-symmetric matrix. We associate to it a weighted and oriented graph, denoted by Q_B, defined as follows:

- the set of vertices is indexed by I;
- given $i, j \in (Q_B)_0$, there is an arrow from i to j if and only if $b_{ij} > 0$;
- if $i, j \in (Q_B)_0$ are such that $b_{ij} \neq 0$, then the edge that links them has weight $|b_{ij}b_{ji}|$. If $b_{ij} = 0$, i.e., if there is no arrow between i and j, we say that they are joined by an edge of weight 0.

Definition 3.6.2. An integer square matrix $B = (b_{ij})$ is called *2-finite* if any matrix $B' = (b'_{ij})$ mutation equivalent to B is sign-skew-symmetric and satisfies $|b'_{ij}b'_{ji}| \leq 3$ for all i and j.

The following results are necessary to prove 3.6.6.

Proposition 3.6.3. [17, Proposition 7.2] *Every 2-finite matrix is skew-symmetrizable.*

Lemma 3.6.4. *If B is 2-finite, then the edges of every triangle in Q_B are oriented in a cyclic way.*

Proof. Suppose, for a contradiction, that there is an unoriented triangle in Q_B, that is, there exist three distinct vertices i, j, k such that $b_{ij}, b_{ik}, b_{kj} > 0$. Consider the matrix $B' = \mu_k(B)$. By definition, we have $b'_{ij} = b_{ij} + b_{ik}b_{kj}$ and $b'_{ji} = b_{ji} - b_{jk}b_{ki}$. Since b_{ij}, b_{ik} and b_{kj} are positive integers, it follows that $b'_{ij} \geq 2$ and $b'_{ji} \leq -2$, and so $|b'_{ij}b'_{ji}| > 3$, contradicting the 2-finiteness. $\square$

We now show how the graph $Q_{\mu_k(B)}$ of $\mu_k(B)$, for any index $k \in I$, is uniquely determined by Q_B, for a given 2-finite matrix B.

Lemma 3.6.5. *Let B be a skew-symmetrizable matrix.*

1. *There exists a diagonal matrix H with positive diagonal entries such that HBH^{-1} is skew-symmetric. Moreover, the matrix $S(B) = (s_{ij}) = HBH^{-1}$ is uniquely determined by B by:*

$$s_{ij} = \begin{cases} \sqrt{|b_{ij}b_{ji}|} & \text{if } b_{ij} \geq 0, \\ -\sqrt{|b_{ij}b_{ji}|} & \text{if } b_{ij} < 0. \end{cases}$$

2. *For any $k \in I$, we have $S(\mu_k(B)) = \mu_k(S(B))$.*

Proof. Let D be a diagonal matrix with positive entries such that DB is skew-symmetric. Consider $H = D^{1/2}$. Since $H = H^{-1}D$, we have $HBH^{-1} = H^{-1}(DB)H^{-1}$, and so $-(HBH^{-1}) = H^{-1}(BT)^t H^{-1} = (H^{-1}(DB)H^{-1})^t = (HBH^{-1})^t$, which means that HBH^{-1} is skew-symmetric. Since $s_{ij} = h_i b_{ij} h_j^{-1}$, we have $sgn(s_{ij}) = sgn(b_{ij})$, because the diagonal entries of H are positive. Moreover:

$$\begin{aligned} s_{ij}^2 &= (h_i b_{ij} h_j^{-1})^2 \\ &= h_i^2 b_{ij} b_{ij} (h_j^{-1})^2 \\ &= d_i(-d_i^{-1} d_j b_{ji}) b_{ij} d_j^{-1} \\ &= -b_{ij} b_{ji} = |b_{ij} b_{ji}|. \end{aligned}$$

This proves the first part of the Lemma.

It is easy to check that $\mu_k(HBH^{-1}) = H(\mu_k(B))H^{-1}$, for any $k \in i$ and any diagonal matrix H with positive entries. Recall that if B is skew-symmetrizable, then $\mu_k(B)$ is skew-symmetrizable as well, with the same skew-symmetrizing matrix. Hence, it follows from (1) that $S(\mu_k(B)) = H\mu_k(B)H^{-1} = \mu_k(HBH^{-1}) = \mu_k(S(B))$, as required. □

Proposition 3.6.6. *Let B be 2-finite and $k \in I$. Then the diagram $Q_{\mu_k(B)}$ is obtained by the graph Q_B as follows:*

- *The orientation of all edges incident to k are reversed, and their weights intact.*
- *If i and j are two vertices such that $i \to k, k \to j \in (Q_B)$, then the edge between i and j in $Q_{\mu_k(B)}$ has weight $c' = (\sqrt{ab} - \sqrt{c})^2$, where a, b and c are the weight of the arrow $i \to k$, $k \to j$ and $j \to i$ in Q_B, respectively, and is directed from i to j if $c' \neq 0$.*
- *The rest of the edges and their weights in Q_B remain unchanged.*

Proof. Denote by B' the matrix obtained from B by matrix mutation in direction k. Since $b'_{ij} = -b_{ij}$, for $i = k$ or $j = k$, it is clear that the orientation of the edges incident to k are reversed and their weights remain intact.

Suppose now that i, j are vertices such that $i \to k$ and $k \to j$ are arrows in Q_B. Write $a = |b_{ik} b_{ki}|$ and $b = |b_{kj} b_{jk}|$. Due to the 2-finiteness of B, the triangles in Q_B are oriented (3.6.4), so either there is no arrow between i and j or $j \to i$ is an edge in Q_B. Let c be the weight of this arrow, with convention that $c = 0$ if it doesn't exist. Since the arrows incident to k are reversed, we have that $j \to k, k \to i \in Q_{B'}$. Because $B' = \mu_k(B)$ is 2-finite as well (by definition), we conclude that the arrow between i and j in $Q_{B'}$, if it exists, must be from i to j. Let us now calculate the weight c' of such arrow. We have $s_{ik} = \sqrt{a}$, $s_{kj} = \sqrt{b}$ and $s_{ij} = -\sqrt{c}$, according to 3.6.5. By the second part of this lemma, we have $S' := S(\mu_k(B)) = \mu_k(S(B))$, and so

$$\sqrt{c'} = s'_{ij} = s_{ij} + \frac{|s_{ik}|s_{kj}| + s_{ik}|s_{kj}|}{2} = -\sqrt{c} + \sqrt{ab},$$

as required.

If i and j are two edges that are not connected in Q_B via a two-edge oriented path going through k, then either $b_{ik} = 0$ or $b_{jk} = 0$, and so $b'_{ij} = b_{ij}$, which concludes the proof of this proposition. $\square$

Fomin and Zelevinsky proved a more general rule for skew-symmetrizable matrices (see [17, Proposition 8.1]). But we are only interested in graphs of matrices corresponding to the clusters of a cluster algebra of finite type, and these matrices are 2-finite, by 3.4.3.

Let C be a cluster of a cluster algebra of simply-laced type and B_C be the exchange matrix in the unique seed that contains the cluster C (the uniqueness comes from 3.5.4). We denote by Q_C the graph corresponding to B_C, for short. Note that the entries of B_C are only $-1, 0$ and 1. Thus, all edges in Q_C have weight 1. For this reason, we ignore the weights in Q_C and the graph Q_C will be called the quiver corresponding to the cluster C.

Remark 3.6.7. The rule for graph mutations given in 3.6.6 can be rewritten as follows:

- the orientations of all the edges incident to k are reversed.
- given two vertices i, j such that $i \to k, k \to j \in (Q_C)$, the arrow $i \to j$ is in the mutated quiver if and only if the arrow $j \to i$ is not in Q_C.
- the rest of the edges remain unchanged.

Chapter 4

Quivers with relations arising from clusters (A_n case)

Gabriel's Theorem (see [2, Theorem VII.5.10], for example) states the following:

Let Q be a finite, connected and acyclic quiver, K be an algebraically closed field and $A = KQ$ be the path K-algebra of Q. Then A is representation-finite if and only if the underlying graph of Q is one of the Dynkin diagrams A_n, D_n, with $n \geq 4$, E_6, E_7 or E_8. Moreover, if this is the case, then there is a bijection $\underline{dim}$ between the set $Ind(A - mod)$ of isomorphism classes of indecomposable A-modules and the set Φ^+ of positive roots of the root system Φ corresponding to the type of the underlying graph of Q. This bijection is given by

$$\underline{dim} : Ind(A - mod) \to \Phi^+$$

$$M \mapsto \underline{dim}(M) := \sum_{i=1}^{n} (dim(M_i))\alpha_i,$$

where M is the representation $(M_i, \varphi_\alpha)_{\substack{i \in Q_0 \\ \alpha \in Q_1}}$ and α_i, for $i \in [n]$, are the simple roots.

This establishes a connection between representation theory and root systems. As we have seen in the last chapter, there is a connection between cluster algebras of finite type and root systems as well.

This chapter is dedicated to the study of [13], one of the papers that investigates the link between representation theory and cluster algebras.

In this paper, to each cluster C of a finite cluster algebra are associated certain relations R_C on the quiver Q_C and the category $Mod\,\mathbb{C}Q_C/R_C$ of representations of Q_C bound by such relations. The main result describes the cluster variables of C in terms of the indecomposables of the category $Mod\,\mathbb{C}Q_C/R_C$.

To define the relations R_C in Q_C we first need the definition of shortest paths. Let $i \to j$ be an arrow in Q_C. A *shortest path* from j to i in Q_C is a path with source j and target i which doesn't repeat vertices and such that the induced subquiver on the vertices of the path is a cycle.

Let $i \to j$ be an arrow in Q_C. If there are two distinct shortest paths ω and ω' from j to i, then we define the relation $Rel_{i,j}$ to be $\omega = \omega'$. If there exists only one shortest path ω from j to i, then $Rel_{i,j}$ is $\omega = 0$. If there is no such shortest path, there is no relation. The relations R_C are the set of all the relations $Rel_{i,j}$ for $i \to j \in (Q_C)_1$.

Denote by $Ind(Q_C)$ the set of isomorphism classes of indecomposables of $Mod\,\mathbb{C}Q_C/R_C$. Recall that each vertex i of Q_C is associated to a simple module S_i. We denote by α_i the isomorphism class of that module.

The main result of the article [13] (cf. 4.4.1) is stated as follows:

Theorem. *Let $C = \{u_1, \ldots, u_n\}$ be a cluster in a cluster algebra of type A_n. Let V be the set of all cluster variables for this cluster algebra. There exists a bijection*

$$\begin{array}{ccc} Ind(Q_C) & \to & V \setminus C \\ \alpha & \mapsto & x[\alpha], \end{array}$$

such that

$$x[\alpha] = \frac{P(u_1, \ldots, u_n)}{\prod_i u_i^{n_i}},$$

where P is a polynomial prime to u_i for all i and where $n_i = n_i(\alpha)$ is the multiplicity of the simple module α_i in the module α.

A key result of this paper ([13, Theorem 2.9], cf. 4.3.1), which is very useful to the proof of the theorem stated above, provides a graphical description of $Mod\,\mathbb{C}Q_C/R_C$, where the objects can be viewed as diagonals in a regular polygon and the morphisms come from rotating these diagonals about their endpoints.

4.1 The category $Mod\,Q_T/Rel_\triangle$

Fix a nonnegative integer n and denote by P_{n+3} the regular $n+3$-gon.

Recall that the clusters of a cluster algebra of type A_n are in bijection with the triangulations of P_{n+3} (3.5.8 (3)). Therefore, it makes sense to 'redefine' the module category $Mod\,\mathbb{C}Q_C/R_C$ in terms of diagonals and triangulations of a regular polygon.

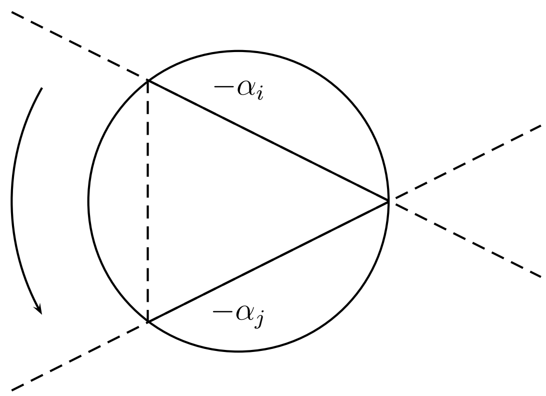
In this section, we associate a quiver to each triangulation of P_{n+3}, which corresponds to the quiver of the cluster associated to the triangulation, and define certain relations, called the triangle relations, which will be seen to be the same as the relations $Rel_{i,j}$ defined above. The category of modules over these quivers satisfying the triangle relations corresponds to $Mod\, \mathbb{C}Q_C/R_C$.

Fix a triangulation T of P_{n+3}. The diagonals of the polygon that belong to T will be called *negative roots* and the others will be the *positive roots*. We denote by Φ_+ the set of these positive roots and I the set of the negative roots. The negative roots are denoted by $-\alpha_i$, with $i \in [n]$ (note that the number of diagonals of T is n).

The *support* Supp $\alpha \subseteq I$ of a positive root α is the set of negative roots that cross α.

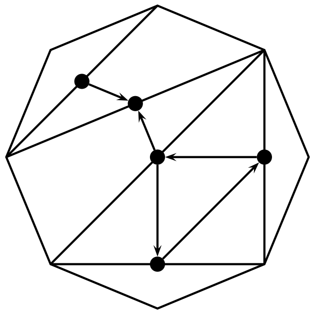
Remark 4.1.1. A positive root is determined by its support. Indeed, given the sequence $-\alpha_1, \dots, -\alpha_k$ of crossed negative roots, one can recover the vertices of the positive root. Denote by T_i and T_{i+1} the triangles corresponding to the diagonal $-\alpha_i$. Let v_i be the vertex of T_i that is not in T_{i+1}. We have that v_1 and v_{k+1} are the vertices of the positive root.

We will now define a relation $<$ on the set of negative roots. Suppose $-\alpha_i$, $-\alpha_j$ are two negative roots that bound the same triangle in T. We say that $-\alpha_i < -\alpha_j$ if the rotation with minimal angle around the common vertex that sends $-\alpha_i$ to $-\alpha_j$ is anti-clockwise, as is illustrated in the following picture:



The quiver Q_T is defined as follows: the vertices are the diagonals of T, i.e., the negative roots, and there is an arrow $j \to i$ if and only if $-\alpha_i < -\alpha_j$. In other words, there is an arrow from the diagonal $-\alpha_i$ to the diagonal $-\alpha_j$ if $-\alpha_j$ is obtained from $-\alpha_i$ by rotating clockwise about a common vertex without passing through another diagonal of T. Note that, by definition, the triangles in these quivers are oriented.

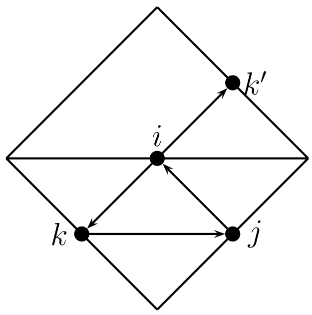
The following picture illustrates an example of a quiver Q_T.



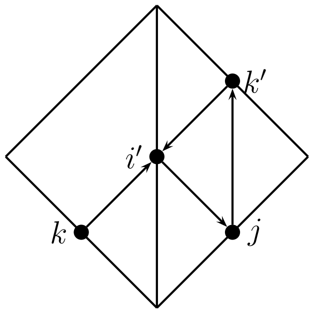
Recall that the mutation of a triangulation at one of its diagonals is the unique triangulation obtained by replacing this diagonal with another one (see section 3.2) and the mutation of quivers is described in 3.6.7. The next lemma guarantees that the mutation of clusters is coherent with the mutation of quivers.

Lemma 4.1.2. *The mutation of quivers corresponds to the mutation of triangulations.*

Proof. We consider only the case



where i, j, k and k' belong to the triangulation T and the other edge is a border edge of the polygon; the other cases are similar. The quiver of the triangulation T' that results from the mutation of the triangulation T by replacing the diagonal i with the other possible, denoted by i', is the following.



As we can see:

- the arrows incident to i were reversed;
- the path $j \to i \to k'$ exists in Q_T, $k' \to j \notin (Q_T)_1$ and $j \to k' \in (Q_{T'})_1$ and
- the path $j \to i \to k$ exists in Q_T as does the arrow $k \to j$, and $j \to k \notin (Q_{T'})_1$.

So $Q_{T'} = \mu_i(Q_T)$, as required. $\square$

Remark 4.1.3. Let $\mathcal{A}$ be a cluster algebra of type A_n. Write $\mathcal{A} = \mathcal{A}(B_0, \underline{p}_0)$, as in 3.4.3. Let C_0 be the cluster for the initial seed $(C_0, B_0, \underline{p}_0)$. The quiver Q_{C_0} is the quiver of type A_n (because $b_{ij} \neq 0$ if and only if $j = i-1, i+1$) with the alternating orientation, i.e., each vertex is either a source or a sink, since $b_{ij}b_{ik} \geq 0$, for all i, j, k.

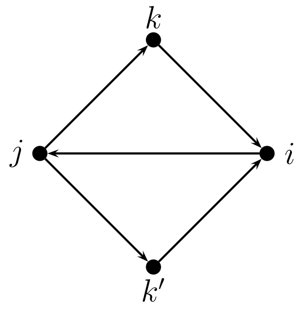
Note that in 3.4.5, C_0 is the cluster related to the negative simple roots of the root system associated to $\mathcal{A}$, i.e., $C_0 = \{x[-\alpha_1], \ldots, x[-\alpha_n]\}$. The negative simple roots correspond to the snake triangulation T_{C_0}. Hence, the quiver $Q_{T_{C_0}}$ is also of type A_n with the same orientation of Q_{C_0}, according to the definition of a quiver of a triangulation.

Let C be any other cluster of $\mathcal{A}$. It follows by induction on the number of mutations needed to get C from C_0, using 4.1.2, that $Q_C = Q_{T_C}$.

Definition 4.1.4. Let T be a triangulation and Q_T its associated quiver.

1. We call the relations of the form $\alpha\beta = 0$, where α, β are two sucessive arrows of a triangle of Q_T, the *triangle relations*.
2. The abelian $\mathbb{C}$-category of modules over the quiver Q_T with the triangle relations will be denoted by $Mod\, Q_T/Rel_\triangle$, where $Rel_\triangle$ is the ideal of $\mathbb{C}Q_T$ generated by the triangle relations.

Note that in Q_T, a cycle that doesn't repeat vertices must have length 3, i.e., is a triangle. Moreover, for each arrow $i \to j$ in Q_T there exists at most one shortest path from j to i, since this situation



cannot happen, by definition of Q_T. These observations permit to conclude that the triangle relations are the relations $Rel_{i,j}$ defined before.

4.2 The category $\mathcal{C}_T$

We will now define a combinatorial additive $\mathbb{C}$-category $\mathcal{C}_T$ which we will be seen to be equivalent to $Mod\, Q_T/Rel_\triangle$.

The objects of $\mathcal{C}_T$ are formal positive linear combinations of positive roots. The indecomposable elements of $Mod\, Q_T/Rel_\triangle$ will correspond to the positive roots and a direct sum of indecomposables correspond to a formal linear combination of positive roots.

It is enough to define the morphisms between positive roots, by additivity. In order to do this we have to define the *pivoting paths* and the *mesh relations*.

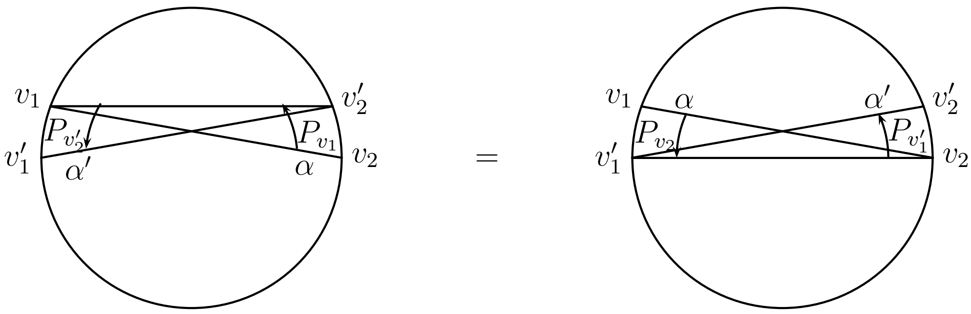
Definition 4.2.1. Let α and α' be two positive roots, i.e., two diagonals of P_{n+3} that doesn't belong to the triangulation T.

1. We say that α is related to α' by a *pivoting elementary move* if the corresponding diagonals have a vertex in common, called the pivot, the other vertices of α and α' are the vertices of a border edge of P_{n+3} and the rotation around the pivot from α to α' is anti-clockwise. The pivoting elementary move with pivot v is denoted by P_v.

2. A *pivoting path* from α to α' is a sequence of pivoting elementary moves starting at α and ending at α'.

Definition 4.2.2. Let α, α' be two positive roots such that α' is obtained from α by two consecutive pivoting elementary moves with distinct pivots. The *mesh relation* for this couple of positive roots is given by $P_{v_2'} P_{v_1} =$

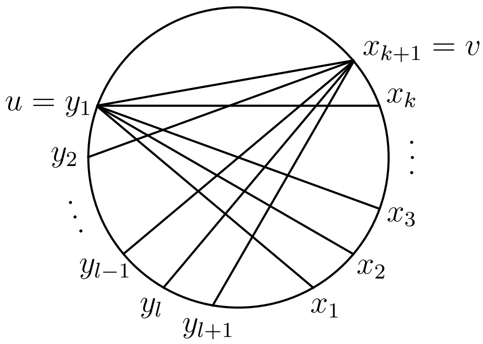
$P_{v'_1} P_{v_2}$, where v_1, v_2 (respectively v'_1, v'_2) are the vertices of α (respectively α') such that $P_{v'_2} P_{v_1}(\alpha) = P_{v'_1} P_{v_2}(\alpha) = \alpha'$. The intermediate arrows $v_1 v'_2$, $v'_1 v_2$ can be negative roots or border edges, but in this case, the corresponding term in the mesh relation is replaced by zero, that is, if $v_1 v'_2$ is either a negative root or a border edge of the polygon, the mesh relation will be $0 = P_{v'_1} P_{v_2}$ and if $v'_1 v_2$ is a negative root or a border edge, the mesh relation is given by $P_{v'_2} P_{v_1} = 0$. The mesh relation $P_{v'_2} P_{v_1} = P_{v'_1} P_{v_2}$ is illustrated in the following figure.



The set of morphisms from α to α', $Hom_{\mathcal{C}_T}(\alpha, \alpha')$, is the quotient of the vector space over $\mathbb{C}$ spanned by pivoting paths from α to α' by the subspace generated by mesh relations.

Lemma 4.2.3. *Let α, α' be two positive roots and $P = P_m P_{m-1} \dots P_1$ be a nonzero pivoting path from α to α'. Then $P(\alpha) = P_v^{m-k} P_u^k(\alpha)$ for some natural k and some vertices u and v.*

Proof. We prove this by induction on m using the mesh relations. If $m = 1$ there is nothing to prove. Let $m > 1$ and suppose the lemma is true for $m-1$. Hence there exists $k \leq m$ such that $P_{m-1} \dots P_1(\alpha) = P_v^{m-k-1} P_u^k(\alpha)$. Write $l = m - k - 1$ for short. The following picture illustrates this pivoting path, where $x_1, x_2, \dots, x_{k+1} = v$ (respectively $u = y_1, y_2, \dots y_l, y_{l+1}$) are the end-points of the positive roots involved in the pivoting path $P_u^k(\alpha)$ (respectively $P_v^l(P_u^k(\alpha))$) other than u (respectively other than v).



If $P_m = P_v$ we are done. If not, then the pivot of P_m must be y_{l+1}. Let w be the endpoint of α' other than y_{l+1}. Observe that w is the vertex of the polygon anticlockwise from v. By the mesh relations, we have

$$\begin{aligned} P_m\, P_{m-1} \ldots P_1(\alpha) &= P_{y_{l+1}}\, P_v^l\, P_u^k(\alpha) \\ &= P_w\, P_{y_l}\, P_v^{l-1}\, P_u^k(\alpha). \end{aligned}$$

Using the mesh relations repeatedly, we get

$$\begin{aligned} P_m \ldots P_1(\alpha) &= P_w\, P_{y_l}\, P_v^{l-1}\, P_u^k(\alpha) \\ &= P_w^2\, P_{y_{l-1}}\, P_v^{l-2}\, P_u^k(\alpha) \\ &\;\vdots \\ &= P_w^l\, P_{y_1}\, P_u^k \\ &= P_w^{m-k-1}\, P_u^{k+1}. \end{aligned}$$

Thus the result holds for m and the lemma is proved. □

4.3 The equivalence between $Mod\, Q_T/Rel_\triangle$ and $\mathcal{C}_T$

In this section we prove the key result of [13], stated below.

Theorem 4.3.1. *$Mod\, Q_T/Rel_\triangle$ and $\mathcal{C}_T$ are equivalent categories.*

The equivalence of the categories will be given by the $\mathbb{C}$-linear additive functor $\Theta : \mathcal{C}_T \to Mod\, Q_T/Rel_\triangle$ defined as follows. For each positive root α, $\Theta(\alpha)$ is the module $(M^\alpha, f^\alpha) = (M_i^\alpha, f_{ij}^\alpha)$ where:

$$M_i^\alpha = \begin{cases} \mathbb{C} & \text{if } i \in Supp\,\alpha \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad f_{ij}^\alpha = \begin{cases} id_{\mathbb{C}} & \text{if } M_i^\alpha = \mathbb{C} = M_j^\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

For a pivoting elementary move $P : \alpha \to \alpha'$, $\Theta(P)$ is defined to be the morphism from (M^α, f^α) to $(M^{\alpha'}, f^{\alpha'})$ such that

$$\Theta(P)_i = \begin{cases} id_{\mathbb{C}} & \text{if } M_i^\alpha = \mathbb{C} = M_i^{\alpha'}, \\ 0 & \text{otherwise.} \end{cases}$$

The definition of Θ is extended to arbitrary objects and morphisms of $\mathcal{C}_T$ by additivity. In order to check that Θ is well defined, we need to prove the following assertions:

1. For each positive root α, (M^α, f^α) is an object of $Mod\, Q_T/Rel_\triangle$, i.e., it satisfies the triangle relations.

2. If $P : \alpha \to \alpha'$ is a pivoting elementary move, then the diagram

$$\begin{tikzcd} M_j^\alpha \arrow[r, "f_{ji}^\alpha"] \arrow[d, "\Theta(P)_j"'] & M_i^\alpha \arrow[d, "\Theta(P)_i"] \\ M_j^{\alpha'} \arrow[r, "f_{ji}^{\alpha'}"] & M_i^{\alpha'} \end{tikzcd}$$

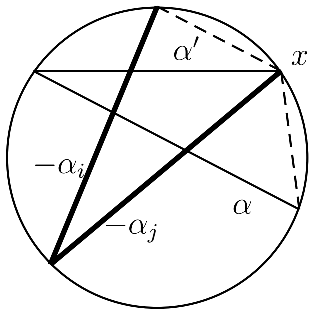
is commutative, for each arrow $j \to i$ in Q_T.

3. The mesh relations hold, i.e., if $P^2P^1(\alpha) = P^4P^3(\alpha) = \gamma$ is a mesh relation and β (resp. β') is the intermediate arrow of P^2P^1 (resp. P^4P^3), then $\Theta(P^2P^1) = \Theta(P^4P^3)$, i.e., $\Theta(P^2)\Theta(P^1) = \Theta(P^4)\Theta(P^3)$.

If there was a nonzero composition of two successive maps in a triangle in Q_T, i.e., if there was a triangle $i \to j \to k \to i$ such that $f_{jk}^\alpha f_{ij}^\alpha \neq 0$, then $f_{ij}^\alpha \neq 0$ and $f_{jk}^\alpha \neq 0$, which means that $M_i^\alpha = M_j^\alpha = M_k^\alpha = \mathbb{C}$. Therefore, α would cross the three sides of the triangle, which is impossible. This proves (1).

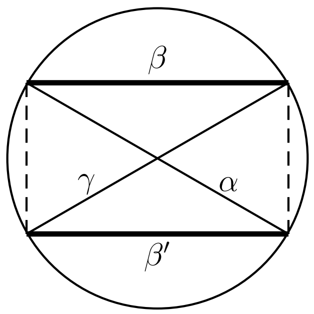
We will now prove (2). Let $P : \alpha \to \alpha'$ be a pivoting elementary move and $j \to i$ be an arrow of Q_T. If $M_j^\alpha = 0$ or $M_i^{\alpha'} = 0$ or if both M_i^α and $M_j^{\alpha'}$ are zero, the commutativity is clear. The commutativity is also obvious when M_i^α, M_j^α, $M_i^{\alpha'}$ and $M_j^{\alpha'}$ are all nonzero, since all the maps are $id_{\mathbb{C}}$.

The only case when the diagram is not commutative is when $M_j^\alpha = \mathbb{C} = M_i^{\alpha'}$ and exactly one of M_i^α, $M_j^{\alpha'}$ is not zero. However, this case is impossible. Indeed, suppose without loss of generality that $M_j^{\alpha'} = 0$ and $M_i^\alpha \neq 0$. So $i, j \in Supp\, \alpha$, $i \in Supp\, \alpha'$ and $j \notin Supp\, \alpha'$. Since $-\alpha_j$ crosses α but doesn't cross α' and $\alpha \to \alpha'$ is a pivoting elementary move, we have that $-\alpha_j$ and α' have a common vertex x. Since there is an arrow between i and j in Q_T, $-\alpha_i$ has a common vertex with $-\alpha_j$, which can't be x because $-\alpha_i$ crosses both α and α', and so we are in this situation:



and so $-\alpha_j < -\alpha_i$, which contradicts the orientation $j \to i$ in Q_T.

It only remains to check (3). Let $\alpha \xrightarrow{P^1} \beta$, $\beta \xrightarrow{P^2} \gamma$, $\alpha \xrightarrow{P^3} \beta'$, $\beta' \xrightarrow{P^4} \gamma$ be pivoting elementary moves with α, γ positive roots and $\beta \neq \beta'$. At least one of β, β' has to be positive, otherwise either α or γ had to be negative, since T is a triangulation (see the following figure).



Suppose, without loss of generality, that β is positive. Assume first that β' is also a positive root. We want to prove that $\Theta(P^2)\,\Theta(P^1) = \Theta(P^4)\,\Theta(P^3)$, that is, the following diagram commutes

$$\begin{tikzcd} M_i^{\alpha} \arrow[r, "\Theta(P^1)_i"] \arrow[d, "\Theta(P^3)_i"'] & M_i^{\beta} \arrow[d, "\Theta(P^2)_i"] \\ M_i^{\beta'} \arrow[r, "\Theta(P^4)_i"] & M_i^{\gamma} \end{tikzcd}$$

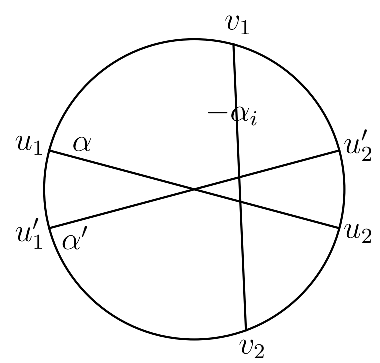
for all i. If $M_i^{\alpha} = 0$ or $M_i^{\gamma} = 0$, this is obvious. Let us consider the case when $i \in Supp\,\alpha \cap Supp\,\gamma$ then. Observe that any diagonal that crosses α and γ has to cross β and β' as well. Therefore, $i \in Supp\,\beta \cap Supp\,\beta'$. So the diagram commutes, since all the maps are the identity in $\mathbb{C}$.

Finally, consider the case when β' is negative or a border edge. In this case, the mesh relation is $P^2\,P^1 = 0$. So we have to show that $\Theta(P^2)_i\,\Theta(P^1)_i : M_i^\alpha \to M_i^\beta \to M_i^\gamma$ is zero for all i. Since β' is a negative root, there can't be a diagonal of T crossing both α and γ. Thus $Supp\,\alpha \cap Supp\,\gamma = \emptyset$, which means that either $M_i^\alpha = 0$ or $M_i^\gamma = 0$ for each i, and so the composition $\Theta(P^2)_i\,\Theta(P^1)_i$ is zero. This concludes the proof of (3) and, as a consequence, the proof that Θ is well defined.

It will be proved that Θ is fully faithful and dense. The equivalence between $\mathcal{C}_T$ and $Mod\,Q_T/Rel_\triangle$ follows by 1.1.14. The following lemmas will be useful to prove that Θ is fully faithful.

Lemma 4.3.2. *Let α and α' be two positive roots. There exists a nonzero morphism from α to α' in $\mathcal{C}_T$ if and only if the following conditions hold.*

- *There is a negative root $-\alpha_i$ that crosses both α and α', i.e., $i \in Supp\,\alpha \cap Supp\,\alpha'$,*
- *Let v_1, v_2 be the endpoints of $-\alpha_i$, u_1, u_2 be the endpoints of α and u_1', u_2' be the endpoints of α'. If we order these vertices in the counterclockwise order (in the positive trigonometric direction) starting at v_1, we get $v_1 < u_1 \leq u_1' < v_2 < u_2 \leq u_2'$. In other words, the relative position of α, α' and $-\alpha'$ are as in the following figure.*



In this case, $Hom_{\mathcal{C}_T}(\alpha, \alpha')$ is of dimension one.

Proof. In the situation of the diagram it is obvious that there exists a pivoting path from α to α' and all its intermediate arrows have to be positive roots, due to the position of $-\alpha_i$ and the fact that negative roots cannot cross each other nor the border edges. Hence there is a non-zero morphism from α to α'.

Conversely, suppose that $Hom_{\mathcal{C}_T}(\alpha, \alpha')$ is non-zero. Then, there is a pivoting path P from α to α'

$$\alpha = \alpha^0 \overset{P_1}{\to} \alpha^1 \overset{P_2}{\to} \dots \overset{P_m}{\to} \alpha^m = \alpha'$$

where the intermediate diagonals α^i are positive roots and the P_i are pivoting elementary moves. By 4.2.3, we have $P_m \dots P_1 = P_v^{m-k} P_u^k$, for a certain k, where u is one of the vertices of α and v is the common vertex of α^k and α'.

The positive roots α and α' must intersect, otherwise there wouldn't be such a pivoting path. Let V_1 (respectively V_2) be the set of vertices of the firsts k arrows of $P = P_v \dots P_v P_u \dots P_u$ other than u (respectively the last $m-k$ arrows of P other than v). It follows by the same argument used in the proof of 4.2.3 and by the fact that $P \neq 0$ that all the diagonals with one vertex in V_1 and the other in V_2 are positive. Hence the quadrilateral in the polygon formed by the vertices u_1, u_2', u_2 and u_1' is not crossed by any negative root from V_1 to V_2. This guarantees the existence of a negative root $-\alpha_i$, whose endpoints don't belong to V_1 nor V_2, crossing the quadrilateral, since T is a triangulation and so we are in the situation of the diagram.

If P' is another non-zero pivoting path from α to α', then P' is obtained from P by mesh relations, which means that P and P' are in the same class. Consequently, the vector space $Hom_{\mathcal{C}_T}(\alpha, \alpha')$ is one-dimensional. $\square$

Lemma 4.3.3. *The support of a positive root and the intersection of the supports of two positive roots are connected as subsets of Q_T.*

Proof. Let us prove first that, for each positive root α, $Supp\,\alpha$ is connected. Given two negative roots $-\alpha_i$ and $-\alpha_j$ that cross α, we want to prove that the corresponding vertices i, j in Q_T are connected by an unoriented path $p : i = i_1, i_2, \dots, i_p = j$, such that $-\alpha_{i_l} \in Supp\,\alpha$ ($l \in [p]$). Since negative roots don't cross each other, $-\alpha_i$ and $-\alpha_j$ divide the polygon into three parts. According to the definition of Q_T, a path from i to j in Q_T must be contained in the middle part, which will be denoted by R_ij. We show the result by induction on the number m of negative roots in R_{ij}. If $m = 1$ then we have nothing to prove since $-\alpha_i = -\alpha_j$. Suppose now that $m > 1$ and the result is valid for $m - 1$. Denote by $\triangle$ the unique triangle in R_{ij} that contains $-\alpha_i$. If α didn't cross any of the borders of $\triangle$ other than $-\alpha_i$ then it wouldn't cross $-\alpha_j$. Since a diagonal cannot cross the three sides of a triangle, α has to cross exactly one of the other two sides of $\triangle$, and it can't be a border edge of the polygon. So it has to be a negative root $-\alpha_k$. As $-\alpha_i$ and $-\alpha_k$ bound the same triangle $\triangle$, there is an arrow between i and k in Q_T. Since $-\alpha_k \in Supp\,\alpha$, and there are $m-1$ negative roots in R_{kj}, where R_{kj} denotes the part of R_{ij} other than $\triangle$, it follows from induction that there is an unoriented path from k to j in $Q_T \cap Supp\,\alpha$, as required.

We now prove that the intersection of supports is connected. Let α and α' be two positive roots and denote by S (respectively S') the support of α (respectively α'). Suppose, for a contradiction, that $S \cap S'$ is not connected. Let $i, k \in S \cap S'$ be two vertices that are not connected. It was seen that S and S' are connected. So there exist two paths: $p : i = i_1, i_2, \ldots, i_p = k$ in S and $p' : i = j_1, j_2, \ldots, j_q = k$ in S' that connect i and k. Let m be the smallest integer such that $i_{m+1} \neq j_{m+1}$. Since p is a path in Q_T, the diagonals $-\alpha_{i_{m-1}}$ and $-\alpha_{i_{m+1}}$ of T have a vertex in common with $-\alpha_{i_m}$ and so has $-\alpha_{j_{m+1}}$ because $-\alpha_{j_m} = -\alpha_{i_m}$ and p' is a path in Q_T. Since $i_{m-1}, i_m, i_{m+1} \in S$ and a positive root can't cross the three sides of a triangle, $-\alpha_{i_{m-1}}, -\alpha_{i_m}, -\alpha_{i_{m+1}}$ cannot bound the same triangle. The same happens to $-\alpha_{i_{m-1}}, -\alpha_{i_m}, -\alpha_{j_{m+1}}$ as $i_{m-1} = j_{m-1}, i_m = j_m, j_{m+1} \in S'$. Hence $-\alpha_{i_m}, -\alpha_{i_{m+1}}$ and $-\alpha_{j_{m+1}}$ form a triangle $\triangle$ in T. So $i_{m+1} \in S \setminus S'$ and $j_{m+1} \in S' \setminus S$. Excluding the triangle $\triangle$, the polygon is divided into three parts: $R_{i_m}, R_{i_{m+1}}$ and $R_{j_{m+1}}$, such that R_l contains $-\alpha_l$, $l = i_m, i_{m+1}, j_{m+1}$. Clearly, $R_{i_{m+1}}$ must contain all $-\alpha_{i_l}$, $l \geq m+1$ and $R_{j_{m+1}}$ must contain $-\alpha_{j_l}$ with $l \geq m+1$. However, $R_{i_{m+1}} \supseteq -\alpha_{i_p} = -\alpha_k = -\alpha_{j_q} \subseteq R_{j_{m+1}}$, a contradiction. □

Lemma 4.3.4. *For two positive roots α and α', there is a nonzero morphism from (M^α, f^α) to $(M^{\alpha'}, f^{\alpha'})$ in $Mod\,Q_T/Rel_\triangle$ if and only if:*

1. *$S \cap S'$ is not empty,*
2. *There is no arrow from $S \setminus S'$ to $S \cap S'$ in Q_T,*
3. *There is no arrow from $S \cap S'$ to $S' \setminus S$ in Q_T,*

where $S = Supp\,\alpha$ and $S' = Supp\,\alpha'$.

In this case, $Hom_{Mod\,Q_T/Rel_\triangle}((M^\alpha, f^\alpha), (M^{\alpha'}, f^{\alpha'}))$ is of dimension one.

Proof. Suppose $Hom_{Mod\,Q_T/Rel_\triangle}((M^\alpha, f^\alpha), (M^{\alpha'}, f^{\alpha'}))$ is not zero. Let $P = (P_i)_i$ be a non-zero element of this vector space. There is at least one $i \in (Q_T)_0$ such that $P_i : M_i^\alpha \to M_i^{\alpha'}$ is not zero. In particular, M_i^α and $M_i^{\alpha'}$ are both non-zero, which means that $-\alpha_i \in S \cap S'$. Hence, the first condition holds. Now, suppose for a contradiction that there exists an arrow $i \to j$ in Q_T with $i \in S \setminus S'$ and $j \in S \cap S'$. So, $M_i^\alpha = M_j^\alpha = M_j^{\alpha'} = \mathbb{C}$, $M_i^{\alpha'} = 0$, $f_{ij}^\alpha = id_{\mathbb{C}}$, $f_{ij}^{\alpha'} = 0$ and according to the definition of morphism of representations, the following square commutes.

$$\begin{tikzcd} M_i^{\alpha} \arrow[r, "id_{\mathbb{C}}"] \arrow[d, "P_i"'] & M_j^{\alpha} \arrow[d, "P_j"] \\ 0 \arrow[r] & M_j^{\alpha'} \end{tikzcd}$$

Therefore P_i and P_j are both zero. Let k be a diagonal of T that crosses both α and α'. Thus, since $k, j \in S \cap S'$, there is an unoriented path $k = k_1, \ldots, k_m = j$ in Q_T with all $k_l \in S \cap S'$, by 4.3.3. Because the diagram

$$\begin{tikzcd} M_j^{\alpha} \arrow[r, dash, "id_{\mathbb{C}}"] \arrow[d, "P_j"'] & M_{k_{m-1}}^{\alpha} \arrow[r, dash, "id_{\mathbb{C}}"] \arrow[d, "P_{k_{m-1}}"] & \cdots \arrow[r, dash, "id_{\mathbb{C}}"] & M_k^{\alpha} \arrow[d, "P_k"] \\ M_j^{\alpha'} \arrow[r, dash, "id_{\mathbb{C}}"] & M_{k_{m-1}}^{\alpha'} \arrow[r, dash, "id_{\mathbb{C}}"] & \cdots \arrow[r, dash, "id_{\mathbb{C}}"] & M_k^{\alpha'} \end{tikzcd}$$

is commutative and $P_j = 0$, it follows that all P_{k_l} are zero, in particular $P_k = 0$. Since $P_l : M_l^{\alpha} \to M_l^{\alpha'}$ is zero for all $l \in (Q_T)_0 \setminus (S \cap S')$, because in this case either $M_l^{\alpha} = 0$ or $M_l^{\alpha'} = 0$, we have that $P = 0$ which contradicts the hypothesis. The argument to prove the third condition is similar.

Conversely, let α, α' be positive roots such that $(1), (2)$ and (3) hold. Consider $P \in Hom_{Mod\,Q_T/Rel_{\triangle}}((M^{\alpha}, f^{\alpha}), (M^{\alpha'}, f^{\alpha'}))$ defined as follows

$$P_i = \begin{cases} id_{\mathbb{C}} & \text{if } i \in S \cap S' \\ 0 & \text{otherwise.} \end{cases}$$

Since $S \cap S' \neq \emptyset$ by (1), P is non-zero. We have to check that P is indeed a morphism of representations, i.e., that the square

$$\begin{tikzcd} M_i^{\alpha} \arrow[r, "f_{ij}^{\alpha}"] \arrow[d, "P_i"'] & M_j^{\alpha} \arrow[d, "P_j"] \\ M_i^{\alpha'} \arrow[r, "f_{ij}^{\alpha'}"] & M_j^{\alpha'} \end{tikzcd}$$

commutes for all $i \to j$ in Q_T. The only cases when this doesn't happen is when $M_i^{\alpha} = M_i^{\alpha'} = M_j^{\alpha'} = \mathbb{C}$ and $M_j^{\alpha} = 0$ or when $M_i^{\alpha} = M_j^{\alpha} = M_j^{\alpha'} = \mathbb{C}$ and $M_i^{\alpha'} = 0$, i.e. when $i \in S \cap S'$ and $j \in S' \setminus S$ or when $i \in S \setminus S'$ and $j \in S \cap S'$. But by (2) and (3) these cases can't happen, so the diagram is commutative.

To prove that the dimension of $Hom_{Mod\,Q_T/Rel_{\triangle}}((M^{\alpha}, f^{\alpha}), (M^{\alpha'}, f^{\alpha'}))$ is one if this vector space is not zero, let $P = (P_i)_{i \in (Q_T)_0}$ be a non-zero morphism between (M^{α}, f^{α}) and $(M^{\alpha'}, f^{\alpha'})$. Thus there exists $i \in S \cap S'$ such that $P_i \neq$

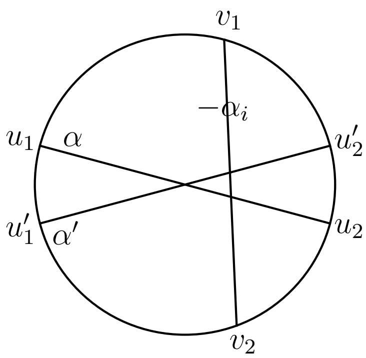
0. Let $j \in S \cap S'$. By 4.3.3 there is an unoriented path $i = k_0, k_1, \dots, k_m = j$ in $S \cap S'$ and the diagram

$$\begin{tikzcd} M_i^{\alpha} \arrow[r, dash, "id_{\mathbb{C}}"] \arrow[d, "P_i"'] & M_{k_1}^{\alpha} \arrow[r, dash, "id_{\mathbb{C}}"] \arrow[d, "P_{k_1}"] & \cdots \arrow[r, dash, "id_{\mathbb{C}}"] & M_j^{\alpha} \arrow[d, "P_j"] \\ M_i^{\alpha'} \arrow[r, dash, "id_{\mathbb{C}}"] & M_{k_1}^{\alpha'} \arrow[r, dash, "id_{\mathbb{C}}"] & \cdots \arrow[r, dash, "id_{\mathbb{C}}"] & M_j^{\alpha'} \end{tikzcd}$$

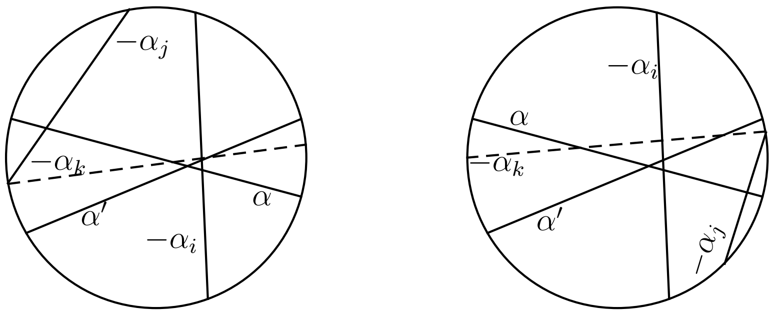
is commutative. Hence $P_{k_l} = P_i$ for $l \in [m]$ and in particular, $P_j = P_i$. For $k \in (Q_T)_0 \setminus S \cap S'$, we have $P_k = 0$. So, the result follows since all vector spaces $M_i^{\alpha}, M_i^{\alpha'}$ are of dimension zero or one. $\square$

Lemma 4.3.5. *The conditions in 4.3.2 and in 4.3.4 are equivalent. As a consequence, for any pair of positive roots α, α', the dimension of $Hom_{\mathcal{C}_T}(\alpha, \alpha')$ equals the dimension of $Hom_{Mod\, Q_T/Rel_{\triangle}}((M^{\alpha}, f^{\alpha}), (M^{\alpha'}, f^{\alpha'}))$, which is either zero or one.*

Proof. Let α, α' be two positive roots such that there is a negative root $-\alpha_i$ that crosses them and such that the relative position of these diagonals are as in the following figure.



It is obvious that $S \cap S' \neq \emptyset$. Suppose now, for a contradiction, that there is an arrow $j \to k$ in Q_T from $S \setminus S'$ to $S \cap S'$. Then $-\alpha_j$ only crosses α, $-\alpha_k$ crosses α and α' and $-\alpha_k < -\alpha_j$. Thus we are in one of these situations

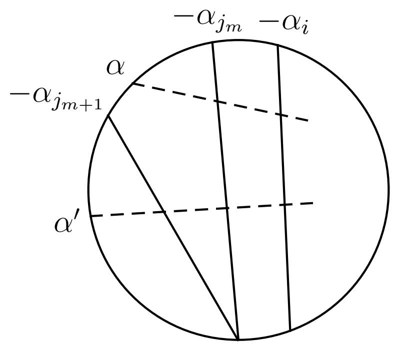


Either way, $-\alpha_k$ has to cross $-\alpha_i$, which is impossible. The proof that there can't be an arrow from $S \cap S'$ to $S' \setminus S$ in Q_T is similar.

Conversely, suppose that α, α' are such that $S \cap S' \neq \emptyset$ and there is no arrow neither from $S \setminus S'$ to $S \cap S'$ nor from $S \cap S'$ to $S' \setminus S$ in Q_T. Then there exists $-\alpha_i$ in $S \cap S'$. This diagonal divides the polygon in two parts. Denote the right-hand side by R_r and the left-hand side by R_l. Since $-\alpha_i$ crosses α and α', each of these parts contains exactly one vertex of α and exactly one vertex of α'. Denote by $S_l = \{-\alpha_i = -\alpha_{i_1}, -\alpha_{i_2}, \ldots, -\alpha_{i_p}\}$ the set of negative roots in R_l that cross α and suppose the elements of S_l are in order, in the sense that there can't be a diagonal $-\alpha_{i_m}$ between $-\alpha_{i_l}$ and $-\alpha_{i_{l+1}}$, for all $l = 1, \ldots p-1$. Analously, denote by $S'_l = \{-\alpha_i = -\alpha_{j_1}, -\alpha_{j_2}, \ldots, -\alpha_{j_q}\}$ the set of negative roots in R_l crossing α' in such an order. Let m be the greatest integer such that $-\alpha_{i_m} = -\alpha_{j_m}$. There are four cases to analyse:

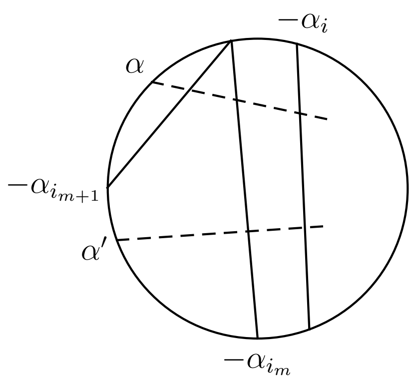
1. $m = p = q$. Suppose, for a contradiction, that α and α' don't have the vertices in R_l in common. Then the vertices of α and α' in R_l and the vertices of the diagonal $-\alpha_{i_m}$ form a quadrilateral, which must have a negative root crossing it due to the triangulation. But such negative root can't intersect $-\alpha_{i_m}$, so it must cross either α or α' or both, contradicting the hypothesis. Therefore α and α' share the same vertex in R_l, in other words, we meet α and α' at the same time going from the endpoints of $-\alpha_i$ in positive direction.

2. If $m = p < q$, then $-\alpha_{j_{m+1}} \in S' \setminus S$ and $-\alpha_{j_m} \in S \cap S'$. Given the order in S'_l, $-\alpha_{j_m}$ and $-\alpha_{j_{m+1}}$ bound the same triangle. Hence the corresponding edge in Q_T is oriented $j_{m+1} \to j_m$ by 4.3.4 (3). As the figure illustrates



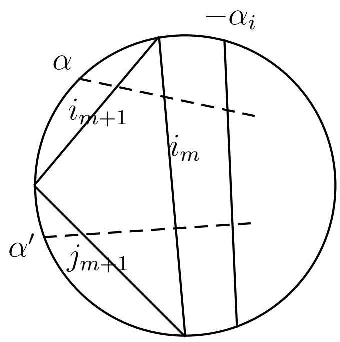
we meet first α and then α' when going from the endpoints of $-\alpha_i$ in positive direction on the boundary of R_l.

3. If $m = q < p$, then $-\alpha_{i_{m+1}} \in S \setminus S'$ and $-\alpha_{i_m} \in S \cap S'$. Given the order in S_l these two diagonals bound the same triangle in T. Hence the corresponding edge in Q_T is oriented $i_m \to i_{m+1}$. So we are in this situation:



Hence, in this situation we meet α first as well.

4. If $m < p$ and $m < q$, then $-\alpha_{i_{m+1}}, -\alpha_{i_m}, -\alpha_{j_{m+1}}$ are three different diagonals that bound the same triangle in T. As in the previous cases, the corresponding edges in Q_T are oriented $i_m \to i_{m+1}$ and $j_{m+1} \to j_m$. This implies once more that going from the endpoints of $-\alpha_i$ in positive direction on the boundary of R_l, we meet α first and then α':



Analogously, using the same results in R_r, we first meet α when going from the endpoints of $-\alpha_i$ anti-clockwise on the boundary of R_r, which proves the relative position of α, α' and $-\alpha_i$ in 4.3.2, as required. □

Proposition 4.3.6. *The functor Θ is fully faithful.*

Proof. By additivity, it is sufficient to prove that

$$\Theta_{\alpha,\alpha'} : Hom_{\mathcal{C}_T}(\alpha, \alpha') \to Hom_{Mod\,Q_T/Rel_\triangle}((M^\alpha, f^\alpha), (M^{\alpha'}, f^{\alpha'}))$$
$$P \mapsto \Theta(P)$$

is bijective, for any two positive roots α, α'. By Lemmas 4.3.2, 4.3.4 and 4.3.5, the vector spaces $Hom_{\mathcal{C}_T}$ and $Hom_{Mod\,Q_T/Rel_\triangle}((M^\alpha, f^\alpha), (M^{\alpha'}, f^{\alpha'}))$ are both zero or one dimensional. So, we only have to check that the image of a non-zero morphism is a non-zero morphism. Let

$$P = \alpha = \alpha^1 \xrightarrow{P_1} \cdots \xrightarrow{P_{m-1}} \alpha^m = \alpha'$$

be a non-zero pivoting path from α to α'. By 4.3.2, α and α' intersect and all the positive roots α^k, with $k \in [m]$, are contained in the quadrilateral formed by the endpoints of α and α', and so the negative root $-\alpha_i$ that crosses α and α' must cross all the α^k as well. Therefore $\Theta(P_l)_i = id_{\mathbb{C}}$, for each $l \in [m-1]$. Consequently, $\Theta(P)_i = id_{\mathbb{C}}$, so $\Theta(P)$ is not zero. □

It now remains to prove that Θ is dense, that is, that each indecomposable module in $Mod\,Q_T$ is the image of a positive root under Θ. This will be done using Auslander-Reiten theory.

Throughout, we set $Q = Q_T$, for short, since the triangulation T is fixed. Before proving the theorem, let us compute the projective and injective modules of $Mod\,Q/Rel_\triangle$, which will be needed in the sequel. Let i, l be two vertices of Q. There is at most one path from i to l modulo the triangle relations. So, according to 2.2.4 (1), $(P_i)_l = \mathbb{C}$ if there exists such an oriented path from i to l and $(P_i)_l = 0$ otherwise. Similarly, by 2.2.4 (2), $(I_i)_l = \mathbb{C}$ if there is an oriented path in Q modulo the triangle relations from l to i and $(I_i)_l = 0$ otherwise. The maps of P_i and of I_i are $id_{\mathbb{C}}$ whenever possible and zero otherwise. Note that these modules are *multiplicity free*, i.e., each vector space of the representation is zero or one dimensional.

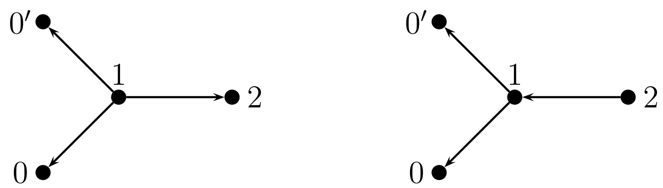
We say that a $Q/Rel_{\triangle}$-module M is of *type* A if the full subquiver of Q on the *support* of M, which is defined to be the set of vertices of Q for which the component of M is nonzero, is of type A_k for some $k \in \mathbb{N}$.

Lemma 4.3.7. *Let M be an indecomposable $Q/Rel_{\triangle}$-module of type A and let N be any indecomposable $Q/Rel_{\triangle}$-module. If there exists an irreducible morphism from N to M or from M to N, then N is of type A.*

Proof. Suppose first that $Hom_{Mod\,Q/Rel_{\triangle}}(N, M)$ contains an irreducible map. Let Q' denote the full subquiver of Q given by the support of M, $Q'_0 := \{1, \ldots, m\}$. By hypothesis, Q' is of type A_m. Renumbering the vertices one can suppose that the extremal vertices of Q' are 1 and m and the edges link i with $i \pm 1$.

Note that as M is an indecomposable Q-module of type A, so it is also an indecomposable Q'-module of type A. In type A_n, $\Phi^+ = \{\alpha_i + \ldots + \alpha_j : 1 \leq i \leq j \leq n\}$, where $\{\alpha_i | i \in [n]\}$ is a simple system, is the set of positive roots. By Gabriel's Theorem, M corresponds to a positive root, i.e., $\underline{dim}(M) := \sum_{i=1}^{n} dim(M_i)\alpha_i = \alpha_i + \ldots + \alpha_j$, for certains i, j such that $1 \leq i \leq j \leq n$. Hence, M is multiplicity free.

Suppose that M is not simple ($m > 1$). Observe that there exists at most one vertex in $Q_0 \setminus Q'_0$, which will be denoted by 0 (resp. $m + 1$), such that $1 \to 0$ (resp. $m \to m + 1$) and such that there exist no other edges between Q'_0 and 0 (resp. $m + 1$). In fact, if there existed two vertices, 0 and $0'$, satisfying such conditions, we would be in one of these situations:

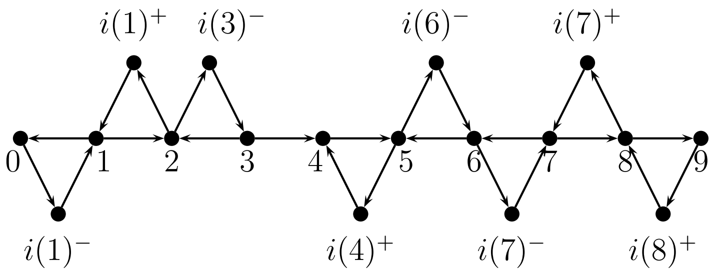


However, the first situation can't happen because a vertex of Q can only be the source of two arrows at most, and in the second situation there must be another arrow , which can't be between 0 and $0'$ since the triangles in Q are oriented, so either 0 or $0'$ is linked to 2, which contradicts the hypothesis.

Let $k \in Q'_0$. Let $i(k)$ be a vertex in $Q_0 \backslash Q'_0$ such that $i(k) \to k$ is an arrow of Q. If k is a sink in Q' and $k \neq 1, m$, then there is no such vertex. For the rest of the vertices, there are at most two vertices satisfying such conditions. If k is a source, then there must be an arrow from $k-1$ to $i(k)$ or from $k+1$ to $i(k)$. Observe that there can't be these two arrows simultaneously. If k is such that $k-1 \leftarrow k \leftarrow k+1$, then there must be an arrow from $k-1$ to $i(k)$ since $k+1$ and $k-1$ are not connected, and finally, if k is such that $k-1 \to k \to k+1$, then $k+1 \to i(k)$ is an arrow of Q.

We denote by $i(k)^+$ (resp. $i(k)^-$) the vertex such that $k+1 \to i(k)^+$ (resp. $k-1 \to i(k)^-$), if it exists.

We follow the argument of the proof on the following example of Q', where $m = 8$.



If M is projective, then N is a direct summand of the radical of M by 2.4.16. In particular, N is a submodule of M. Hence $Supp\, N \subseteq Q'_0$. As N is indecomposable, N must be of type A.

Suppose now that M is not projective. Let us prove that the sequence $P^1 \to P^0 \to M \to 0$, with $P^0 := \oplus_{i \in R} P_i$, where R is the set of sources of Q', and $P^1 := \oplus_{j \in R'} P_j$, where R' is the union of $\{0, m+1\}$ with the sinks of Q' not equal to 1 or m, is a minimal projective presentation of M. We have that for each $i \in R$ and $k \in Q_0$, $dim(P_i)_k = 1$ if there is a path from i to k in Q, not going through two sides of a triangle and $dim(P_i)_k = 0$ else. So $dim(P^0)_k$ equals the number of such paths from sources of Q' to k. It is obvious that P^0 and P^1 are projective modules. For each $i \in R$ consider the morphism $\psi_i : P_i \to M$ such that $(\psi_i)_k : (P_i)_k \to M_k$ is the identity in $\mathbb{C}$ when possible and zero else. Observe that each ψ_i is nonzero since $(P_i)_i = \mathbb{C}$ for each source i in Q'. The morphism $p_0 = \oplus_{i \in R} \psi_i : P^0 \to M$ is surjective since for each $k \in Q'_0$, there exists at least one source i in Q'_0 such that $(P_i)_k \neq 0$. Note that it is necessary to take all P_i, with $i \in R$, as summands of P^0 for the surjectivity of p_0. Otherwise, if there was a source j in Q' such that P_j is not a summand of P^0, then $(P^0)_j = 0$ and p_0 wouldn't be surjective. It remains to prove that $Ker\, p_0$ is superfluous in P^0 in order to see that P^0 is a projective cover of M. Write $K = Ker\, p_0$, for short. Let us compute K. For $j \in R$ and $j \in Q'_0$ such that $j-1 \to j \to j+1$ or $j-1 \leftarrow j \leftarrow j+1$, $dim(P^0)_j = 1$ and so $dim\, K_j = dim(P^0)_j - dim\, Im\,(p_0)_j = 0$. If j is a sink in Q' different from 1 or m, then it is reached by two sources and so $dim(P^0)_k = 2$. Thus, $dim\, K_j = 1$. Let us analyse now the vertices j outside Q'_0. In this case the dimension of K_j equals the dimension of $(P^0)_j$. Let $k \in Q'_0$ such that $k-1 \to k \to k+1$ (resp. $k-1 \leftarrow k \leftarrow k+1$) and denote by $j(k)$ the successor of k outside Q', if such a vertex exists. Observe that there is at most one vertex satisfying such conditions. We must have $k-1 \leftarrow j(k)$ (resp. $k+1 \leftarrow j(k)$). Consequently, $(P^0)_{j(k)} = 0$, due to the triangle relations. So the multiplicity of K is zero at any vertex along a path out of Q' with origin in $j(k)$. Let now k be a sink in Q'_0 other than 1 and m. Denote by $j(k)^-$ (resp. $j(k)^+$) the successor of k such that $j(k)^- \to k-1$ (resp. $j(k)^+ \to k+1$), if there exist such vertices. The source of Q' preceeding k gives rise to non-zero spaces for K along the path beginning at $j(k)^+$, but not along the path beginning at $j(k)^-$, due to the triangle relations. Analogously, the source of Q' succeeding k gives rise to non-zero spaces for K along the path beginning at $j(k)^-$, but not along the path beginning at $j(k)^+$. Hence, we get multiplicity 1 in K at all vertices along the paths (that doesn't go through two sides of a triangle) beginning at $j(k)^+$ and at $j(k)^-$. Let us consider now the extremes of Q'. Suppose that the vertex 1 is a source of Q'. Then the only successor of 1 outside Q' is 0. If there exist two paths starting at 0, one of them doesn't count for the multiplicity of K due to the triangle relations and the fact that there must be an arrow from one of the successors of 0 to 1. Along the other path, the multiplicity of the kernel is one. Finally, suppose 1 is a sink of Q'. Then the first source of Q' gives rise to non-zero spaces for K along the path beginning at 0. Thus we get multiplicity 1 along the path with origin in 0. If there exists another successor $0'$ of 1, then there must be an arrow from $0'$ to 2, and so the multiplicity of K at the vertices along the path starting at $0'$ is zero. For m is similar. So, $dim\, K_i$ is zero or one at every vertex i. As far as maps are concerned, they are the identity in $\mathbb{C}$ where possible and 0 else. We check this only for maps between sinks in Q' and their successors out of Q', the other cases are trivial. Let then k be a sink in Q'. Let k^+ (resp. k^-) be the source of Q' preceding (resp. succeeding) k. In P^0 we have this situation:

$$\begin{tikzcd}
 & & \mathbb{C} \\
\mathbb{C} \arrow[r] & \mathbb{C}^2 \arrow[ur, "{[1\,0]}"] \arrow[dl, "{[0\,1]}"'] & \mathbb{C} \arrow[l] \\
\mathbb{C} & &
\end{tikzcd}$$

Note that $(1,-1) \in \mathbb{C}^2$ belongs to the kernel of $(p_0)_k$. So $(1,-1)$ spans K_k and maps to a generator 1 of $K_{j(k)^+}$ and to a generator -1 of $K_{j(k)^-}$. Thus the Kernel is of the form:

$$\begin{tikzcd} & & \mathbb{C} \\ 0 \arrow[r] & \mathbb{C} \arrow[ur, "1"] \arrow[dl, "-1"] & 0 \arrow[l] \\ \mathbb{C} & & \end{tikzcd}$$

We can choose $\{-1\}$ to be the basis of each copy of $\mathbb{C}$ in the path starting at $j(k)^-$, since the tree structure ensures that we never get back to Q' or any other such path. So all the maps are the identity. In order to see that K is superfluous in P^0, let X be a submodule of P^0 such that $K + X = P^0$. We want to prove that $X = P^0$. At vertices of Q' that are not sinks of Q', the component of X must be $\mathbb{C}$. Note also that X cannot have multiplicity zero at sinks of Q'. Hence, the multiplicity of X at vertices along the paths (that doesn't go through two sides of a triangle) starting at 0, $m+1$ and at the successors of the sinks of Q', is one, since X embeds in P^0. It remains to prove that $X_k = \mathbb{C}^2$, for the sinks k of Q'. Let k be a sink in Q'. Suppose, for a contradiction, that $X_k = \mathbb{C}$. Consider the following commutative square from the embedding of X into P^0.

$$\begin{tikzcd} \mathbb{C} = X_{k-1} \arrow[r] \arrow[d] & \mathbb{C} = X_k \arrow[d] \\ \mathbb{C} = P^0_{k-1} \arrow[r] & \mathbb{C}^2 = P^0_k \end{tikzcd}$$

Let (a,b) in $\mathbb{C}^2 = P^0_k$ be the image of 1 in X_k, $x \in P^0_{k-1}$ be the image of 1 in X_{k-1} and y be the image of 1 under the map $X_{k-1} \to X_k$. Note that $x \neq 0$ as X_{k-1} embeds in P^0_{k-1}, but y can be zero. By the commutativity of the diagram, we have $(x,0) = (ya, yb)$. Hence $x = ya$ and $yb = 0$. If $y = 0$, then $x = 0$, a contradiction. So b must be 0. Similarly, we obtain that $a = 0$ from the commutativity of the diagram

$$\begin{tikzcd} \mathbb{C} = X_k \arrow[d] & \mathbb{C} = X_{k+1} \arrow[l] \arrow[d] \\ \mathbb{C}^2 = P^0_k & \mathbb{C} = P^0_{k+1}. \arrow[l] \end{tikzcd}$$

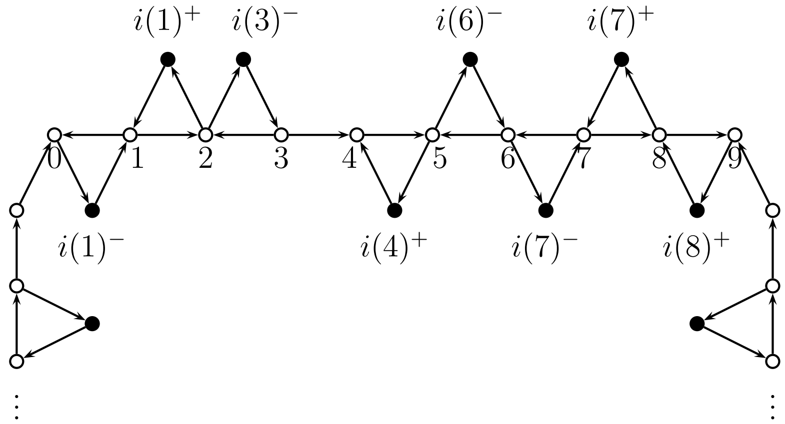
Therefore, the map from X_k to P^0_k os zero, which contradicts the fact that X embeds in P^0_k. We have already observed that $X_k \neq 0$, so X_k has dimension 2, as required.

The argument to prove that P^1 is a projective cover of $Ker\, p_0$ is similar to the computation of the projective cover of M just done.

By 2.4.10, the sequence $0 \to \tau M \to I^1 \to I^0$ is a minimal injective presentation of τM where $I^1 := \oplus_{j \in R'} I_j = \nu P^1$ and $I^0 := \oplus_{i \in R} I_i = \nu P^0$ (by 2.4.2).

Let us introduce some terminology. Denote by S_0, resp. S_{m+1}, the support of the injective module associated to 0, resp. $m+1$, in the full subquiver of Q with set of vertices $Q \setminus 1$, resp. $Q \setminus m$.

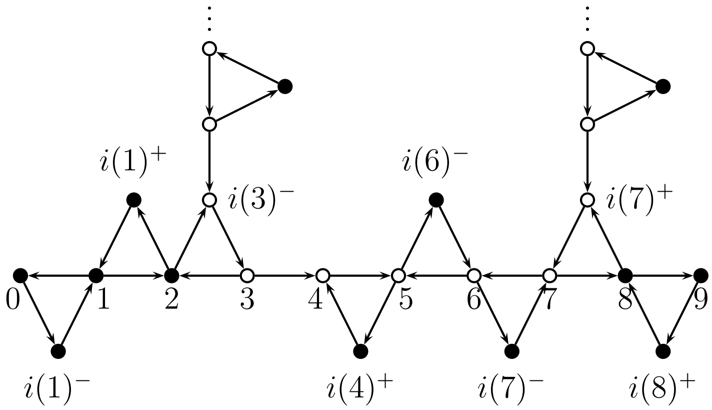
From the tree structure of Q, the set $Q_0'' := S_0 \cup Q_0' \cup S_{m+1}$ is the set of vertices of a full subquiver Q'' of Q of type A. In the following figure, Q'' is the set of the open nodes. The support S_0, resp. S_9, is formed by the open nodes not labelled on the left, resp. on the right.



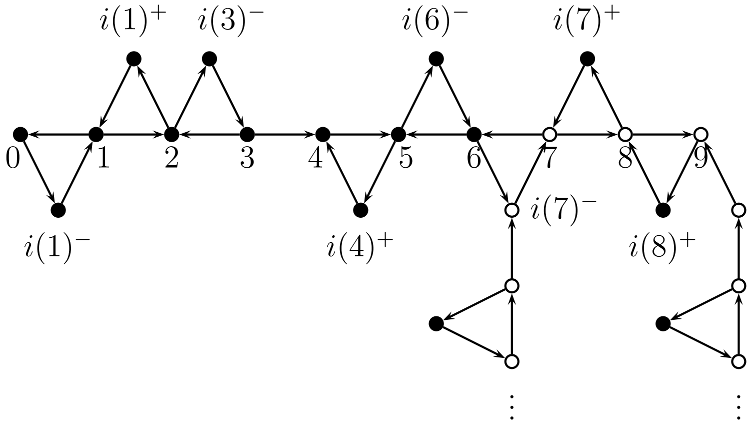
For each source k or $k = 1, m$, denote by $S_k^{\pm}$ the support of the injective module associated to $i(k)^{\pm}$ in the full subquiver of Q with set of vertices $Q_0 \setminus Q_0'$.

For k in R', let k^+, resp. k^-, be the source of Q' succeeding, resp. preceding, k. If there is no such source, we set $k^+ = m$, resp. $k^- = 1$.

Then, for each sink k of Q_0', the support of I_k is given by $Supp(I_k) = S_{k^-}^- \cup S_{k^+}^+ \cup (Supp(I_k) \cap Q_0')$. The vertices of the full subquivers $S_{k^-}^+$, $S_{k^+}^-$ and $S_l^{\pm}$, where $l \in \{k^- + 1, \ldots, k-1, k+1, \ldots, k^+ - 1\}$, don't belong to $(Supp(I_k))_0$ due to the triangle relations. In the example, consider the sink $k = 5$ of Q_0'. We have $k^- = 3$ and $k^+ = 7$. The support of I_5 is given by the set of the open nodes.



The support of I_0 is $S_l^+ \cup (Supp(I_0) \cap Q_0'')$, where l is the lowest source of Q_0'. Similarly, the support of I_{m+1} is given by $Supp(I_{m+1}) = S_g^- \cup (Supp(I_{m+1}) \cap Q_0'')$, where g is the greatest source of Q_0'. In our example, we have $g = 7$ and the support of 9 is given by the open nodes of the following picture.

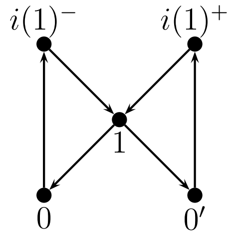


It is obvious that, for each source k of Q_0', the support of I_k contains $S_k^{\pm}$. Consequently, the support of τM is contained in Q_0''[1].

By 2.4.13 (a), there exists an almost split sequence $0 \to \mathcal{A}(M) \to X \to M \to 0$. Note that, in particular, the map $X \to M$ of this sequence is right minimal almost split. Since $Hom_Q(N, M) \neq 0$, we deduce that N is a direct summand of X, by 2.3.13 (b). Therefore $Supp(N) \subseteq Supp(X) \subseteq Supp(\mathcal{A}(M)) \cup Supp(M) \subseteq Q_0''$. Thus, N is an indecomposable Q_0''-module and the Lemma follows.

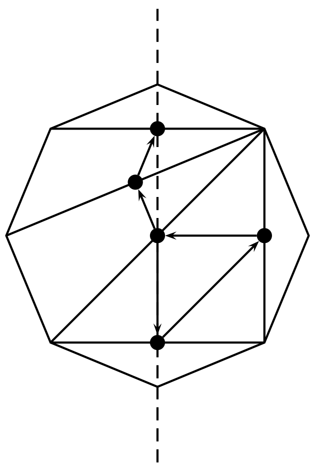
If M is simple, we are in the situation illustrated as follows.

[1] See 4.3.8 (1)

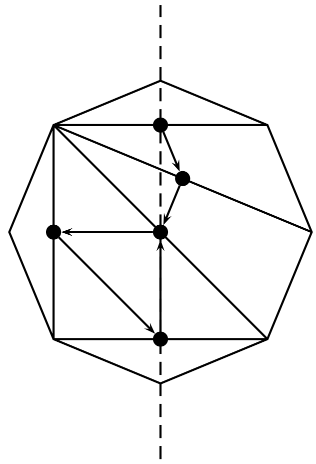


The vertex $0'$ plays the role of $m+1$ in the nonsimple case. The projectives of the minimal projective presentation of M, if this is nonprojective, are $P^0 = P_1$ and $P^1 = P_0 \oplus P_{0'}$. The proof of the Lemma for this case is exactly the same.

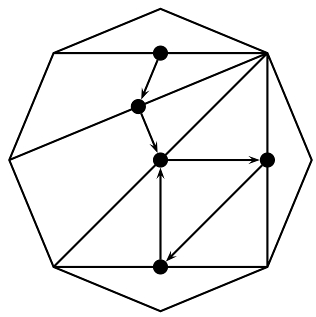
We have just proved the Lemma to the case when $Hom_{Mod\,Q_T/Rel_{\triangle}}(N, M)$ contains an irreducible. Suppose now that there is an irreducible morphism $f : M \to N$. Observe first that $Mod\,Q_T^{op} = Mod\,Q_{T^*}$, where T^* is the *mirror triangulation*, that is, T^* is T reflected in a line of symmetry of the polygon. Take as an example the following quiver Q_T:



Reflecting by the dashed line of symmetry and using the rule for the orientation of the arrows in the quiver Q_T, we obtain T^*:



This quiver is the same as the following (obtained by reflecting again by the previous line of symmetry):



which is the opposite of Q_T.

Moreover, Q_T^{op} is bound by the same relations. So $Q_T^{op} = Q_{T^*}$. The duality functor $D = Hom_{\mathbb{C}}(-, \mathbb{C})$ gives an (anti)-equivalence between $Mod\, Q_T$ and $Mod\, Q^{op} = Mod\, Q_{T^*}$. Let $f \in Hom_Q(M, N)$ be an irreducible. We claim that $D(f) \in Hom_{Q_{T^*}}(D(N), D(M))$ is irreducible. Indeed, if $D(f)$ was a section, there would exist a morphism $h : D(M) \to D(N)$ such that $hD(f) = id_{D(N)}$. But D is an (anti)-equivalence, so $h = D(g)$ for some $g \in Hom_{Q_T}(N, M)$ and so $id_{D(N)} = hD(f) = D(g)D(f) = D(fg)$ which implies that $fg = id_N$, i.e., f would be a retraction. Similarly, $D(f)$ is not a retraction. Finally, suppose $D(f) = g_1 g_2$, with $g_1 \in Hom_{Q_{T^*}}(D(X), D(M))$ and $g_2 \in Hom_{Q_{T^*}}(D(N), D(X))$, for some $X \in Mod_{Q_{T^*}}$. There is $f_1 \in Hom_{Q_T}(M, X)$ and $f_2 \in Hom_{Q_T}(X, M)$ such that $g_1 = D(f_1), g_2 = D(f_2)$. Thus, $D(f) = D(f_1)D(f_2) = D(f_2 f_1)$, which implies that $f = f_2 f_1$. Since f is irreducible, either f_2 is a retraction or f_1 is a section. By what we have just proved, we conclude that either $g_2 = D(f_2)$ is a section or $g_1 = D(f_1)$ is a retraction, as we wanted. Therefore, $Hom_{Q_{T^*}}(D(N), D(M))$ contains an irreducible. Since $D(M)$ is an indecomposable Q_{T^*}-module of type A and $D(N)$ is indecomposable, we conclude that $D(N)$ has type A, from what we have proved. Hence, N has type A. □

Remark 4.3.8.

1. I have been unable to prove that $Supp\, \tau M \subseteq Q_0''$. Here is one attempt. We want to prove that $(Ker\,(I^1 \to I^0))_j = 0$ for $j \notin Q_0''$, as $\tau M \cong Ker\,(I^1 \to I^0)$. Let $k \in R'$. Then the map $I^1 \to I^0$ cannot kill all of I_k. Otherwise, I_k would be contained in the Kernel of $I^1 \to I^0$, and so it would be a summand of τM. Since τM is indecomposable, it follows that $\tau M = I_k$ is injective, which implies that $I^0 = 0$, a contradiction. Therefore, the restriction of $I^1 \to I^0$ to I^k is non-zero. But $dim\, Hom(I_k, I_l) = dim\,(I_k)_l$, where $l \in R$. Since $Supp\, I_k \cap R = \{k^+, k^-\}$, we conclude that the only possible nonzero maps from I_k to summands of I^0 are to I_{k^+} and I_{k^-}. Let f be a nonzero map from I_k to I_{k^-} (if such a map exists). Suppose the map $f_j : (I_k)_j \to (I_{k^-})_j$ is zero, for $j \in S^-_{k^-}$. Then all the maps f_l, with $l \in S^-_{k^-}$, are zero, which implies that $f = 0$, a contradition. Therefore, as $dim\,(I_k)_l = 1 = dim\,(I_{k^-})_l$ for $l \in S^-_{k^-}$, we have $dim\, Ker\,(I_k \to I_{k^-})_l = 0$. Consequently, the map $I^1 \to I^0$ is nonzero on any vertex in $S^-_{k^-}$, since I_k is the only summand of I^1 that has support in $S^-_{k^-}$, which implies that the jth-component of the map $I^1 \to I^0$, for $j \in S^-_{k^-}$, is the same as the jth-component of the restriction to I_k. Similarly, if there exists a nonzero map from I_k to I_{k^+}, then the map $I^1 \to I^0$ is nonzero on any vertex in $S^+_{k^+}$. Hence, $Ker\,(I^1 \to I^0)$ has no support on $S^-_{k^-}$ or $Ker\,(I^1 \to I^0)$ has no support on $S^+_{k^+}$, but this argument doesn't prove that these two situation occur, as there could be a nonzero map from I_k to I_{k^+} while the map from I_k to I_{k^-} is zero, or vice versa.

   Here is another approach that was considered. Because $0 \to \tau M \to I^1 \xrightarrow{u} I^0$ is a minimal injective presentation of τM, The morphism $Im\, u \hookrightarrow I^0$ is minimal, that is, every nonzero submodule of I^0 must intersect the image of $u : I^1 \to I^0$. But I can't think of any useful nonzero submodule of I^0.

2. Lemma 4.3.7 was already known from a different context, the string algebras (cf. [12]). A path algebra $A = KQ/I$ is said to be a *string algebra* if it satisfies the following assertions:

   (a) Any vertex of Q is the source of at most two arrows and the target of at most two arrows, as well.

   (b) For each arrow α of Q, there is at most one arrow β whose source is the target of α and such that $\alpha\beta \notin I$, and there is at most one arrow γ whose target is the source of α and such that $\gamma\alpha \notin I$.

   (c) For each arrow α of Q, there is $n_\alpha \in \mathbb{N}$ such that any path $\alpha\alpha_2 \ldots \alpha_{n_\alpha}$ of Q whose first arrow is α contains a subpath in I, and there is $n'_\alpha \in \mathbb{N}$ such that any path $\alpha_1 \ldots \alpha_{n_\alpha - 1}\alpha$ of Q whose last arrow is α contains a subpath in I.

   It is easy to check that the path algebra associated to a quiver Q_T (where T is a triangulation of P_{n+3}) bound by the triangle relations is a string algebra.

   A *string* of length $n \geq 1$ is a path $c_1 \ldots c_n$, where each c_i is an arrow of Q or $c_i = \alpha^{-1}$ for some $\alpha : a \to b \in Q_1$, where α^{-1} is the arrow $b \to a$, such that the source of c_{i+1} is the target of c_i, for all $i = 1 \ldots n-1$, and such that the following conditions hold:

   (a) $c_{i+1} \neq c_i^{-1}$, for all $i = 1 \ldots n-1$,

   (b) There are no subpaths $c_i c_{i+1} \ldots c_{i+k}$ such that $c_i c_{i+1} \ldots c_{i+k}$ or $(c_i c_{i+1} \ldots c_{i+k})^{-1} := c_{i+k} \ldots c_{i+1} c_i$ belongs to I.

   We define also two strings of length 0 for each vertex a of Q, denoted by $1_{(a,1)}$ and $1_{(a,-1)}$, whose both source and target is a, and we define $1_{(a,i)}^{-1} = 1_{(a,-i)}$, for $i = -1, 1$.

   Let $C = c_1 \ldots c_n$ or $C = 1_{(a,t)}$, with $t = 1, -1$, $a \in Q_0$, be a string. Denote by $u(i)$ the target of c_{i-1}, $i = 2, \ldots, n+1$ and by $u(1)$ the source of C. The *string module* $M(C)$ associated to C is the representation $M(C) = (M(C)_v, f_\alpha)$ of Q bound by the relations in I defined as follows:

   (a) For each vertex v of Q, $M(C)_v$ is the K-vector space with dimension equal to the cardinality of the set $I_v = \{i \mid u(i) = v\}$.

   (b) Let $\alpha : x \to y$ be an arrow of Q and let $\{z_i \mid i \in I_x\}$ be a basis of $M(C)_x$ and $\{z'_i \mid i \in I_y\}$. Suppose $\alpha = c_i$ for some $i \in \{2, \ldots, n+1\}$. We have that x is the target of c_{i-1}, so $x = u(i)$, and y is the target of c_i, so $y = u(i+1)$. The K-linear map $f_\alpha : M(C)_x \to M(C)_y$ is defined by

$$f_\alpha(z_j) = \begin{cases} z'_{i+1} & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

Suppose now $\alpha = c_i^{-1}$ for some $c_i : x \to y$, $i \in \{2, \ldots, n+1\}$. Then $f_\alpha : M(C)_y \to M(C)_x$ is defined by $f_\alpha(z'_{i+1}) = z_i$ and $f_\alpha(z'_j) = 0$, for $j \neq i$.

If α is any another arrow of Q, then $f_\alpha := 0$.

It is easy to check that $M(C)$ satisfies the relations in I. Furthermore, $M(C^{-1})$ is isomorphic to $M(C)$ and $M(1_{(a,t)})$ is the simple representation corresponding to the vertex a.

Consider the path algebra $\mathbb{C}Q_T/Rel_\triangle$. In this case, none of the strings contains a cycle, because any cycle of Q_T contains a triangle relation, and according to the definition, the strings cannot contain any relation. We can deduce that the string modules are the same as the modules of type A for the path algebra $\mathbb{C}Q_T/Rel_\triangle$.

In [12] is given a list of almost split sequences, called canonical exact sequences, where the start and end terms and all summands of the middle terms are string modules, and it is proved that these canonical exact sequences are exactly the almost split sequences containing string modules. Lemma 4.3.7 follows from this result for the irreducible morphisms are components of morphisms in almost split sequences (see 2.3.2, 2.3.13 and 2.4.13).

Proof of Theorem 4.3.1.

We are now able to finish the proof of the Theorem 4.3.1. The objective is to prove that each indecomposable module in $Mod\, Q_T$ is the image of a positive root under Θ.

Let M be an indecomposable Q_T-module bound by the triangle relations of type A. By 4.3.7, the component of M in the Auslander-Reiten quiver contains only modules of type A. Thus, this component is finite and by 2.5.10, every indecomposable module is of type A and in particular is multiplicity free. So there exists a one-to-one correspondence between indecomposable $Q_T/Rel_\triangle$-modules and full subquivers of Q of type A with no paths going through two sides of a triangle. Since there is exactly one full subquiver of type A of Q_T, with no paths going through two sides of a triangle, with extremes i and j, for each pair i, j of vertices of Q_T, the number of these full subquivers of type A is equal to $\frac{n(n+1)}{2}$. So there are $\frac{n(n+1)}{2}$ indecomposable Q_T-modules.

Let α be a positive root and let $M = (M^\alpha, f^\alpha)$ be the image of α under Θ. We know that α can't intersect all the sides of a triangle T and the support of α is connected, by 4.3.3. Hence, according to the definition of M^α, M is an indecomposable of type A. Moreover, by 4.1.1, different positive roots have different supports. Thus, the map $\Theta : \Phi_+ \to \{ M \mid M = \Theta(\alpha) \text{ for some } \alpha \in \Phi_+ \}$ is injective. Since the number of positive roots is equal to $\frac{n(n+1)}{2}$, the same as the number of indecomposable Q_T-modules, we conclude that this map is bijective as required. $\square$

Denote by $Ind(Q_T)$ the set of the indecomposable Q_T-modules bound by the triangle relations. Note that we showed in this proof that $Ind(Q_T) = \{(M^\alpha, f^\alpha) \mid \alpha \text{ diagonal of } P_{n+3} \text{ not in } T\}$.

We end this section with a straightforward corollary of 4.3.1.

Corollary 4.3.9. *There exists a bijection φ between $Ind(Q_T)$ and the diagonals of the polygon not in T. Moreover, for M in $Ind(Q_T)$ and any vertex i of Q_T, the multiplicity of the simple module S_i in the module M is 1 if $\varphi(M)$ crosses the i^{th} diagonal of T and 0 if not. In particular, for two isoclasses M, M' in $Ind(Q_T)$, we have $M = M'$ if and only if $n_i(M) = n_i(M')$ for all i.*

4.4 The exponents in the Laurent polynomials

Recall the theorem 3.4.5, which gives a bijection between the cluster variables of a cluster algebra of finite type and the set of almost positive roots of the corresponding root system, such that each cluster variable $x[\alpha]$ is described as a Laurent polynomial in a certain cluster $\underline{x}_0$, with the denominator given by the linear combination of α in terms of the simple roots.

The main result of [13] gives a description of any cluster variable of a cluster algebra of type A_n as a Laurent polynomial in an arbitrary cluster C, where the denominators are related to the indecomposable modules of $Mod\, Q_T/Rel_\triangle$ (or equivalently, $Mod\, Q_C/R_C$), and is stated as follows.

Theorem 4.4.1. *Let $C = \{u_1, \ldots, u_n\}$ be a cluster of a cluster algebra of type A_n and let V be the set of all cluster variables of the algebra. Let Q_C be the quiver with relations associated to C and $Ind(Q_C)$ the set of isoclasses of* *indecomposable modules. Then there is a bijection*

$$\begin{array}{ccc} Ind(Q_C) & \to & V \setminus C \\ \alpha & \mapsto & x[\alpha], \end{array}$$

such that

$$x[\alpha] = \frac{P(u_1,\ldots,u_n)}{\prod_{i=1}^n u_i^{n_i(\alpha)}},$$

where P is a polynomial not divisible by any u_i ($i = 1, \ldots, n$) and $n_i(\alpha)$ is the multiplicity of the simple module α_i in the module α.

This last section is devoted to the proof of this theorem. The proof of the existence of the bijection uses 4.3.9 and 3.4.5. The results presented in this section are related to the calculation of the exponents in the denominators of the Laurent polynomials.

Firstly, let us introduce the terminology used in the sequel. Let $\mathcal{A}$ be a cluster algebra of finite type and rank n. Denote by $\Phi_{\geq -1}$ the set of almost positive roots, and by $x[\alpha]$ the cluster variable corresponding to α, by the bijection of 3.4.5. Fix a cluster $C = \{u_1, \ldots, u_n\}$ and let $\beta_1, \ldots, \beta_n$ be the almost positive roots such that $x[\beta_i] = u_i$.

By the Laurent phenomenon 3.3.1, each cluster variable $x[\alpha]$ ($\alpha \in \Phi_{\geq -1}$) can be written in the form:

$$x[\alpha] = \frac{R_{\alpha,C}}{\prod_{i=1}^n u_i^{[\alpha,\beta_i,C]}}, \tag{4.1}$$

where $R_{\alpha,C}$ is a polynomial in the variables $u_1, \ldots, u_n$ such that none of the u_i divides $R_{\alpha,C}$, and $[\alpha, \beta_i, C] \in \mathbb{Z}$.

Lemma 4.4.2. *For any pair of almost positive roots α, β_i and any pair of clusters C, C' such that $u_i = x[\beta_i] \in C \cap C'$, we have $[\alpha, \beta_i, C] = [\alpha, \beta_i, C']$.*

Hence, we can denote $[\alpha, \beta, C]$ by $[\alpha, \beta]$, for any pair α, β of almost positive roots.

Proof. Let us prove first that the exchange graph for clusters containing the given cluster variable u_i is connected. Let C_1 and C_2 be two clusters containing a fixed cluster variable and suppose they are adjacent, and write $C_2 = C_1 \setminus \{\beta\} \cup \{\beta'\}$. Let τ be one of the involutions τ_+, τ_-. Applying τ to these clusters, we get two clusters $\tau(C_1)$ and $\tau(C_2)$ satisfying $\tau(C_2) \cap \tau(C_1) = \tau(C_1) \setminus \{\tau(\beta)\}$, for τ is a bijection. Recall that in finite type, two clusters are adjacent in the exchange graph if they agree on $n-1$ elements (where n is the rank of the cluster algebra). Thus, $\tau(C_1)$ and $\tau(C_2)$ are adjacent.

Hence, the exchange graph of clusters containing a fixed cluster variable is mapped to an isomorphic graph by the action of τ.

The clusters in Φ containing $-\alpha_i$ are the union of $\{-\alpha_i\}$ with a cluster C of Φ_i, where Φ_i is the root subsystem of Φ obtained from Φ by removing $-\alpha_i$ (cf. 3.5.5 (2)). Two such clusters $\{-\alpha_i\} \cup C$ and $\{-\alpha_i\} \cup C'$ are adjacent in the exchange graph if and only if C and C' have the same elements except one, i.e., C and C' are adjacent in the exchange graph for Φ_i. Let $\Psi_1, \dots, \Psi_r$ be the connected components of Φ_i. Then, the clusters of Φ_i are given by disjoint unions of clusters for $\Psi_1, \dots, \Psi_r$ (cf. 3.5.5 (3)). Two clusters are adjacent in Φ_i if and only if their components in Ψ_j are the same for all j except one, where they are adjacent in the exchange graph of Ψ_j. Therefore, the exchange graph of Φ_i is the product of the exchange graphs of the Ψ_j. Each of these exchange graphs is connected, as it is the exchange graph of a cluster algebra of finite type corresponding to its Dynkin diagram. Hence the direct product of these graphs is connected as well and so is the exchange graph of Φ_i. Consequently, the induced subgraph of the exchange graph of Φ on the vertices corresponding to clusters containing $-\alpha_i$, which is isomorphic to the exchange graph of Φ_i, is connected.

If we show now that the exponent of u_i in the denominator of $x[\alpha]$ is unchanged in mutations which do not exchange u_i, the lemma is proved. Consider the mutation $C \to C'$ that exchanges the cluster variables $u_j \in C$ and $u'_j \in C'$. The exchange relation gives $u_j = \frac{M_1+M_2}{u'_j}$, where M_1 and M_2 are monomials without common divisors in the variables $C \setminus \{u_j\}$. From equation 4.1, we obtain by substitution

$$x[\alpha] = \frac{R_{\alpha,C}(u_1, \dots, u_{j-1}, \frac{M_1+M_2}{u'_j}, u_{j+1}, \dots, u_n)}{\prod_{l \neq j} u_l^{[\alpha,\beta_l,C]} (\frac{M_1+M_2}{u'_j})^{[\alpha,\beta_j,C]}}.$$

By the Laurent phenomenon, $x[\alpha]$ is also a Laurent polynomial in the variables $u_1, \dots, u'_j, \dots, u_n$:

$$x[\alpha] = \frac{f}{\left(\prod_{k \neq j} u_k^{d_k}\right) (u'_j)^{d_j}}, \tag{4.2}$$

where f is a polynomial in $u_1, \dots, u'_j, \dots, u_n$, not divisible by any of the variables. We want to show that $d_i = [\alpha, \beta_i, C]$. We have

$$u_i^{[\alpha,\beta_i,C]} x[\alpha] = \frac{R_{\alpha,C}(u_1, \dots, u_{j-1}, \frac{M_1+M_2}{u'_j}, u_{j+1}, \dots, u_n)}{\prod_{l \neq i,j} u_l^{[\alpha,\beta_l,C]} \left(\frac{M_1+M_2}{u'_j}\right)^{[\alpha,\beta_j,C]}}. \tag{4.3}$$

The denominator is non-zero at $u_i = 0$, as the first part does not involve u_i and M_1, M_2 have no common divisors, in particular u_i doesn't divide both M_1 and M_2, Hence $M_1 + M_2|_{u_i=0} \neq 0$. On the other hand we have

$$R_{\alpha,C}(u_1,\ldots,u_n)|_{u_i=0} = R_{\alpha_C}(u_1,\ldots,u_{j-1},\frac{M_1+M_2}{u'_j},u_{j+1},\ldots,u_n)|_{u'_j\leftarrow\frac{M_1+M_2}{u_j}}|_{u_i=0}$$

$$= R_{\alpha,C}(u_1,\ldots,u_{j-1},\frac{M_1+M_2}{u'_j},u_{j+1},\ldots,u_n)|_{u_i=0}|_{u'_j\leftarrow\frac{M_1+M_2}{u_j}|_{u_i=0}}.$$

By assumption, $R_{\alpha,C}(u_1,\ldots,u_n)|_{u_i=0} \neq 0$. Hence

$$R_{\alpha,C}(u_1,\ldots,u_{j-1},\frac{M_1+M_2}{u'_j},u_{j+1},\ldots,u_n)|_{u_i=0} \neq 0.$$

Therefore, the right-hand side of 4.3 is defined and non-zero at $u_i = 0$. Using 4.2, the left-hand side of 4.3 is given by

$$u_i^{[\alpha,\beta_i,C]}x[\alpha] = \frac{f}{\prod_{k\neq j,i} u_k^{d_k}(u'_j)^{d_j}u_i^{d_i-[\alpha,\beta_i,C]}}. \tag{4.4}$$

Suppose $d_i > [\alpha,\beta_i,C]$. Then, the right-hand side of 4.4 is not defined at $u_i = 0$, a contradiction. If $d_i < [\alpha,\beta_i,C]$, the right-hand side of 4.4 is zero at $u_i = 0$, a contradiction too. Therefore, d_i must be equal to $[\alpha,\beta_i,C]$, and the Lemma is proved. □

Lemma 4.4.3. [17, Lemma 4.8] *Let $(\underline{x},\underline{p},B)$ be a seed. Denote by $b_{\alpha,\beta}(C)$ the entry $b_{x[\alpha],x[\beta]}$ of B, where $x[\alpha], x[\beta]$ are two cluster variables of $\underline{x}$. For $\tau \in \{\tau_-,\tau_+\}$, we have $b_{\tau(\alpha),\tau(\beta)}(\tau(C)) = -b_{\alpha,\beta}(C)$.*

From now on we only consider simply-laced root systems.

Lemma 4.4.4. *Let α,β be two almost positive roots. Then*

$$[\alpha,\beta] = [\tau_\pm\alpha,\tau_\pm\beta].$$

Proof. Let C_0 be a root cluster that contains α and C_m be a root cluster that contains β. These two clusters are connected by a sequence of adjacent root clusters

$$C_0 \leftrightarrow C_1 \leftrightarrow \ldots \leftrightarrow C_m, \tag{4.5}$$

in the exchange graph. Applying τ, where τ is one of the involutions τ_+,τ_-, we get another sequence of adjacent root clusters

$$\tau(C_0) \leftrightarrow \tau(C_1) \leftrightarrow \ldots \leftrightarrow \tau(C_m), \tag{4.6}$$

where $\tau(\alpha) \in \tau(C_0)$ and $\tau(\beta) \in \tau(C_m)$ (this was already observed in the proof of 4.4.2). We can obtain an expression for the cluster variable $x[\alpha]$ (resp. $x[\tau(\alpha)]$) in terms of the cluster variables of C_m (resp. $\tau(C_m)$) using the exchange relations associated to the mutations of the sequence 4.5 (resp. 4.6). Note that these exchange relations depend only on the matrices $B(C_0), \dots, B(C_m)$ (resp. $B(\tau(C_0)), \dots B(\tau(C_m))$) associated to the clusters $C_0, \dots, C_m$ (resp. $\tau(C_0), \dots, \tau(C_m)$). Thus, by 4.4.3 and because the exchange relations are not changed by the minus sign (cf. 3.1.23), those expressions are the same. In particular, $[\alpha, \beta] = [\alpha, \beta, C_m] = [\tau(\alpha), \tau(\beta), \tau(C_m)] = [\tau(\alpha), \tau(\beta)]$, by 4.4.2. $\square$

Lemma 4.4.5. *Let $-\alpha_i$ be a simple negative root and α an almost positive root. Then*

$$[\alpha, -\alpha_i] = n_i(\alpha).$$

Proof. Let C_0 be the cluster whose cluster variables correspond to the negative simple roots, i.e., $C_0 = \{x[-\alpha_i], \dots, x[-\alpha_n]\}$. By 4.1, $[\alpha, -\alpha_i, C_0]$ is the exponent of $x[-\alpha_i]$ in the denominator of the Laurent polynomial expression of $x[\alpha]$ in terms of the elements of C_0. On the other hand, by 3.4.5, which gives an expression for $x[\alpha]$ in terms of the cluster C_O, $[\alpha, -\alpha_i, C_0]$. By 4.4.2, $[\alpha, -\alpha_i, C_0] = [\alpha, -\alpha_i]$, and we are done. $\square$

Proposition 4.4.6. *Let α, β be two distinct almost positive roots. Then*

$$[\alpha, \beta] = (\alpha \| \beta).$$

Proof. Consider the function $b : \Phi_{\geq -1} \times \Phi_{\geq -1} \to \mathbb{Z}_{\geq 0}$ defined by

$$b(\alpha, \beta) = \begin{cases} [\beta, \alpha] & \text{if } \alpha \neq \beta, \\ 0 & \text{if } \alpha = \beta. \end{cases}$$

If we prove that b is such that $b(-\alpha_i, \beta) = max(n_i(\beta), 0)$, where $n_i(\beta)$ is the coefficient of α_i on the expression of β written as a linear combination of the simple roots, and $b(\tau_\pm(\alpha), \tau_\pm(\beta)) = b(\alpha, \beta)$, for any almost positive roots α, β, then we conclude that $b(\alpha, \beta) = (\beta \| \alpha)$, since the compatibility degree is the unique function satisfying these properties, and the lemma follows since $(\|)$ is symmetric in the simply-laced case.. Indeed, we have $[\beta, -\alpha_i] = n_i(\beta)$, by 4.4.5, and so $b(-\alpha_i, \beta) = max(n_i(\beta), 0)$. The second property follows directly from 4.4.4. $\square$

Now we are able to prove the theorem 4.4.1.

Let $\mathcal{A}$ be a cluster algebra of type A_n. Let D be the set of diagonals of P_{n+3}. The almost positive roots correspond to these diagonals, as it was observed in the end of section 5 of the chapter 3. Moreover, the root clusters correspond to the triangulations of P_{n+3} (cf. 3.5.8 (3)).

On the other hand, theorem 3.4.5 gives a bijection $\alpha \mapsto x[\alpha]$ between the set of almost positive roots and the set V of the clusters variables. So there is a bijection, say ϕ, between D and V.

Fix a cluster $C = \{u_1, \dots, u_n\}$ of $\mathcal{A}$. Let $\{\beta_1, \dots, \beta_n\}$ be the corresponding root cluster, i.e., $u_i = x[\beta_i]$, $\forall i \in [n]$. Denote by T_C the triangulation of P_{n+3} that corresponds to this cluster. Thus, ϕ takes the elements of T_C to the elements of C. Since ϕ is a bijection, it maps $D \setminus T_C$ to $V \setminus C$.

By corollary 4.3.9, there is a bijection φ between $Ind(Q_{T_C}) = Ind\, Q_C$ (cf. 4.1.3) and $D \setminus T_C$. Therefore, the composition of φ and ϕ yields a bijection from $Ind\, Q_C$ to $V \setminus C$.

Let M^α be an element of $Ind\, Q_C$, with the corresponding diagonal α in $D \setminus T_C$. By 4.1 and 4.4.2,

$$x[\alpha] = \frac{P(u_1, \dots, u_n)}{\prod_{i=1}^{n} u_i^{[\alpha, \beta_i]}},$$

where $P(u_1, \dots, u_n)$ is not divisible by any of u_i, $i = 1, \dots, n$. We want to show that $[\alpha, \beta_i] = n_i(\alpha)$ for each $i \in [n]$, where $n_i(\alpha)$ denotes the multiplicity of the simple module corresponding to the vertex i of Q_C in the module M^α.

Note that as $\alpha \notin T_C$, $\alpha \neq \beta_i$. Hence, by applying 4.4.6, we get $[\alpha, \beta_i] = (\alpha \| \beta_i)$. By 3.5.8 (2), $(\alpha \| \beta_i)$ is one if the diagonals α and β_i are crossing, which is to say that β_i lies in the support of α, and zero otherwise. Hence

$$[\alpha, \beta_i] = \begin{cases} 1 & \text{if } \beta_i \in Supp(\alpha), \\ 0 & \text{otherwise,} \end{cases}$$

which is equal to $n_i(\alpha)$, since β_i belongs to the support of α if and only if the simple corresponding to the vertex i of Q_C occurs in M^α. $\square$

Bibliography

[1] Anderson, F. W.; Fuller, K. R., *Rings and Categories of Modules*, Graduate Texts in Mathematics **13**, Springer-Verlag, New York, Heidelberg, Berlin, 1973.

[2] Assem, I.; Simson, D.; Skowroński, A., *Elements of the Representation Theory of Associative Algebras, vol 1 techniques of Representation Theory*, London Mathematical Society student texts, **65**, Cambridge University Press, Cambridge, 2006.

[3] Auslander, M.; Platzeck, M. I.; Reiten, I., Coxeter functors without diagrams, *Trans. Amer. Math. Soc.*, **250**, (1979), 1–46.

[4] Auslander, M.; Reiten, I; Smalø, S. O., *Representation theory of Artin Algebras*, Cambridge Studies in Advanced Mathematics, **36**, Cambridge University Press, Cambridge, 1995.

[5] Berenstein, A.; Fomin, S.; Zelevinsky, A., Cluster algebras III: Upper bounds and double Bruhat cells. *Duke Math. J.*, **126**, (2005), No. 1, 1–52.

[6] Bourbaki, N., *Lie groups and Lie Algebras*, Springer-Verlag, Berlin, 2002.

[7] Buan, A. B.; Marsh, R. J., Cluster-tilting theory, In Trends in Representation Theory of Algebras and Related Topics, *Contemp. Math.* **406** (2006), 1–30.

[8] Buan, A. B.; Marsh, R. J.; Reiten, I., Cluster-tilted algebras, *Trans. Amer. Math. Soc.*, **359**, (2007), 323–332

[9] Buan, A. B.; Marsh, R. J.; Reiten, I., Cluster mutation via quiver representations, *Comm. Math. Helv.* **83** Issue 1, (2008), 143–177.

[10] Buan, A. B.; Marsh, R. J.; Reiten, I., Cluster-tilted algebras of finite representation type, *J. Algebra* **306** (2006), no. 2, 412–431.

[11] Buan, A. B.; Marsh, R. J.; Reineke, M.; Reiten, I.; Todorov, G., Tilting theory and cluster combinatorics, *Adv. Math.*, **204**, (2006), No. 2, 572–618.

[12] Butler, M. C. R., Ringel, C. M., Auslander-Reiten sequences with few middle terms and apllications to string algebras, *Communications in Algebra* **15** (1 & 2), 145–179, 1987.

[13] Caldero, P.; Chapoton, F.; Schiffler, R., Quivers with relations arising from clusters (A_n case), *Trans. Amer. Math. Soc.*, **358**, (2006), 1347–1364.

[14] Carter, R. W., *Cluster Algebras*, Textos de Matemática, série B, **37**, Departamento de Matemática da Universidade de Coimbra, Portugal, 2006.

[15] Crawley-Boevey, W., *Lectures on representations of quivers*, http://www.amsta.leeds.ac.uk/ pmtwc/quivlecs.pdf

[16] Fomin, S.; Zelevinsky, A., Cluster algebras I: Foundations, *J. Amer. Math. Soc.*, **15** (2002), no.2, 497–529.

[17] Fomin, S.; Zelevinsky, A., Cluster Algebras II: Finite type classification, *Invent. Math.*, **154** (2003), no.1, 63–121.

[18] Fomin, S.; Zelevinsky, A., Y-systems and generalized associahedra, *Annals of Mathematics*, **158** (2003), no.3, 977–1018.

[19] Hilton, P. J., Stammbach, U., *A course in homological algebra*, New York : Springer-Verlag, 1997.

[20] Humphreys, J. E., *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, **29**, Cambridge University Press, Cambridge, 1990.

[21] Kasch, F., *Modules and Rings*, London Mathematical Society Monographs, **17**, Academic Press, 1982.

[22] Marsh, R. J.; Reineke, M; Zelevinsky, A., Generalized associahedra via quiver representations, *Trans. Amer. Math. Soc.*, **355**, (2003), no. 10, 4171–4186.

[23] Ringel, C. M., *Some remarks concerning tilting modules and tilted algebras, Origin. Relevance. Future.* (An appendix to the Handbook of Tilting Theory) http://www.mathematik.uni-bielefeld.de/ ringel/opus/616.pdf

[24] Vermani, L. R., *An elementary approach to homological algebra*, Chapman & Hall/CRC, Monographs and Surveys in Pure and Applied Mathematics, **130**, 2003.

Index