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On the Proof Theory of Modal Logics

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Resumo

Esta dissertação tem como objetivo apresentar uma análise, do ponto de vista da teoria da demonstração, de lógicas modais.

Lógica Modal
A Lógica modal é um ramo da lógica que estuda o uso de modalidades, isto é, expressões usadas para qualificar a veracidade de uma afirmação. Estas lógicas, estendem a lógica proposicional clássica através de operadores unários duais $\Box$ e $\Diamond$, geralmente interpretados como necessidade e possibilidade, respetivamente. É possível remeter o início do estudo da lógica modal, enquanto raciocínio sobre necessidade e possibilidade à Grécia antiga, nomeadamente a Aristóteles, que no livro I do seu trabalho *Primeiros Analíticos* desenvolveu um sistema de silogística modal que lidava exatamente com estas noções de modalidade. Ainda assim, o estudo desta disciplina nos termos em que a conhecemos hoje, não foi desenvolvido até ao século passado, por matemáticos e filósofos como Clarence I. Lewis e Saul Kripke. Os dois foram cruciais para o desenvolvimento desta disciplina: Clarence Lewis é considerado recorrentemente como o 'pai' da lógica modal, tendo apresentado no seu trabalho [27] os primeiros sistemas axiomáticos formais modernos para lógicas modais $\mathbf{S1}$ a $\mathbf{S5}$; por outro lado, Saul Kripke revolucionou o estudo da semântica, com trabalhos como [21, 22], ao introduzir os 'possible world models'. A sua influência na lógica modal foi tão relevante que muitas noções semânticas como a de estrutura e modelo são ainda apelidadas com o seu nome ("estrutura de Kripke" e "modelo de Kripke").

Neste trabalho iremos focar-nos na lógica modal básica $\mathbf{K}$ e nas restantes lógicas do cubo-$\mathbf{S5}$, isto é, as lógicas construídas a partir de $\mathbf{K}$ ao restringir a relação de acessibilidade considerada, impondo combinações de condições como serialidade, reflexividade, simetria, transitividade e euclidianidade.

Teoria da Demonstração
A Teoria da Demonstração é o ramo da matemática que lida com o estudo de demonstrações enquanto objetos formais. Foi introduzida, na forma como a conhecemos, por David Hilbert no século passado. Esta disciplina foi desenvolvida no contexto do programa de Hilbert, que ambicionava formalizar toda a matemática através de axiomas, e provar a sua consistência. Assim, o trabalho de Hilbert insere-se maioritariamente na teoria da demonstração axiomática, sendo uma das suas grandes contribuições o *sistema de Hilbert*, isto é, um sistema de dedução formal formado por axiomas e regras de inferência. Desde então vários outros sistemas de demonstração formais têm sido desenvolvidos e estudados exaustivamente. Destacamos a dedução natural, introduzida por Gentzen [12] e Jaśkowski [17], os sistemas tableaux, atribuídos a Beth [4], e o cálculo de sequentes, ao qual iremos prestar especial atenção. O cálculo de sequentes foi introduzido por Gentzen em 1930, essencialmente como ferramenta para provar a consistência da Aritmética de Peano. É um tipo de sistema de dedução formal que permite a manipulação de sequentes, isto é, expressões do tipo $\Gamma \Rightarrow \Delta$ onde Γ e Δ são multiconjuntos, através de

regras de inferência. Ao contrário dos sistemas de Hilbert, constituídos maioritariamente por axiomas e um número reduzido de regras de inferência, os sistemas de dedução natural e de cálculo de sequentes contêm menos (ou nenhuns) axiomas, assentando principalmente nas regras que permitem manipular as fórmulas.

Cálculo de sequentes para lógicas modais

Neste trabalho focamo-nos no estudo de cálculo de sequentes para lógicas modais. Mais concretamente, queremos obter sistemas que tenham um comportamento semelhante ao cálculo de Gentzen $\mathbf{G3cp}$ para a lógica proposicional clássica, no que diz respeito à admissibilidade de regras estruturais, incluindo a regra do corte, à correção e completude. Para abordar este problema podemos tanto enriquecer a estrutura dos sequentes ou enriquecer a linguagem dos mesmos. A primeira abordagem consiste em permitir outros conetivos estruturais para além da seta " $\Rightarrow$ " e vírgulas "$,$" que habitualmente estão presentes nos sequentes. Destacamos o cálculo de hipersequentes ou os sequentes entrelaçados. Iremos seguir a segunda abordagem, mais especificamente, estudaremos os sistemas de sequentes apresentados por Sara Negri em [29]. Nestes sistemas, enriquecemos a linguagem anotando as fórmulas com etiquetas $x, y, z, ...$ que denotam mundos numa Kripke-estrutura. Além disso, consideramos átomos relacionais xRy que relacionam os mundos nesta mesma estrutura. Assim, um sequente etiquetado é da forma $\Gamma \Rightarrow \Delta$, em que Γ e Δ são multiconjuntos de fórmulas da forma $x : \phi$, onde x é uma etiqueta e ϕ é uma fórmula modal, Γ pode ainda conter átomos relacionais da forma xRy. Intuitivamente uma fórmula $x : \phi$ significa que ϕ é verdadeiro no mundo x. Note-se ainda que esta abordagem é muito mais vasta do que o apresentado nesta dissertação. Primeiramente, não se restringe a sistemas de cálculo de sequentes, podendo ser aplicada, entre outros, a sistemas de dedução natural, destacando-se os trabalhos de [2, 3], e a sistemas tableau, presente em [10]. Mais ainda, as etiquetas que anotam as fórmulas podem ter outros significados, que não sejam mundos numa Kripke-estrutura, por exemplo, valores de verdade [28].

Os sistemas que apresentamos neste trabalho são construídos a partir de um sistema base para a lógica $\mathbf{K}$, que denotamos $\mathbf{LabK}$. São constituídos por duas regras sem premissas, a que nos referimos como *sequentes iniciais*, um par de regras para cada conetivo $\wedge, \vee, \rightarrow, \Box, \Diamond$ e, no caso dos cálculos para extensões de $\mathbf{K}$, regras estruturais que representam as condições impostas à relação de acessibilidade da lógica correspondente. Dizemos que um sequente etiquetado $\Gamma \Rightarrow \Delta$ é derivável no sistema $\mathbf{X}$ ($\mathbf{X}$ é uma lógica do cubo) se existe uma árvore cujas folhas são sequentes iniciais, a raiz é $\Gamma \Rightarrow \Delta$, e em que cada nodo se obtém através de outro aplicando alguma regra lógica do cálculo.

Estes sistemas apresentam propriedades notáveis, nomeadamente, são modulares, isto é, conseguimos facilmente definir cálculos para uma família vasta de lógicas modais ao adicionarmos regras estruturais - regras que não afetam um conetivo lógico em particular mas sim a estrutura dos sequentes - ao sistema básico $\mathbf{LabK}$. Esta família vai para além do cubo-$\mathbf{S5}$, em particular, é possível definir sistemas a partir de $\mathbf{LabK}$ para qualquer lógica modal que seja caracterizada por restrições à sua relação de acessibilidade que sejam exprimíveis através de fórmulas da lógica de primeira ordem [9]. Além disso, os sistemas são corretos e completos em relação à lógica modal correspondente, e admitem a regra do corte. A última propriedade afirma que se conseguimos derivar os sequentes $\Gamma \Rightarrow \Delta, x : \phi$ e $x : \phi, \Gamma' \Rightarrow \Delta'$, então podemos ainda derivar, no mesmo número de passos, o sequente $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. É especialmente importante uma vez que permite afirmar que, lendo uma derivação começando na raiz, nunca são introduzidas fórmulas que, a menos de atómos relacionais, não sejam subfórmulas da fórmula na raiz da árvore. Este facto é essencial para garantir um certo grau de eficiência na procura de demonstrações e em particular,

para garantir que as procuras por estas demonstração terminam num número finito de passos. Nesta dissertação, esta propriedade tem um papel central para provar que as lógicas $\mathbf{K}$ e $\mathbf{S4}$ são decidíveis. Este resultado é demonstrado exibindo um algoritmo que, dado um sequente $\Rightarrow x : \phi$ tenta construir uma derivação para o mesmo e, caso falhe, termina e permite construir um contramodelo (finito) para ϕ. Mais ainda, o processo de decisão que apresentamos permite-nos assim estabelecer a propriedade do modelo finito, que diz que para qualquer fórmula que não seja satisfazível num modelo infinito, existe um modelo finito onde também não é satisfazível. Apesar destes resultados serem conhecido na literatura, a abordagem que utilizamos, baseada estritamente em ferramentas da teoria da demonstração e no cálculo de sequentes etiquetados, constitui uma contribuição da presente dissertação.

Palavras-chave: teoria da demonstração, lógica modal, cálculo de sequentes, dedução etiquetada, decidibilidade

Abstract

This thesis aims at presenting a proof-theoretical analysis of modal logics.

Modal logics extend classical propositional logic by adding to the language operators $'\Box'$ and $'\Diamond'$, expressing *necessity* and *possibility*. In this work, we will be focusing on the modal logics in the $\mathbf{S5}$-cube, built from the basic modal logic $\mathbf{K}$ by considering combinations of certain frame conditions such as reflexivity, symmetry and transitivity. We are interested in the study of sequent systems for this family of logics. The systems we present are based on Gentzen's calculus $\mathbf{G3cp}$, with two additional pairs of rules for the modal operators and where the language has been extended with labels. These labels annotate formulas denoting worlds in a Kripke-model where they are satisfied. Note that this idea is not limited to sequent calculi, in fact, it has been studied for other formal systems such as natural deduction [2, 3] and tableau [10]. Moreover, labels can represent, not only worlds in a model, but also truth values [28]. We discuss several results that have been obtained in the literature for this family of modal logics, such as admissibility of weakening, contraction, and most notably cut-admissibility, which ensures the subformula property. Furthermore, we investigate proof-search termination strategies, which allows us to obtain countermodels for non-derivable sequents, and prove, via proof-theoretical tools, decidability and the finite model property for the logics in the cube, in particular for $\mathbf{K}$ and $\mathbf{S4}$ which we take as a case study.

Keywords: proof theory, modal logic, sequent calculi, labelled deduction, decidability

Contents

List of Figures

List of Tables

Introduction

General aim

This thesis aims at presenting a proof-theoretical analysis of modal logics.

Modal logics is the branch of logics concerned with the study of modalities, that is, expressions used to qualify the truth of a judgement [11]. These logics extend classical propositional logic to include operators expressing, for instance, necessity ($\Box$) and possibility ($\Diamond$). While this subject date back to Aristotle, the study of modal logic was not really developed until the 20th century, due to mathematicians and philosophers such as Clarence I. Lewis and Saul Kripke. While Lewis is often considered the founder of modern modal logics, having introduced the first modern formal axiom systems for modal logics $\mathbf{S1}$ to $\mathbf{S5}$ in [27], Kripke revolutionized the study of semantics of modal logics in works such as [21, 25], by introducing possible world models.

Proof theory is the branch of mathematical logic which concerns the study of proofs as formal objects. It was introduced, in the form in which it exists today, by David Hilbert in the past century. It was developed in the context of Hilbert's formalist program aiming at showing consistency of arithmetic. Hilbert focused primarily on *axiomatic* proof-theory, where proofs are constructed according to the axioms and rules of inference of a given logical system. Hilbert's work led to the creation of the Hilbert system, a formal deduction system that uses logical axioms and rules of inference. There are various types of proof systems that have been studied extensively. These include natural deduction, tableux systems and sequent calculus, a type of formal system allowing for the manipulation of sequents, which was introduced by the German mathematician and logician Gerhard Gentzen in the 1930s.

We are interested in the study of sequent calculi for modal logics. Namely, in obtaining cut-free sequent calculi for modal logics, behaving similarly to Gentzen's calculus $\mathbf{G3cp}$. However, there is not a unique approach to this problem. Two main approaches which appear in the literature consist of either enriching the structure of sequents [6, 1] or enriching the language. We follow the second approach, specifically, we study the sequent calculi presented by Sara Negri in [29]. In these systems, formulas are annotated with labels which denote worlds in a Kripke-frame. Hence, a labelled formula is of the form $x : \phi$ which intuitively means that ϕ holds at world x. It should be noted that the labelled approach is not restricted to sequent calculi and may be applied to other kinds of proof systems, namely natural deduction [2, 36, 3] and tableux [10]. Furthermore, it is also possible to annotate formulas with labels representing, for instance, truth values [28], and not only worlds in a model.

The logical systems presented in this work present remarkable properties. They are **modular**, that is, we can easily define calculi for a very broad family of modal logics by adding rules to the basic system

$\mathbf{LabK}$, going way beyond the modal cube. In particular, we can define sequent systems for any modal logic defined by first-order frame conditions [9]. Moreover, they are **sound**, **complete** and **cut-free**. Admissibility of the cut rule is particularly important as it guarantees that at each step of a derivation, no new formulas (modulo labels) are introduced. This fact is crucial to ensure proof-search **termination**, which ultimately allows us to devise **decision algorithms** for the logics in the cube. Furthermore, these decision algorithms provide us with a method to construct a finite **countermodels** for non-derivable formulas, hence establishing the **finite model property**.

Structure of the thesis

This thesis is organized as follows. In Chapter 1 we begin by introducing modal logics, focusing on the modal logics in the $\mathbf{S5}-$cube that are built from the basic modal logic $\mathbf{K}$ by considering combinations of certain frame conditions: reflexivity, symmetry and transitivity. We prove several relevant results, namely soundness and completeness of all the logics in the cube. Moreover, we show the *frame correspondence* result, establishing a correspondence between the characteristic axioms of these systems and the frame conditions which define them. In Chapter 2 we lay the groundwork for the study of sequent calculi, focusing on Gentzen's calculi $\mathbf{G3cp}$ for classical propositional logic. We study the properties of this calculus, such as admissibility of weakening, contraction and cut (to which we devote special attention), as well as soundness and completeness. Furthermore, we attempt at devising a cut-free sequent system for the logics in the cube and discuss the reasons why the proposed solution is not suitable, motivating us to change our strategy in the following chapter. In Chapter 3 we present a solution to this problem by following the labelled approach, as presented in [29]. We internalize the Kripke semantics by enriching the language of sequents with *labels*, which index the formulas and denote worlds in a Kripke frame. We begin by presenting the calculus $\mathbf{LabK}$ for $\mathbf{K}$ and then introduce structural rules, in correspondence to frame conditions, to extend the calculus to every logic in the cube. For each such logic $\mathbf{X}$, we obtain a calculus $\mathbf{LabX}$. We show these systems preserve all the mentioned properties of $\mathbf{G3cp}$, namely, they are cut-free, sound and complete. We give special emphasis to the completeness proof and prove this result in two different ways. The first one is syntactical: we show that given an Hilbert system H_X for a modal logic $\mathbf{X}$ in the cube, all the axioms in H_K are derivable in the sequent system, and the rules in H_K are admissible. This proof requires the use of cut to simulate the rule of *modus ponens*. Section 3.3 is then devoted to providing a proof of semantic completeness: we show that by building a proof-search tree for a labelled sequent $\Gamma \Rightarrow \Delta$ we either obtain a derivation, or we obtain an infinite tree from which we can extract a countermodel for the sequent at the root. Finally, Chapter 4 focuses on investigating termination strategies through proof search and proving decidability, taking $\mathbf{S4}$ as a case study. We improve the proof-search strategy used to prove semantic completeness, obtaining an always terminating algorithm which returns a derivation for the formula at the root of the tree if and only if it is valid. Furthermore, the countermodel obtained from a failed proof-search tree is now finite, allowing us to prove the finite model property.

While the decidability results established in this thesis are known in the literature, the method we follow, adapted from the work in [30] and [31], together with proofs of the termination of the decision algorithms, had not been detailed.

Chapter 1

Preliminaries of Modal Logics

The primary aim behind the development of modal logics is to formalize reasoning about necessity, possibility, and related concepts using specialized logical operators. The roots of this subject date back to Aristotle, who discussed the concepts of necessity and contingency within a logical framework in Book I of his work *Prior Analytics* (for a translated version of this book by Robin Smith refer to [32]). Later on, in the 20th century, Clarence Lewis made notable contributions to this subject, in fact, he may be considered to have founded modern modal logic. In his book [27] published in 1932, Lewis defines the logics $S1$ to $S5$ axiomatically, therefore introducing the first modern formal axiom systems for modal logics. Nonetheless, it is unavoidable to mention the name of Saul Kripke. The American philosopher revolutionized modal logic and semantics in the mid-20th century with his groundbreaking work, including "Semantical Considerations on Modal logic" ([21]) and "Naming and Necessity" ([22]). Kripke's contributions to modal logic are particularly associated with the development of Kripke semantics, a powerful framework for understanding and interpreting modal logic. A discussion on Kripke's results is presented in [24] and subsequent developments can be found in the paper [35] by Johan van Benthem. This paper also presents a concise contextualization of modal logics and highlights its expansion into various fields.

In this chapter, we provide an introduction to modal logic, covering several basic definitions and fundamental results, laying the groundwork for the following chapters. In Section 1.1, dedicated solely to syntax, we introduce the *modal language* and the operators $\Box$ and $\Diamond$, discussing their possible meanings together with other modal operators for different logics. Section 1.2 introduces several semantical notions, namely *Kripke-frames* and *models*, and the *frame conditions* which ultimately will allow us to define the *basic modal system* $\mathbf{K}$ and all the other logics in the $\mathbf{S5}$-cube, that is, the modal logics built from $\mathbf{K}$ by considering certain combinations of these frame conditions, namely reflexivity, symmetry and transitivity. We go on to provide an alternative characterization for these logics in Section 1.3, discussing suitable Hilbert systems for each one of them. Finally, in Section 1.4 we prove the two main theorems of the chapter which are soundness and completeness of $\mathbf{K}$, providing detailed proofs for each of these results. At last, we discuss the correspondence between the first-order frame conditions presented in Section 1.2, and the characteristic axioms for the logics in the modal cube and then prove soundness and completeness of these logics.

1.1 Syntax of the Modal Language

Syntax is the backbone of any formal system. It encompasses the grammar and structure of the language and allows us to precisely formulate and analyze modal statements. This section introduces us to the language of the *modal logic*, to its symbols, including the modal operators 'Box' $\Box$ and 'Diamond' $\Diamond$, and to the formulas that can be expressed in it. In the following Ξ will denote a countably infinite set of propositional symbols and the letter p denotes an arbitrary element of this set. We begin by introducing the language $\mathcal{L}$ of classical propositional logic, generated by the grammar:

$$\phi ::= p \mid \bot \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \to \psi.$$

We can enrich this language with the universal and existential quantifiers, obtaining the language of first-order logic $\mathcal{L}^{fo}$:

$$\phi ::= p \mid \bot \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \to \psi \mid \forall \phi \mid \exists \phi.$$

If instead we enrich $\mathcal{L}$ with the modal symbols $\Box$ and $\Diamond$, we get the *modal language*, denoted $\mathcal{L}^{\Box,\Diamond}$. A *well-formed formula* ϕ in $\mathcal{L}^{\Box,\Diamond}$ is generated by the grammar:

$$\phi ::= p \mid \bot \mid \phi \vee \psi \mid \phi \wedge \psi \mid \phi \to \psi \mid \Box \phi \mid \Diamond \phi.$$

Other Boolean connectives that are not defined as primitives can be defined as follows:

$$\begin{aligned} \neg \phi &:= \phi \to \bot \\ \top &:= \neg \bot \\ \phi \leftrightarrow \psi &:= (\phi \to \psi) \wedge (\psi \to \phi) \end{aligned}$$

For the sake of simplicity, we chose to have both modal operators as primitives, however, $\Box$ and $\Diamond$ can be defined interchangeably:

$$\Box \phi := \neg \Diamond \neg \phi$$
$$\Diamond \phi := \neg \Box \neg \phi$$

Note that the relation between these operators is much like the relation between the universal quantifier $\forall$ and the existential qualifier $\exists$. This parallel will be further evidenced in the following section when we define the satisfiability relation for formulas with the Box and Diamond.

Originally, modal logic was developed to express formulas that assert *necessity* or *possibility*. While this is still the most common interpretation for $\Box$ and $\Diamond$, often called the *alethic* modalities for that reason, these operators can be given other interpretations, the only requirement being that they must respect the relation stated above. As the study of modal logic evolved, other modalities came to be of interest and gave rise to an entire family of logics, some of which are summarized in Table 1.1 along with their denotation and the interpretation of the modal operators.

A comprehensive overview of the development of modal logic and evolution over the past century can be found in the paper [13] published by Robert Goldblatt.

Logic	Modalities	Interpreted as
Epistemic	K_α	Agent α *knows* that...
	B_α	Agent α *believes* that...
Deontic	O	It is *obligatory* that...
	P	It is *permitted* that...
Temporal	F	It *will* be the case that...
	P	It *was* the case that...

Table 1.1: Modal Logics, modalities and interpretations.

1.2 Introduction to Kripke semantics

In a context where symbols can have intuitive meanings that we assign to them, there must be tools that provide a clear and rigorous way to interpret and understand the meaning of modal formulas and operators. Here is where *Kripke semantics* comes into play, ensuring that there is no place for ambiguities or misinterpretations. This section will define several notions as *Kripke frame* and *Kripke model*, and will ultimately allow us to introduce several well-known modal systems, starting with the most simple one: the *basic modal system*.

Definition 1.1. A *Kripke frame* is a pair $\mathcal{F} = (W, R)$ where W is a non-empty set called the *universe* and R is a binary relation such that $R \subseteq W \times W$, satisfying a (possibly empty) subset of the conditions in Table 1.2. We will refer to the elements of W as *states* or *worlds* and to the relation R as the *accessibility relation*. We will write sRt to denote that t is accessible from s.

Seriality	*For all* $x \in W$ *there is* $y \in W$ *such that* xRy	d
Reflexivity	*For all* $x \in W$ *we have* xRx	t
Symmetry	*For all* $x, y \in W$ *if* xRy *then* yRx	b
Transitivity	*For all* $x, y, z \in W$ *if* xRy *and* yRz *then* xRz	$\mathsf{4}$
Euclideaness	*For all* $x, y, z \in W$ *if* xRy *and* xRz *then* yRz	$\mathsf{5}$

Table 1.2: Classes of frames and frame conditions.

Table 1.2 summarizes several *frame conditions* as well as the classes of frames they induce. The class of all frames is absent from the table, as the frames within this class don't obey any particular condition, and is denoted by k.

Definition 1.2. For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$, an X-frame is a Kripke frame satisfying the conditions specified in X.

Example 1.3. Consider Figure 1.1.
This is a graphical representation of a Kripke frame with a set of worlds $W = \{1, 2, 3, 4\}$. The accessibility relation is represented by the arrows between states, for instance, we have $3R3$ and $2R4$. Note that this frame does not satisfy any of the properties in Table 1.2. In Figures 1.2 and 1.3, we have modified the accessibility relation of the last frame in order to produce frames that are within t and $\mathsf{4}$, respectively.

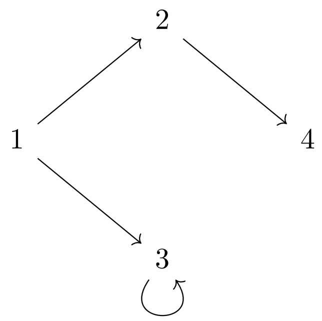


Figure 1.1: Simple example of a Kripke frame.

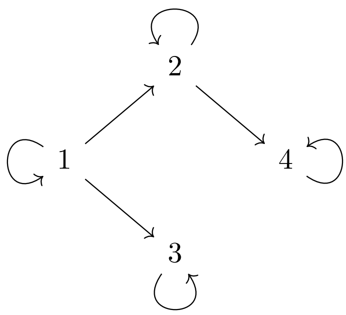


Figure 1.2: Example of a reflexive frame.

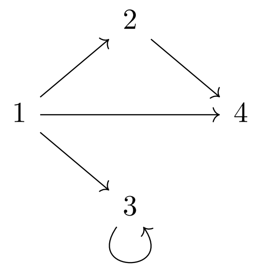


Figure 1.3: Example of a transitive frame.

Definition 1.4. A *Kripke model* based on $\mathcal{F}$ is a pair $\mathcal{M} = (\mathcal{F}, [\![\cdot]\!])$ where $\mathcal{F}$ is a frame and $[\![\cdot]\!] : \Xi \longrightarrow \mathcal{P}(W)$ is a map called the *valuation*. For $p \in \Xi$, $[\![p]\!]$ is the set of worlds where p holds.

Example 1.5. Consider Figure 1.4. This is a model based on the frame from Figure 1.3. The valuation is expressed by writing next to each state the propositional variables which hold there. There are only two propositional symbols p and q, and we have that $[\![p]\!] = \{1, 2\}$ and $[\![q]\!] = \{4\}$.

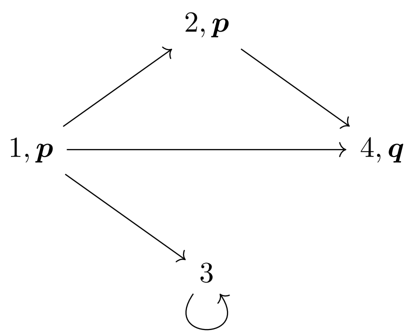


Figure 1.4: Kripke-model based on the frame from Figure 1.1.

Definition 1.6 (Satisfiability)**.** Consider a model $\mathcal{M} = (W, R, [\![\cdot]\!])$ and some state s. We inductively define the notion of a formula ϕ being *satisfiable in $\mathcal{M}$ at state s*, written $\mathcal{M}, s \Vdash \phi$, as follows:

$$
\begin{array}{ll}
\mathcal{M}, s \nVdash \bot & \\
\mathcal{M}, s \Vdash p & \text{iff } s \in [\![p]\!] \\
\mathcal{M}, s \Vdash \phi \vee \psi & \text{iff } \mathcal{M}, s \Vdash \phi \textit{ or } \mathcal{M}, s \Vdash \psi \\
\mathcal{M}, s \Vdash \phi \wedge \psi & \text{iff } \mathcal{M}, s \Vdash \phi \textit{ and } \mathcal{M}, s \Vdash \psi \\
\mathcal{M}, s \Vdash \phi \rightarrow \psi & \text{iff } \mathcal{M}, s \nVdash \phi \textit{ or } \mathcal{M}, s \Vdash \psi \\
\mathcal{M}, s \Vdash \Box \phi & \text{iff for all } t \in W \text{ such that } sRt \text{ we have } \mathcal{M}, t \Vdash \phi \\
\mathcal{M}, s \Vdash \Diamond \phi & \text{iff for some } t \in W \text{ such that } sRt \text{ we have } \mathcal{M}, t \Vdash \phi
\end{array}
$$

A set of formulas Σ is satisfiable in $\mathcal{M}$ at state s, written $\mathcal{M}, s \Vdash \Sigma$, if for every formula $\phi \in \Sigma$, it holds that $\mathcal{M}, s \Vdash \phi$.

Example 1.7. Recall the model from example 1.5, which we will denote by $\mathcal{M}$. It is clear that the modal formula $\Diamond p$ is true at state 1, since $\mathcal{M}, 2 \Vdash p$. The formula $\Diamond (p \vee q)$ is true at worlds 1 and 2 since $1R4$, $2R4$ and $\mathcal{M}, 4 \Vdash p \vee q$. Another example is $\Box \bot$ which is vacuously true at state 4 as there is no other state accessible from there.

Definition 1.8. A formula ϕ is *satisfiable in a model* $\mathcal{M}$, written $\mathcal{M} \Vdash \phi$, if and only if for all states s in $\mathcal{M}$ we have that $\mathcal{M}, s \Vdash \phi$. A set of formulas Σ is satisfiable in $\mathcal{M}$ if and only if $\mathcal{M}, s \Vdash \Sigma$ for all states s in $\mathcal{M}$, and we write $\mathcal{M} \Vdash \Sigma$.

Definition 1.9 (Logical Consequence)**.** A formula ϕ is said to be a *logical consequence* of a set of formulas Σ at a class of frames F, written $\Sigma \models_{\mathsf{F}} \phi$, if for all models $\mathcal{M}$ based on $\mathcal{F}$ and all worlds s in $\mathcal{M}$, if $\mathcal{M}, s \Vdash \Sigma$ then $\mathcal{M}, s \Vdash \phi$.

Definition 1.10. A formula ϕ is valid at a frame $\mathcal{F}$, denoted $\Vdash_{\mathcal{F}} \phi$, if for every model $\mathcal{M}$ based on $\mathcal{F}$ and every state s in $\mathcal{M}$, it holds that $\mathcal{M}, s \Vdash \phi$. A formula ϕ is valid in a class of frames F, denoted $\Vdash_{\mathsf{F}} \phi$ if it is valid in every $\mathcal{F} \in \mathsf{F}$.

We say a formula is *valid* if it is valid in the class of all frames.

Example 1.11. Here follow two examples of valid formulas:

$$
\begin{gathered}
\Diamond (\phi \vee \psi) \leftrightarrow (\Diamond \phi \vee \Diamond \psi) \\
\Box (\phi \wedge \psi) \leftrightarrow (\Box \phi \wedge \Box \psi)
\end{gathered}
$$

Let's look at the second one and try to understand why it is valid. Let $\mathcal{F}$ be a frame in the class of all frames.

($\rightarrow$) Consider an arbitrary model $\mathcal{M}$ based on $\mathcal{F}$ and an arbitrary state s such that $\mathcal{M}, s \Vdash \Box (\phi \wedge \psi)$. By the definition of satisfiability for $\Box$, we get that for any state t such that sRt, $\mathcal{M}, t \Vdash \phi \wedge \psi$. Then, for the satisfiability condition for $\wedge$, $\mathcal{M}, t \Vdash \phi$ and $\mathcal{M}, t \Vdash \psi$ Therefore we conclude that $\mathcal{M}, s \Vdash \Box \phi$ and $\mathcal{M}, s \Vdash \Box \psi$, i.e., $\mathcal{M}, s \Vdash \Box \phi \wedge \Box \psi$, thus $\Vdash_{\mathcal{F}} \Box \phi \wedge \Box \psi$.

($\leftarrow$) Conversely, consider any model $\mathcal{M}$ based on $\mathcal{F}$ and any state s such that $\mathcal{M}, s \Vdash \Box \phi \wedge \Box \psi$. By the satisfiability condition for $\wedge$, We have that $\mathcal{M}, s \Vdash \Box \phi$, hence for any state t accessible from s we have $\mathcal{M}, t \Vdash \phi$. On the other hand $\mathcal{M}, s \Vdash \Box \psi$, thus for any state t' such that sRt' it is the case that $\mathcal{M}, t' \Vdash \psi$. We can conclude that for any state t'' accessible from s we have $\mathcal{M}, t'' \Vdash \phi \wedge \psi$, therefore $\mathcal{M}, s \Vdash \Box(\phi \wedge \psi)$, and so $\Vdash_{\mathcal{F}} \Box(\phi \wedge \psi)$.

Since $\mathcal{F}$ was arbitrary, we conclude that the formula $\Box(\phi \wedge \psi) \leftrightarrow (\Box\phi \wedge \Box\psi)$ is valid in the class of all frames, i.e., is valid.

Note that, just as remarked in Section 1.1, $\Box$ and $\Diamond$ behave much like $\forall$ and $\exists$. We can see in Definition 1.6 that the satisfiability of the box imposes a condition on *all* states while the satisfiability of the diamond imposes a condition on *some* states. Moreover, the distributivity principles from the last example also resemble the ones for the universal and existential quantifiers which further illustrates the relation.

Example 1.12. Consider the formula:

$$\Diamond(\phi \wedge \psi) \leftrightarrow (\Diamond\phi \wedge \Diamond\psi)$$

Consider the following Kripke model $\mathcal{M}$:

$$\begin{tikzcd} & 2, \phi \\ 1 \arrow[ur] \arrow[dr] & \\ & 3, \psi \end{tikzcd}$$

It is easy to see that $\Diamond\phi$ is satisfied at state 1 since we have $1R2$ and $\mathcal{M}, 2 \Vdash \phi$. Similarly $\Diamond\psi$ is also satisfied at state 1, hence $\Diamond\phi \wedge \Diamond\psi$ is satisfied at state 1. However $\Diamond(\phi \wedge \psi)$ is not satisfied at state 1 since $\phi \wedge \psi$ is not satisfied in state 2 or 3. Since there is a state in $\mathcal{M}$ where the formula is not satisfied, we conclude that $\mathcal{M} \nVdash \Diamond(\phi \wedge \psi) \leftrightarrow (\Diamond\phi \wedge \Diamond\psi)$, therefore the formula is not valid.

Definition 1.13. We define the *basic modal system*, denoted by $\mathbf{K}$, as the set of formulas which are valid in the class of all frames, that is, the set of formulas ϕ such that $\Vdash_{\mathsf{K}} \phi$.

There are several interesting modal systems, among which $\mathbf{K}$ is the weakest, being included in any other. These other systems can be defined by a class of frames or by specific axioms. For now, we just focus on the first approach. Recall Table 1.2. The frame conditions there presented can be combined, in which case the corresponding class of frames will be denoted in a natural way, for example, the class of frames that are both reflexive and symmetric will be denoted tb. Even though the logics most commonly considered are represented in Figure 1.5, we can actually consider any arbitrary combinations of the frame conditions. Note however that several of these will give rise to the same system, for instance, we can easily check that the classes of frames dt and t coincide.

Definition 1.14. Let $\mathsf{X} \in \{\mathsf{4}, \mathsf{5}, \mathsf{45}, \mathsf{b}, \mathsf{b5}, \mathsf{d}, \mathsf{db}, \mathsf{d4}, \mathsf{d5}, \mathsf{d45}, \mathsf{t}, \mathsf{tb}, \mathsf{t4}, \mathsf{t5}\}$ be a class of frames. We define system $\mathbf{KX}$ as the set of formulas ϕ such that $\Vdash_{\mathsf{X}} \phi$.

The modal logics introduced in Definition 1.14 are represented in Figures 1.5 and 1.6. Note that the two figures represent precisely the same systems apart from a slight notation change. In Figure 1.5, which is a simplified version of the chart introduced by Brian Chellas in [7], we can see the original denotations of the systems, corresponding to the combinations of the frame conditions as we presented in Definition

1.14. However, for various reasons, other names became more common in the literature, for instance, KT4 and KT5 are frequently referred to as S4 and S5, respectively, following the notation introduced by I. Lewis in [27]. Figure 1.6, commonly referred to as the **S5** cube, shows us the names to which the logics from Figure 1.5 we renamed. From now on we will adopt these denotations. Other than representing the logical systems, the cube(s) provide a visual representation of how the various frame conditions from Table 1.2 are related. To interpret the figure clearly, we imagine within the cube a Cartesian referential with $\mathbf{K}$ at the origin and three axes: the x-axis represents transitivity, the y-axis represents symmetry and the z-axis represents seriality. With this in mind, we can infer several facts about the systems in the cube, for instance, we can see that since $\mathbf{T}$ is above $\mathbf{D}$ in the seriality axis we have that $\Vdash_{\mathsf{t}} \phi$ whenever $\Vdash_{\mathsf{d}} \phi$.

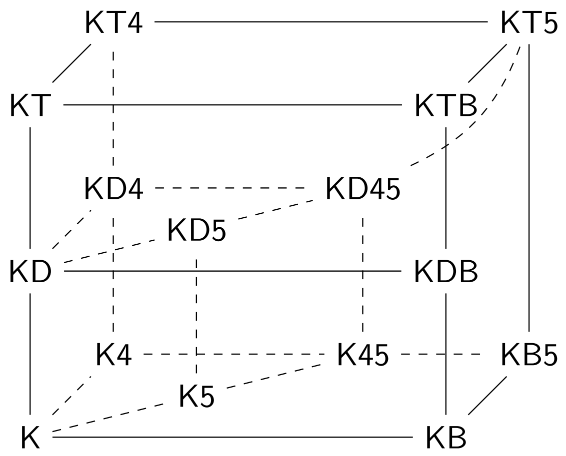


Figure 1.5: Modal cube, original denotations from [7].

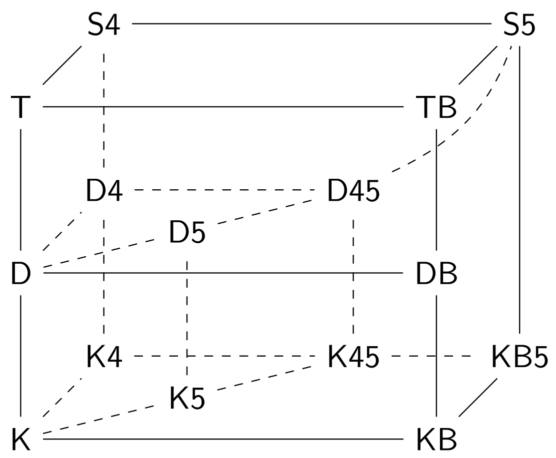


Figure 1.6: Modal cube with commonly used denotations.

Definition 1.15. Let $\mathbf{\Lambda}$ be a logic defined by a class of frames Λ. We say that $\mathbf{\Lambda}$ has the *finite model property* (f.m.p) if for every ϕ such that $\nVdash_{\Lambda} \phi$, there is a finite model based on an Λ-frame where ϕ is not satisfiable.

For now, we merely present this property without discussing it. While it is fundamentally a semantic concept, it will be revisited later in this work in a different setting, when we explore proofs and decidability. In particular, we will show the following result for $\mathbf{K}$ and $\mathbf{S4}$, using only proof-theoretical tools. A semantical proof of this theorem can be found in [16].

Theorem 1.16. *Let $\mathbf{X}$ be one of the logics in the cube. Then, $\mathbf{X}$ has the finite model property.*

1.3 An Hilbert System for K

Now that we are familiar with the semantics for the systems in the cube, we want to move on and study what can be proved within these systems. The first proof systems we shall introduce to this purpose is Hilbert-style axiom systems.

We begin by defining what is a *proof*: it is a finite sequence of formulas where each is either an instance of an *axiom* or the result of applying *inference rules* to the previous formulas. A set of axioms and allowed inference rules that we use to prove logical statements is what we call an *Hilbert-style proof system*. We begin by presenting a Hilbert system for Classical Propositional Logic, and then a system for $\mathbf{K}$ which extends the first one.

Definition 1.17. We define a Hilbert system H_{cp} as the set consisting of the following axioms:

Ax 1. $\bot \to \phi$

Ax 2. $\phi \to (\psi \to \phi)$

Ax 3. $(\phi \to (\psi \to \zeta)) \to ((\phi \to \psi) \to (\phi \to \zeta))$

Ax 4. $(\phi \to \psi) \to ((\zeta \to \phi) \to (\zeta \to \psi))$

Ax 5. $(\phi \wedge \psi) \to \phi$

Ax 6. $(\phi \wedge \psi) \to \psi$

Ax 7. $\phi \to (\psi \to (\phi \wedge \psi))$

Ax 8. $\phi \to (\psi \vee \phi)$

Ax 9. $\psi \to (\psi \vee \phi)$

Ax 10. $(\phi \to \zeta) \to ((\psi \to \zeta) \to ((\phi \vee \psi) \to \zeta))$

Furthermore, we consider the following inference rule:

- *Modus ponens*: $\dfrac{\phi \quad \phi \to \psi}{\psi}$;

Definition 1.18. We define a Hilbert system H_K as the set consisting of the following axioms:

- The axioms from H_{cp};
- Modal distribution (K): $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$;

and having as inference rules:

- *Modus ponens*: $\dfrac{\phi \quad \phi \to \psi}{\psi}$;

- *Necessitation rule*: $\dfrac{\phi}{\Box\,\phi}$;

The next definitions take into account Table 1.3, which summarizes the *characteristic axiom(s)* of the logics in the cube.

D	$\Box\,\phi \to \Diamond\,\phi$
T	$\Box\,\phi \to \phi$
B	$\phi \to \Box\Diamond\,\phi$
4	$\Box\,\phi \to \Box\Box\,\phi$
5	$\Diamond\,\phi \to \Box\Diamond\,\phi$

Table 1.3: Axioms for the extensions of $\mathbf{K}$ within the modal cube.

Definition 1.19. Let $X \subseteq \{D, T, B, 4, 5\}$ denoting a non-empty set of axioms from Table 1.3. We define the system H_X as H_K to which we add the axioms corresponding in X. We have that H_X is a Hilbert system for $\mathbf{X}$.

Definition 1.20. A formula ϕ is *derivable* in $\mathbf{X}$, written $\vdash_{\mathbf{X}} \phi$, if there is a proof ending in ϕ constructed using the axioms and inference rules in H_X.

Definition 1.21. A formula ϕ is *derivable* from Σ in $\mathbf{X}$, written $\Sigma \vdash_{\mathbf{X}} \phi$, if and only if there is finite subset $\{\sigma_1, \ldots, \sigma_n\}$ of Σ such that $\vdash_{\mathbf{X}} (\sigma_1 \wedge \cdots \wedge \sigma_n) \to \phi$.

As we will see in the next section, the axioms in H_K give us an axiomatization of the basic modal system, in the sense that the formulas in $\mathbf{K}$ are precisely those that you can derive from the axioms in H_K using the allowed inference rules. In the final section of this chapter, we will see that the same applies to any logic in the cube and we will be able to characterize them using axiom systems. For example, we have H_T as the Hilbert system which adds axiom T to H_K and thus we have the alternative characterization for system $\mathbf{T}$ which is that $\mathbf{T} = \mathbf{K} + T$.

1.4 Soundness and Completeness

Soundness and completeness are two fundamental concepts in logic, as they establish the relation between syntax and semantics. If a system is sound, it preserves truth, in the sense that manipulating formulas using the syntactic rules of the system can only originate valid conclusions, which is a semantic notion. On the other hand, completeness asserts that a system can capture all valid formulas, as in, every valid formula can be derived within that system using the allowed inference rules.

Definition 1.22. Let F be a class of frames and $\mathbf{X}$ one of the logics in the cube. We say that $\mathbf{X}$ is *sound* with respect to F if $\vdash_{\mathbf{X}} \phi$ implies $\Vdash_{\mathsf{F}} \phi$, for all formulas ϕ.

Proving that the basic modal system $\mathbf{K}$ is sound is simple and can be done directly by working with the axioms and inference rules in H_K.

Theorem 1.23 (Soundness of $\mathbf{K}$). *The basic modal system* $\mathbf{K}$ *is sound with respect to the class of all frames. This is, for all formulas* ϕ *we have that* $\vdash_{\mathbf{K}} \phi$ *implies* $\Vdash_{\mathsf{k}} \phi$.

Proof. To prove soundness we must:

1. Prove the axioms of H_K are valid.
   The proof for the validity of the axioms from H_{cp} is standard, we illustrate this with the proof for **Ax** 5.

   $((\phi \to (\psi \to \zeta)) \to ((\phi \wedge \psi) \to \zeta))$: Let $\mathcal{F}$ be some frame in the class of all frames, $\mathcal{M}$ be any model based on $\mathcal{F}$ and s be any state. Suppose that $\mathcal{M}, s \Vdash \phi \to (\psi \to \zeta)$, that is, that either $\mathcal{M}, s \nVdash \phi$ or $\mathcal{M}, s \Vdash \psi \to \zeta$. In the first case we also have that $\mathcal{M}, s \nVdash \phi \wedge \psi$ and thus $\mathcal{M}, s \Vdash (\phi \wedge \psi) \to \zeta$. In the second case, we have that either $\mathcal{M}, s \nVdash \psi$, from which we conclude that $\mathcal{M}, s \nVdash \phi \wedge \psi$, or we have that $\mathcal{M}, s \Vdash \zeta$. In every scenario, we get that $\mathcal{M}, s \Vdash (\phi \wedge \psi) \to \zeta$ therefore $\mathcal{M}, s \Vdash ((\phi \to (\psi \to \zeta)) \to ((\phi \wedge \psi) \to \zeta))$. Since $\mathcal{M}$ and s were arbitrary we get that $\Vdash_{\mathcal{F}} ((\phi \to (\psi \to \zeta)) \to ((\phi \wedge \psi) \to \zeta))$, hence this axiom is valid in k.

   Modal Distribution: Let $\mathcal{F}$ be some frame in the class of all frames, $\mathcal{M}$ be any model based on $\mathcal{F}$ and s be any state. Suppose that $\mathcal{M}, s \Vdash \Box(\phi \to \psi)$. Then, for any state t, accessible from s, we have $\mathcal{M}, t \Vdash \phi \to \psi$, that is, either $\mathcal{M}, t \nVdash \phi$ or $\mathcal{M}, t \Vdash \psi$. If there is a state t, accessible from s, such that $\mathcal{M}, t \nVdash \psi$, then by assumption we must have $\mathcal{M}, t \nVdash \phi$, hence $\mathcal{M}, s \nVdash \Box\phi$. We conclude that $\mathcal{M}, s \Vdash \Box\phi \to \Box\psi$. On the other hand, if for all states t accessible from s we have $\mathcal{M}, t \Vdash \psi$, then $\mathcal{M}, s \Vdash \Box\psi$, and therefore $\mathcal{M}, s \Vdash \Box\phi \to \Box\psi$. In either case we conclude $\mathcal{M}, s \Vdash \Box\phi \to \Box\psi$, and since $\mathcal{M}$ and s were arbitrary, we get that $\Vdash_{\mathcal{F}} \Box\phi \to \Box\psi$. We conclude that the Modal Distribution axiom is valid in k.

2. Prove the inference rules of $H_{\mathbf{K}}$ preserve validity - we must show that if the premises of the rule are valid, then the conclusion is still valid. In the following, let $\mathcal{F}$ be any frame.

   Modus Ponens: Let $\mathcal{M}$ be any model based on $\mathcal{F}$, s any state and assume $\mathcal{M}, s \Vdash \phi$ and $\mathcal{M}, s \Vdash \phi \to \psi$, that is, the premisses are valid. From $\mathcal{M}, s \Vdash \phi \to \psi$ we conclude that either $\mathcal{M}, s \nVdash \phi$, which can't be the case because $\mathcal{M}, s \Vdash \phi$, or $\mathcal{M}, s \Vdash \psi$. Hence $\mathcal{M}, s \Vdash \psi$. Since $\mathcal{M}$ and s were arbitrary, we conclude that $\Vdash_{\mathsf{k}} \psi$.

   Necessitation Rule: Let $\mathcal{M}$ be any model based on $\mathcal{F}$, assume that $\mathcal{M}, s \Vdash \phi$ for any state s in $\mathcal{M}$. Let t be any state accessible from s, then t is still a state within the model, hence by the arbitrariness of s, $\mathcal{M}, t \Vdash \phi$. Thus $\mathcal{M}, s \Vdash \Box\phi$. Again, because $\mathcal{M}$ and s were arbitrary, we conclude that $\Vdash_{\mathsf{k}} \Box\phi$.

We have seen that the axioms in H_K are valid and that the inference rules preserve validity. Let ϕ be any formula such that $\vdash_{\mathbf{K}} \phi$, that is, there is a proof for ϕ constructed from the axioms and inference rules of H_K. The conclusion that $\Vdash_{\mathsf{k}} \phi$ follows by induction on the length of the proof for ϕ.

QED.

Soundness is especially useful if we want to prove that a certain formula ϕ is *not* derivable. Note that without this property this could be very hard, even unfeasible, as we would need to analyse *every* possible proof for ϕ and argue it did not work. In the presence of a sound logical system, we may reduce this task to proving that ϕ is not valid, that is, that there is *some* model that does not satisfy it.

Example 1.24. Show that $\nvdash_{\mathbf{K}} \Diamond(\phi \wedge \psi) \leftrightarrow (\Diamond \phi \wedge \Diamond \psi)$.
Looking at example 1.12 we can see a model where the formula is not satisfiable and thus it is not valid in k. The result follows by the soundness of $\mathbf{K}$.

We move on to *completeness*, which as discussed before is another essential property of a logical system. To show that system $\mathbf{K}$ is complete is way more complex than showing soundness. Several new definitions and results must be presented in order to prove this result.

Definition 1.25. Let $\mathbf{X}$ be one of the logics in the cube. We say $\mathbf{X}$ is *(weakly) complete* with respect to a class of frames F if $\Vdash_{\mathsf{F}} \phi$ implies $\vdash_{\mathbf{X}} \phi$, for every formula ϕ.

Definition 1.26. Let $\mathbf{X}$ be one of the logics in the cube. We say $\mathbf{X}$ is *strongly complete* with respect to a class of frames F, if for every set of formulas $\Sigma \cup \{\phi\}$, we have that $\Sigma \models_{\mathsf{F}} \phi$ implies $\Sigma \Vdash_{\mathbf{X}} \phi$.

We will show a stronger result than we first aimed to, which is that $\mathbf{K}$ is strongly complete. This immediately implies the system's (weak) completeness as well. However, tackling this property directly from the definition presents considerable challenges due to the absence of tools that establish connections between validity and provability. Consequently, we will introduce an alternative characterization of strong completeness which simplifies our problem by allowing us to analyse a particular kind of set of formulas - *consistent sets*.

In the following, let Λ denote some logical system, $\vdash_{\Lambda}$ denote derivability in the axiom system for Λ and F some class of frames.

Definition 1.27. A set of formulas Σ is Λ-*consistent* if $\Sigma \nvdash_{\Lambda} \bot$. Otherwise, we say Σ is Λ-*inconsistent*.

Proposition 1.28. A logic Λ is strongly complete with respect to a class of frames F *if and only if* every Λ-consistent set of formulas is satisfiable in some world in some model based on an F-frame.

Proof.

($\rightarrow$) Assume that Λ is strongly complete and let Σ be a consistent set of formulas, i.e., such that $\Sigma \nvdash_{\Lambda} \bot$. By strong completeness of Λ we have that $\Sigma \not\models_{\mathsf{F}} \bot$, hence by the definition of logical consequence, there is a model $\mathcal{M}$ based on $\mathcal{F} \in \mathsf{F}$ such that $\mathcal{M} \Vdash \Sigma$ but $\mathcal{M} \nVdash \bot$, in particular Σ is satisfiable as we wanted to show.

($\leftarrow$) Assume Λ is not strongly complete. Then there is a set $\Sigma \cup \{\phi\}$ such that $\Sigma \models_{\mathsf{F}} \phi$ and $\Sigma \nvdash_{\Lambda} \phi$. It follows from $\Sigma \models_{\mathsf{F}} \phi$ that for every model $\mathcal{M}$ based on an F-frame and every world s in $\mathcal{M}$ we have that $\mathcal{M}, s \Vdash \Sigma$ implies $\mathcal{M}, s \Vdash \phi$. On the other hand, since $\Sigma \nvdash_{\Lambda} \phi$, we have that $\Sigma \cup \{\neg\phi\}$ is consistent. Suppose this set of formulas in satisfiable in some world s of some model $\mathcal{M}$ based on an F-frame, that is, $\mathcal{M}, s \Vdash \Sigma \cup \{\neg\phi\}$. Then, it would be the case that $\mathcal{M}, s \Vdash \Sigma$, and thus, by the definition of logical consequence $\mathcal{M}, s \Vdash \phi$, contradicting the fact that $\mathcal{M}, s \Vdash \neg\phi$.

QED.

Consequently, to establish the strong completeness of any logical system Λ, it suffices to construct models that satisfy each $\Lambda-$consistent set. The question we need to answer is: how do we build these models? Before answering this question, consider the following definition and results.

Definition 1.29. A set of formulas Σ is Λ-*maximal consistent* (Λ-MCS), if it is consistent, i.e. $\Sigma \nvdash_{\Lambda} \bot$ and any set Γ containing Σ is inconsistent.

Proposition 1.30. Let Γ be a Λ-MCS, the following hold:

1. $\phi \in \Gamma \leftrightarrow \Gamma \vdash_\Lambda \phi$;
2. $\bot \notin \Gamma$;
3. $\Lambda \subseteq \Gamma$;
4. $\neg\,\phi \in \Gamma \leftrightarrow \phi \notin \Gamma$;
5. $\phi \wedge \psi \in \Gamma \;\leftrightarrow\; \phi \in \Gamma$ *and* $\psi \in \Gamma$;

Similar properties for other logical connectives can be derived from these last ones.

The following lemma, although very significant to establishing our main result, is presented without proof since there are no relevant additions to the proof of the propositional case. A comprehensive proof can be found in [5].

Lemma 1.31 (Lindenbaum's Lemma)**.** Let Λ be some modal logic. For every Λ-consistent set of formulas Σ, there is a Λ-maximally consistent set Σ^+ such that $\Sigma \subseteq \Sigma^+$.

From now on we state our results only for logic $\mathbf{K}$ but the following definition and lemmas work for all other systems within the modal cube.

Definition 1.32 (Canonical Model)**.** The canonical model for $\mathbf{K}$ in the basic language is the triple $\mathcal{M}^{\mathbf{K}} = \left(W^{\mathbf{K}}, R^{\mathbf{K}}, [\![\cdot]\!]^{\mathbf{K}}\right)$ where:

- $W^{\mathbf{K}}$ is the set of all the $\mathbf{K}$-maximally consistent sets;
- The *canonical* relation $R^{\mathbf{K}} \subseteq W^{\mathbf{K}} \times W^{\mathbf{K}}$ is defined by the condition:

$$uR^{\mathbf{K}}v \text{ if and only if } \phi \in v \text{ implies } \Diamond\,\phi \in u, \text{ for any formula } \phi \in \mathcal{L}^{\Box,\Diamond}$$

- The *canonical* valuation $[\![\,\cdot\,]\!]^{\mathbf{K}}$ is such that $[\![\,p\,]\!]^{\mathbf{K}} = \{u \in W^{\mathbf{K}} | \, p \in u\}$;

Lemma 1.33 (Existence Lemma)**.** Let $u \in W^{\mathbf{K}}$, if $\Diamond\,\phi \in u$ then there is a state $v \in W^{\mathbf{K}}$ such that $uR^{\mathbf{K}}v$ and $\phi \in v$.

Proof. Let $u \in W^{\mathbf{K}}$ and ϕ any formula such that $\Diamond\,\phi \in u$. Let's construct a state v satisfying the desired conditions.
Let $v' = \{\phi\} \cup \{\psi \,|\, \Box\,\psi \in u\}$. We prove by contraposition that this set is $\mathbf{K}$-consistent. Assume it is not, then there exist $\psi_1, \ldots, \psi_n \in v'$ such that $\vdash_{\mathbf{K}} (\psi_1 \wedge \cdots \wedge \psi_n) \to \neg\,\phi$. By necessitation we have

$$\vdash_{\mathbf{K}} \Box\,((\psi_1 \wedge \cdots \wedge \psi_n) \to \neg\,\phi)\,.$$

Then, applying axiom *K* and modus ponens it follows that:

$$\vdash_{\mathbf{K}} \Box\,(\psi_1 \wedge \cdots \wedge \psi_n) \to \Box\,(\neg\,\phi)\,.$$

We have proved before that $\Box$ distributes over the conjunction, hence:

$$\vdash_{\mathbf{K}} (\Box\,\psi_1 \wedge \cdots \wedge \Box\,\psi_n) \to \Box\,(\neg\,\phi)\,.$$

By the definition of v', we have that $\Box\,\psi_1, \ldots, \Box\,\psi_n \in u$ thus the conjunction $\Box\,\psi_1 \wedge \cdots \wedge \Box\,\psi_n$ is also in u, as it is a $\mathbf{K}$-maximally consistent set. Therefore $\Box\,(\neg\,\phi) \in u$, or equivalently, $\neg\,\Diamond\,\phi \in u$. This contradicts our initial claim that $\Diamond\,\phi \in u$, thus v' must be $\mathbf{K}$-consistent.

Let v be a MCS such that $v' \subseteq v$ (this exists by Lindenbaum's Lemma), clearly $\phi \in v$. Assume that for some $\zeta \in v$, we have $\Diamond\,\zeta \notin u$, i.e. that $\Box\,\neg\,\zeta \in u$. Note that ζ cannot be ϕ, hence by the definition of v it follows that $\neg\,\zeta \in v$, i.e., $\zeta \notin v$, which contradicts our assumption. Hence $uR^{\mathbf{K}}v$.

QED.

Lemma 1.34 (Truth Lemma)**.** Consider the canonical model $\mathcal{M}^{\mathbf{K}}$, and u some state. Let ϕ be any formula. Then:

$$\mathcal{M}^{\mathbf{K}}, u \Vdash \phi \ \textit{ iff } \ \phi \in u.$$

Proof. The proof goes by induction on the structure of ϕ. We only show some of the boolean cases, which follow essentially from the properties of MCS's.

- ϕ is $\bot$. It follows immediately since $\bot \notin u$ and $\mathcal{M}^{\mathbf{K}}, u \nVdash \bot$;
- ϕ is p. Then $\mathcal{M}^{\mathbf{K}}, u \Vdash p$ *iff* $u \in [\![\, p \,]\!]^{\mathbf{K}} = \{v \in W^{\mathbf{K}} \mid p \in v\}$ *iff* $p \in u$;
- ϕ is $\psi_1 \wedge \psi_2$. Then $\mathcal{M}^{\mathbf{K}}, u \Vdash \psi_1 \wedge \psi_2$ *iff* $\mathcal{M}^{\mathbf{K}}, u \Vdash \psi_1$ and $\mathcal{M}^{\mathbf{K}}, u \Vdash \psi_2$. By the inductive hypothesis, this is equivalent to having $\psi_1 \in u$ and $\psi_2 \in u$ *iff* $\psi_1 \wedge \psi_2 \in u$ because it is a $\mathbf{K}$-MCS;
- ϕ is $\Diamond\,\psi$. First, we show the implication from left to right. Assume $\mathcal{M}^{\mathbf{K}}, u \Vdash \Diamond\,\psi$. By definition there is $v \in W^{\mathbf{K}}$ such that $uR^{\mathbf{K}}v$ and $\mathcal{M}^{\mathbf{K}}, v \Vdash \psi$. By induction hypothesis, $\psi \in v$. By definition of the canonical relation, this implies that $\Diamond\,\psi \in u$.
  Suppose now that $\Diamond\,\psi \in u$. The Existence Lemma guarantees the existence of a state $v \in W^{\mathbf{K}}$ such that $uR^{\mathbf{K}}v$ and $\psi \in v$. Hence $\mathcal{M}^{\mathbf{K}}, u \Vdash \Diamond\,\psi$;
- ϕ is $\Box\,\psi$. From right to left, we do the proof by contraposition. Assume $\mathcal{M}^{\mathbf{K}}, u \nVdash \Box\,\psi$, our goal is to prove that $\Box\,\psi \notin u$. By assumption, there is a state $v \in W^{\mathbf{K}}$, such as $uR^{\mathbf{K}}v$ and $\mathcal{M}, v \nVdash \psi$. By induction hypothesis, we get that $\psi \notin v$, and since v is a $\mathbf{K}$-MCS, $\neg\,\psi \in v$. By the definition of $R^{\mathbf{K}}$, since $uR^{\mathbf{K}}v$ and $(\neg\,\psi) \in v$, we get that $\Diamond(\neg\,\psi) \in u$. Using maximal consistency of u we get that $\neg(\Diamond\neg\,\phi) \notin u$, hence $\Box\,\psi \notin u$.
  From left to right, assume $\mathcal{M}^{\mathbf{K}}, u \Vdash \Box\,\phi$. Then for all v such that $uR^{\mathbf{K}}v$ we have $\mathcal{M}^{\mathbf{K}}, v \Vdash \phi$. Assume, in order to reach a contradiction, that $\Box\,\phi \notin u$. By maximal consistency $\neg\Box\,\phi \in u$, that is, $\Diamond\neg\phi \in u$. By the inductive hypothesis $\mathcal{M}^{\mathbf{K}}, u \Vdash \Diamond\neg\phi$, thus $\mathcal{M}^{\mathbf{K}}, v \Vdash \neg\phi$ for some v such that $uR^{\mathbf{K}}v$. In particular, we would have that $\mathcal{M}^{\mathbf{K}}, v \nVdash \phi$ contradicting our initial hypothesis. Thus $\Box\,\phi \in u$.

QED.

Theorem 1.35 (Canonical Model Theorem)**.** $\mathbf{K}$ *is strongly complete with respect to the class of all frames.*

Proof. Let Σ be any $\mathbf{K}$-consistent set, which by Lindenbaum's Lemma can be extended to a $\mathbf{K}$-MCS Σ^+. Lemma 1.34 ensures that $\mathcal{M}^{\mathbf{K}}, \Sigma^+ \Vdash \Sigma$, i.e., Σ is satisfiable in some world in the canonical model. By Proposition 1.28, $\mathbf{K}$ is strongly complete.

QED.

Theorem 1.36. $\mathbf{K}$ *is complete.*

Proof. By theorem 1.35, for every set $\Sigma \cup \{\phi\}$, $\Sigma \models_{\mathsf{k}} \phi$ implies $\Sigma \vdash_{\mathbf{K}} \phi$. Taking the empty set as Σ we conclude that $\Vdash_{\mathsf{k}} \phi$ implies $\vdash_{\mathbf{K}} \phi$, i.e, we conclude that $\mathbf{K}$ is complete.

QED.

Having successfully established soundness and completeness, we can now justify that the axioms in H_K provide an axiomatization for $\mathbf{K}$. Let ϕ be any formula in $\mathbf{K}$, that is, any formula such $\Vdash_{\mathsf{k}} \phi$. By soundness and completeness, this is equivalent to $\vdash_{\mathbf{K}} \phi$, i.e., any formula in $\mathbf{K}$ is derivable from the axioms in H_K.

1.5 Beyond K and into the modal cube

Section 1.2 introduced us to several frame conditions and to the logical systems that can be defined by them, with the modal cube illustrating the relation between these logics and the basic modal system $\mathbf{K}$. Later, in Section 1.3, we noted that there is an alternative way to define the logics in the cube, that is, through axioms. The present section aims to show that these two characterizations are equivalent by proving the *frame correspondence* result. Furthermore, we extend soundness and completeness to these systems.

Definition 1.37. If a class of frames can be simultaneously defined by a modal formula ϕ and by a first-order formula α, we say ϕ and α are *frame correspondents*.

Table 1.4 displays side-by-side the axioms and frame conditions for the classes of frames in Table 1.2, as well as the corresponding extension of $\mathbf{K}$.

K	$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$	
D	$\Box\phi \to \Diamond\phi$	*for all* $x \in W$ *there is* $y \in W$ *such that* xRy
T	$\Box\phi \to \phi$	*for all* $x \in W$ *we have* xRx
B	$\phi \to \Box\Diamond\phi$	*for all* $x, y \in W$ *if* xRy *then* yRx
4	$\Box\phi \to \Box\Box\phi$	*for all* $x, y, z \in W$ *if* xRy *and* yRz *then* xRz
5	$\Diamond\phi \to \Box\Diamond\phi$	*for all* $x, y, z \in W$ *if* xRy *and* xRz *then* yRz

Table 1.4: Frame correspondents.

We are now ready to state the main result of the section, which allows us to gain a deeper insight into which structural properties of frames are required in order to have a given logic satisfy specific modal axioms. We begin by illustrating this result by proving two particular lemmas for the cases of axioms T and 4.

Lemma 1.38. Let $\mathcal{F}$ be a frame and ϕ be any formula. We have that

$$\Vdash_{\mathcal{F}} \Box\phi \to \phi \textit{ if and only if } \mathcal{F} \textit{ is reflexive}.$$

Proof.

($\leftarrow$) Suppose $\mathcal{F} = (W, R)$ is a reflexive frame, i.e., that for all $s \in W$, we have sRs. Let $\mathcal{M}$ be any model based on $\mathcal{F}$ and s be any state. Assume that $\mathcal{M}, s \Vdash \Box\phi$, then by the satisfiability condition of the $\Box$ we have that for any state t accessible from s we have that $\mathcal{M}, t \Vdash \phi$. But since $\mathcal{F}$ is reflexive, s is accessible from itself, hence $\mathcal{M}, s \Vdash \phi$. We conclude that $\mathcal{M}, s \Vdash \Box\phi \to \phi$. Therefore the formula $\Box\phi \to \phi$ is valid at $\mathcal{F}$.

($\to$) Let's prove the contrapositive. Assume that $\mathcal{F}$ is not reflexive, i.e., that there is some state s such that sRs does not hold. We must find a model $\mathcal{M}$ based on $\mathcal{F}$ and a state t such that $\mathcal{M}, t \Vdash \Box\phi$ but $\mathcal{M}, t \nVdash \phi$. Let $\mathcal{M} = (\mathcal{F}, [\![\cdot]\!])$ and let's choose t to be an irreflexive state s. Define $[\![\phi]\!] = W \backslash \{s\}$, then it is clear that $\mathcal{M}, s \nVdash \phi$. On the other hand, let u be a state accessible from s, u must be different from s since $\mathcal{F}$ is not reflexive, hence $\mathcal{M}, u \Vdash \phi$. As u was arbitrary, $\mathcal{M}, s \Vdash \Box\phi$, and therefore $\mathcal{M}, s \nVdash \Box\phi \to \phi$.

QED.

Lemma 1.39. Let $\mathcal{F}$ be a frame and ϕ be any formula. We have that

$$\Vdash_{\mathcal{F}} \Box\phi \to \Box\Box\phi \textit{ if and only if } \mathcal{F} \textit{ is transitive.}$$

Proof.

($\leftarrow$) Suppose $\mathcal{F} = (W, R)$ is a transitive frame, i.e., for all $s, t, u \in W$, if it holds that sRt and tRu then sRu. Let $\mathcal{M}$ be a model based on $\mathcal{F}$ and let s be some state such that $\mathcal{M}, s \Vdash \Box\phi$. Then, we have that for any state t such that sRt, $\mathcal{M}, t \Vdash \phi$. Let u be any state such that tRu, since $\mathcal{F}$ is transitive, we have that sRu and thus $\mathcal{M}, u \Vdash \phi$. Therefore, $\mathcal{M}, t \Vdash \Box\phi$, and so $\mathcal{M}, s \Vdash \Box\Box\phi$. We conclude that $\mathcal{M}, s \Vdash \Box\phi \to \Box\Box\phi$ and hence $\Vdash_{\mathcal{F}} \Box\phi \to \Box\Box\phi$.

($\to$) We prove the contrapositive. Assume that $\mathcal{F}$ is not transitive, i.e., that there are states s, t, u such that sRt and tRu but it does not hold that sRu. We must find a model $\mathcal{M}$ based on $\mathcal{F}$ and a state v such that $\mathcal{M}, v \Vdash \Box\phi$ but $\mathcal{M}, v \nVdash \Box\Box\phi$. Let $\mathcal{M} = (\mathcal{F}, [\![\cdot]\!])$ and v be the state s. Define $[\![\phi]\!] = W \setminus \{u\}$. Then, for all states t accessible from s, $\mathcal{M}, t \Vdash \phi$ since u is not accessible from s, and thus $\mathcal{M}, s \Vdash \Box\phi$. On the other hand, $\mathcal{M}, t \nVdash \Box\phi$, since tRu and $\mathcal{M}, u \nVdash \phi$, and thus $\mathcal{M}, s \nVdash \Box\Box\phi$. Hence, $\mathcal{M}, s \nVdash \Box\phi \to \Box\Box\phi$.

QED.

The main result is stated as follows. The proof for the theorem is similar to the cases above and will be omitted.

Theorem 1.40 (Frame Correspondence)**.** *Let* X *be a class of frames and* X *the corresponding axiom(s) from table 1.4. Let* $\mathcal{F}$ *be any frame. We have* $\Vdash_{\mathcal{F}} X$ *if and only if* $\mathcal{F} \in \mathsf{X}$.

We now aim at establishing soundness and completeness for all the logics in the cube. The proof strategies follow what was done in Section 1.4. Soundness follows almost immediately from Theorem 1.40.

Theorem 1.41. *Let* $\mathbf{X}$ *be any of the logics in the cube and* X *the corresponding class of frames. Then* $\mathbf{X}$ *is sound with respect to* X.

Proof. Consider the Hilbert system H_X for $\mathbf{X}$. In order to prove soundness we must check that the axioms are valid in the class of frames X and that the inference rules preserve validity. The validity of the axioms in the corresponding class of frames is an immediate consequence of lemma 1.40. The inference rules are the same as in H_K so there is nothing new to prove.

QED.

In order to prove completeness, we make use of the Canonical Model and Lemma 1.34. Note that the only necessary change in the definition of the Canonical Model for any logic $\mathbf{X}$ in the cube is regarding the set of states $W^{\mathbf{X}}$ which is now the set of $\mathbf{X}$-MCS. Naturally, the Canonical Model for $\mathbf{X}$ is denoted $\mathcal{M}^{\mathbf{X}} = \left(W^{\mathbf{X}}, R^{\mathbf{X}}, [\![\cdot]\!]^{\mathbf{X}}\right)$. Other than adapting this definition, the crucial change in the completeness proof is that we now need to ensure that the model we are constructing is within the desired class of frames. This result is first presented in the following two lemmas which illustrate it regarding systems $\mathbf{T}$ and $\mathbf{S4}$ and then generalized for every logic in the cube.

Lemma 1.42. If $\vdash_{\mathbf{T}} \Box\,\phi \to \phi$ then $\mathcal{M}^{\mathbf{T}}$ is reflexive.

Proof. Assume that $\vdash_{\mathbf{T}} T$ and consider any $\mathbf{T}$-MCS set v, we have that $T \in v$. Note that v is a state from the canonical model $\mathcal{M}^{\mathbf{T}}$. Then, from Lemma 1.34 we get that $\mathcal{M}^{\mathbf{T}}, v \Vdash \Box\,\phi \to \phi$, hence $\Box\,\phi \in v$ which implies $\phi \in v$. Note that in the definition of the accessibility relation in $\mathcal{M}^{\mathbf{T}}$, we have that a state s is accessible from itself if for every formula ψ, $\psi \in s$ implies $\Diamond\,\psi \in s$, which is equivalent to $\Box\,\neg\,\psi \in s$ implies $\neg\,\psi \in s$. Therefore, we conclude that $vR^{\mathbf{T}}v$.

QED.

Lemma 1.43. If $\vdash_{\mathbf{S4}} \Box\,\phi \to \Box\Box\,\phi$ then $\mathcal{M}^{\mathbf{S4}}$ is transitive.

Proof. Assume that $\vdash_{\mathbf{S4}} 4$ and consider any u, v, w $\mathbf{S4}$-MCS such that $uR^{\mathbf{S4}}v$ and $vR^{\mathbf{S4}}w$. We have that $\Box\,\phi \to \Box\Box\,\phi \in u$ hence $\mathcal{M}, u \Vdash \Box\,\phi \to \Box\Box\,\phi$, analogously for y and z. Note that this condition is equivalent to $\mathcal{M}, u \Vdash \neg\Diamond\neg\,\phi \to \neg\Diamond\neg\Box\,\phi$ which is equivalent to $\mathcal{M}, u \Vdash \neg\Diamond\neg\,\phi \to \neg\Diamond\Diamond\neg\,\phi$ and $\mathcal{M}, u \Vdash \Diamond\Diamond\neg\,\phi \to \Diamond\neg\,\phi$. Let $\psi \in w$, we want to prove that $uR^{\mathbf{S4}}w$, that is, we want to show that $\Diamond\,\psi \in u$. Since $vR^{\mathbf{S4}}w$ we have that $\Diamond\,\psi \in v$ and hence $\Diamond\Diamond\,\psi \in u$ since $uR^{\mathbf{S4}}v$. By Lemma 1.34 we have that $\mathcal{M}, u \Vdash \Diamond\Diamond\,\psi$, thus $\mathcal{M}, u \Vdash \Diamond\,\psi$ as we wanted to show.

QED.

Lemma 1.44. For $\mathbf{X}$ one of the logics in the cube and X its characteristic axiom(s), we have that $\vdash_{\mathbf{X}} X$ if and only if $\mathcal{M}^{\mathbf{X}}$ satisfies the frame condition corresponding to X.

We now move on to completeness, which is illustrated for $\mathbf{T}$.

Lemma 1.45. $\mathbf{T}$ is complete with respect to the class of reflexive frames.

Proof. Let Σ be some $\mathbf{T}$-consistent set, we want to show that this is satisfiable in some $\mathcal{T} \in \mathsf{t}$. By lemma 1.31, Σ can be extended to a $\mathbf{T}$-MCS that we will denote by Σ^+. By Lemma 1.39 we have that $\mathcal{M}^{\mathbf{T}}$ is reflexive, and by Lemma 1.34 we have that $\mathcal{M}^{\mathbf{T}}, \Sigma^+ \Vdash \Sigma$. Hence, by proposition 1.28 we have that $\mathbf{T}$ is strongly complete, and consequently $\mathbf{T}$ is complete.

QED.

Theorem 1.46. *Let $\mathbf{X}$ be any of the logics in the cube and X the corresponding class of frames. Then $\mathbf{X}$ is complete with respect to X.*

In this chapter, we have succeeded at relating, for any logic, validity within a class of frames and derivability from a Hilbert system. This result is depicted in the following diagram, which will be completed with more information throughout the next chapters:

$$\begin{tikzcd} \Vdash_{\mathsf{X}} \phi \arrow[rr, bend left, Leftrightarrow] & & \vdash_{\mathbf{X}} \phi \end{tikzcd}$$

Chapter 2

A gentle introduction to sequent calculus

Proof theory is the branch of mathematical logic which concerns the study of *proofs* as formal objects. The concept of proof dates back to Ancient Greece, with its first instances attributed to Thales (cf. [19], [20]), who proved that every circle is bisected by a diameter. However, the formal study of proofs is relatively recent. In fact, the study of proof theory, as it exists today, has emerged only in the past century with roots in the work of David Hilbert. The German mathematician aimed at solving an ongoing problem concerning the foundations of mathematics, seeking a consistent, complete, finite set of axioms that could formulate *all* mathematics. This goal was known as *Hilbert's program*. Even though Gödel later demonstrated the impossibility of achieving Hilbert's ambitious goal, his studies started the research field of proof theory, leading to subsequent developments.

This chapter serves as an introduction to a specific style of formal systems used to express line-by-line arguments called *sequent calculus*. These systems were introduced by Gerhard Gentzen in the early 20th century, marking a crucial moment in the development of proof theory. We discuss such formal systems for classical propositional logic and how to possibly extend them to $\mathbf{K}$ and all the logics within the modal cube. Section 2.1 focuses solely on the Gentzen calculus denoted $\mathbf{G3cp}$, as presented in [33]. We present some basic definitions along with the rules of the calculus. We prove the invertibility of all the rules, admissibility of weakening, contraction, and most importantly cut. This result, as stated in Theorem 2.11, can be regarded as one of the most notorious results in this domain as we will briefly evidentiate in Section 2.1. The section finishes by proving soundness and completeness of the calculus. Moving on to Section 2.2, we aim at devising rules, in strict articulation with the characteristic axioms from Table 1.3, in order to develop calculi for the modal logics in the $\mathbf{S5}$-cube, extending $\mathbf{G3cp}$. However, as we shall discuss, the solution proposed has some undesirable consequences, resulting in the loss of several desirable properties that the system $\mathbf{G3cp}$ had, namely the loss of the admissibility of cut in the calculus for $\mathbf{S5}$.

2.1 Gentzen calculus for classical propositional logic

In this section, we will focus on the Gentzen style calculus for classical propositional logic, denoted $\mathbf{G3cp}$, as presented by Troelstra and Schwichtenberg in [33]. Recall the language of classical propositional calculus introduced in Section 1.1. In order to properly define our calculus, we must introduce what are the rules we can use to build arguments, but firstly we define what exactly is a sequent.

Definition 2.1. A sequent in the calculus $\mathbf{G3cp}$ is an expression $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite multisets [1] of formulas. Γ is called the *antecedent* and Δ the *succedent*.

Initial sequents

$$\dfrac{}{p, \Gamma \Rightarrow \Delta, p}\,\mathsf{init} \qquad \dfrac{}{\bot, \Gamma \Rightarrow \Delta}\,\bot_{\mathsf{L}}$$

Logical rules

$$\dfrac{\phi, \psi, \Gamma \Rightarrow \Delta}{\phi \wedge \psi, \Gamma \Rightarrow \Delta}\,\wedge_{\mathsf{L}} \qquad \dfrac{\Gamma \Rightarrow \Delta, \phi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \wedge \psi}\,\wedge_{\mathsf{R}}$$

$$\dfrac{\phi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\phi \vee \psi, \Gamma \Rightarrow \Delta}\,\vee_{\mathsf{L}} \qquad \dfrac{\Gamma \Rightarrow \Delta, \phi, \psi}{\Gamma \Rightarrow \Delta, \phi \vee \psi}\,\vee_{\mathsf{R}}$$

$$\dfrac{\psi, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \phi}{\phi \to \psi, \Gamma \Rightarrow \Delta}\,\to_{\mathsf{L}} \qquad \dfrac{\phi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \to \psi}\,\to_{\mathsf{R}}$$

Table 2.1: The sequent calculus $\mathbf{G3cp}$.

The sequent calculus $\mathbf{G3cp}$ is composed of the axioms and rules depicted in Table 2.1. Refering to the rules, we call Γ and Δ *contexts*, that is, they are multisets of formulas which are not affected by the rules. The formulas in the conclusion of each rule which are not in the context are called *principal formulas*. The formulas in the premise(s) which are not in the context are called *active formulas*. We also distinguish two types of rules, *logical rules* affect the connectives in the sequents, and *structural rules* that do not. All the rules in Table 2.1 are logical rules. Proofs in these formal systems, also called *derivations*, can be identified with *labelled finite trees*.

Definition 2.2. A *directed graph* G is a pair (V, E) where V is a set of nodes and E a set of edges, that is, ordered pairs (v_i, v_j) where $v_i, v_j \in V$. A *path* is a sequence of nodes $p = v_1 \dots v_n$ such that for every $i \in \{1, \dots, n-1\}$ there is an edge from v_i to v_{i+1}. We say a path p is a *cycle* if there are two elements in the path v_i and v_j, with $i \neq j$, such that $v_i = v_j$.

A *tree* is an acyclic-directed graph in which any two nodes are connected by at most one path. Furthermore, every node v has at most one *parent*, that is, there is at most one v' such that $(v, v') \in E$. A *child* of a node v is a v' such that $(v', v) \in E$. The *root* is a node which has no parent and the *leaves are nodes with no children*. We call a tree with a unique root a *rooted tree*. A *branch* is a path starting in the root and ending in one of the leaves. If the tree is finite and has a root, we can define the *height* of the tree as the number of nodes in the longest branch. In the context of sequent calculus, we are interested in *labelled trees*, that is, trees where every node has a unique *label*.

Definition 2.3. A *derivation* is a finite rooted labelled tree, such that each node is labelled by a sequent. The leaves of the tree are initial sequents. Every sequent occurring in a node that is not a leaf is obtained by the sequents occurring in the children-nodes of that node, by the application of an inference rule. The sequent occupying the root node is called the conclusion of the derivation. A sequent $\Gamma \Rightarrow \Delta$ is *derivable*

[1]A multiset is a collection of objects that allows repetition of its elements and where order is not relevant. We use the same notation for sets and multisets relying on context to differentiate them.

in $\mathbf{G3cp}$, written $\vdash_{\mathbf{G3cp}} \Gamma \Rightarrow \Delta$ if there is a derivation whose root is labelled by $\Gamma \Rightarrow \Delta$. An arbitrary formula $\phi \in \mathcal{L}$ is derivable if there is a derivation for $\Rightarrow \phi$, and we write $\vdash_{\mathbf{G3cp}} \Rightarrow \phi$.

Example 2.4. Let p, q be propositional symbols. The formula $p \to (q \to p)$ is derivable in $\mathbf{G3cp}$:

$$\dfrac{\dfrac{\dfrac{}{p, q \Rightarrow p}\,\mathsf{init}}{p \Rightarrow q \to p}\,\to_{\mathsf{R}}}{\Rightarrow p \to (q \to p)}\,\to_{\mathsf{R}}$$

At this point, we have properly introduced the $\mathbf{G3cp}$ calculus we move on to analysing some additional structural rules. Let R be some rule of a sequent calculus $\mathbf{S}$. We say that R is:

- *admissible* whenever derivability of the premisses implies derivability of the conclusion;
- *height-preserving admisible* whenever there exists a derivation of height h of the premisses then also exists a derivation for the conclusion with height at most h;
- *invertible* whenever derivability of the conclusion implies derivability of all the premisses;
- *height-preserving invertible* whenever if there exists a derivation of height h of the conclusion then also exists a derivation for the premisses with height at most h;

Definition 2.5. Let $\circ$ be any propositional connective. The only subformula of p is p. The subformulas of $\neg\,\phi$ are $\neg\,\phi$ and the subformulas of ϕ. The subformulas of $\phi \circ \psi$ are $\phi \circ \psi$ and the subformulas of ϕ and ψ.

We say that the sequent calculus $\mathbf{S}$ has the *subformula property* if for all the rules in $\mathbf{S}$ the formulas in the premisses are all subformulas of the formulas in the conclusion.

$$\dfrac{\Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta}\,\mathsf{wk_L} \qquad \dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \phi}\,\mathsf{wk_R}$$

$$\dfrac{\phi, \phi, \Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta}\,\mathsf{Ctr_L} \qquad \dfrac{\Gamma \Rightarrow \Delta, \phi, \phi}{\Gamma \Rightarrow \Delta, \phi}\,\mathsf{Ctr_R}$$

$$\dfrac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\mathsf{cut}$$

Table 2.2: Structural Rules of $\mathbf{G3cp}$.

Other than the logical rules we can consider the *structural rules* of *weakening*, *contraction* and *cut* in table 2.2. We will now show that these rules, although not explicitly part of the calculus, are admissible and can be used freely in derivations.

Proposition 2.6. Rules $\mathsf{wk_L}$ and $\mathsf{wk_R}$ height-preserving admissible in $\mathbf{G3cp}$.

Proof. Let's assume that $\Gamma \Rightarrow \Delta$ is derivable with derivation height equal to h. We reason by induction on h. If $h = 1$, $\Gamma \Rightarrow \Delta$ is an initial sequent, hence $\phi, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, \phi$ are too and therefore are derivable with derivation height $h = 1$.
If $h > 1$, we must consider all the cases for what the last rule applied may be. We only show this for $\wedge_{\mathsf{L}}$. Assume that $\zeta \wedge \psi, \Gamma \Rightarrow \Delta$ is derivable with derivation height h and that the last rule applied was $\wedge_{\mathsf{L}}$. Hence, $\zeta, \psi, \Gamma \Rightarrow \Delta$ is derivable with height $h - 1$. By induction hypothesis, for any formula ϕ, we have that $\vdash_{\mathbf{G3cp}} \phi, \zeta, \psi, \Gamma \Rightarrow \Delta$ with height at most $h - 1$, and by $\wedge_{\mathsf{L}}$ we get $\vdash \phi, \zeta \wedge \psi, \Gamma \Rightarrow \Delta$ with derivation height at most h. This shows that $\mathsf{wk_L}$ is height-preserving admissible. The proof for $\mathsf{wk_R}$ is analogous.

QED.

The following lemma will be useful in subsequent proofs.

Lemma 2.7. Let ϕ be some formula, Γ, Δ finite multisets. Then

$$\vdash_{\mathbf{G3cp}} \phi, \Gamma \Rightarrow \Delta, \phi$$

Proof. The proof is by induction on the structure of ϕ. Let Γ, Δ be finite multisets.

(Base) If ϕ is atomic the result follows from rule init.

(Step) We shall show the case where ϕ is $\psi \wedge \zeta$. Consider the derivation:

$$\dfrac{\dfrac{\dfrac{\vdots}{\psi, \Gamma \Rightarrow \Delta, \psi}}{\psi, \zeta, \Gamma \Rightarrow \Delta, \psi}\,\mathsf{w_L} \qquad \dfrac{\dfrac{\vdots}{\zeta, \Gamma \Rightarrow \Delta, \zeta}}{\psi, \zeta, \Gamma \Rightarrow \Delta, \zeta}\,\mathsf{w_L}}{\dfrac{\psi, \zeta, \Gamma \Rightarrow \Delta, \psi \wedge \zeta}{\psi \wedge \zeta, \Gamma \Rightarrow \Delta, \psi \wedge \zeta}\,\wedge_{\mathsf{L}}}\,\wedge_{\mathsf{R}}$$

By the induction hypothesis, $\psi, \Gamma \Rightarrow \Delta, \psi$ and $\zeta, \Gamma \Rightarrow \Delta, \zeta$ are derivable. Hence, by the steps shown above, we conclude that there is a derivation ending in $\psi \wedge \zeta, \Gamma \Rightarrow \Delta, \psi \wedge \zeta$, as we wanted to show.

QED.

Although negation is not a primitive symbol of the language, recall that it can be expressed using implication $\neg \phi := \phi \to \bot$, and thus, we have that the rules for left and right negation are admissible in **G3cp**. Consider the following derivations:

$$\dfrac{\Gamma \Rightarrow \phi, \Delta \qquad \dfrac{}{\bot, \Gamma \Rightarrow \Delta}\,\bot_{\mathsf{L}}}{\phi \to \bot, \Gamma \Rightarrow \Delta}\,\to_{\mathsf{R}} \qquad \dfrac{\dfrac{\phi, \Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta, \bot}\,\mathsf{wk_R}}{\Gamma \Rightarrow \Delta, \phi \to \bot}\,\to_{\mathsf{R}}$$

Accordingly, the rules $\neg_{\mathsf{L}}$ and $\neg_{\mathsf{R}}$ would be, respectively:

$$\frac{\Gamma \Rightarrow \phi, \Delta}{\neg \phi, \Gamma \Rightarrow \Delta} \neg_{\mathsf{L}} \qquad \frac{\phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg \phi} \neg_{\mathsf{R}}$$

Lemma 2.8 (Inversion lemma)**.** The rules in $\mathbf{G3cp}$ are height-preserving invertible.

Proof. The proof is by induction on the height of the derivation of the conclusion of each rule. We must distinguish between several cases, one for every possible rule applied. We only present two of them:

($\vee_{\mathsf{L}}$) Let $\phi \vee \psi, \Gamma \Rightarrow \Delta$ be derivable with derivation height h. Let's show that $\phi, \Gamma \Rightarrow \Delta$ and $\psi, \Gamma \Rightarrow \Delta$ are derivable, also with derivation height equal to h. For the base case, $h = 1$, note that if $\phi \vee \psi, \Gamma \Rightarrow \Delta$ is an initial sequent, then it is either the result of init or $\bot_{\mathsf{L}}$ and then also $\phi, \Gamma \Rightarrow \Delta$ and $\psi, \Gamma \Rightarrow \Delta$ must be initial sequents. For the inductive step assume height-preserving inversion up to height h and suppose that $\phi \vee \psi, \Gamma \Rightarrow \Delta$ is derivable with height $h + 1$. We distinguish two cases:

1. If $\phi \vee \psi$ is principal in the last rule then the sequent $\phi \vee \psi, \Gamma \Rightarrow \Delta$ results from applying $\vee_{\mathsf{L}}$ to $\phi, \Gamma \Rightarrow \Delta$ and $\psi, \Gamma \Rightarrow \Delta$, both derivable with height h.
2. If $\phi \vee \psi$ is not principal in the last rule then $\phi \vee \psi, \Gamma \Rightarrow \Delta$ results from applying some rule $\mathcal{R}$ to the sequent, generating one or two premisses $\phi \vee \psi, \Gamma' \Rightarrow \Delta'$ and $\phi \vee \psi, \Gamma'' \Rightarrow \Delta''$, each derivable with derivation height at most h. By the induction hypothesis, there are derivations of height at most h of $\phi, \Gamma' \Rightarrow \Delta'$, $\psi, \Gamma' \Rightarrow \Delta'$, $\phi, \Gamma'' \Rightarrow \Delta''$ and $\psi, \Gamma'' \Rightarrow \Delta''$. Applying rule $\mathcal{R}$ to $\phi, \Gamma' \Rightarrow \Delta'$ and $\phi, \Gamma'' \Rightarrow \Delta''$ we derive $\phi, \Gamma \Rightarrow \Delta$ and to $\psi, \Gamma' \Rightarrow \Delta'$ and $\psi, \Gamma'' \Rightarrow \Delta''$ we conclude $\psi, \Gamma \Rightarrow \Delta$, both with derivation height at most $h + 1$.

($\rightarrow_{\mathsf{R}}$) Let $\Gamma \Rightarrow \Delta, \phi \rightarrow \psi$ be derivable with derivation height h. Let's show that $\phi, \Gamma \Rightarrow \Delta, \psi$ is derivable, also with derivation height equal to h.

For the base case, $h = 1$, then $\Gamma \Rightarrow \Delta, \phi \rightarrow \psi$ is an initial sequent, thus $\phi, \Gamma \rightarrow \Delta, \psi$ must be too. Assume height-preserving invertibility up to height h and suppose that we can derive $\Gamma \Rightarrow \Delta, \phi \rightarrow \psi$ with height $h + 1$. We distinguish two cases:

1. If $\phi \rightarrow \psi$ is principal in the last rule applied, then $\Gamma \Rightarrow \Delta, \phi \rightarrow \psi$ results from applying $\rightarrow_{\mathsf{R}}$ to $\phi, \Gamma \Rightarrow \Delta, \psi$ which is then derivable with height h.
2. If $\phi \rightarrow \psi$ is not principal in the last rule applied then this sequent results from applying some rule $\mathcal{R}$ to one or two premisses $\Gamma' \Rightarrow \Delta', \phi \rightarrow \psi$ and $\Gamma'' \Rightarrow \Delta'', \phi \rightarrow \psi$, each derivable with height at most h. By the induction hypothesis $\Gamma', \phi \Rightarrow \Delta', \psi$ and $\Gamma'', \phi \Rightarrow \Delta'', \psi$ are derivable with derivation height at most h. Hence, applying again rule $\mathcal{R}$ we conclude that $\phi, \Gamma \Rightarrow \Delta, \psi$ is derivable with height at most $h + 1$.

QED.

Proposition 2.9. The rules $\mathsf{Ctr_L}$ and $\mathsf{Ctr_R}$ are height-preserving admissible in $\mathbf{G3cp}$.

Proof. We reason by induction on the height h of a derivation of the premisses. The proof for $\mathsf{Ctr_L}$ and $\mathsf{Ctr_R}$ is done simultaneously. If $h = 1$, the premiss is derived by init or by $\bot_{\mathsf{L}}$, in which case the conclusion must be obtained from that same rule, with derivation height equal to 1, regardless of where contraction is applied. For the inductive step, assume that $\phi, \phi, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, \phi, \phi$ are derivable with height h. We have the following inductive hypotheses:

IH1: If $\vdash_{\mathbf{G3cp}} \psi, \psi, \Gamma' \Rightarrow \Delta'$ with height $h' < h$ then $\vdash_{\mathbf{G3cp}} \psi, \Gamma' \Rightarrow \Delta'$ with height at most h'.

IH2: $\vdash_{\mathbf{G3cp}} \Gamma' \Rightarrow \Delta', \psi, \psi$ with height $h' < h$ then $\vdash_{\mathbf{G3cp}} \Gamma' \Rightarrow \Delta', \psi$ with height at most h'.

We distinguish two cases:

1. ϕ is not principal in the last rule applied. Then $\phi, \phi, \Gamma \Rightarrow \Delta$ is obtained from applying some rule $\mathcal{R}$ to the sequent yielding one or two premisses $\phi, \phi, \Gamma' \Rightarrow \Delta'$ and $\phi, \phi, \Gamma'' \Rightarrow \Delta''$ derivable with height at most $h-1$, both containing the contraction formula twice in the antecedent. The result follows by applying IH1 to each premiss and then applying rule $\mathcal{R}$ to both of them. Similarly, using IH2 we show the conclusion of $\mathsf{Ctr_R}$ is derivable.

2. If ϕ is principal in the last rule applied we must split the proof into several cases, one for each rule in $\mathbf{G3cp}$. We only show the cases where the last rule applied is $\vee_\mathsf{L}$ or $\rightarrow_\mathsf{R}$.

   Let $\vee_\mathsf{L}$ be the last rule applied in the derivation of $\phi, \phi, \Gamma \Rightarrow \Delta$, with $\phi = \psi \vee \zeta$. Then $\psi, \psi \vee \zeta, \Gamma \Rightarrow \Delta$ and $\zeta, \psi \vee \zeta, \Gamma \Rightarrow \Delta$ are derivable, each with derivation height at most $h-1$. By Lemma 2.8 we get that $\psi, \psi, \zeta, \Gamma \Rightarrow \Delta$ and $\zeta, \zeta, \psi, \Gamma \Rightarrow \Delta$ are also derivable with height at most $h-1$. Applying IH1 to both sequents we get that there are derivations for $\zeta, \Gamma \Rightarrow \Delta$ and $\psi, \Gamma \Rightarrow \Delta$ with height at most $h-1$. Then, by $\vee_\mathsf{L}$ we conclude that $\phi, \Gamma \Rightarrow \Delta$ is derivable, with height at most h. Analogously, using IH2 we conclude that $\Gamma \Rightarrow \Delta, \phi$ is derivable with height at most h.

   Let $\rightarrow_\mathsf{R}$ be last rule applied in the derivation of $\Gamma \Rightarrow \Delta, \phi, \phi$, with $\phi = \psi \rightarrow \zeta$. We have that $\psi, \Gamma \Rightarrow \Delta, \zeta, \psi \rightarrow \zeta$ is derivable with height $h-1$. By Lemma 2.8 there is a derivation with height at most $h-1$ of $\psi, \psi, \Gamma \Rightarrow \Delta, \zeta, \zeta$. Applying IH1 and then IH2 we get that we can derive $\psi, \Gamma \Rightarrow \Delta, \zeta$ with derivation height at most $h-1$, hence by applying $\rightarrow_\mathsf{R}$ we conclude that $\Gamma, \Rightarrow \Delta, \psi \rightarrow \zeta$ is derivable with height at most h.

QED.

The next theorem was introduced by Gentzen in the 1930s in his paper [12], and is a crucial result in the context of sequent calculus. It establishes the admissibility of the cut, a tool for simplifying proofs, while attesting that its incorporation into the logical system does not originate any contradictions or inconsistencies. It is often referred to as the Hauptsatz, or ”main theorem”, attesting to its importance. Essentially, this result states that any sequent that can be proven by employing the cut rule can also be established without resorting to it. Moreover, it guarantees that the calculus $\mathbf{G3cp}$ satisfies the subformula property, meaning that the complexity of any formula in a derivation is bounded above by the complexity of the formulas in the root. Ultimately, the admissibility of cut leads to completeness.

Define the *cut-height* as the sum of the heights of the derivations of the two premisses of the cut rule.

Definition 2.10. Let ϕ, ψ be formulas in the language $\mathcal{L}$, p any atom and $\circ \in \{\wedge, \vee, \rightarrow\}$. We inductively define the *weight* of ϕ, denoted $w(\phi)$ as follows:

- $w(\bot) = 0$;
- $w(p) = 1$;
- $w(\phi \circ \psi) = w(\phi) + w(\psi) + 1$.

Theorem 2.11 (Cut Admissibility)**.** *The* cut *rule is admissible in the calculus* $\mathbf{G3cp}$.[2]

Proof. Suppose both premisses of a cut $\Gamma \Rightarrow \Delta, \phi$ and $\phi, \Gamma' \Rightarrow \Delta'$ are derivable and that the sum of the heights of their derivations is h. We shall show how to construct a derivation of the conclusion of the rule $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. The proof is organized into several cases and sub-cases. Intuitively, in the first case, which is cut with a premiss derived from init of $\bot_{\mathsf{L}}$ we "eliminate" the cut. In the other cases, we show how to apply cut to formulas of lesser weight than ϕ, or we permute cut upwards in the derivation.

The proof follows by primary induction on the *weight* of the cut-formula ϕ and by secondary induction on the *cut-height* of the application of cut. Assume the premisses of the cut rule are derivable and that the sum of their derivation heights is h. We have the following inductive hypotheses:

IH1: Cut is admissible whenever the cut formula ψ is such that $w(\psi) < w(\phi)$.

IH2: Cut is admissible whenever the cut formula ψ is such that $w(\psi) = w(\phi)$, and the cut-height is strictly smaller than h.

Cut with an initial sequent

1. The **left premiss** $\Gamma \Rightarrow \Delta, \phi$ is an initial sequent

   1.1 If ϕ is in Γ we apply weakening to the right premiss $\phi, \Gamma' \Rightarrow \Delta'$ to obtain $\Gamma, \Gamma' \Rightarrow \Delta', \Delta$.

   1.2 If Γ and Δ have some propositional formula in common then $\Gamma, \Gamma' \Rightarrow \Delta', \Delta$ is an axiom.

   1.3 If $\Gamma \Rightarrow \Delta, \phi$ is the conclusion of $\bot_{\mathsf{L}}$ then $\bot$ is in Γ. Then also $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ is the conclusion of $\bot_{\mathsf{L}}$.

2. The **right premiss** $\phi, \Gamma' \Rightarrow \Delta'$ is an initial sequent:

   2.1 If ϕ is in Δ' we apply weakening to the left premiss $\Gamma \Rightarrow \Delta, \phi$ to obtain $\Gamma, \Gamma' \Rightarrow \Delta', \Delta$.

   2.2 If Γ and Δ have some propositional formula in common then $\Gamma, \Gamma' \Rightarrow \Delta', \Delta$ is an axiom.

   2.3 If $\phi, \Gamma' \Rightarrow \Delta'$ is the conclusion of $\bot_{\mathsf{L}}$ and ϕ is not $\bot$, then $\bot$ is in Γ. Then also $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ is the conclusion of $\bot_{\mathsf{L}}$.

   2.4 If $\phi = \bot$ and the left premiss is not an initial sequent, we have that $\Gamma \Rightarrow \Delta, \bot$ has been derived by some logical rule. Note that $\bot$ cannot be active in the last rule applied. The proof strategy is similar to that in the case of cut where neither premiss is an initial sequent and the cut formula ϕ is not principal in the left premiss with the particularity that ϕ is $\bot$.

Cut where neither premiss is an initial sequent

In the following derivations, the topsequents have heights $n, m, k, \ldots$. Assume that the premisses of cut are derivable. If we can transform the derivations into one with a smaller cut-height we can conclude that $\Gamma, \Gamma' \Rightarrow \Delta', \Delta$ is derivable by applying the inductive hypothesis and then the appropriate logical rule.

3. The cut formula ϕ is **not principal in the left premiss**, $\Gamma \Rightarrow \Delta, \phi$. We distinguish several cases according to the rule that was used to derive this premiss. We only show the details for the left rules.

[2] In the proof that follows we use a slight abuse of notation by using occurrences of the cut rule. These are actually intended as applications of the inductive hypothesis of the cut admissibility result, and not as explicit rules of the system.

3.1 $\wedge_{\mathsf{L}}$ and $\Gamma = \psi_1 \wedge \psi_2, \Gamma''$. Consider the following derivation, where the bottom-most instance of cut has cut-height $h = n + m + 1$.

$$\dfrac{\dfrac{\psi_1, \psi_2, \Gamma'' \Rightarrow \Delta, \phi}{\psi_1 \wedge \psi_2, \Gamma'' \Rightarrow \Delta, \phi}\wedge_{\mathsf{L}} \qquad \phi, \Gamma' \Rightarrow \Delta'}{\psi_1 \wedge \psi_2, \Gamma'', \Gamma' \Rightarrow \Delta', \Delta}\,\mathsf{cut}$$

This derivation is transformed into the one that follows, by applying $\wedge_{\mathsf{L}}$ to the sequent at the root and then applying cut:

$$\dfrac{\dfrac{\psi_1, \psi_2, \Gamma'' \Rightarrow \Delta, \phi \quad \phi, \Gamma' \Rightarrow \Delta'}{\psi_1, \psi_2, \Gamma'', \Gamma' \Rightarrow \Delta', \Delta}\,\mathsf{cut}}{\psi_1 \wedge \psi_2, \Gamma'', \Gamma' \Rightarrow \Delta', \Delta}\wedge_{\mathsf{L}}$$

The instance of cut displayed in the derivation above has cut-height $n + m < h$, hence IH2 applies.

3.2 $\vee_{\mathsf{L}}$ and $\Gamma = \psi_1 \vee \psi_2, \Gamma''$. Consider the following derivation, where the bottom-most instance of cut has cut-height $h = max(n, m) + k + 1$.

$$\dfrac{\dfrac{\psi_1, \Gamma'' \Rightarrow \Delta, \phi \quad \psi_2, \Gamma'' \Rightarrow \Delta, \phi}{\psi_1 \vee \psi_2, \Gamma'' \Rightarrow \Delta, \phi}\vee_{\mathsf{L}} \qquad \phi, \Gamma' \Rightarrow \Delta'}{\psi_1 \vee \psi_2, \Gamma'', \Gamma' \Rightarrow \Delta', \Delta}\,\mathsf{cut}$$

This derivation is transformed into the one that follows, by applying $\vee_{\mathsf{L}}$ to the sequent at the root and then applying cut:

$$\dfrac{\dfrac{\psi_1, \Gamma'' \Rightarrow \Delta, \phi \quad \phi, \Gamma' \Rightarrow \Delta'}{\psi_1, \Gamma'', \Gamma' \Rightarrow \Delta', \Delta}\,\mathsf{cut} \qquad \dfrac{\psi_2, \Gamma'' \Rightarrow \Delta, \phi \quad \phi, \Gamma' \Rightarrow \Delta'}{\psi_2, \Gamma'', \Gamma' \Rightarrow \Delta', \Delta}\,\mathsf{cut}}{\psi_1 \vee \psi_2, \Gamma'', \Gamma' \Rightarrow \Delta', \Delta}\vee_{\mathsf{L}}$$

The uppermost instances of cut in this derivation have height $n + k$ and $m + k$, both less than h, hence IH2 applies.

3.3 $\rightarrow_{\mathsf{L}}$ and $\Gamma = \psi_1 \rightarrow \psi_2, \Gamma''$. Consider the following derivation, where the bottom-most instance of cut has cut-height $h = max(n, m) + k + 1$.

$$\dfrac{\dfrac{\psi_2, \Gamma'' \Rightarrow \Delta, \phi \quad \psi_1, \Gamma'' \Rightarrow \Delta, \phi}{\psi_1 \rightarrow \psi_2, \Gamma'' \Rightarrow \Delta, \phi}\rightarrow_{\mathsf{L}} \qquad \phi, \Gamma' \Rightarrow \Delta'}{\psi_1 \rightarrow \psi_2, \Gamma'', \Gamma' \Rightarrow \Delta', \Delta}\,\mathsf{cut}$$

This derivation is transformed into the one that follows, by applying $\rightarrow_{\mathsf{L}}$ to the sequent at the root and then applying cut:

$$\dfrac{\dfrac{\psi_2, \Gamma'' \Rightarrow \Delta, \phi \quad \phi, \Gamma' \Rightarrow \Delta'}{\psi_2, \Gamma'', \Gamma' \Rightarrow \Delta', \Delta}\,\mathsf{cut} \qquad \dfrac{\Gamma'' \Rightarrow \Delta, \psi_1, \phi \quad \phi, \Gamma' \Rightarrow \Delta'}{\Gamma'', \Gamma' \Rightarrow \Delta', \Delta, \psi_1}\,\mathsf{cut}}{\psi_1 \rightarrow \psi_2, \Gamma'', \Gamma' \Rightarrow \Delta', \Delta}\rightarrow_{\mathsf{L}}$$

The uppermost instances of cut in this derivation have height $n+k$ and $m+k$, both less than h, hence IH2 applies.

4. The cut formula ϕ is **principal in the left premiss only** $\Gamma \Rightarrow \Delta, \phi$. We distinguish several cases according to the rule that was used to derive the right premises $\phi, \Gamma' \Rightarrow \Delta'$, we only provide the details for the right rules.

   4.1 $\wedge_{\mathsf{R}}$ and $\Delta' = \psi_1 \wedge \psi_2, \Delta''$. Consider the following derivation, where the down-most instance of cut has cut-height $h = n + max(m,k) + 1$.

$$\dfrac{\Gamma \Rightarrow \Delta, \phi \qquad \dfrac{\phi, \Gamma' \Rightarrow \psi_1, \Delta'' \quad \phi, \Gamma' \Rightarrow \psi_2, \Delta''}{\phi, \Gamma' \Rightarrow \psi_1 \wedge \psi_2, \Delta''}\wedge_{\mathsf{R}}}{\Gamma, \Gamma' \Rightarrow \psi_1 \wedge \psi_2, \Delta'', \Delta}\mathsf{cut}$$

   This derivation is transformed into the one that follows, by applying $\wedge_{\mathsf{R}}$ to the sequent at the root and then applying cut:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \Gamma' \Rightarrow \psi_1, \Delta''}{\Gamma, \Gamma' \Rightarrow \psi_1, \Delta'', \Delta}\mathsf{cut} \qquad \dfrac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \Gamma' \Rightarrow \psi_2, \Delta''}{\Gamma, \Gamma' \Rightarrow \psi_2, \Delta'', \Delta}\mathsf{cut}}{\Gamma, \Gamma' \Rightarrow \psi_1 \wedge \psi_2, \Delta'', \Delta}\wedge_{\mathsf{R}}$$

   The uppermost instances of cut in this derivation have height $n+m$ and $n+k$, both less than h, hence IH2 applies.

   4.2 $\vee_{\mathsf{R}}$ and $\Delta' = \psi_1 \vee \psi_2, \Delta''$. Consider the following derivation, where the instance of cut has cut-height $h = n + m + 1$.

$$\dfrac{\Gamma \Rightarrow \Delta, \phi \qquad \dfrac{\phi, \Gamma' \Rightarrow \psi_1, \psi_2, \Delta''}{\phi, \Gamma' \Rightarrow \psi_1 \vee \psi_2, \Delta''}\vee_{\mathsf{R}}}{\Gamma, \Gamma' \Rightarrow \psi_1 \vee \psi_2, \Delta'', \Delta}\mathsf{cut}$$

   This derivation is transformed into the one that follows, by applying $\vee_{\mathsf{R}}$ to the sequent at the root and then cut:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \Gamma' \Rightarrow \psi_1, \psi_2, \Delta''}{\Gamma, \Gamma' \Rightarrow \psi_1, \psi_2, \Delta'', \Delta}\mathsf{cut}}{\Gamma, \Gamma' \Rightarrow \psi_1 \vee \psi_2, \Delta'', \Delta}\vee_{\mathsf{R}}$$

   This derivation has cut-height $h = n + m < h$ hence IH2.

   4.3 $\to_{\mathsf{R}}$ and $\Delta' = \psi_1 \to \psi_2, \Delta''$. The derivation with cut-height $n + m + 1$

$$\dfrac{\Gamma \Rightarrow \Delta, \phi \qquad \dfrac{\phi, \psi_1, \Gamma' \Rightarrow \psi_2, \Delta''}{\phi, \Gamma' \Rightarrow \psi_1 \to \psi_2, \Delta''}\to_{\mathsf{R}}}{\Gamma, \Gamma' \Rightarrow \psi_1 \to \psi_2, \Delta'', \Delta}\mathsf{cut}$$

is transformed into another derivation by applying $\vee_{\mathsf{R}}$ to the conclusion and then cut:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \psi_1, \Gamma' \Rightarrow \psi_2, \Delta''}{\psi_1, \Gamma, \Gamma' \Rightarrow \psi_2, \Delta'', \Delta}\,\mathsf{cut}}{\Gamma, \Gamma' \Rightarrow \psi_1 \to \psi_2, \Delta'', \Delta}\,\to_{\mathsf{R}}$$

The instance of cut displayed in the derivation above has cut-height $n + m < h$ hence IH2.

5. The cut formula ϕ is **principal in both premisses**. We have three subcases according to the structure of ϕ. Note that in the following cases, an application of a rule of contraction to a certain sequent, might actually denote several such steps.

5.1 $\phi = \psi_1 \wedge \psi_2$. The cut formula in the down-most instance of cut in the derivation

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \psi_1 \quad \Gamma \Rightarrow \Delta, \psi_2}{\Gamma \Rightarrow \Delta, \psi_1 \wedge \psi_2}\,\wedge_{\mathsf{R}} \qquad \dfrac{\psi_1, \psi_2, \Gamma' \Rightarrow \Delta'}{\psi_1 \wedge \psi_2, \Gamma' \Rightarrow \Delta'}\,\wedge_{\mathsf{L}}}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta}\,\mathsf{cut}$$

has weight $w(\psi_1) + w(\psi_2) + 1$. This derivation is transformed into the one that follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \psi_2 \quad \psi_2, \psi_1, \Gamma' \Rightarrow \Delta'}{\psi_1, \Gamma, \Gamma' \Rightarrow \Delta', \Delta}\,\mathsf{cut} \qquad \Gamma \Rightarrow \Delta, \psi_1}{\Gamma, \Gamma, \Gamma' \Rightarrow \Delta', \Delta, \Delta}\,\mathsf{cut}}{\Gamma, \Gamma, \Gamma' \Rightarrow \Delta', \Delta}\,\mathsf{Ctr_R}}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta}\,\mathsf{Ctr_L}$$

Both instances of cut in this derivation have cut formulas with weight less than $w(\psi_1) + w(\psi_2) + 1$, hence IH1 can be applied.

5.2 $\phi = \psi_1 \vee \psi_2$. The cut formula in the down-most instance of cut in the derivation

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \psi_1, \psi_2}{\Gamma \Rightarrow \Delta, \psi_1 \vee \psi_2}\,\vee_{\mathsf{R}} \qquad \dfrac{\psi_1, \Gamma' \Rightarrow \Delta' \quad \psi_2, \Gamma' \Rightarrow \Delta'}{\psi_1 \vee \psi_2, \Gamma' \Rightarrow \Delta'}\,\vee_{\mathsf{L}}}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta}\,\mathsf{cut}$$

has weight $w(\psi_1) + w(\psi_2) + 1$. This derivation is transformed into the one that follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \psi_1, \psi_2 \quad \psi_2, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta, \psi_1}\,\mathsf{cut} \qquad \psi_1, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma', \Gamma' \Rightarrow \Delta', \Delta', \Delta}\,\mathsf{cut}}{\Gamma, \Gamma', \Gamma' \Rightarrow \Delta', \Delta}\,\mathsf{Ctr_R}}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta}\,\mathsf{Ctr_L}$$

Both instances of cut in this derivation have cut formulas with weight less than $w(\psi_1) + w(\psi_2) + 1$, hence IH1 can be applied.

5.3 $\phi = \psi_1 \to \psi_2$. The cut formula in the down-most instance of cut in the derivation

$$\dfrac{\dfrac{\psi_1, \Gamma \Rightarrow \Delta, \psi_2}{\Gamma \Rightarrow \Delta, \psi_1 \to \psi_2} \to_R \quad \dfrac{\psi_2, \Gamma' \Rightarrow \Delta' \quad \Gamma' \Rightarrow \Delta', \psi_1}{\psi_1 \to \psi_2, \Gamma' \Rightarrow \Delta'} \to_L}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta} \text{cut}$$

has weight $w(\psi_1) + w(\psi_2) + 1$. This derivation is transformed into the one that follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\psi_1, \Gamma \Rightarrow \Delta, \psi_2 \quad \psi_2, \Gamma' \Rightarrow \Delta'}{\psi_1, \Gamma, \Gamma' \Rightarrow \Delta', \Delta} \text{cut} \quad \Gamma' \Rightarrow \Delta', \psi_1}{\Gamma, \Gamma', \Gamma' \Rightarrow \Delta', \Delta', \Delta} \text{cut}}{\Gamma, \Gamma', \Gamma' \Rightarrow \Delta', \Delta} \text{Ctr}_R}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta} \text{Ctr}_L$$

Both instances of cuts in this derivation have cut formulas with weight less than $w(\psi_1) + w(\psi_2) + 1$, hence IH1 can be applied.

QED.

Now that we have an understanding of how the calculus **G3cp** works we move on to relating derivability in this calculus with derivability in the Hilbert system for classical propositional logic. We can interpret a sequent $\Gamma \Rightarrow \Delta$ as:

$$\bigwedge \Gamma \to \bigvee \Delta$$

where $\bigwedge \Gamma$ stands for the conjunction of the formulas in Γ, and $\bigvee \Delta$ stands for the disjunction of formulas in Δ. This means that if the sequent is derivable we have that from *everything* in Γ follows *something* in Δ. Note that the antecedent and the succedent may be empty. In these cases, we use the usual convention for disjunction and conjunction of empty sets of formulas as follows:

$$\bigwedge_{\phi \in \emptyset} \phi \equiv \top \quad \text{and} \quad \bigvee_{\phi \in \emptyset} \phi \equiv \bot.$$

It is especially important that we can prove that our system is sound and complete with respect to classical propositional logic. If we achieve this, we guarantee that for any formula ϕ, we have that ϕ is valid in X if and only if it is derivable in **X**. So, in order to decide validity we just need to construct derivations, done by *root first proof search*, where the formula that is to be proved is placed at the bottom and then we go on applying the inference rules of the calculus bottom-up.

Recall the Hilbert system H_{cp} that was defined in Section 1.3.

Theorem 2.12 (Completeness)**. G3cp** *is complete with respect to* H_{cp}.

Proof. Assume that a formula ϕ has a proof in H_{cp}. The goal is to show that we can construct a derivation for $\Rightarrow \phi$ in **G3cp**. If suffices to show that there is an admissible rule simulating Modus Ponens and that the axioms in H_{cp} are derivable. Consider the following rule:

$$\frac{\Rightarrow \phi \to \psi \quad \Rightarrow \phi}{\Rightarrow \psi}$$

Assume that the premisses of the rule are derivable. By invertibility of $\to_{\mathsf{R}}$ we have that $\phi \Rightarrow \psi$. By the rule of cut:

$$\dfrac{\phi \Rightarrow \psi \quad \Rightarrow \phi}{\Rightarrow \psi}\ \mathsf{cut}$$

Hence we conclude that ψ is derivable which shows the admissibility of the rule. We prove the axioms in H_{cp} are derivable by constructing derivations for them. Consider the following derivation, which shows that $(\neg\phi \to \neg\psi) \to (\psi \to \phi)$ is derivable in **G3cp**:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\psi \Rightarrow \psi, \phi}\,\mathsf{init}}{\psi, \neg\psi \Rightarrow \phi}\,\neg_{\mathsf{L}} \quad \dfrac{\dfrac{}{\phi, \psi \Rightarrow \phi}\,\mathsf{init}}{\psi \Rightarrow \phi, \neg\phi}\,\neg_{\mathsf{R}}}{\psi, \neg\phi \to \neg\psi \Rightarrow \phi}\,\to_{\mathsf{L}}}{\neg\phi \to \neg\psi \Rightarrow \psi \to \phi}\,\to_{\mathsf{R}}}{\Rightarrow (\neg\phi \to \neg\psi) \to (\psi \to \phi)}\,\to_{\mathsf{R}}$$

The other axioms can be shown to be derivable in a similar way, hence this part of the proof will be omitted.

QED.

Theorem 2.13 (Soundness)**.** *The calculus* **G3cp** *is sound with respect to classical propositional calculus.*

Proof. Assume that a sequent $\Gamma \Rightarrow \Delta$ is derivable in **G3cp**. We must show that $\bigwedge\Gamma \to \bigvee\Delta$ is derivable using the axioms and inference rule in H_{cp}. The proof is by induction on the height h of the derivation of $\Gamma \Rightarrow \Delta$. If $h = 1$, then $\Gamma \Rightarrow \Delta$ is an initial sequent. If it is of the form $p, \Gamma' \Rightarrow \Delta', p$, its formula interpretation is $p \wedge \bigwedge\Gamma' \to p \vee \bigvee\Delta'$, which is derivable from H_{cp} since $p \to p$ is derivable. If it is of the form $\bot, \Gamma' \Rightarrow \Delta$, then its formula interpretation is $\bot \wedge \bigwedge\Gamma \to \bigvee\Delta$, which is also clearly derivable. If $h > 1$, then we distinguish cases according to the last rule applied. We show only the following case:

($\wedge_{\mathsf{L}}$) Let ϕ, ψ be formulas and Γ, Δ finite multisets. Assume that $\phi, \psi, \Gamma \Rightarrow \Delta$ is derivable. By inductive hypothesis, the formula interpretation of this sequent, $\phi \wedge \psi \wedge \bigwedge\Gamma \to \bigvee\Delta$ is derivable from the axioms in H_{cp}. The conclusion is immediate since this is equivalent to $(\phi \wedge \psi) \wedge \bigwedge\Gamma \to \bigvee\Delta$ which is the formula interpretation of the conclusion of $\phi \wedge \psi, \Gamma \Rightarrow \Delta$.

QED.

We get the following corollary:

Corollary 2.14. The sequent $\Rightarrow \bot$ is not derivable in **G3cp**.

Proof. This follows from the fact that no rule of **G3cp** can be applied to this sequent.

QED.

2.2 Sequent calculi for modal logics: first approach

We are now interested in extending the sequent calculi we just presented for the modal logics in the cube in Figure 1.6, hence we must have rules capable of dealing with the modal operators $\Box$ and $\Diamond$. In this section, we present rules that can extend the calculus we presented in the last section, as well as a brief discussion on why other systems may be preferable over the one we now present.

Consider a multiset $\Gamma = \{\psi_1, \psi_2, \ldots, \psi_n\}$. Then, $\Box\,\Gamma$ denotes the multiset $\{\Box\,\psi_1, \Box\,\psi_2 \ldots, \Box\,\psi_n\}$. The following rule K extends $\mathbf{G3cp}$ to a sequent calculus for $\mathbf{K}$:

$$\frac{\Gamma \Rightarrow \phi}{\Gamma', \Box\,\Gamma \Rightarrow \Box\,\phi, \Delta}\,\mathsf{K}$$

This rule has the nice property of allowing us to derive the axioms and simulate the rules in H_K. Furthermore, if Γ', Γ and Δ are all empty multisets, we get the following rule, corresponding to the rule of necessitation in H_K:

$$\frac{\Rightarrow \phi}{\Rightarrow \Box\,\phi}$$

Also, in Section 2.1 we have already seen how Modus Ponens can be simulated and that the axioms from classical propositional logic can be derived. At last, we show that the axiom of modal distributivity is derivable. Consider the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\phi, \psi \Rightarrow \psi}\,\mathsf{init} \qquad \dfrac{}{\phi \Rightarrow \phi, \psi}\,\mathsf{init}}{\phi \to \psi, \phi \Rightarrow \psi}\,\to_{\mathsf{L}}}{\Box(\phi \to \psi), \Box\,\phi \Rightarrow \Box\,\psi}\,\mathsf{K}}{\Box(\phi \to \psi) \Rightarrow \Box\,\phi \to \Box\,\psi}\,\to_{\mathsf{R}}}{\Rightarrow \Box(\phi \to \psi) \to (\Box\,\phi \to \Box\,\psi)}\,\to_{\mathsf{R}}$$

We can extend the calculus for $\mathbf{K}$ to some other logics in the cube by adding the rules in table 2.3. With these rules we have that, for instance, a system for $\mathbf{S5}$ would be $\mathbf{G3cp} + \mathsf{K} + \mathsf{T} + 45$.

These rules are all sound with respect to the respective class of frames, which will lead to the soundness of the calculus. Let's show this, for instance, for T. Suppose that the formula interpretation of the conclusion of the rule is not valid in the class of reflexive frames, i.e., there is a model $\mathcal{M}$ based on a reflexive frame, and a state s such that $\mathcal{M}, s \nVdash \Box\,\phi \wedge \bigwedge \Gamma \to \bigvee \Delta$. Then, we must have that $\mathcal{M}, s \Vdash \Box\,\phi \wedge \bigwedge \Gamma$

$$\frac{\phi, \Gamma \Rightarrow \Delta}{\Box\,\phi, \Gamma \Rightarrow \Delta}\,\mathsf{T} \qquad \frac{\Box\,\Gamma \Rightarrow \phi}{\Gamma', \Box\,\Gamma \Rightarrow \Box\,\phi, \Delta}\,4 \qquad \frac{\Box\,\Gamma \Rightarrow \phi, \Box\,\Delta}{\Gamma', \Box\,\Gamma \Rightarrow \Box\,\phi, \Box\,\Delta, \Delta'}\,45$$

Table 2.3: Rules for extensions of the calculus for $\mathbf{K}$.

but $\mathcal{M}, s \nVdash \bigvee \Delta$, that is:

$$\begin{aligned}
&\mathcal{M}, s \Vdash \Box\,\phi \\
&\mathcal{M}, s \Vdash \gamma \text{ for every } \gamma \in \Gamma \\
&\mathcal{M}, s \nVdash \delta \text{ for any } \delta \in \Delta
\end{aligned}$$

In particular, the first condition implies that $\mathcal{M}, s \Vdash \phi$ since sRs because $\mathcal{M}$ is based on a reflexive frame. Hence, $\mathcal{M}, s \Vdash \phi \wedge \bigwedge \Gamma$, thus $\mathcal{M}, s \nVdash \phi \wedge \bigwedge \Gamma \to \bigvee \Delta$, that is, the formula interpretation of the premises of the rule is not valid with respect to the class of frames t.

Furthermore, we can show that the systems with these rules are complete by providing derivations for the characteristic axioms of the corresponding logics. For instance, the following derivation shows that axiom T is derivable in $\mathbf{G3cp} + \mathsf{K} + \mathsf{T}$:

$$\dfrac{\dfrac{\dfrac{\vdots}{\phi \Rightarrow \phi}}{\Box\,\phi \Rightarrow \phi}\,\mathsf{T}}{\Rightarrow \Box\,\phi \to \phi}\,\to_{\mathsf{R}}$$

A less simple example is to derive axiom B, $\phi \to \Box\Diamond\,\phi$ in the system for $\mathbf{S5}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\phi \Rightarrow \phi}}{\neg\,\phi, \phi \Rightarrow}\,\neg_{\mathsf{L}}}{\Box\,\neg\,\phi, \phi \Rightarrow}\,\mathsf{T} \qquad \dfrac{\dfrac{\dfrac{\vdots}{\Box\,\neg\,\phi \Rightarrow \Box\,\neg\,\phi}}{\Rightarrow \Box\,\neg\,\phi, \neg\,\Box\,\neg\,\phi}\,\neg_{\mathsf{R}}}{\Rightarrow \Box\,\neg\phi, \Box\,\neg\,\Box\,\neg\,\phi}\,45}{\phi \Rightarrow \Box\,\neg\,\Box\,\neg\,\phi}\,\mathsf{cut}}{\Rightarrow \phi \to \Box\,\neg\,\Box\,\neg\,\phi}\,\to_{\mathsf{R}}$$

Note how the previous derivation requires the use of the cut rule. It is not clear how this rule could be eliminated from the calculus or even if it is possible. The standard procedure, as shown in the proof for Theorem 2.11, does not seem to work. Recall, according to the structure of the proof, the case where the cut formula is principal only in the left premises, which has been derived by T. Suppose the right premiss has been derived by 45:

$$\dfrac{\dfrac{\phi, \Gamma'' \Rightarrow \Delta''}{\Box\,\phi, \Gamma'' \Rightarrow \Delta''}\,\mathsf{T} \qquad \dfrac{\Box\,\Gamma \Rightarrow \Box\,\Delta, \psi, \Box\,\phi}{\Box\,\Gamma, \Gamma' \Rightarrow \Box\,\Delta, \Delta', \Box\,\psi, \Box\,\phi}\,45}{\Box\,\Gamma, \Gamma', \Gamma'' \Rightarrow \Box\,\Delta, \Delta', \Delta'', \Box\,\psi}\,\mathsf{cut}$$

This application of cut has height $h + 1 + m + 1$. Following the proof strategy presented before, the standard approach would be to transform the last derivation into another one where the application of cut is permuted with 45, in order to have less cut-height. The first step of the resulting derivation would be as follows:

$$\frac{\Box\,\Gamma \Rightarrow \Box\,\Delta, \psi, \Box\,\phi \quad \Box\,\phi, \Gamma'' \Rightarrow \Delta''}{\Box\,\Gamma, \Gamma'' \Rightarrow \Box\,\Delta, \Delta'', \psi}\,\mathsf{cut}$$

Note that, from this point, it is impossible to apply rule 45 since this rule does not allow the presence of unboxed formulas in its premiss. Hence, the proof fails. Of course, this doesn't mean that it is impossible to prove that this system is actually cut-free, but only that there doesn't seem to be a practical way to do so. In fact, no proof is known of this result yet.

The main issue that comes with a system where cut cannot be eliminated is that of decidability. In the absence of the rule of cut, root first proof search can be done easily as every step is determined by the rule we choose to apply. However, in the presence of cut it is necessary to predict an appropriate cut-formula in order to continue searching for the proof. A wrong choice of cut-formula may lead to an unnecessarily long derivation or may even make it impossible to find said proof. This discussion motivates the next chapter, where we will formulate a cut-free sequent calculus for K and for the logics in the cube. To do so, we shall enrich the Gentzen-style sequent calculus by adding some relevant semantic information.

Chapter 3

Labelled Sequent Calculi

The main idea behind labelled systems is to internalize semantical information of the logics within the calculus by annotating formulas with labels. This annotation, which can represent worlds in a Kripke-model, truth values, names of agents, etc., allows the tracking of semantic information associated with each formula. It's essential to note that while the labelled approach integrates semantics, the treatment of the calculus is still purely syntactic, that is, proofs depends solely on the structure of formulas. This underlying idea has been around since the mid-20th century, with Stig Kanger [18] as its earliest promotor. Amongst many others who have studied proof systems with internalized semantics we highlight Kripke [25], Fitting [10] and Viganò [36]. Although our focus will be on labelled sequent calculi as presented by Sara Negri in [29], labelled systems are not exclusive to sequent calculi and can be extended to many other formal deductive systems, such as *natural deduction* ([2, 3]) or *tableaux* ([14, 15]).

In this chapter, we begin by presenting a Gentzen-style calculus for the basic modal logic $\mathbf{K}$, which behaves similarly to **G3cp**, based on the work of Sara Negri in [29]. We prove several pertinent results, including Gentzen's Hauptsatz, as we did in Chapter 2 concerning $\mathbf{G3cp}$. In Section 3.2, in order to extend the system for $\mathbf{K}$ to the other logics in the cube, we present several rules in correspondence to the rules for the accessibility relation. This strategy allows us to devise systems which, as we shall prove, preserve the structural properties of $\mathbf{LabK}$. We also prove soundness and completeness (via cut-adimissibility), giving a particular emphasis to the second result. At last, Section 3.3 is dedicated to providing proof for *semantic* completeness of our systems, via the construction of a (possibly infinite) countermodel.

3.1 The proof system LabK

This section introduces us to the system $\mathbf{LabK}$, a labelled sequent calculus for the basic modal system $\mathbf{K}$. In this system, we internalize the Kripke semantics by considering a set of countably many variables $x, y, z \ldots$, the *labels*, which index the formulas and intuitively denote the worlds in a Kripke frame. Instead of the forcing relation $x \Vdash \phi$ that we considered in Section 1.2, we now consider an internal relation $x : \phi$, where ϕ is any formula in $\mathcal{L}^{\Box,\Diamond}$. An expression of this type is called a *labelled formula*. An atomic formula in the language of labelled sequents is $x : p$ where x is a label and $p \in \mathcal{L}^{\Box,\Diamond}$ is atomic. Other than these formulas, we have *relational atoms* xRy, where R denotes the accessibility relation between the worlds of the frame, as usual.

Definition 3.1. A *labelled sequent* is an expression $\Gamma \Rightarrow \Delta$ where Γ is a finite multiset of labelled formulas and relational atoms and Δ is a finite multiset of labelled formulas.

Initial sequents

$$\frac{}{x : p, \Gamma \Rightarrow \Delta, x : p}\,\mathsf{init} \qquad \frac{}{x : \bot, \Gamma \Rightarrow \Delta}\,\bot_{\mathsf{L}}$$

Logical rules

$$\frac{x : \phi, x : \psi, \Gamma \Rightarrow \Delta}{x : \phi \wedge \psi, \Gamma \Rightarrow \Delta}\,\wedge_{\mathsf{L}} \qquad \frac{\Gamma \Rightarrow \Delta, x : \phi \quad \Gamma \Rightarrow \Delta, x : \psi}{\Gamma \Rightarrow \Delta, x : \phi \wedge \psi}\,\wedge_{\mathsf{R}}$$

$$\frac{x : \phi, \Gamma \Rightarrow \Delta \quad x : \psi, \Gamma \Rightarrow \Delta}{x : \phi \vee \psi, \Gamma \Rightarrow \Delta}\,\vee_{\mathsf{L}} \qquad \frac{\Gamma \Rightarrow \Delta, x : \phi, x : \psi}{\Gamma \Rightarrow \Delta, x : \phi \vee \psi}\,\vee_{\mathsf{R}}$$

$$\frac{x : \psi, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, x : \phi}{x : \phi \to \psi, \Gamma \Rightarrow \Delta}\,\to_{\mathsf{L}} \qquad \frac{x : \phi, \Gamma \Rightarrow \Delta, x : \psi}{\Gamma \Rightarrow \Delta, x : \phi \to \psi}\,\to_{\mathsf{R}}$$

$$\frac{x : \Box\,\phi, xRy, y : \phi, \Gamma \Rightarrow \Delta}{x : \Box\,\phi, xRy, \Gamma \Rightarrow \Delta}\,\Box_{\mathsf{L}} \qquad \frac{xRy, \Gamma \Rightarrow \Delta, y : \phi}{\Gamma \Rightarrow \Delta, x : \Box\,\phi}\,\Box_{\mathsf{R}} \quad (y!)$$

$$\frac{xRy, y : \phi, \Gamma \Rightarrow \Delta}{x : \Diamond\,\phi, \Gamma \Rightarrow \Delta}\,\Diamond_{\mathsf{L}} \quad (y!) \qquad \frac{xRy, \Gamma \Rightarrow \Delta, y : \phi, x : \Diamond\,\phi}{xRy, \Gamma \Rightarrow \Delta, x : \Diamond\,\phi}\,\Diamond_{\mathsf{R}}$$

Table 3.1: The sequent calculus $\mathbf{LabK}$. Notation $(y!)$ means that y does not occur in the conclusion of the rule.

The system $\mathbf{LabK}$, for the modal logic $\mathbf{K}$, is composed of the initial sequents and rules in Table 3.1. In the leftmost initial sequent, $x : p$ is an atomic formula. In $\Diamond_{\mathsf{L}}$ and $\Box_{\mathsf{R}}$, the label y, called the *eigenvariable*, does not occur in the conclusion of the rule, as indicated by the side condition $(y!)$. The definitions of derivation tree, height of a derivation and derivability are analogous to the ones in Section 2.1. We write $\vdash_{\mathbf{LabK}} \Gamma \Rightarrow \Delta$ to denote that the labelled sequent $\Gamma \Rightarrow \Delta$ is derivable in $\mathbf{LabK}$. Just like in $\mathbf{G3cp}$, we can come up with rules for negation by considering that $x : \neg\,\phi$ is equivalent to $x : \phi \to \bot$, as follows:

$$\frac{\Gamma \Rightarrow x : \phi, \Delta}{x : \neg\,\phi, \Gamma \Rightarrow \Delta}\,\neg_{\mathsf{L}} \qquad \frac{x : \phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x : \neg\,\phi}\,\neg_{\mathsf{R}}$$

It is important to note that the logical rules in $\mathbf{LabK}$ are obtained from the definition of satisfiability in a Kripke model for the correspondent connective. Then, it is clear how the rules for the propositional connectives are obtained. These rules remain the same as in $\mathbf{G3cp}$ with the exception that the formulas are now labelled. It is the rules for the modal operators that require explanation. Recall the truth condition for a boxed formula:

$$x \Vdash \Box\,\phi \textit{ iff for all } y \textit{ such that } xRy \textit{ we have } y \Vdash \phi.$$

The rule $\Box_{\mathsf{R}}$, with added contexts, corresponds to the right-to-left implication of the last condition: if $y : \phi$ can be derived for *any* y accessible from x, then $x : \Box\,\phi$ can be derived. In this rule, the arbitrariness of y is assured since y must not be in $\Gamma \cup \Delta$. On the other hand, if $x : \Box\,\phi$ is derivable and y is such that xRy, then $y : \phi$ must be derivable, with corresponds to $\Box_{\mathsf{L}}$. The formula $x : \Box\,\phi$ is kept in the premise of this rule to ensure invertibility. The same arguments could be made in order to semantically explain the rules for $\Diamond$.

Consider now the following definition of subformula, adapted to take into consideration the labels:

Definition 3.2. Let $\circ$ be any propositional connective. The subformulas of $x : \phi \circ \psi$ are $x : \phi \circ \psi$ and the subformulas of $x : \phi$ and $x : \psi$. The subformulas of $x : \Box \phi$ are $x : \Box \phi$ and the subformulas of $y : \phi$ for an arbitrary y. The subformulas of $x : \Diamond \phi$ are $x : \Diamond \phi$ and the subformulas of $y : \phi$ for an arbitrary y.

Note that, strictly following this definition, $\mathbf{LabK}$ does not have the subformula property. The relational atoms xRy in the premises of the right rules for the modal operators are not subformulas of the ones in the conclusion, in fact, they are not subformulas of any other formula according to this definition. We can slightly alter this property to the following one, which $\mathbf{LabK}$ satisfies: we say that a labelled proof system $\mathbf{S}$ has the *weak subformula property* if all the formulas in a derivation are subformulas of the formulas in the endsequent or relational atoms of the form xRy.

The goal is now to show the admissibility of the structural rules of weakening, contraction and cut as we did in Section 2.1. We begin by presenting some auxiliary lemmas and definitions. The following definition defines substitution for relational atoms and labelled formulas.

Definition 3.3. Let $x,\ y,\ z,\ w$ be labels and $\phi \in \mathcal{L}^{\Box,\Diamond}$. We define substitution as follows:

$xRy\,[w/z] = xRy$ if $w \neq x$ and $w \neq y$;

$xRy\,[x/z] = zRy$ if $x \neq y$;

$xRy\,[y/z] = xRz$ if $x \neq y$;

$xRx\,[x/z] = zRz$;

$x : \phi\,[y/z] = x : \phi$ if $x \neq y$;

$x : \phi\,[x/z] = z : \phi$

Additionally, substitution over a multiset Γ of labelled formulas (with relational atoms) is the multiset resultant from applying substitution to each formula in Γ.

The next result is needed in order to prove height-preserving admissibility of weakening.

Lemma 3.4 (Substitution Lemma)**.** Let Γ, Δ be multisets and x, y labels. If $\Gamma \Rightarrow \Delta$ is derivable in $\mathbf{LabK}$, then so is $\Gamma\,[x/y] \Rightarrow \Delta\,[x/y]$.

Proof. The proof is by induction on the height h of a derivation for $\Gamma \Rightarrow \Delta$. If $h = 1$, $\Gamma \Rightarrow \Delta$ results from init or $\bot_{\mathsf{L}}$, in either case $\Gamma\,[x/y] \Rightarrow \Delta\,[x/y]$ results from the same rule. If $h > 1$, assume that up to height $h-1$ we have that if a sequent $\Gamma' \Rightarrow \Delta'$ is derivable, so is $\Gamma'\,[x/y] \Rightarrow \Delta'\,[x/y]$. If the substitution is vacuous, i.e. if x does not appear in $\Gamma' \cup \Delta'$, there is nothing to prove. From now on, assume that it is not vacuous. We distinguish several cases according to the last rule applied:

1. $\Gamma \Rightarrow \Delta$ results from the application of a propositional rule. Assume the last rule applied was $\wedge_{\mathsf{L}}$. The premiss of this rule $x : \phi, x : \psi, \Gamma' \Rightarrow \Delta$ is derivable with height $h-1$, hence, we can apply the inductive hypothesis to conclude that $y : \phi, y : \psi, \Gamma'\,[x/y] \Rightarrow \Delta\,[x/y]$ is also derivable. By applying $\wedge_{\mathsf{L}}$ again we conclude that $y : \phi \wedge \psi, \Gamma'\,[x/y] \Rightarrow \Delta\,[x/y]$ is derivable. The method is the same for proving the result for any other rule concerning a propositional connective.

2. If $\Gamma \Rightarrow \Delta$ results from $\Box_{\mathsf{L}}$ or $\Diamond_{\mathsf{R}}$ the strategy is similar. Suppose $\Gamma \Rightarrow \Delta$ is the conclusion of the application of $\Box_{\mathsf{L}}$, with premiss $x : \Box\,\phi, xRz, z : \phi, \Gamma' \Rightarrow \Delta$, which is derivable with height $h-1$. We apply the inductive hypothesis and then $\Box_{\mathsf{L}}$ to conclude that $y : \Box\,\phi, yRz, \Gamma'\,[x/y] \Rightarrow \Delta\,[x/y]$ is derivable. Similarly for $\Diamond_{\mathsf{R}}$.

3. $\Gamma \Rightarrow \Delta$ results from $\Box_{\mathsf{R}}$ or $\Diamond_{\mathsf{L}}$, that is, modal rules with a variable condition. Suppose $\Gamma \Rightarrow \Delta$ is the conclusion of applying $\Diamond_{\mathsf{L}}$ with premiss $xRz, z : \phi, \Gamma' \Rightarrow \Delta$, derivable with height $h-1$, where $z \neq x$ and z is not in $\Gamma' \cup \Delta$. Note that in this case, we have that x is the label of the principal formula. We have two cases:
   - 3.1 If $y \neq z$, we apply the inductive hypothesis to the premises of the rule, getting that $yRz, z : \phi, \Gamma'\,[x/y] \Rightarrow \Delta\,[x/y]$ is derivable. By $\Diamond_{\mathsf{L}}$, since z is not in $\Gamma' \cup \Delta$, we have that $y : \Diamond\,\phi, \Gamma'\,[x/y] \Rightarrow \Delta\,[x/y]$ is derivable.
   - 3.2 If $y = z$, the premiss of the rule is $xRy, y : \phi, \Gamma' \Rightarrow \Delta$. We begin by applying the inductive hypothesis in order to substitute y for some w which is not in $\{x\} \cup \Gamma' \cup \Delta$, getting that $xRw, w : \phi, \Gamma'\,[y/w] \Rightarrow \Delta\,[y/w]$ is derivable. Since y is not in $\Gamma' \cup \Delta$, this sequent is equivalent to $xRw, w : \phi, \Gamma' \Rightarrow \Delta$. We apply the inductive hypothesis again, substituting x for y, which results in $yRw, w : \phi, \Gamma'\,[x/y] \Rightarrow \Delta\,[x/y]$ being derivable. By $\Diamond_{\mathsf{L}}$ we conclude that $y : \Diamond\,\phi, \Gamma'\,[x/y] \Rightarrow \Delta\,[x/y]$ is derivable.

   If x is not the label of the principal formula the proof is similar.

QED.

$$\frac{\Gamma \Rightarrow \Delta}{x : \phi, \Gamma \Rightarrow \Delta}\,\mathsf{wk_L} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x : \phi}\,\mathsf{wk_R}$$

$$\frac{\Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta}\,\mathsf{wk_L}$$

Table 3.2: Rules of Weakening of $\mathbf{LabK}$.

Proposition 3.5. The structural rules of weakening in Table 3.2 are height-preserving admissible in $\mathbf{LabK}$.

Proof. The proof is by induction on the height h of a derivation of the premisses of the rule. If $h = 1$, then $\Gamma \Rightarrow \Delta$ is an initial sequent or the conclusion of $\bot_{\mathsf{L}}$, and hence the derivability of the conclusion of weakening with height $h = 1$ follows immediately. If $h > 1$, we distinguish several cases, one for each possible last rule applied in the derivation of $\Gamma \Rightarrow \Delta$. We show the case for $\mathsf{wk_L}$, the case for $\mathsf{wk_R}$ being similar. The cases for the propositional connectives are analogous to Proposition 2.6, taking into account that the formulas are now labelled. We show the cases for the $\Box$, the proofs for the $\Diamond$ rules are analogous.

($\Box_{\mathsf{L}}$) Assume that $z : \Box\,\zeta, \Gamma \Rightarrow \Delta$ is derivable with height h. Then $zRw, z : \Box\,\zeta, w : \zeta, \Gamma \Rightarrow \Delta$ is derivable with height $h-1$. Let σ be xRy or $x : \phi$. By the inductive hypothesis there is a derivation of height at most $h-1$ of $\sigma, zRw, z : \Box\,\zeta, w : \zeta, \Gamma \Rightarrow \Delta$. By $\Box_{\mathsf{L}}$ we conclude that $\sigma, z : \Box\,\zeta, \Gamma \Rightarrow \Delta$ is derivable with height at most h.

($\Box_{\mathsf{R}}$) Assume that $\Gamma \Rightarrow \Delta, z : \Box\,\zeta$ is derivable with height h. Then, $zRw, \Gamma \Rightarrow \Delta, w : \zeta$ is derivable with height $h-1$, where $w \neq z$ and does not occur in $\Gamma \cup \Delta$. Let σ be $x : \phi$ or xRy. If w is x or y, apply Lemma 3.4 to $zRw, \Gamma \Rightarrow \Delta, w : \zeta$ and substitute w for some variable k such that $k \neq x, y, z$ and it does not occur in $\Gamma \cup \Delta$. We obtain the sequent $zRk, \Gamma \Rightarrow \Delta, k : \zeta$, which is derivable with height at most $h-1$. By the inductive hypothesis $\sigma, zRk, \Gamma \Rightarrow \Delta, k : \zeta$ is derivable also with height at most $h-1$. Since $k \neq z$ and it does not occur in $\{\sigma\} \cup \Gamma \cup \Delta$, we can apply $\Box_{\mathsf{R}}$ to conclude that $\sigma, \Gamma \Rightarrow \Delta, z : \Box\zeta$ is derivable with height at most h.

QED.

The next lemma will be useful in subsequent proofs.

Lemma 3.6. Let ϕ be some formula in $\mathcal{L}^{\Box,\Diamond}$, Γ be a finite multiset of labelled formulas and relational atoms and Δ a finite multiset of labelled formulas. Then:

$$\vdash_{\mathbf{LabK}} x : \phi, \Gamma \Rightarrow \Delta, x : \phi.$$

Proof. The proof is by induction on the structure of ϕ.

(Base) The base case, where ϕ is atomic, follows from init, if ϕ is $\bot$ it follows from $\bot_{\mathsf{L}}$.

(Step) The cases for the propositional connectives follow as in Lemma 2.7, taking into account that the formulas are now labelled. For the cases concering modalities we explicitely show derivations to prove our claim.

Let $\phi \equiv \Box\,\psi$. Consider the following derivation:

$$\dfrac{\dfrac{\dfrac{\vdots}{y : \psi, x : \Box\,\psi, xRy, \Gamma \Rightarrow \Delta, y : \psi}}{x : \Box\,\psi, xRy, \Gamma \Rightarrow \Delta, y : \psi}\,\Box_{\mathsf{L}}}{x : \Box\,\psi, \Gamma \Rightarrow \Delta, x : \Box\,\psi}\,\Box_{\mathsf{R}}$$

Where the topmost sequent is derivable by the inductive hypothesis.

Let $\phi \equiv \Diamond\,\psi$. Consider the derivation:

$$\dfrac{\dfrac{\dfrac{\vdots}{y : \psi, xRy, \Gamma \Rightarrow \Delta, x : \Diamond\,\psi, y : \psi}}{y : \psi, xRy, \Gamma \Rightarrow \Delta, x : \Diamond\,\psi}\,\Diamond_{\mathsf{R}}}{x : \Diamond\,\psi, \Gamma \Rightarrow \Delta, x : \Diamond\,\psi}\,\Diamond_{\mathsf{L}}$$

Where, once more, the topmost sequent is derivable by the inductive hypothesis.
By displaying derivations for the sequent $x : \phi, \Gamma \Rightarrow \Delta, x : \phi$ we conclude the proof for the modal cases.

QED.

Lemma 3.7 (Inversion lemma)**.** The rules in $\mathbf{LabK}$ are height-preserving invertible.

Proof. The only relevant cases are $\Box_{\mathsf{R}}$ and $\Diamond_{\mathsf{L}}$. The cases for propositional connectives are analogous to Lemma 2.8 and the cases of $\Box_{\mathsf{L}}$ and $\Diamond_{\mathsf{R}}$ are immediate since the premisses of these rules are obtained by

weakening the conclusion, and weakening is height-preserving.
We show a proof for invertibility of $\Diamond_{\mathsf{L}}$. Suppose that $x : \Diamond\phi, \Gamma \Rightarrow \Delta$ is derivable with derivation height h. We shall prove that $xRy, y : \phi, \Gamma \Rightarrow \Delta$ is derivable, also with height at most h, for any y such that $y \neq x$ and $y \notin \Gamma \cup \Delta$. The proof is by induction on the height h of the derivation for $x : \Diamond\phi, \Gamma \Rightarrow \Delta$.

(Base) If $h = 1$, $x : \Diamond\phi, \Gamma \Rightarrow \Delta$ is obtained by init or $\bot_{\mathsf{L}}$. Since $\Diamond\phi$ is not an atomic formula or $\bot$, we can conclude the same about $xRy, y : \phi, \Gamma \Rightarrow \Delta$, which means it can also be derived with height equal to 1.

(Step) Let $h > 1$ and assume height-preserving inversion of rule $\Diamond_{\mathsf{L}}$ up to derivation height $h - 1$.
If $x : \Diamond\phi, \Gamma \Rightarrow \Delta$ was obtained from some rule $\mathcal{R}$ different from $\Box_{\mathsf{R}}$ and $\Diamond_{\mathsf{L}}$, then we have that the premisses of this rule $x : \Diamond\phi, \Gamma' \Rightarrow \Delta'$ and $x : \Diamond\phi, \Gamma'' \Rightarrow \Delta''$ are derivable with height at most $h - 1$. By the inductive hypothesis, $xRy, y : \phi, \Gamma' \Rightarrow \Delta'$ and $xRy, y : \phi, \Gamma'' \Rightarrow \Delta''$ are derivable with height at most $h - 1$. Hence by applying $\mathcal{R}$ we conclude that $xRy, y : \phi, \Gamma \Rightarrow \Delta$ is derivable with height at most h.

Now suppose that $x : \Diamond\phi, \Gamma \Rightarrow \Delta$ was concluded by $\Box_{\mathsf{R}}$. Then it must be of the form $x : \Diamond\phi, \Gamma \Rightarrow \Delta', y : \Box\phi$ and have as premiss the sequent $x : \Diamond\phi, yRz, \Gamma \Rightarrow \Delta', z : \phi$, where $z \neq x, y$ and $z \notin \Gamma \cup \Delta'$,derivable with height $h - 1$. By the inductive hypothesis, we get that $xRw, w : \phi, yRz, \Gamma \Rightarrow \Delta', z : \phi$ is derivable with at most height $h - 1$, where $w \neq x, y, z$ and $w \notin \Gamma \cup \Delta'$, and hence, by applying $\Box_{\mathsf{R}}$ we get $xRw, w : \phi, \Gamma \Rightarrow \Delta', y : \Box\phi$. Therefore, $xRw, w : \phi, \Gamma \Rightarrow \Delta$ is derivable with height at most h.

If $x : \Diamond\phi, \Gamma \Rightarrow \Delta$ was concluded from $\Diamond_{\mathsf{L}}$, distinguish two subcases. If the principal formula is in Γ, we proceed as in the last case. Otherwise, if the principal formula is $x : \Diamond\phi$, the premise of this rule must be of the form $xRy, y : \phi, \Gamma \Rightarrow \Delta$, with y not occuring up in $x : \Diamond\phi, \Gamma \Rightarrow \Delta$, up to the naming of the world y, which can be renamed to any other world but x by Lemma 3.4. We conclude that $xRy, y : \phi, \Gamma \Rightarrow \Delta$ is derivable with height at most h.

QED.

$$\frac{x : \phi, x : \phi, \Gamma \Rightarrow \Delta}{x : \phi, \Gamma \Rightarrow \Delta}\,\mathsf{Ctr_L} \qquad \frac{\Gamma \Rightarrow \Delta, x : \phi, x : \phi}{\Gamma \Rightarrow \Delta, x : \phi}\,\mathsf{Ctr_R}$$

$$\frac{xRy, xRy, \Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta}\,\mathsf{Ctr_L}$$

Table 3.3: Rules of Contraction of $\mathbf{LabK}$.

Proposition 3.8. The structural rules of contraction in Table 3.3 are height-preserving admissible in $\mathbf{LabK}$.

Proof. We reason by induction on the height h of the derivation of the premisses of the rules of contraction on the left and on the right. Admissibility of the three contraction rules is proved simultaneously. Observe that if the contraction formula ψ is in the antecedent of the premise of the rule, it can be both a labelled formula $\psi = x : \phi$ or a relational atom $\psi = xRy$. However, if it is on the succedent it must be a labelled formula $\psi = x : \phi$. If $h = 1$ it is immediate, if $\psi, \psi, \Gamma \Rightarrow \Delta$ follows from init or $\bot_{\mathsf{L}}$ then clearly so does $\psi, \Gamma \Rightarrow \Delta$, and the same if the contraction formula is on the right. For $h > 1$ assume that $\psi, \psi, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, \psi, \psi$ are derivable with height h and consider three inductive hypothesis:

IH1: If $\vdash_{\mathbf{G3cp}} x : \phi, x : \phi, \Gamma \Rightarrow \Delta$ with height $h' < h$ then $\vdash_{\mathbf{G3cp}} x : \phi, \Gamma \Rightarrow \Delta$ with height at most h'.

IH2: $\vdash_{\mathbf{G3cp}} \Gamma \Rightarrow \Delta, x : \phi, x : \phi$ with height $h' < h$ then $\vdash_{\mathbf{G3cp}} \Gamma \Rightarrow \Delta, x : \phi$ with height at most h'.

IH3: If $\vdash_{\mathbf{G3cp}} xRy, xRy, \Gamma \Rightarrow \Delta$ with height $h' < h$ then $\vdash_{\mathbf{G3cp}} xRy, \Gamma \Rightarrow \Delta$ with height at most h'.

We consider the following cases:

1. If ψ was not principal in the last rule applied $\mathcal{R}$, then the contraction formula appears twice in the premiss of this rule, which has height $h-1$. By applying the appropriate inductive hypothesis and then $\mathcal{R}$ we conclude that $\psi, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, \psi$ can be derived with height at most h.

2. If the contraction formula is principal in the last rule applied we have three subcases:

   2.1. The principal formula appears both in the premisses and conclusion of $\mathcal{R}$, which means this rule is either $\Box_{\mathsf{L}}$ or $\Diamond_{\mathsf{R}}$. In the case of $\Box_{\mathsf{L}}$ we have $\psi = x : \Box\phi$ the conclusion of the application of the rule is $x : \Box\phi, x : \Box\phi, xRy, \Gamma \Rightarrow \Delta$, hence the premiss $x : \Box\phi, x : \Box\phi, xRy, y : \phi, \Gamma \Rightarrow \Delta$ is derivable with height $h-1$. By IH1, $x : \Box\phi, xRy, y : \phi, \Gamma \Rightarrow \Delta$ is derivable with height at most $h-1$, hence by applying $\Box_{\mathsf{L}}$ again we conclude that $x : \Box\phi, xRy, \Gamma \Rightarrow \Delta$ is derivable with height at most h. The case for $\Diamond_{\mathsf{R}}$ is similar.

   2.2. $\mathcal{R}$ is some rule other than $\Box_{\mathsf{L}}$ and $\Diamond_{\mathsf{R}}$, and the active formulas in the premiss(es) of the rule are not relational atoms. This means that $\mathcal{R}$ is a rule for $\wedge$, $\vee$ or $\rightarrow$, hence by applying the appropriate inductive hypothesis followed by $\mathcal{R}$ twice we get that the conclusion of the contraction rule is derivable with height at most h.

   2.3. $\mathcal{R}$ is some rule other than $\Box_{\mathsf{L}}$ and $\Diamond_{\mathsf{R}}$ but now the active formulas in the premises(es) of $\mathcal{R}$ may be relational atoms or labelled formulas. This means that this rule is either $\Box_{\mathsf{R}}$ or $\Diamond_{\mathsf{L}}$. In the case of $\Box_{\mathsf{R}}$, we have $\psi = x : \Box\phi$ and the conclusion of the application of the rule is $\Gamma \Rightarrow \Delta, x : \Box\phi, x : \Box\phi$, hence the premiss $xRy, \Gamma \Rightarrow \Delta, x : \Box\phi, y : \phi$, where $y \neq x$ and $y \notin \Gamma \cup \Delta$ is derivable with height $h-1$. Again, by height-preserving invertibility the sequent $xRy, xRz, \Gamma \Rightarrow, y : \phi, z : \phi$ is derivable with height at most $h-1$. We can substitute z for y in this sequent, resulting in the sequent $xRy, xRy, \Gamma\,[y/z] \Rightarrow \Delta\,[y/z]\,, y : \phi, y : \phi$, that it, $xRy, xRy, \Gamma \Rightarrow \Delta, y : \phi, y : \phi$, also derivable with height at most $h-1$. Applying IH2, IH3 and $\Box_{\mathsf{R}}$ we conclude that $\Gamma \Rightarrow \Delta, x : \Box, \phi$ is derivable with height at most h.

QED.

At last, we can state and prove the most important result of this section regarding the rule of cut, formulated as follows:

$$\frac{\Gamma \Rightarrow \Delta, x : \phi \quad x : \phi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta}\,\mathsf{cut}$$

Before stating and proving the admissibility of cut, we must define the weight of labelled formulas. Let ϕ be some formula in $\mathcal{L}^{\Box,\Diamond}$ and $\circ \in \{\Box, \Diamond\}$. We have that $w(\circ\phi) = w(\phi) + 1$. Furthermore, ψ be a formula in $\mathcal{L}^{\Box,\Diamond}$. Then, the weight of $w(x : \psi)$ is equal to $w(\psi)$.

Theorem 3.9. *The* cut *rule is admissible in* $\mathbf{LabK}$.[1]

Proof. The proof follows by primary induction on the *weight* of the cut-formula $x : \phi$ and by secondary induction on the *cut-height* of the application of cut. Assume the premisses of the cut rule are derivable and that the sum of their derivation heights is h. We have:

IH1: Cut is admissible whenever the cut formula $y : \psi$ is such that $w(y : \psi) < w(x : \phi)$.

IH2: Cut is admissible whenever the cut formula $y : \psi$ is such that $w(y : \psi) = w(x : \phi)$, and the cut-height is strictly smaller than h.

The proof follows the structure and numbering of the proof of Theorem 2.11. The proof is organized into five cases and sub-cases. In the first two cases, one of the premisses of cut is derived by init or $\bot_{\mathsf{L}}$, i.e., one of the premisses is an initial sequent, and we proceed in a way analogous to the proof in Section 2.1, eliminating the cut rule from the derivations. In the other cases some new sub-cases arise since we now introduce modal formulas. We only display some of these sub-cases.

Cut where neither premiss is an initial sequent

In the following derivations, the topsequents have heights $n, m, k, \ldots$. Assume that the premisses of cut are derivable.

3. The cut formula $x : \phi$ is **not principal in the left premiss**, $\Gamma \Rightarrow \Delta, x : \phi$. We distinguish several cases according to the rule that was used to derive the left premiss $x : \phi, \Gamma \Rightarrow \Delta$. If it was a logical rule, or a modal rule without a variable condition, proceed as in Theorem 2.11 permuting the application of the logical rule with the application of cut. Otherwise we consider two cases for $\Diamond_{\mathsf{L}}$ and $\Box_{\mathsf{R}}$. We only show one of these cases.

    3.1. $\Diamond_{\mathsf{L}}$ and $\Gamma = y : \Diamond\psi, \Gamma''$. Consider the following derivation, where the bottom-most instance of cut has cut-height $h = n + 1 + m$, and where $z \neq x, y$ and $z \notin \Gamma'' \cup \Delta$.

$$\dfrac{\dfrac{yRz, z : \psi, \Gamma'' \Rightarrow \Delta, x : \phi}{y : \Diamond\psi, \Gamma'' \Rightarrow \Delta, x : \phi}\,\Diamond_{\mathsf{L}} \qquad x : \phi, \Gamma' \Rightarrow \Delta'}{y : \Diamond\psi, \Gamma'', \Gamma' \Rightarrow \Delta', \Delta}\,\mathsf{cut}$$

    This derivation is transformed into the one that follows.

$$\dfrac{\dfrac{yRw, w : \psi, \Gamma'' \Rightarrow \Delta, x : \phi \quad x : \phi, \Gamma' \Rightarrow \Delta'}{yRw, w : \psi, \Gamma'', \Gamma' \Rightarrow \Delta', \Delta}\,\mathsf{cut}}{y : \Diamond\psi, \Gamma'', \Gamma' \Rightarrow \Delta', \Delta}\,\Diamond_{\mathsf{L}}$$

    In this derivation, the left premiss of cut is obtained from substituting the label z by w in the sequent $yRz, z : \psi, \Gamma'' \Rightarrow \Delta, x : \phi$, hence it has derivation height equal to n, following Lemma 3.4. Therefore, this instance of cut has height $n + m < h$ and IH2 applies.

[1] Similarly to what was noted in the proof of cut admissibility in Chapter 2, we will be using a slight abuse of notation in this proof by using occurrences of the cut rule. These should be understood simply as applications of the inductive hypothesis.

4. The cut formula $x : \phi$ is **principal in the left premiss only** $\Gamma \Rightarrow \Delta, x : \phi$. We distinguish several cases according to the rule that was used to derive the right premiss $\Gamma' \Rightarrow \Delta', x : \phi$. If it was a logical rule, or a modal rule without a variable condition, proceed as in Theorem 2.11 permuting the application of the logical rule with the application of cut. We must consider two cases for $\Diamond_{\mathsf{L}}$ and $\Box_{\mathsf{R}}$. We only show one of these cases.

   4.1 $\Box_{\mathsf{R}}$ and $\Delta' = y : \Box\psi, \Delta''$. Consider the following derivation, where the bottom-most instance of cut has cut-height $h = n + m + 1$, and where $z \neq x, y$ and $z \notin \Gamma' \cup \Delta''$.

$$\dfrac{\Gamma \Rightarrow \Delta, x : \phi \qquad \dfrac{yRz, x : \phi, \Gamma' \Rightarrow z : \psi, \Delta''}{x : \phi, \Gamma' \Rightarrow y : \Box\psi, \Delta''}\,\Box_{\mathsf{R}}}{\Gamma, \Gamma' \Rightarrow y : \Box\psi, \Delta'', \Delta}\,\mathsf{cut}$$

   This derivation is transformed into the one that follows.

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, x : \phi \qquad yRw, x : \phi, \Gamma' \Rightarrow w : \psi, \Delta''}{yRw, \Gamma, \Gamma' \Rightarrow w : \psi, \Delta'', \Delta}\,\mathsf{cut}}{\Gamma, \Gamma' \Rightarrow y : \Box\psi, \Delta'', \Delta}\,\Box_{\mathsf{R}}$$

   In this derivation, the right premiss of cut is obtained from substituting the label z for w in the sequent $yRz, x : \phi, \Gamma' \Rightarrow \Delta''$, hence it has derivation height equal to m, following Lemma 3.4. Therefore, this instance of cut has height $n + m < h$ and IH2 applies.

5. The cut formula $x : \phi$ is **principal in both premisses**. We have two new subcases according to the structure of $x : \phi$.

   5.1 $\phi = \Box\psi$ and $\Gamma' = \Gamma'' \cup \{xRy\}$. Consider the following derivation, where the bottom-most instance of cut has cut-height $h = n + 1 + m + 1$, and where $z \neq x$ and $z \notin \Gamma \cup \Delta$.

$$\dfrac{\dfrac{xRz, \Gamma \Rightarrow \Delta, z : \psi}{\Gamma \Rightarrow \Delta, x : \Box\psi}\,\Box_{\mathsf{R}} \qquad \dfrac{xRy, x : \Box\psi, y : \psi, \Gamma'' \Rightarrow \Delta'}{xRy, x : \Box\psi, \Gamma'' \Rightarrow \Delta'}\,\Box_{\mathsf{L}}}{xRy, \Gamma, \Gamma'' \Rightarrow \Delta', \Delta}\,\mathsf{cut}$$

   The cut formula $x : \Box\psi$ has weight $w(x : \Box\psi) = w(x : \psi) + 1$. This derivation is transformed into the one that follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{xRy, \Gamma \Rightarrow \Delta, y : \psi \qquad \dfrac{\Gamma \Rightarrow \Delta, x : \Box\psi \qquad xRy, y : \psi, x : \Box\psi, \Gamma'' \Rightarrow \Delta'}{xRy, y : \psi, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}\,\mathsf{cut}}{\Gamma, \Gamma, xRy, xRy, \Gamma'' \Rightarrow \Delta, \Delta, \Delta'}\,\mathsf{cut}}{\Gamma, \Gamma, xRy, \Gamma'' \Rightarrow \Delta, \Delta, \Delta'}\,\mathsf{Ctr_L}}{\Gamma, \Gamma, xRy, \Gamma'' \Rightarrow \Delta, \Delta'}\,\mathsf{Ctr_R}}{\Gamma, xRy, \Gamma'' \Rightarrow \Delta, \Delta'}\,\mathsf{Ctr_L}$$

   In this derivation, the uppermost instance of cut has cut-height $n - 1 + m < h$, hence IH2 applies. In the other instance of cut, the right premiss $xRy, \Gamma \Rightarrow \Delta, y : \psi$ is obtained from substituting the label z for y in the sequent $xRz, \Gamma \Rightarrow \Delta, z : \psi$, hence it is derivable with height n, following Lemma 3.4. Furthermore, in this instance of cut, the cut formula $y : \psi$ which has weight less than $w(x : \Box\psi)$, hence IH1 applies.

QED.

We end this section by proving soundness and completeness for $\mathbf{LabK}$. The proof that we give of completeness is analogous to the one in Chapter 2 for the completeness of $\mathbf{G3cp}$ and relies on deriving the characteristic axiom of $\mathbf{K}$ and simulating the rule of necessitation and Modus Ponens, which forces us to use cut. The proof for soundness is quite different since the labelled sequents don't have an interpretation as sequents did in Chapter 2. Thus, proving soundness requires discussing the semantics of labelled calculi. The definitions we now introduce have been presented in [30].

Definition 3.10 (Satisfiability)**.** Let $\mathcal{M} = (W, R, [\![\cdot]\!])$ be a Kripke model, and $\mathcal{S} = \Gamma \Rightarrow \Delta$ a labelled sequent. Furthermore let $\mathsf{Lb}(\mathcal{S})$ denote the set of all the labels that appear in $\mathcal{S}$, and $\rho : \mathsf{Lb}(\mathcal{S}) \to W$ be a map, called an *interpretation*, that matches labels to worlds in a model.
We define the *satisfiability of labelled formulas and atoms at $\mathcal{M}$ under ρ* as:

$$\mathcal{M}, \rho \Vdash xRy \text{ iff } (\rho(x), \rho(y)) \in R$$
$$\mathcal{M}, \rho \Vdash x : \phi \text{ iff } \mathcal{M}, \rho(x) \Vdash \phi$$

We define the *satisfiability of a sequent at $\mathcal{M}$ under ρ* as:

$$\mathcal{M}, \rho \Vdash \mathcal{S} \text{ iff if for all } xRy \in \Gamma,\ z : \phi \in \Gamma \text{ we have } \mathcal{M}, \rho \Vdash xRy \text{ and } \mathcal{M}, \rho \Vdash z : \phi$$
$$\text{then for some } w : \psi \in \Delta,\ \mathcal{M}, \rho \Vdash w : \psi.$$

Definition 3.11 (Validity)**.** We define *validity of a sequent $\mathcal{S}$ in a class of frames* F as:

$$\models_{\mathsf{F}} \mathcal{S} \text{ iff for any } \rho \text{ and any } \mathcal{M} \text{ based on } \mathsf{F} \text{ we have } \mathcal{M}, \rho \Vdash \mathcal{S}.$$

Theorem 3.12 (Soundness)**.** *Let $\phi \in \mathcal{L}^{\Box,\Diamond}$. If $\vdash_{\mathbf{LabK}} \Gamma \Rightarrow \Delta$ then $\Vdash_{\mathsf{k}} \Gamma \Rightarrow \Delta$.*

Proof. The proof is by induction on the height h of a derivation for $\Gamma \Rightarrow \Delta$.

(Base) If $h = 1$, $\Gamma \Rightarrow \Delta$ can either be the result of init or $\bot_{\mathsf{L}}$. If it follows from $\bot_{\mathsf{L}}$ then we have $\Vdash_{\mathsf{k}} x : \bot, \Gamma' \Rightarrow \Delta$ if and only if for all models $\mathcal{M}$ and interpretations ρ we have $\mathcal{M}, \rho \Vdash x : \bot, \Gamma' \Rightarrow \Delta$, which is always the case since $\mathcal{M}, \rho \Vdash x : \bot$ is always false. If it follows from init we have $\Vdash_{\mathsf{k}} x : p, \Gamma' \Rightarrow \Delta', x : p$ if and only if for all models $\mathcal{M}$ and interpretations ρ we have $\mathcal{M}, \rho \Vdash x : p, \Gamma' \Rightarrow \Delta', x : p$. This is always true, assuming that every formula in the antecedent is satisfied, in particular $x : p$, then there is always a formula $x : \psi$ in the succedent of the sequent such that $\mathcal{M}, \rho \Vdash x : \psi$, namely $x : p$.

(Step) Now assume that $\vdash_{\mathbf{LabK}} \Gamma \Rightarrow \Delta$ is derivable with height $h > 1$. We distinguish two cases according to the last rule applied:

1. $\Gamma \Rightarrow \Delta$ is the conclusion of a propositional rule, for instance $\vee_{\mathsf{R}}$, with premiss $\Gamma \Rightarrow \Delta', x : \phi, x : \psi$. By the inductive hypothesis we have that $\Vdash_{\mathsf{k}} \Gamma \Rightarrow \Delta', x : \phi, x : \psi$, that is, for every model $\mathcal{M}$ and interpretation ρ we have $\mathcal{M}, \rho \Vdash \Gamma \Rightarrow \Delta', x : \phi, x : \psi$. Assume that $\mathcal{M}, \rho \Vdash \gamma$, for every labelled formula or relational atom $\gamma \in \Gamma$. By the definition of satisfiability of a sequent at a model over an interpretation we have that $\mathcal{M}, \rho \Vdash \zeta$ for

some $\zeta \in \Delta' \cup \{x : \phi, x : \psi\}$. Note that $\mathcal{M}, \rho \Vdash x : \phi \vee \psi$ if and only if $\mathcal{M}, \rho \Vdash x : \phi$ or $\mathcal{M}, \rho \Vdash \psi$. If $\mathcal{M}, \rho \Vdash \delta$ for some $\delta \in \Delta'$ then $\mathcal{M}, \rho \Vdash \Gamma \Rightarrow \Delta', x : \phi \vee \psi$. If $\mathcal{M}, \rho \Vdash x : \phi$ or $\mathcal{M}, \rho \Vdash x : \psi$ then $\mathcal{M}, \rho \Vdash \Gamma \Rightarrow \Delta', x : \phi \vee \psi$. By the arbitrariness of $\mathcal{M}$ and ρ we conclude that $\Vdash_{\mathsf{k}} \Gamma \Rightarrow \Delta', x : \phi \vee \psi$ as we wanted to show.

2. $\Gamma \Rightarrow \Delta$ is the conclusion of a modal rule, for instance $\Diamond_{\mathsf{L}}$, with premiss $xRy, y : \phi, \Gamma' \Rightarrow \Delta$, where $y \neq x$ and y does not appear in $\Gamma' \cup \Delta$. By the inductive hypothesis $\Vdash_{\mathsf{k}} xRy, y : \phi, \Gamma' \Rightarrow \Delta$. In order to reach a contradiction, assume that $\nVdash_{\mathsf{k}} x : \Diamond \phi, \Gamma' \Rightarrow \Delta$, that is, there is a model $\mathcal{M}$ and an interpretation ρ such that $\mathcal{M}, \rho \nVdash x : \Diamond \phi, \Gamma' \Rightarrow \Delta$. Thus:

$$\begin{array}{ll} \mathcal{M}, \rho \Vdash x : \Diamond \phi; & \\ \mathcal{M}, \rho \Vdash \gamma & \text{for every } \gamma \in \Gamma'; \\ \mathcal{M}, \rho \nVdash \delta & \text{for every } \delta \in \Delta. \end{array}$$

By the first condition, we have a z in $\mathcal{M}$ such that $\mathcal{M}, z \Vdash \phi$ and $\rho(x)Rz$. Let ρ' be an interpretation that coincides with ρ except possibly on y and let $\rho'(y) = z$. We have that:

$$\begin{array}{ll} \mathcal{M}, \rho' \Vdash xRy; & \\ \mathcal{M}, \rho' \Vdash y : \phi; & \\ \mathcal{M}, \rho' \Vdash \gamma & \text{for every } \gamma \in \Gamma'; \\ \mathcal{M}, \rho' \nVdash \delta & \text{for every } \delta \in \Delta. \end{array}$$

These conditions imply that $\nVdash_{\mathsf{k}} xRy, y : \phi, \Gamma' \Rightarrow \Delta$, thus reaching a contradiction.

QED.

In particular, if Γ is empty and Δ is composed by a single labelled formula $x : \phi$ we get that if $\vdash_{\mathbf{LabK}} \Rightarrow x : \phi$ then $\Vdash_{\mathsf{k}} \phi$.

Theorem 3.13 (Completeness)**.** *Let $\phi \in \mathcal{L}^{\Box,\Diamond}$. If $\vdash_{\mathbf{K}} \phi$ then $\vdash_{\mathbf{LabK}} \Rightarrow x : \phi$.*

Proof. We can show that the axiom of Modal distributivity is derivable in $\mathbf{LabK}$ by considering the following derivation:

$$\begin{prooftree}
\AxiomC{}
\RightLabel{Lemma 3.14}
\UnaryInfC{$xRy, x : \Box(\phi \to \psi), y : \psi, x : \Box\phi, y : \phi \Rightarrow y : \psi$}
\AxiomC{}
\RightLabel{Lemma 3.14}
\UnaryInfC{$xRy, x : \Box(\phi \to \psi), x : \Box\phi, y : \phi \Rightarrow y : \psi, y : \phi$}
\RightLabel{$\to_{\mathsf{L}}$}
\BinaryInfC{$xRy, x : \Box(\phi \to \psi), y : \phi \to \psi, x : \Box\phi, y : \phi \Rightarrow y : \psi$}
\RightLabel{$\Box_{\mathsf{L}}$}
\UnaryInfC{$xRy, x : \Box(\phi \to \psi), x : \Box\phi, y : \phi \Rightarrow y : \psi$}
\RightLabel{$\Box_{\mathsf{L}}$}
\UnaryInfC{$xRy, x : \Box(\phi \to \psi), x : \Box\phi \Rightarrow y : \psi$}
\RightLabel{$\Box_{\mathsf{R}}$}
\UnaryInfC{$x : \Box(\phi \to \psi), x : \Box\phi \Rightarrow x : \Box\psi$}
\RightLabel{$\to_{\mathsf{R}}$}
\UnaryInfC{$x : \Box(\phi \to \psi) \Rightarrow x : \Box\phi \to \Box\psi$}
\RightLabel{$\to_{\mathsf{R}}$}
\UnaryInfC{$\Rightarrow x : \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$}
\end{prooftree}$$

Furthermore, we show how to simulate the necessitation rule by showing that the following rule is admissible:

$$\frac{\Rightarrow x : \phi}{\Rightarrow x : \Box\phi}$$

Assume that $\Rightarrow x : \phi$ is derivable. By Lemma 3.4 there is a derivation of the sequent $y : \phi$ and, by weakening, of $xRy \Rightarrow y : \phi$. Applying $\Box_{\mathsf{R}}$ we get that $\Rightarrow x : \Box\phi$ is derivable. Similarly to what was done in Theorem 2.12, we can simulate the rule of Modus Ponens using the cut rule:

$$\frac{\Rightarrow x : \phi \to \psi \quad \Rightarrow x : \phi}{\Rightarrow x : \psi}.$$

QED.

3.2 Labelled calculi for extensions of K

This section aims to extend the labelled calculus for $\mathbf{K}$ that was presented to the other logics in the modal cube. The rules that we are going to present concern the accessibility relation between worlds in a model and translate the frame conditions in Table 1.2. We also extend the results that were presented in the last section to the systems that we are now going to adress.

$$\frac{xRx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\textit{Ref} \qquad \frac{xRy, yRx, \Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta}\,\textit{Sym}$$

$$\frac{xRy, yRz, xRz, \Gamma \Rightarrow \Delta}{xRy, yRz, \Gamma \Rightarrow \Delta}\,\textit{Trans} \qquad \frac{xRy, xRz, yRz, \Gamma \Rightarrow \Delta}{xRy, xRz, \Gamma \Rightarrow \Delta}\,\textit{Euc}$$

$$\frac{xRy, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\textit{Ser} \quad (y!)$$

Table 3.4: Sequent rules for R for extending $\mathbf{LabK}$. Notation $(y!)$ means that y does not occur in the conclusion of the rule.

Let $\mathbf{X}$ be one of the logics in the cube and $\mathsf{X} \in \{\mathsf{4}, \mathsf{5}, \mathsf{45}, \mathsf{b}, \mathsf{b5}, \mathsf{d}, \mathsf{db}, \mathsf{d4}, \mathsf{d5}, \mathsf{d45}, \mathsf{t}, \mathsf{tb}, \mathsf{t4}, \mathsf{t5}\}$ the corresponding class of frames. The labelled system for $\mathbf{X}$ is denoted by $\mathbf{LabX}$ and is obtained by extending $\mathbf{LabK}$ with the appropriate rule(s) for R that can be found in table 3.4, which are the *structural rules*. Among these systems are:

$\mathbf{LabD} = \mathbf{LabK} + \textit{Ser}$

$\mathbf{LabT} = \mathbf{LabK} + \textit{Ref}$

$\mathbf{LabKB} = \mathbf{LabK} + \textit{Sym}$

$\mathbf{LabS4} = \mathbf{LabK} + \textit{Ref} + \textit{Trans}$

$\mathbf{LabS5} = \mathbf{LabK} + \textit{Ref} + \textit{Sym} + \textit{Trans}$

The following results extend the ones in the last section to the systems that were just presented. The majority of the proofs are analogous to what was done, and hence we only provide sketches of these proofs.

Lemma 3.14. Let ϕ be some formula in $\mathcal{L}^{\Box,\Diamond}$. Then:

$$\vdash_{\mathbf{LabX}} x : \phi, \Gamma \Rightarrow \Delta, x : \phi.$$

Lemma 3.15 (Substitution Lemma)**.** Let Γ, Δ be multisets and x, y labels. If $\Gamma \Rightarrow \Delta$ is derivable in $\mathbf{LabX}$, then $\Gamma [x/y] \Rightarrow \Delta [x/y]$ is also derivable with the same height.

Proof. The proof proceeds by induction on the height h of a derivation for $\Gamma \Rightarrow \Delta$. The base case $h = 1$, and the case $h > 1$ where the substitution is vacuous proceed as in the proof of Lemma 3.4. We assume that the substitution is not vacuous and distinguish several cases, according to the last rule applied to $\Gamma \Rightarrow \Delta$. We only show this for the case of *Ref*, which is analogous to the propositional cases, and for *Ser*. Suppose that $\Gamma \Rightarrow \Delta$ is derivable in $\mathbf{LabK}$ with derivation height $h > 1$ and that for any sequent $\Gamma' \Rightarrow \Delta'$ derivable with height less than h we have that $\Gamma' [x/y] \Rightarrow \Delta' [x/y]$ is also derivable.

(Ref) Assume that $\Gamma \Rightarrow \Delta$ results from an application of the *Ref* rule to a premiss of the form $zRz, \Gamma \Rightarrow \Delta$, which is derivable with height $h - 1$. By the inductive hypothesis, we have that $zRz [x/y], \Gamma [x/y] \Rightarrow \Delta [x/y]$ is derivable. Applying *Ref* to this sequent we conclude that $\Gamma [x/y] \Rightarrow \Delta [x/y]$ is derivable, as we wanted to show. Note that it is not relevant if the substitution is vacuous or not as the proof works in any case.

(Ser) Assume that $\Gamma \Rightarrow \Delta$ results from an application of the Ser rule to a premiss of the form $zRw, \Gamma \Rightarrow \Delta$, derivable with height $h - 1$, where w is not z or in $\Gamma \cup \Delta$. By the inductive hypothesis, we have that $zRw [x/y], \Gamma [x/y] \Rightarrow \Delta [x/y]$ is derivable. Assume the substitution is not vacuous. If w is neither x or y, we can apply seriality to $zRw [x/y], \Gamma [x/y] \Rightarrow \Delta [x/y]$, whence we conclude that $\Gamma [x/y] \Rightarrow \Delta [x/y]$ is derivable. If w is x or y, pick w' such that $w' \neq x, y, z$ and w' does not occur in $\Gamma \cup \Delta$. We can apply the inductive hypothesis to $zRw, \Gamma \Rightarrow \Delta$ to conclude that the sequent $zRw [w/w'], \Gamma [w/w'] \Rightarrow \Delta [w/w']$, which is $zRw', \Gamma \Rightarrow \Delta$, is derivable. Note that since $w' \neq x, y, z$ and w' does not occur in $\Gamma \cup \Delta$, we can apply seriality to $zRw [x/y], \Gamma [x/y] \Rightarrow \Delta [x/y]$. Therefore we conclude that $\Gamma [x/y] \Rightarrow \Delta [x/y]$ is derivable.

QED.

Proposition 3.16. The structural rules of weakening in Table 3.2 are height-preserving admissible in $\mathbf{LabX}$.

Proof. The proof proceeds as in Proposition 3.5, by induction on the height of the premiss of weakening. We consider the cases where the last rule applied is a rule for R. The proofs for the R rules without any variable conditions are analogous to the cases for the propositional rules. In the case where the premiss of weakening is obtained from applying *Ser* to $xRy, \Gamma \Rightarrow \Delta$, such that y does not occur in $\Gamma \cup \Delta$. If the eigenvariable y is in the conclusion of weakening, we apply Lemma 3.15 to rename this variable in the premiss of *Ser* to another not in $\Gamma \cup \Delta$. We then apply the inductive hypothesis and *Ser*.

QED.

Lemma 3.17 (Inversion lemma)**.** The rules in $\mathbf{LabX}$ are height-preserving invertible.

Proof. The premiss of every structural rule from Table 3.4 can be obtained by weakening the conclusion. Since weakening is height-preserving these rules are height-preserving invertible.

QED.

Proposition 3.18. The structural rules of contraction in Table 3.3 are height-preserving admissible in $\mathbf{LabX}$.

Proof. The proof for the admissibility of the three contraction rules is done simultaneously. It proceeds by induction as in Proposition 3.8. We follow the same numbering of cases and of the inductive hypotheses. The only relevant case to add is when the contraction formula is principal in the last rule applied and this is a structural rule from Table 3.4. We only display one of the possible sub-cases according to the structural rule that was applied, all other cases are analogous:

2.4.1 The last rule applied is Sym. We have that $xRy, xRy, \Gamma \Rightarrow \Delta$is derivable with height h, hence $xRy, xRy, yRx, \Gamma \Rightarrow \Delta$ is derivable with height $h-1$. By IH3 applied to the premiss, we have that $xRy, yRx, \Gamma \Rightarrow \Delta$ is derivable with at most that same height. By applying Sym to this sequent, we get the desired result.

QED.

Theorem 3.19. *The* cut *rule is admissible in* $\mathbf{LabX}$.

Proof. The cut formula cannot be a relational atom hence all cases but 3 and 4 remain the same as in the proof of Theorem 3.9. In 3 we consider the cases in which the left rule was derived using a rule for R and in 4 the cases in which the right rule was derived using a rule for R. In either case, the proofs are analogous to the ones for the one premiss rules for connectives without variable conditions.

QED.

Proposition 3.20. The following hold:

1. $\vdash_{\mathbf{LabD}} \Rightarrow x : (\Box \phi \to \Diamond \phi)$
2. $\vdash_{\mathbf{LabT}} \Rightarrow x : (\Box \phi \to \phi)$
3. $\vdash_{\mathbf{LabKB}} \Rightarrow x : (\phi \to \Box\Diamond \phi)$
4. $\vdash_{\mathbf{LabS4}} \Rightarrow x : (\Box \phi \to \Box\Box \phi)$
5. $\vdash_{\mathbf{LabS5}} \Rightarrow x : (\Diamond \phi \to \Box\Diamond \phi)$

Proof. By root first proof search of the sequent we want to derive.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{xRy, yRx, x : \phi \Rightarrow y : \Diamond \phi, x : \phi}\ \text{Lemma } 3.14}{xRy, yRx, x : \phi \Rightarrow y : \Diamond \phi}\ \Diamond_{\mathsf{R}}}{xRy, x : \phi \Rightarrow y : \Diamond\phi}\ \mathit{Sym}}{x : \phi \Rightarrow x : \Box\Diamond\phi}\ \Box_{\mathsf{R}}}{\Rightarrow x : (\phi \to \Box\Diamond \phi)}\ \to_{\mathsf{R}}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{xRy, xRz, yRx, yRz, z : \phi \Rightarrow y : \Diamond \phi, z : \phi}\text{Lemma 3.14}}{xRy, xRz, yRx, yRz, z : \phi \Rightarrow y : \Diamond \phi}\Diamond_{\mathsf{R}}}{xRy, xRz, yRx, z : \phi \Rightarrow y : \Diamond \phi}\textit{Trans}}{xRy, xRz, z : \phi \Rightarrow y : \Diamond \phi}\textit{Sym}}{xRy, x : \Diamond \phi \Rightarrow y : \Diamond \phi}\Diamond_{\mathsf{L}}}{x : \Diamond \phi \Rightarrow x : \Box\Diamond \phi}\Box_{\mathsf{R}}}{\Rightarrow x : (\Diamond \phi \to \Box\Diamond \phi)}\to_{\mathsf{R}}
$$

QED.

We now show soundness and completeness of the systems for the logics in the cube, extending $\mathbf{LabK}$.

Theorem 3.21 (Soundness)**.** *Let* $\phi \in \mathcal{L}^{\Box,\Diamond}$. *If* $\vdash_{\mathbf{LabX}} \Gamma \Rightarrow \Delta$ *then* $\Vdash_{\mathsf{X}} \Gamma \Rightarrow \Delta$.

Proof. We must prove that the structural rules from Table 3.4 preserve the validity of sequents. We illustrate this with the proofs for *Ref* and *Ser*.

(Ref) Let $\mathcal{M}$ and ρ be an arbitrary reflexive model and interpretation, respectively. Recall the rule

$$
\dfrac{xRx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\textit{Ref}.
$$

Assume that $\mathcal{M}, \rho \Vdash \gamma$ for every $\gamma \in \Gamma$. By reflexivity of the model, $\mathcal{M}, \rho \Vdash yRy$ for every y in $\mathcal{M}$, in particular $\mathcal{M}, \rho \Vdash xRx$ thus the premiss of the *Ref* rule is valid. It follows that there is a $\delta \in \Delta$ such that $\mathcal{M}, \rho \Vdash \delta$. Thus by the arbitrariness of $\mathcal{M}$ and ρ we conclude that $\Vdash_{\mathsf{t}} \Gamma \Rightarrow \Delta$.

(Ser) Let $\mathcal{M}$ and ρ be an arbitrary serial model and interpretation, respectively. Recall the rule

$$
\dfrac{xRy, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\textit{Ser},
$$

where $y \neq x$ and y does not occur in $\Gamma \cup \Delta$. Assume that $\mathcal{M}, \rho \Vdash \gamma$ for every $\gamma \in \Gamma$. By seriality of the model, for any x occuring in $\Gamma \cup \Delta$, there is k in $\mathcal{M}$ such that $\rho(x)Rk$. Take another ρ' such that ρ and ρ' coincide in every label except possibly on y and set $\rho'(y) = k$. We have $\mathcal{M}, \rho' \Vdash \gamma$ for every $\gamma \in \Gamma$ since y is not in Γ and $\mathcal{M}, \rho' \Vdash xRy$ because $\rho'(x)R\rho'(y)$. By the validity of the premiss, $\mathcal{M}, \rho' \Vdash z : \zeta$ for some $z : \zeta \in \Delta$. Note that ρ and ρ' coincide in every label in Δ hence $\mathcal{M}, \rho \Vdash z : \delta$. Thus by the arbitrarianess of $\mathcal{M}$ and ρ we conclude that $\Vdash_{\mathsf{b}} \Gamma \Rightarrow \Delta$.

QED.

Theorem 3.22 (Completeness)**.** *Let* $\phi \in \mathcal{L}^{\Box,\Diamond}$. *If* $\vdash_{\mathbf{X}} \phi$ *then* $\vdash_{\mathbf{LabX}} \Rightarrow x : \phi$.

Proof. Follows from the derivability of the characteristic axiom(s) of $\mathbf{X}$, as shown in Proposition 3.20 and from the proof of Theorem 3.13.

QED.

At the beginning of this chapter, we introduced the labelled calculus $\mathbf{LabK}$ and in the present section presented rules that extend it to the other logics in the cube. The last two theorems state soundness and completeness for all the sequent systems we introduced. Note that the two proofs have different strategies: soundness is proven by showing that the rules of the calculus preserve validity within a class of frame, while completeness is proven by showing that the axioms in H_X are derivable and that the inference rules can be simulated using the rules in $\mathbf{LabX}$. Furthermore, the proof we provided for completeness uses the cut rule to simulate Modus Ponens. Note that our systems are cut-free, but Theorem 3.19 states that the cut rule can be simulated within each sys. We can now recall and complete Diagram 1.5 in which validity within a class of frames and derivability from a Hilbert system were related:

$$
\begin{tikzcd}
\Vdash_{X} \phi \arrow[rr, Leftrightarrow, bend left, "1.41+1.46"] & & \vdash_{\mathbf{X}} \phi \arrow[dl, Rightarrow, bend left, "3.22"] \\
& \vdash_{\mathbf{LabX}} \Rightarrow x : \phi \arrow[ul, Rightarrow, bend left, "3.21"] &
\end{tikzcd}
$$

3.3 Completeness: a semantic approach

The goal of this section is to provide a semantical proof of Theorem 3.22, following Kripke's arguments of completeness for several systems of modal logic. In his work [23], Kripke proves completeness of the $\mathbf{S5}$ system by showing that for every formula which is not derivable in a given tableau system, there must be a model where it is not satisfiable while providing the construction of this *countermodel*. In later work [24] Kripke extends that result to a broader family of normal modal logics. We begin by describing this procedure for $\mathbf{K}$, and then extend it to the other systems. The construction we describe is based on the work presented in [30]. In this paper Negri proves completeness for every logic in the cube by presenting an algorithm according to which we can conduct proof-search and then details how to extract a possibly infinite countermodel from the resulting tree.

3.3.1 Semantic completeness of $\mathbf{K}$

We devote this section to proving the following result:

Theorem 3.23. *Let $\Gamma \Rightarrow \Delta$ be a sequent in the language of $\mathbf{LabK}$. Then, either this sequent is derivable in $\mathbf{LabK}$ or there is a Kripke model where it is not satisfiable.*

The countermodel is obtained by directly attempting to construct a derivation for $\Gamma \Rightarrow \Delta$. The proof search for this sequent can give rise to one of two scenarios: either we successfully construct a derivation for this sequent, which means we found a proof, or we fail at doing so. This can be the case if we have exhaustively applied every rule of the calculus with no more possibilities left but don't obtain a tree with initial sequents as leaves, or if we get stuck in a loop where we keep applying the same rule(s). In order to prove Theorem 3.23, we begin by describing an algorithm according to which we try to obtain said derivation, constructing a *reduction tree*. If we cannot find a derivation, the resulting tree will be infinite and will be the starting point to construct the countermodel. Otherwise, the reduction tree is finite and can be read from the leaves to the root, giving a derivation for $\Gamma \Rightarrow \Delta$.

Constructing the reduction tree.
Begin by placing $\Gamma \Rightarrow \Delta$ at the root of the tree. If this sequent is initial, the construction ends. Otherwise, the construction will continue by writing on top of every non-initial top sequent the result of performing a certain step corresponding to at least one (possibly several) root first application of a rule of $\mathbf{LabK}$, in the following order:

$\wedge_\mathsf{L}$: Apply this step to each topmost sequent $\Gamma' \Rightarrow \Delta'$, containing in the antecedent formulas having $\wedge$ as their outermost logical connective. Each such sequent will have the following form, for some $m > 0$, and where we explicitly display all the $\wedge$-formulas in the antecedent:

$$x_1 : \psi_1 \wedge \zeta_1, \dots, x_m : \psi_m \wedge \zeta_m, \Gamma'' \Rightarrow \Delta'.$$

We write on top of it the following (single) sequent:

$$x_1 : \psi_1, x_1 : \zeta_1, \dots, x_m : \psi_m, x_m : \zeta_m, \Gamma'' \Rightarrow \Delta'.$$

$\wedge_\mathsf{R}$: Apply this step to each topmost sequent $\Gamma' \Rightarrow \Delta'$, containing in the succedent formulas having $\wedge$ as their outermost logical connective. Each such sequent will have the following form, for some $m > 0$, and where we explicitly display all the $\wedge$-formulas in the succedent:

$$\Gamma' \Rightarrow \Delta'', x_1 : \psi_1 \wedge \zeta_1, \dots, x_m : \psi_m \wedge \zeta_m.$$

Note that applying $\wedge_\mathsf{R}$ would result in adding on top of it two sequents, yielding two branches in the tree. Applying the rule m times to cover every formula in the sequent above originates 2^m new sequents of the form:

$$\Gamma' \Rightarrow \Delta'', x_1 : \xi_1, \dots, x_m : \xi_m$$

where each $\xi_i \in \{\psi_i, \zeta_i\}$. Write every possible sequent of that form on top of $\Gamma' \Rightarrow \Delta'$.

$\vee_\mathsf{L}$: Apply this step to each topmost sequent $\Gamma' \Rightarrow \Delta'$, containing in the antecedent formulas having $\vee$ as their outermost logical connective. Each such sequent will have the following form, for some $m > 0$, and where we explicitly display all the $\vee$-formulas in the antecedent:

$$x_1 : \psi_1 \vee \zeta_1, \dots, x_m : \psi_m \vee \zeta_m, \Gamma'' \Rightarrow \Delta'.$$

Following the reasoning from Stage 2, write every 2^m possible sequents of the form

$$x_1 : \xi_1, \dots, x_m : \xi_m, \Gamma'' \Rightarrow \Delta'$$

where $\xi_i \in \{\psi_i, \zeta_i\}$, on top of $\Gamma' \Rightarrow \Delta'$.

$\vee_\mathsf{R}$: Apply this step to each topmost sequent $\Gamma' \Rightarrow \Delta'$, containing in the succedent formulas having $\vee$ as their outermost logical connective. Each such sequent will have the following form, for some $m > 0$, and where we explicitly display all the $\vee$-formulas in the succedent:

$$\Gamma' \Rightarrow \Delta'', x_1 : \psi_1 \vee \zeta_1, \dots, x_m : \psi_m \vee \zeta_m.$$

We write on top of it the following (single) sequent:

$$\Gamma' \Rightarrow \Delta'', x_1 : \psi_1, x_1 : \zeta_1, \ldots, x_m : \psi_m, x_m : \zeta_m.$$

$\to_\mathsf{L}$: Apply this step to each topmost sequent $\Gamma' \Rightarrow \Delta'$, containing in the antecedent formulas having $\to$ as their outermost logical connective. Each such sequent will have the following form, for some $m > 0$, and where we explicitly display all the $\to$-formulas in the antecedent:

$$x_1 : \psi_1 \to \zeta_1, \ldots, x_m : \psi_m \to \zeta_m, \Gamma'' \Rightarrow \Delta'.$$

Each application of $\to_\mathsf{L}$ gives rise to two sequents where one of them has $x_i : \zeta_i$ in the antecedent and the other has $x_i : \psi_i$ in the succedent. We write on top of $\Gamma' \Rightarrow \Delta'$ every sequent of the form:

$$\begin{aligned} x_1 : \zeta_1, \ldots, x_m : \zeta_m, \Gamma'' &\Rightarrow \Delta' \\ x_{i_1} : \zeta i_1, \ldots, x_{i_k} : \zeta i_k, \Gamma'' &\Rightarrow \Delta', x_{i_{k+1}} : \psi_{i_{k+1}}, \ldots, x_{i_m} : \psi_{i_m} \\ \Gamma'' &\Rightarrow \Delta', x_1 : \psi_1, \ldots, x_m : \psi_m \end{aligned}$$

where $\{i_1, \ldots, i_k\}$ is any non-empty proper subset of $\{1, \ldots, m\}$ and $\{i_{k+1}, \ldots, i_m\}$ is the set $\{1, \ldots, m\} \setminus \{i_1, \ldots, i_k\}$. Note that there are $2^m - 2$ possibilities for $\{i_1, \ldots, i_k\}$, meaning that in total we have 2^m sequents corresponding to applying the $\to_\mathsf{L}$ rule m times.

$\to_\mathsf{R}$: Apply this step to each topmost sequent $\Gamma' \Rightarrow \Delta'$, containing in the succedent formulas having $\to$ as their outermost logical connective. Each such sequent will have the following form, for some $m > 0$, and where we explicitly display all the $\to$-formulas in the succedent:

$$\Gamma' \Rightarrow \Delta'', x_1 : \psi_1 \to \zeta_1, \ldots, x_m : \psi_m \to \zeta_m.$$

We write on top of it the following (single) sequent:

$$x_1 : \psi_1, \ldots, x_m : \psi_m, \Gamma' \Rightarrow \Delta'', x_1 : \zeta_1, \ldots, x_m : \zeta_m.$$

$\Box_\mathsf{L}$: Apply this step to each topmost sequent $\Gamma' \Rightarrow \Delta'$, containing in the antecedent formulas having $\Box$ as their outermost logical connective. Each such sequent will have the following form, for some $m > 0$, and where we explicitly display all the $\Box$-formulas in the antecedent:

$$x_1 R y_{11}, \ldots, x_1 R y_{1k_1}, \ldots, x_m R y_{m1}, \ldots, x_m R y_{mk_m}, x_1 : \Box\,\psi_1, \ldots, x_m : \Box\,\psi_m, \Gamma'' \Rightarrow \Delta'.$$

Write on top of $\Gamma' \Rightarrow \Delta'$ the following sequent:

$$\begin{aligned} &x_1 R y_{11}, \ldots, x_1 R y_{1k_1}, \ldots, x_m R y_{m1}, \ldots, x_m R y_{mk_m}, x_1 : \Box\,\psi_1, \ldots, x_m : \Box\,\psi_m, \\ &\qquad y_{11} : \psi_1, \ldots, y_{1k_1} : \psi_1, \ldots, y_{m1} : \psi_m, \ldots, y_{mk_m} : \psi_m, \Gamma'' \Rightarrow \Delta'. \end{aligned}$$

$\Diamond_\mathsf{R}$: Apply this step to each topmost sequent $\Gamma' \Rightarrow \Delta'$, containing in the succedent formulas having $\Diamond$ as their outermost logical connective. Each such sequent will have the following form, for some

$m > 0$, and where we explicitly display all the $\Diamond$-formulas in the succedent:

$$x_1Ry_{11}, \dots, x_1Ry_{1k_1}, \dots, x_mRy_{m1}, \dots, x_mRy_{mk_m}, \Gamma' \Rightarrow \Delta'', x_1 : \Diamond\,\psi_1, \dots, x_m : \Diamond\,\psi_m.$$

Write on top of $\Gamma' \Rightarrow \Delta'$ the following sequent:

$$x_1Ry_{11}, \dots, x_1Ry_{1k_1}, \dots, x_mRy_{m1}, \dots, x_mRy_{mk_m}, \Gamma' \Rightarrow \Delta'', x_1 : \Diamond\,\psi_1, \dots, x_m : \Diamond\,\psi_m,$$
$$y_{11} : \psi_1, \dots, y_{1k_1} : \psi_1, \dots, y_{m1} : \psi_m, \dots, y_{mk_m} : \psi_m.$$

$\Box_{\mathsf{R}}$: Apply this step to each topmost sequent $\Gamma' \Rightarrow \Delta'$, containing in the succedent formulas having $\Box$ as their outermost logical connective. Each such sequent will have the following form, for some $m > 0$, and where we explicitly display all the $\Box$-formulas in the succedent:

$$\Gamma' \Rightarrow \Delta'', x_1 : \Box\,\psi_1, \dots, x_m : \Box\,\psi_m.$$

Let $y_1, \dots, y_m$ be variables which have not appeared in the reduction tree up to Stage n, that is, that satisfy the variable condition in $\Box_{\mathsf{R}}$. Write on top of $\Gamma' \Rightarrow \Delta'$ the following sequent:

$$x_1Ry_1, \dots, x_mRy_m, \Gamma' \Rightarrow \Delta'', y_1 : \psi_1, \dots, y_m : \psi_m.$$

$\Diamond_{\mathsf{L}}$: Apply this step to each topmost sequent $\Gamma' \Rightarrow \Delta'$, containing in the antecedent formulas having $\Diamond$ as their outermost logical connective. Each such sequent will have the following form, for some $m > 0$, and where we explicitly display all the $\Diamond$-formulas in the antecedent:

$$x_1 : \Diamond\,\psi_1, \dots, x_m : \Diamond\,\psi_m, \Gamma'' \Rightarrow \Delta'.$$

Let $y_1, \dots, y_m$ be variables which have not appeared in the reduction tree up to Stage n, that is, that satisfy the variable condition in $\Diamond_{\mathsf{L}}$. Write on top of $\Gamma' \Rightarrow \Delta'$ the following sequent:

$$x_1Ry_1, \dots, x_mRy_m, y_1 : \psi_1, \dots, y_m : \psi_m, \Gamma'' \Rightarrow \Delta'.$$

Then, for each leaf sequent, check if it is initial. If every such sequent is initial, the construction stops and the resulting tree gives rise to a derivation for the sequent at the root. Otherwise, for each non initial leaf sequents to which no rule can be applied, write the sequent on top of itself. Restart the algorithm by perform all the rules again, starting in $\wedge_{\mathsf{L}}$. This process is schematized in Figure 3.1. Notation $(\star)$ in Figure 3.1 stands for each of the logical rules in $\mathbf{LabK}$ according to the order in which they were specified above.

Constructing the countermodel.
If the resulting reduction tree is infinite, then by Kőnig's lemma[2], it has an infinite branch $\mathcal{B}^{\times} = (\Gamma_i \Rightarrow \Delta_i)_{i<\omega}$, such that $\Gamma \Rightarrow \Delta \equiv \Gamma_0 \Rightarrow \Delta_0$. We will $\mathcal{B}^{\times}$ use to construct the countermodel. Consider the sets:

[2] Kőnig's lemma states that every connected, locally finite, infinite graph has an infinite path.

input: sequent $\Gamma \Rightarrow \Delta$;

Place $\Gamma \Rightarrow \Delta$ at the root;
If this sequent is initial, stop;

Stages $1-10$:
For $\star \in \{\wedge_{\mathsf{L}}, \wedge_{\mathsf{R}}, \vee_{\mathsf{L}}, \vee_{\mathsf{R}}, \rightarrow_{\mathsf{L}}, \rightarrow_{\mathsf{R}}, \Box_{\mathsf{L}}, \Diamond_{\mathsf{R}}, \Box_{\mathsf{R}}, \Diamond_{\mathsf{L}}\}$ apply step $\star$;

If every leaf is an initial sequent, stop;
For each non initial leaf, such that none of the Stages can be applied to the sequent, write it on top of itself;
Go to Stage 1;

output: a tree with $\Gamma \Rightarrow \Delta$ at the root.

Figure 3.1: Algorithm for the construction of the reduction tree in $\mathbf{LabK}$- non terminating.

$$\mathbf{\Gamma} = \bigcup_{i \geq 0} \Gamma_i \qquad \mathbf{\Delta} = \bigcup_{i \geq 0} \Delta_i$$

The countermodel is given by $\mathcal{M}^\times = (W^\times, R^\times, [\![\cdot]\!]^\times)$ where:

- The set of states $W^\times$ consists of the labels in $\Gamma \Rightarrow \Delta$ plus the labels appearing in the relation atoms in $\mathbf{\Gamma}$;
- The accessibility relation $R^\times$ is given by the relational atoms in Γ, that is, xRy is in $\mathbf{\Gamma}$ if and only if $(x, y) \in R^\times$;
- The valuation $[\![\cdot]\!]^\times$ is such that for any $x \in W^\times$, and p propositional symbol, we have that $x \in [\![p]\!]^\times$ if and only if $x : p$ is in $\mathbf{\Gamma}$.

Let $\rho^\times$ be the interpretation such that for every label x appearing in $\mathcal{B}^\times$ we have $\rho^\times(x) = x$.

Remark 1. Note that the definition of the valuation is coherent since, for every atomic formula $x : p$, it is never the case that $x : p \in \mathbf{\Gamma}$ and $x : p \in \mathbf{\Delta}$ happen simultaneously. If this were the case, there would some sequent along the branch where $x : p$ appears both in the antecedent and the succedent, that is, there would be an initial sequent in $\mathcal{B}^\times$, contradicting the fact that this is an infinite branch.

Lemma 3.24 (Truth Lemma)**.** Let $\mathcal{M}^\times$ be the model and $\rho^\times$ the interpretation constructed as described above. The following hold:

$$\text{For every } yRz \in \mathbf{\Gamma} \text{ we have that } \mathcal{M}^\times, \rho^\times \Vdash yRz; \tag{3.1}$$

$$\text{For every } x : \phi \in \mathbf{\Gamma} \text{ we have that } \mathcal{M}^\times, \rho^\times \Vdash x : \phi; \tag{3.2}$$

$$\text{For every } x : \phi \in \mathbf{\Delta} \text{ we have that } \mathcal{M}^\times, \rho^\times \nVdash x : \phi. \tag{3.3}$$

Proof. Note that for every $yRz \in \mathbf{\Gamma}$ it follows from the definition of $R^\times$ that $\mathcal{M}, \rho^\times \Vdash yRz$. The proof is by induction on the weight w of ϕ.

(Base) If $w = 0$ we have $\phi \equiv \bot$.
Note that $x : \bot$ can not be in $\mathbf{\Gamma}$. If that were the case, there would be a sequent in $\mathcal{B}^{\times}$ which was the conclusion of $\bot_{\mathsf{L}}$, that is, there would be an initial sequent in $\mathcal{B}^{\times}$.
If $x : \bot$ is in $\mathbf{\Delta}$, by the definition of satisfiability, we have that $\mathcal{M}^{\times}, x \nVdash \bot$ hence $\mathcal{M}^{\times}, \rho^{\times} \nVdash x : \bot$.

(Step) If $w \geq 1$ consider the following sub-cases:

· ϕ is atomic
This case follows from the definition of the valuation underlying $\mathcal{M}^{\times}$ and from Remark 1.

· $\phi \equiv \psi \vee \zeta$
If $x : \psi \vee \zeta$ is in $\mathbf{\Gamma}$, the reduction which applies to $\psi \vee \zeta$ yields two branches in the tree, such that one of them belongs in $\mathcal{B}^{\times}$. Hence, either $x : \psi \in \mathbf{\Gamma}$ or $x : \zeta \in \mathbf{\Gamma}$. By the inductive hypothesis $\mathcal{M}^{\times}, x \Vdash \psi$ or $\mathcal{M}^{\times}, x \Vdash \zeta$, so $\mathcal{M}^{\times}, \rho^{\times} \Vdash x : \psi \vee \zeta$
If $x : \psi \vee \zeta$ is in $\mathbf{\Delta}$, it appears in the succedent of some sequent in $\mathcal{B}^{\times}$. Suppose its first occurrence is in Δ_i. Then, for some $j \geq i$ we have that $x : \psi, x : \zeta \in \Delta_j$. By the inductive hypothesis $\mathcal{M}^{\times}, x \nVdash \psi$ and $\mathcal{M}^{\times}, x \nVdash \zeta$, hence $\mathcal{M}^{\times}, \rho^{\times} \nVdash x : \psi \vee \zeta$.

· $\phi \equiv \psi \wedge \zeta$
Proceed symmetrically to the last case. If $x : \psi \wedge \zeta$ is in $\mathbf{\Gamma}$ proceed as if $x : \psi \vee \zeta$ is in $\mathbf{\Delta}$. If $x : \psi \wedge \zeta$ is in $\mathbf{\Delta}$ proceed as if $x : \psi \vee \zeta$ is in $\mathbf{\Gamma}$.

· $\phi \equiv \psi \to \zeta$
If $x : \psi \to \zeta$ is in $\mathbf{\Gamma}$, by the reduction which applies to this formula, we get that either $x : \zeta \in \mathbf{\Gamma}$ or $x : \psi \in \mathbf{\Delta}$. By the inductive hypothesis we have that $\mathcal{M}^{\times}, x \Vdash \zeta$ and $\mathcal{M}, x \nVdash \psi$. Hence, $\mathcal{M}, \rho^{\times} \Vdash x : \psi \to \zeta$.
If $x : \psi \to \zeta$ is in $\mathbf{\Delta}$, it appears first in some Δ_i, the succedent of some sequent in $\mathcal{B}^{\times}$. Then, for some $j \geq i$ we have that $x : \psi \in \Gamma_j$ and $x : \zeta \in \Delta_j$. By the inductive step $\mathcal{M}^{\times}, x \Vdash \psi$ and $\mathcal{M}^{\times}, x \nVdash \zeta$, so $\mathcal{M}^{\times}, \rho^{\times} \nVdash \psi \to \zeta$.

· $\phi \equiv \Diamond \psi$
If $x : \Diamond \psi$ is in $\mathbf{\Gamma}$, by the reduction which applied to this formula, we can find $y : \psi \in \mathbf{\Gamma}$ for some y such that xRy is also in $\mathbf{\Gamma}$. By the inductive hypothesis $\mathcal{M}^{\times}, y \Vdash \psi$, hence $\mathcal{M}^{\times}, \rho^{\times} \Vdash x : \Diamond \psi$.
If $x : \Diamond \psi$ is in $\mathbf{\Delta}$ and there is no y such that a relational atom xRy can be found in $\mathbf{\Gamma}$, then $\mathcal{M}^{\times}, x \nVdash \Diamond \psi$ is true, hence $\mathcal{M}^{\times}, \rho^{\times} \nVdash x : \Diamond \psi$. Else, for any y such that xRy is in $\mathbf{\Gamma}$, by the construction of the reduction tree, we find $y : \phi$ in $\mathbf{\Delta}$ so $\mathcal{M}^{\times}, x \nVdash \Diamond \psi$. Hence $\mathcal{M}^{\times}, \rho^{\times} \nVdash x : \Diamond \psi$

· $\phi \equiv \Box \psi$
Proceed symmetrically to the last case. If $x : \Box \psi$ is in $\mathbf{\Gamma}$ proceed as if $x : \Diamond \psi$ is in $\mathbf{\Delta}$. If $x : \Box \psi$ is in $\mathbf{\Delta}$ proceed as if $x : \Diamond \psi$ is in $\mathbf{\Gamma}$.

QED.

As previously discussed, for a sequent $\Gamma \Rightarrow \Delta$ we have that either the reduction tree is finite, in which case we conclude the sequent is derivable, or it is infinite and we construct the model $\mathcal{M}^{\times}$. Finally, Lemma 3.24 shows that this model is indeed a countermodel for $\Gamma \Rightarrow \Delta$. Therefore, we have proven Theorem 3.23.

3.3.2 Semantic Completeness of other logics in the cube

The construction we described was suited for establishing semantical completeness for the logical system $\mathbf{K}$. Nevertheless, we can modify it and obtain a more general construction which covers every logic in the cube. We now resume the construction of the reduction tree in order to specify how to apply the rules for R in the proof search. Recall, that for any logic in the cube $\mathbf{X}$, the labelled system $\mathbf{LabX}$ can be obtained from $\mathbf{LabK}$ by adding to this calculus certain rules for R from Table 3.4, as described in Section 3.2.

Constructing the reduction tree.
Consider a logic $\mathbf{X}$ within the cube, the corresponding labelled system $\mathbf{LabX}$ and the corresponding m rules for R. Place a sequent $\Gamma \Rightarrow \Delta$ at the root of the tree. If this sequent it initial, the construction ends. Otherwise, continue the construction by performing the relevant steps corresponding to the propositional rules (Stages 1 to 6 from Figure 3.1). Next, consider m steps corresponding to the rules for R present in $\mathbf{LabX}$, applied as follows:

Ser: Consider all topsequents $\Gamma' \Rightarrow \Delta'$. Write on top of each one the sequent $\Gamma'' \Rightarrow \Delta'$ where Γ'' is obtained by adding to Γ' every relational atom of the form xRy with y a fresh variable, for each label x in $\Gamma' \Rightarrow \Delta'$.

Ref: Consider all topsequents $\Gamma' \Rightarrow \Delta'$. Write on top of each one the sequent $\Gamma'' \Rightarrow \Delta'$ where Γ'' is obtained by adding to Γ' the relational atom xRx, for each label x in $\Gamma' \Rightarrow \Delta'$.

Sym: Consider all topsequents $\Gamma' \Rightarrow \Delta'$. Write on top of each one the sequent $\Gamma'' \Rightarrow \Delta'$ where Γ'' is obtained by adding to Γ' the relational atom yRx, for each atom xRy in Γ'.

Trans: Consider all topsequents $\Gamma' \Rightarrow \Delta'$. Write on top of each one the sequent $\Gamma'' \Rightarrow \Delta'$ where Γ'' is obtained by adding to Γ' the relational atom xRz, for each pair of atoms xRy and yRz in Γ'.

Euc: Consider all topsequents $\Gamma' \Rightarrow \Delta'$. Write on top of each one the sequent $\Gamma'' \Rightarrow \Delta'$ where Γ'' is obtained by adding to Γ' the relational atom yRz, for each pair of atoms xRy and xRz in Γ'.

Finally, perform the steps corresponding to the rules with modal operators in the same order as in the algorithm from Figure 3.1. For each leaf sequent, check if it is initial. If every such sequent is initial, the construction stops and the resulting tree gives rise to a derivation for the sequent at the root. Otherwise, for each non initial leaf sequents to which no rule applied, write the sequent on top of itself[3]. Perform all the rules again, starting in $\wedge_{\mathsf{L}}$.

Consider the system $\mathbf{LabS4} = \mathbf{LabK}$+*Ref*+*Trans*. The algorithm for the construction of the reduction tree, following the process just described, is depicted in Figure 3.2. Note that the numbering of the stages differs from Figure 3.1 due to the addition of new rules.

Note that adding stages to the construction of the reduction tree does not affect the construction of the countermodel. Therefore, it is not necessary to make any adjustments to the definition of the countermodel but only to verify that for any sequent that is not derivable in $\mathbf{LabX}$, the structure constructed

[3]Note that this step omitted in the algorithm from Figure 3.2 due to the fact that *Ref* can always be applied to any labelled sequent.

input: sequent $\Gamma \Rightarrow \Delta$;

Place $\Gamma \Rightarrow \Delta$ at the root;
If this sequent is initial, stop;

Stages $1-6$:
For $\star \in \{\wedge_\mathsf{L}, \wedge_\mathsf{R}, \vee_\mathsf{L}, \vee_\mathsf{R}, \to_\mathsf{L}, \to_\mathsf{R}\}$ apply step $\star$;

Stage $7-8$:
For each leaf sequent apply step *Ref*, *Trans*;

Stage $9-12$:
For $\star \in \{\Box_\mathsf{L}, \Diamond_\mathsf{R}, \Box_\mathsf{R}, \Diamond_\mathsf{L}\}$ apply step $\star$;

If every leaf is an initial sequent, stop;
Else, go to Stage 1;

output: a tree with $\Gamma \Rightarrow \Delta$ at the root.

Figure 3.2: Algorithm for the construction of the reduction tree in $\mathbf{S4}$- non terminating.

from the failed proof search tree respects the frame conditions that characterize $\mathbf{X}$ are satisfied. For instance, consider a sequent $\Gamma \Rightarrow \Delta$, not derivable in $\mathbf{LabT}$, and let $\mathcal{M}^\times$ be the resulting countermodel for this sequent, constructed from an infinite branch $\mathcal{B}^\times$ of the reduction tree. We must verify that $R^\times$ is reflexive. Consider the multisets $\mathbf{\Gamma}$ and $\mathbf{\Delta}$ following the previous notation. Let $x \in W^\times$, then, x appears first in some Γ_i, that is, the antecedent of some $\Gamma_i \Rightarrow \Delta_i$ in $\mathcal{B}^\times$. The application of Stage ref in the construction of the reduction tree assures that the atom xRx was added to Γ_i, yielding a new sequent in that branch. Therefore, xRx is in $\mathbf{\Gamma}$, and hence $xR^\times x$. We have just proved a particular case of the following result:

Lemma 3.25. Let $\mathbf{X}$ be any logic in the cube and $\Gamma \Rightarrow \Delta$ a sequent not derivable in $\mathbf{LabX}$. Let $\mathcal{M}^\times$ be obtained from the algorithm presented above, and $\mathcal{F}^\times$ the underlying frame. Then $\mathcal{F}^\times$ is an $\mathsf{X}-$frame.

The last lemma, together with Lemma 3.24 give rise to the following theorem, which generalizes Theorem 3.23.

Theorem 3.26. *Let $\Gamma \Rightarrow \Delta$ be a sequent in the language of $\mathbf{LabX}$. Then, either this sequent is derivable in $\mathbf{LabX}$ or there is a Kripke model where it is not satisfiable.*

As a consequence of this theorem, we get an explicit (semantic) completeness result.

Theorem 3.27. *Let $\phi \in \mathcal{L}^{\Box,\Diamond}$. If $\Vdash_\mathsf{X} \phi$ then $\vdash_{\mathbf{LabX}} \Rightarrow x : \phi$.*

Proof. Let $\phi \in \mathcal{L}^{\Box,\Diamond}$. By Theorem 3.26 we have that if $\nvdash_{\mathbf{LabX}} \Rightarrow x : \phi$ then there exists a Kripke model $\mathcal{M}$, based on a frame $\mathcal{F}$ in X, such that $\mathcal{M}, x \nVdash \phi$, and thus $\mathcal{M} \nVdash \phi$. That is, if $\nvdash_{\mathbf{LabX}} \Rightarrow x : \phi$ then $\nVdash_\mathcal{F} \phi$, and hence $\nVdash_\mathsf{X} \phi$. By the counterpositve of this statement we conclude that if $\Vdash_\mathsf{X} \phi$ then $\vdash_{\mathbf{LabX}} \Rightarrow x : \phi$. QED.

To conclude this section, we present the following diagram which captures all the soundness and completeness results that were proven throughout this work, in particular it now includes the arrow corresponding to Theorem 3.27.

$$
\begin{tikzcd}
\Vdash_{\mathsf{X}} \phi \arrow[rr, Leftrightarrow, bend left, "1.41+1.46"] \arrow[dr, Rightarrow, bend left, "3.27"] & & \vdash_{\mathbf{X}} \phi \arrow[dl, Rightarrow, bend left, "3.22"] \\
& \vdash_{\mathbf{LabX}} \Rightarrow x : \phi \arrow[ul, Rightarrow, bend left, "3.21"] &
\end{tikzcd}
$$

Chapter 4

Decidability of Modal Logics

Decidability is one of the main properties that one should attempt to understand when studying a logical system. In informal terms, we consider a logical system as *decidable* if there exists an algorithm capable of determining whether any given formula is valid or not in that system, equivalently, if that formula is a *theorem* in that system. While this problem is not new, dating back to Leibniz in the 17th century, and his quest for a universal machine, it gained special attention in the 20th century due to several mathematicians, namely, David Hilbert. In 1928 Hilbert formally formulated the decision problem concerning first-order logic, which came to be known as *Hilbert's Entscheidungsproblem*. Gödel's incompleteness results, published in 1931, shed light into this problem, hinting at a negative answer. However, the definitive answer to the problem was given some years later by Alonzo Church [8] and Alan Turing, who proved it independently, heavily influenced by Gödel's work.

Modal logic lays somewhere between propositional logic and first order logic in terms of expressive power. However, it is not straightforward where it would fall regarding decidability. Mathematicians such as Kripke and Arthur Prior, with a specific focus on tense logic, have tackled this intricate problem. It is noteworthy to mention that all the logics within the modal cube are known to be decidable. We can find detailed proofs of the decidability of $\mathbf{K}$ in [34] via several methods, such as filtration. Proofs of decidability of $\mathbf{S4}$ can be found in [26] by reduction to other problems and in [6] using nested sequents.

In this chapter, we focus on proof-theoretical methods to establish decidability. Specifically, in Section 4.1 we start from the algorithm we used in Section 3.3 as a tool to prove semantic completeness of the labelled system $\mathbf{LabK}$, and refine it to provide a concrete method capable of determining the validity of any formula within the basic modal system $\mathbf{K}$. Furthermore, in Section 4.2 we extend this method to the logic $\mathbf{S4}$, offering insights into the potential obstacles and solutions that this approach presents for the question of decidability across various logics within the modal cube. While the decidability results concerning the logics in the cube are known in the literature, the method we follow in this chapter, adapted from the work in [30] and [31], together with proofs of their termination, represent an original contribuition of this work.

4.1 A decision algorithm for $\mathbf{K}$

The algorithm presented in Section 3.3 hints at a possible solution for the validity problem for $\mathbf{K}$ and the other logics in the cube. However, one must note that this algorithm only allows to build a semi-decidable decision procedure. In fact, if we start the algorithm from a given formula and it does not stop, we have

no way of determining if it would eventually terminate (in which case the formula would be valid) or if it will go on infinitely (resulting in an infinite countermodel). Therefore, aiming at a proof of decidability of **K**, we must alter the construction of the reduction tree so that it is always finite.
The first challenge to termination arises when no rule can be applied to a sequent. In this case, the leaf sequent to which no rule can be applied would be re-written on top of itself indeterminately, giving rise to an infinite branch. To address this, a straightforward solution is to simply avoid rewriting this sequent onto itself repeatedly. This adjustment ensures that the construction concludes along a branch after exhaustively attempting to perform all possible Stages. To formalize this concept, we introduce the notion of a *saturated branch*, which, intuitively, denotes a branch where every possible rule has been systematically applied to each sequent.

Definition 4.1. Consider a branch $\mathcal{B}$ in a reduction tree up to a sequent $\Gamma \Rightarrow \Delta$. Let $\downarrow\Gamma$ (resp. $\downarrow\Delta$) denote the union of all formulas occuring in the antecedents (resp. succedents) of every sequent in $\mathcal{B}$ from the root up to $\Gamma \Rightarrow \Delta$. We say that $\mathcal{B}$ is *saturated* (up to $\Gamma \Rightarrow \Delta$) if $\Gamma \Rightarrow \Delta$ is not initial and the following hold:

($\wedge_\mathsf{L}$) If $x : \psi \wedge \zeta$ is in $\downarrow\Gamma$ then both $x : \psi$ and $x : \zeta$ are in $\downarrow\Gamma$;

($\wedge_\mathsf{R}$) If $x : \psi \wedge \zeta$ is in $\downarrow\Delta$ then either $x : \psi$ or $x : \zeta$ are in $\downarrow\Delta$;

($\vee_\mathsf{L}$) If $x : \psi \vee \zeta$ is in $\downarrow\Gamma$ then either $x : \psi$ or $x : \zeta$ is in $\downarrow\Gamma$;

($\vee_\mathsf{R}$) If $x : \psi \vee \zeta$ is in $\downarrow\Delta$ then both $x : \psi$ and $x : \zeta$ are in $\downarrow\Delta$;

($\rightarrow_\mathsf{L}$) If $x : \psi \rightarrow \zeta$ is in $\downarrow\Gamma$ then either $x : \zeta$ is in $\downarrow\Gamma$ or $x : \psi$ is in $\downarrow\Delta$;

($\rightarrow_\mathsf{R}$) If $x : \psi \rightarrow \zeta$ is in $\downarrow\Delta$ then $x : \psi$ is in $\downarrow\Gamma$ and $x : \zeta$ is in $\downarrow\Delta$;

($\Box_\mathsf{L}$) If $x : \Box\phi$ is in $\downarrow\Gamma$, then for all xRy in $\downarrow\Gamma$ we have that $y : \phi$ is in $\downarrow\Gamma$;

($\Box_\mathsf{R}$) If $x : \Box\phi$ is in $\downarrow\Delta$, then for some y, we have that xRy is in $\downarrow\Gamma$ and $y : \phi$ is in $\downarrow\Delta$;

($\Diamond_\mathsf{L}$) If $x : \Diamond\phi$ is in $\downarrow\Gamma$, then for some y, we have that xRy and $y : \phi$ is in $\downarrow\Delta$;

($\Diamond_\mathsf{R}$) If $x : \Diamond\phi$ is in $\downarrow\Delta$, then for all xRy in $\downarrow\Gamma$ we have that $y : \phi$ is in $\downarrow\Delta$.

The informal idea we discussed previously translates into altering the algorithm from Figure 3.1 in such a way that the construction of the reduction tree continues until we only have initial sequents at the leaves or until we reach a saturated branch. As we can see in the following derivation, this would not be sufficient to make the construction finite as the Stages corresponding to rules which do not delete the principal formulas could still pose an issue. Here follows an example:

$$
\cfrac{\cfrac{\cfrac{\cfrac{\vdots}{xRy, x : \Box\phi, y : \phi, y : \phi, y : \phi, \Gamma \Rightarrow \Delta}\,\Box_\mathsf{L}}{xRy, x : \Box\phi, y : \phi, y : \phi, \Gamma \Rightarrow \Delta}\,\Box_\mathsf{L}}{xRy, x : \Box\phi, y : \phi, \Gamma \Rightarrow \Delta}\,\Box_\mathsf{L}}{xRy, x : \Box\phi, \Gamma \Rightarrow \Delta}\,\Box_\mathsf{L}
$$

Unlike the other rules, the rules $\Box_{\mathsf{L}}$ and $\Diamond_{\mathsf{R}}$ do not remove the principal formula from the premisses[1]. Moreover, since no rule removes relational atoms, $\Box_{\mathsf{L}}$ and $\Diamond_{\mathsf{R}}$ can be applied as many times as desired to the same pairs of formulas. Hence, to obtain termination we must modify the stages corresponding to rules $\Box_{\mathsf{L}}$ and $\Diamond_{\mathsf{R}}$ in the algorithm from Figure 3.1. Informally, we must ensure that if there is a top sequent $xRy, x : \Box\,\phi, \Gamma' \Rightarrow \Delta'$ such that $y : \phi$ is already in the antecedent of some sequent along that branch, i.e. if the saturation condition $\Box_{\mathsf{L}}$ is already met, then we skip the application of rule $\Box_{\mathsf{L}}$ to $xRy, x : \Box\,\phi$. Analogously, if there is a top sequent $xRy, \Gamma' \Rightarrow \Delta', x : \Diamond\phi$ such that $y : \phi$ is already in the succedent of some sequent along that branch, then we skip the application of rule $\Diamond_{\mathsf{R}}$ to $xRy, x : \Box\,\phi$. Consider the new version of the $\Box_{\mathsf{L}}$ and $\Diamond_{\mathsf{R}}$ step (which give rise to new Stages $7'$ and $8'$ in the algorithm for $\mathbf{K}$ in Figure 4.1):

$\Box_{\mathsf{L}}'$: This step might be applied to each topmost sequent $\Gamma' \Rightarrow \Delta'$, containing in the antecedent formulas having $\Box$ as their outermost logical connective. Each such sequent will have the following form, for some $m > 0$, and where we explicitly display all the $\Box$-formulas in the antecedent:

$$x_1Ry_{11}, \ldots, x_1Ry_{1k_1}, \ldots, x_mRy_{m1}, \ldots, x_mRy_{mk_m}, x_1 : \Box\,\psi_1, \ldots, x_m : \Box\,\psi_m, \Gamma'' \Rightarrow \Delta'.$$

If for all y_{ik_j}, we have that $y_{ik_j} : \psi_i$ is already in the antecedent of some sequent along this branch, skip this Stage. Otherwise, let Σ be the set of formulas $y_{ik_j} : \psi_i \in \{y_{11} : \psi_1, \ldots, y_{1k_1} : \psi_1, \ldots, y_{m1} : \psi_m, \ldots, y_{mk_m} : \psi_m\}$ such that $y_{ik_j} : \psi_i$ is not in the antecedent of any sequent along this branch. Write on top of $\Gamma' \Rightarrow \Delta'$ the sequent

$$x_1Ry_{11}, \ldots, x_1Ry_{1k_1}, \ldots, x_mRy_{m1}, \ldots, x_mRy_{mk_m}, x_1 : \Box\,\psi_1, \ldots, x_m : \Box\,\psi_m, \Sigma, \Gamma'' \Rightarrow \Delta'.$$

$\Diamond_{\mathsf{R}}\prime$: This step might be applied to each topmost sequent $\Gamma' \Rightarrow \Delta'$, containing in the succedent formulas having $\Diamond$ as their outermost logical connective. Each such sequent will have the following form, for some $m > 0$, and where we explicitly display all the $\Diamond$-formulas in the succedent:

$$x_1Ry_{11}, \ldots, x_1Ry_{1k_1}, \ldots, x_mRy_{m1}, \ldots, x_mRy_{mk_m}, \Gamma' \Rightarrow \Delta'', x_1 : \Diamond\,\psi_1, \ldots, x_m : \Diamond\,\psi_m.$$

If for all y_{ik_j}, we have that $y_{ik_j} : \psi_i$ is already in the succedent of some sequent along this branch, skip this Stage. Otherwise, let Σ be the set of formulas $y_{ik_j} : \psi_i \in \{y_{11} : \psi_1, \ldots, y_{1k_1} : \psi_1, \ldots, y_{m1} : \psi_m, \ldots, y_{mk_m} : \psi_m\}$ such that $y_{ik_j} : \psi_i$ is not in the succedent of any sequent along this branch. Write on top of $\Gamma' \Rightarrow \Delta'$ the sequent

$$x_1Ry_{11}, \ldots, x_1Ry_{1k_1}, \ldots, x_mRy_{m1}, \ldots, x_mRy_{mk_m}, \Gamma' \Rightarrow \Delta'', \Sigma, x_1 : \Diamond\,\psi_1, \ldots, x_m : \Diamond\,\psi_m.$$

We now describe the decision algorithm for $\mathbf{K}$, by modifying the algorithm from Section 3.3.

[1]In this work we opted for defining the Stages of the algorithms as well as the rules in a non-cumulative way. This means that we allow (principal) formulas to be deleted by application of the rules. This choice simplifies the specification of the terminating algorithm for $\mathbf{K}$ and subsequently, for $\mathbf{S4}$. If we chose to always keep the principal formulas in the sequents, the modification being made to Stages $\Box_{\mathsf{L}}$ and $\Diamond_{\mathsf{R}}$ should be made to every Stage, adding more difficulty to the comprehension of this algorithm. This choice implies that when analyzing the reduction tree we look not only at the leaf sequents but at the branches they belong to, in particular, we must define saturation with respect to branches and not sequents.

Modified construction of the reduction tree for $\Gamma \Rightarrow \Delta$.
Begin by placing $\Gamma \Rightarrow \Delta$ at the root of the tree. If this sequent it initial, the construction ends. Otherwise, the construction will continue by writing on top of every non-initial top sequent the result of performing a step corresponding to at least one (possibly several) root first application of a rule of $\mathbf{LabK}$, where the application of $\Box_{\mathsf{L}}$ and $\Diamond_{\mathsf{R}}$ follows the restrictions above. If every resulting top sequent is initial the construction stops and the resulting tree gives rise to a derivation for the sequent at the root. If there is a top sequent which is not initial but none of the steps applied, the construction stops and the search for a proof of $\Gamma \Rightarrow \Delta$ has *failed*. Furthermore, the branch going from the root of the tree up to this sequent is saturated, and will be used to construct the countermodel. Otherwise, perform all the rules again, starting from $\wedge_{\mathsf{L}}$. The modified algorithm is schematized in Figure 4.1.

---

input: sequent $\Gamma \Rightarrow \Delta$;

Place $\Gamma \Rightarrow \Delta$ at the root;
If this sequent is initial, stop;

Stages $1-6$:
    For $\star \in \{\wedge_{\mathsf{L}}, \wedge_{\mathsf{R}}, \vee_{\mathsf{L}}, \vee_{\mathsf{R}}, \to_{\mathsf{L}}, \to_{\mathsf{R}}\}$ apply step $\star$;

Stage $7'-8'$:
    Apply steps $\Box'_{\mathsf{L}}$, $\Diamond'_{\mathsf{R}}$;

Stages $9-10$:
    Apply steps $\Box_{\mathsf{R}}$, $\Diamond_{\mathsf{L}}$;

If every leaf is an initial sequent, stop;
If there is a non initial leaf sequent to which none of the stages were applied, stop;
Else, go to Stage 1;

output: a finite tree with $\Gamma \Rightarrow \Delta$ at the root. The tree is derivation if and only if the sequent is valid.

Figure 4.1: Algorithm for the construction of the reduction tree for $\mathbf{LabK}$- terminating.

Example 4.2. Consider the sequent $\Rightarrow x : (\Box r \wedge \Diamond(p \to q)) \to \Box\Box(p \wedge q)$. The following corresponds to one branch of the reduction tree obtained via the algorithm from Figure 4.1:

$$
\begin{prooftree}
\AxiomC{$xRy_1, xRy_2, y_1Ry_3, x : \Box r, y_1 : r, y_2 : r, y_2 : q \Rightarrow y_3 : p$}
\AxiomC{$\vdots$}
\RightLabel{$\wedge_{\mathsf{R}}$}
\BinaryInfC{$xRy_1, xRy_2, y_1Ry_3, x : \Box r, y_1 : r, y_2 : r, y_2 : q \Rightarrow y_3 : p \wedge q$}
\RightLabel{$\Box_{\mathsf{R}}$}
\UnaryInfC{$xRy_1, xRy_2, x : \Box r, y_1 : r, y_2 : r, y_2 : q \Rightarrow y_1 : \Box(p \wedge q)$}
\RightLabel{$\Box_{\mathsf{L}}$}
\UnaryInfC{$xRy_1, xRy_2, x : \Box r, y_2 : q \Rightarrow y_1 : \Box(p \wedge q)$}
\AxiomC{$\vdots$}
\RightLabel{$\to_{\mathsf{L}}$}
\BinaryInfC{$xRy_1, xRy_2, x : \Box r, y_2 : p \to q \Rightarrow y_1 : \Box(p \wedge q)$}
\RightLabel{$\Diamond_{\mathsf{L}}$}
\UnaryInfC{$xRy_1, x : \Box r, x : \Diamond(p \to q) \Rightarrow y_1 : \Box(p \wedge q)$}
\RightLabel{$\Box_{\mathsf{R}}$}
\UnaryInfC{$x : \Box r, x : \Diamond(p \to q) \Rightarrow x : \Box\Box(p \wedge q)$}
\RightLabel{$\wedge_{\mathsf{L}}$}
\UnaryInfC{$x : \Box r \wedge \Diamond(p \to q) \Rightarrow x : \Box\Box(p \wedge q)$}
\RightLabel{$\to_{\mathsf{R}}$}
\UnaryInfC{$\Rightarrow x : (\Box r \wedge \Diamond(p \to q)) \to \Box\Box(p \wedge q)$}
\end{prooftree}
$$

It is clear that the topsequent of the left branch of the tree is not initial, but no rule can be applied to it.

Let's denote this sequent by $\Gamma \Rightarrow \Delta$. Then $\downarrow\Gamma = \{x : \Box r \wedge \Diamond(p \to q), x : \Box r, x : \Diamond(p \to q), y_2 : p \to q, y_2 : q,$ $y_1 : r, y_2 : r, xRy_1, xRy_2, y_1Ry_3\}$ and $\downarrow\Delta = \{x : (\Box r \wedge \Diamond(p \to q)) \to \Box\Box(p \wedge q), x : \Box\Box(p \wedge q),$ $y_1 : \Box(p \wedge q), y_3 : p \wedge q, y_3 : p\}$. Let's check that the set $\downarrow\Delta$ satisfies all the conditions in Definiton 4.1:

- $x : (\Box r \wedge \Diamond(p \to q))$ is in $\downarrow\Delta$, $x : \Box r \wedge \Diamond(p \to q)$ is in $\downarrow\Gamma$ and $x : \Box\Box(p \to q)$ is in $\downarrow\Delta$;
- $x : \Box\Box(p \wedge q)$ is in $\downarrow\Delta$, the relational atom xRy_1 is in $\downarrow\Gamma$ and $y_1 : \Box(p \wedge q)$ is in $\downarrow\Delta$;
- $y_1 : \Box(p \wedge q)$ is in $\downarrow\Delta$, the relational atom y_1Ry_3 is in $\downarrow\Gamma$ and $y_3 : p \wedge q$ is in $\downarrow\Delta$;
- $y_3 : p \wedge q$ is in $\downarrow\Delta$ and $y_3 : p$ is in $\downarrow\Delta$.

Similarly, it is easy to check that every formula in $\downarrow\Gamma$ also fulfills every condition. Hence the branch is saturated and the construction of the reduction tree has terminated. Whence the formula at the root is not valid.

Constructing the countermodel.
Consider some sequent $\Gamma \Rightarrow \Delta$ for which the algorithm outputs a finite reduction tree which is not a derivation. In Section 3.3, the countermodel for this sequent would be obtained from an infinite branch in the reduction tree constructed following our algorithm. However, note that in the modified algorithm, that branch will be saturated and finite. Consider such branch $\mathcal{B}^\times = (\Gamma_i \Rightarrow \Delta_i)_{0 \leq i \leq k}$ where $\Gamma \Rightarrow \Delta \equiv \Gamma_0 \Rightarrow \Delta_0$. We use $\mathcal{B}^\times$ to construct the countermodel. Let $\mathcal{M}^\times = (W^\times, R^\times, [\![\cdot]\!]^\times)$ where the interpretation $\rho^\times$, set of worlds $W^\times$, accessibility relation $R^\times$ and valuation $[\![\cdot]\!]^\times$ are constructed as in Section 3.3.

Lemma 4.3 (Truth Lemma for **LabK**)**.** Let $\mathcal{M}^\times$ be the model and $\rho^\times$ the interpretation constructed from a saturated branch $\mathcal{B}^\times$ in the reduction tree built for $\Gamma \Rightarrow \Delta$. Furthermore, let $\mathbf{\Gamma}$ (resp. $\mathbf{\Delta}$) denote the union of the antecedents (resp. succedents) of the sequents in $\mathcal{B}^\times$. The following hold:

$$\text{For every } x : \phi, yRz \in \mathbf{\Gamma} \text{ we have that } \mathcal{M}^\times, \rho^\times \Vdash x : \phi \text{ and } \mathcal{M}^\times, \rho^\times \Vdash yRz; \tag{4.1}$$

$$\text{For every } x : \phi \in \mathbf{\Delta} \text{ we have that } \mathcal{M}^\times, \rho^\times \nVdash x : \phi; \tag{4.2}$$

Proof. The proof is by induction on the weight of ϕ. Note that the fact that we avoid rewriting a sequent on top of itself if no rule applies to it does not affect the construction of the countermodel, since this does not affect the set of formulas that appear in the corresponding branch. Furthermore, since the only changes in the construction of the reduction tree are regarding $\Box_\mathsf{L}$ and $\Diamond_\mathsf{R}$, we only need to show that for every formula $x : \Box\psi \in \mathbf{\Gamma}$ we have $\mathcal{M}^\times, \rho^\times \Vdash x : \Box\psi$, and that for every formula $x : \Diamond\psi \in \mathbf{\Delta}$ we have $\mathcal{M}^\times, \rho^\times \nVdash x : \Diamond\psi$. Let $x : \Box\psi \in \mathbf{\Gamma}$. If there is no y such that xRy, then $\mathcal{M}^\times, x \Vdash \Box\psi$ is vacuously true. Otherwise, take such y and let Γ_i be the antecedent of the first sequent were $x : \Box\psi$ and xRy appear simultaneously. If there is no formula $y : \psi \in\downarrow\Gamma_i$, then then we apply $\Box_\mathsf{L}$, hence for some $j > i$ we get that $y : \psi \in \Gamma_j$. Otherwise, we have that for some $j \leq i$, $y : \psi \in \Gamma_j$. In either case, $y : \psi \in \mathbf{\Gamma}$, hence, by the inductive hypothesis, $\mathcal{M}^\times, \rho^\times \Vdash y : \psi$. Therefore $\mathcal{M}^\times, \rho^\times \Vdash x : \Box\phi$. The proof for $x : \Diamond\psi \in \mathbf{\Delta}$ is analogous. QED.

Example 4.4. Recall the reduction tree from Example 4.2 and the sets $\downarrow\Gamma$ and $\downarrow\Delta$. We define $\mathcal{M}^\times$ as follows:

- $W^\times$ is the set of labels appearing in the root sequent or in the relational atoms in $\downarrow\Gamma$, that is, $W^\times = \{x, y_1, y_2, y_3\}$;
- The accessibility relation is given by the relational atoms in $\downarrow\Gamma$, hence, $R^\times = \{xRy_1, xRy_2, y_1Ry_3\}$;
- The only atomic formulas in $\downarrow\Gamma$ are $y_1 : r$, $y_2 : r$ and $y_2 : q$, hence $[\![r]\!] = \{y_1, y_2\}$, $[\![q]\!] = \{q\}$ and $[\![p]\!]$ is empty.

We represent $\mathcal{M}^\times$ as follows:

$$\begin{tikzcd}
 & y_1, \boldsymbol{r} \arrow[r] & y_3 \\
x \arrow[ur] \arrow[dr] & & \\
 & y_2, \boldsymbol{r}, \boldsymbol{q} &
\end{tikzcd}$$

Note that $\mathcal{M}^\times, x \not\Vdash (\Box r \wedge \Diamond(p \to q)) \to \Box\Box(p \wedge q)$ if and only if $\mathcal{M}^\times, x \Vdash \Box r \wedge \Diamond(p \to q)$ and $\mathcal{M}^\times, x \not\Vdash \Box\Box(p \wedge q)$. To check the first condition note that $\mathcal{M}^\times, x \Vdash \Box r$, since the only states accessible from x are y_1 and y_2, and we have that $y_1, y_2 \in [\![r]\!]$. Furthermore, $\mathcal{M}^\times, y_1 \Vdash p \to q$, since $\mathcal{M}^\times, y_1 \not\Vdash p$, and hence we have that $\mathcal{M}^\times, x \Vdash \Diamond(p \to q)$. To see the the second condition holds, note that y_3 is accessible from y_1 and $\mathcal{M}^\times, y_3 \not\Vdash p \wedge q$, hence $\mathcal{M}^\times, y_1 \not\Vdash \Box(p \wedge q)$, and therefore $\mathcal{M}^\times, x \not\Vdash \Box\Box(p \wedge q)$. Hence, we conclude that $\mathcal{M}^\times$ is indeed a countermodel to $(\Box r \wedge \Diamond(p \to q)) \to \Box\Box(p \wedge q)$, that is, $\not\Vdash_{\mathsf{K}} (\Box r \wedge \Diamond(p \to q)) \to \Box\Box(p \wedge q)$.

It remains to prove the following result, crucial to obtain a decision algorithm.

Proposition 4.5. The algorithm presented in Figure 4.1 always terminates.

Proof. In order to prove our result we show that every branch in the reduction tree outputted by the algorithm in Figure 4.1 is finite. We can show this is the case by proving two things:

1. There are only finitely many formulas (modulo labels) occurring in a branch.
   Let $x : \phi$ be the formula at the root of this branch. The number of subformulas of $x : \phi$ is bounded by the weight of ϕ. Furthermore, any rule in **LabK** applied to a formula ϕ labelled with some label x will introduce only subformulas of this formula, labelled with some label y, and having weight strictly smaller than the weight of ϕ. Hence, we can be more precise and state that if $x : \psi$ is the formula at the root of the tree, then the number of formulas (modulo labels) in the branch is not only finite, but bounded by the weight of ψ.

2. There are only finitely many labels in a branch.
   We always start the construction with a formula with a single label at the root of the tree. Apart from $\Box_{\mathsf{R}}$ and $\Diamond_{\mathsf{L}}$ no rule introduces new labels. However, the introduction of these new labels does not pose a problem since both these rules remove the principal formula and introduces a labelled formula having a strictly smaller weight. That is, we can apply these rules only once to a formula $x : \phi$. Hence, an application of $\Box_{\mathsf{R}}$ or $\Diamond_{\mathsf{L}}$ generates exactly one new label. Therefore, the number of labels in the branch is bounded by the number of boxes and diamonds of the formula at the root.

QED.

We get a (purely) proof-theoretical proof of the following results.

Theorem 4.6. *The basic modal logic* $\mathbf{K}$ *is decidable.*

Proof. Let $\phi \in \mathcal{L}^{\Box,\Diamond}$. By Proposition 4.5, by running the algorithm from Figure 4.1 on the sequent $\Rightarrow x : \phi$ we obtain a finite reduction tree.
If all the leaves of this tree are initial, then we have obtained a $\mathbf{LabK}$ derivation for $\Rightarrow x : \phi$. By Theorem 3.12 we conclude that ϕ is valid.
Otherwise, there is a branch of the reduction tree which is saturated. We construct a countermodel $\mathcal{M}^\times$ and an interpretation $\rho^\times$ from this branch. By Lemma 4.3, we have that $\mathcal{M}^\times, \rho^\times \not\Vdash x : \phi$, whence $\mathcal{M}^\times, x \not\Vdash \phi$. We conclude that ϕ is not valid and that $\mathcal{M}^\times$ is a countermodel for it. QED.

Because the countermodel is finite, we obtain the following corollary:

Corollary 4.7. The basic modal logic $\mathbf{K}$ has the finite model property.

4.2 A decision algorithm for S4

Similarly to what we did in Section 3.3, we would like to extend our decision algorithm to other logics in the cube. It is not as straightforward to adapt this algorithm as in the case of $\mathbf{K}$, as the structural rules play a crucial role in what could prevent the algorithm from terminating. The obstacles we need to overcome in order to have termination vary greatly according to the structural rules. In this section, we only show what happens in the case of $\mathbf{S4}$. Our starting point is the algorithm for $\mathbf{LabK}$ in Figure 4.1 to which we add *Ref* and *Trans* from Section 3.3. Recall Definition 4.1, we now redefine the notion of saturated branch by adding conditions that deal with reflexivity and transitivity.

Definition 4.8. We say that a branch $\mathcal{B}$ in a proof search tree up to $\Gamma \Rightarrow \Delta$ is saturated if $\Gamma \Rightarrow \Delta$ is not initial and the conditions $(\wedge_\mathsf{L})$ to $(\Diamond_\mathsf{R})$ from Definition 4.1 are met, plus the following:

(ref) : If x is a label in $\downarrow\Gamma$ or $\downarrow\Delta$ then xRx is in $\downarrow\Gamma$;

(trans) : If xRy, yRz are relational atoms in $\downarrow\Gamma$ then xRz is in $\downarrow\Gamma$;

The first clear obstacle to termination in the algorithm form Figure 3.2 is the possibility to endlessly create new relational atoms using reflexivity and transitivity. Therefore, we should modify how to apply these rules when constructing the reduction tree in order to avoid making copies of relational atoms that had been created before. Consider the updated formulations as follows:

Ref′: For each leaf sequent apply rule *Ref* with respect to a label x if and only if xRx is not already in the antecedent of the sequent;

Trans′: For each leaf sequent apply rule *Trans* to a pair of relational atoms xRy, yRz if and only if xRz is not already in the antecedent of the sequent;

However, as showcased in the following example, this is not enough to obtain termination.

Example 4.9. Consider the formula $\Box\Diamond p \to \bot$. This is not valid in $\mathbf{S4}$ since there exist models in the $\mathsf{t}\cup\mathsf{4}$ class of frames where the formula is not satisfiable. For instance, it is not satisfiable in the following reflexive and transitive countermodel $\mathcal{M}$:

$$
\begin{tikzcd}
x \arrow[loop left] \arrow[rr] & & y, \boldsymbol{p} \arrow[loop right]
\end{tikzcd}
$$

We have that $\mathcal{M}, x \Vdash \Box\Diamond p$ since $\mathcal{M}, y \Vdash p$ but $\mathcal{M}, x \nVdash \bot$, and so $\mathcal{M}, x \nVdash \Box\Diamond p \to \bot$. Since the formula is not valid, our algorithm should be able to output a failed reduction tree, in a finite number of steps. However, this is not the case:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{xRx, xRy, yRy, yRz, zRz, xRz, x : \Box\Diamond p, y : p, z : p \Rightarrow x : \bot}}{xRx, xRy, yRy, yRz, x : \Box\Diamond p, y : p, z : p \Rightarrow x : \bot}\,\mathit{Ref} + \mathit{Trans}}{xRx, xRy, yRy, x : \Box\Diamond p, y : p, y : \Diamond p \Rightarrow x : \bot}\,\Diamond_{\mathsf{L}}}{xRx, xRy, yRy, x : \Box\Diamond p, y : p \Rightarrow x : \bot}\,\Box_{\mathsf{L}}}{xRx, xRy, x : \Box\Diamond p, y : p \Rightarrow x : \bot}\,\mathit{Ref}}{xRx, x : \Box\Diamond p, x : \Diamond p \Rightarrow x : \bot}\,\Diamond_{\mathsf{L}}}{xRx, x : \Box\Diamond p \Rightarrow x : \bot}\,\Box_{\mathsf{L}}}{x : \Box\Diamond p \Rightarrow x : \bot}\,\mathit{Ref}}{\Rightarrow x : \Box\Diamond p \to \bot}\,\to_{\mathsf{R}}
$$

Note that when we reached the sequent $xRx, xRy, yRy, x : \Box\Diamond p, y : p, y : \Diamond p \Rightarrow x : \bot$ in the derivation, we could already extract the countermodel we showed above since the saturation conditions for every connective are already met at that point in the branch.

The last example shows one of the several obstacles that arise if we allow the rules for R in proof search. Essentially, even though the branch of the derivation was already saturated, we kept the construction going and ended up in a loop, which makes the tree infinite. This was not an issue with $\mathbf{LabK}$ due to the absence of structural rules but must now be prevented. Note that the same could happen with the propositional connectives, in fact, we allow the application of $\wedge_{\mathsf{L}}$ to a formula $x : \psi \wedge \zeta$ in the antecedent even if $x : \psi$, $x : \zeta$ are already in the antecedent of some sequent in that branch. However, this is not problematic since these rules are not generating new labels, and hence, while some steps might be redundant, they do not increase the complexity of the tree or the countermodel. The crucial aspect is that we prevent application of rules $\Box_{\mathsf{R}}$ and $\Diamond_{\mathsf{L}}$ to formulas that already meet the saturation conditions, that is, if we have $x : \Box\phi$ in the succedent of a leaf, we do not create a new label y so that xRy and $y : \psi$ if there was already z, in the succedent of some sequent along that branch, that met those conditions. Similarly for $\Diamond_{\mathsf{L}}$. Consider the following steps which detail the application of rules $\Box_{\mathsf{R}}$ and $\Diamond_{\mathsf{L}}$ in the construction of the reduction tree, giving rise to the stages $11'$ and $12'$, substituting stages 11 and 12:

$\Box'_{\mathsf{R}}$: This step might be applied to each topmost sequent $\Gamma' \Rightarrow \Delta'$, containing in the succedent formulas having $\Box$ as their outermost logical connective. Each such sequent will have the following form, for some $m > 0$, and where we explicitly display all the $\Box$-formulas in the succedent:

$$
\Gamma' \Rightarrow \Delta'', x_1 : \Box\psi_1, \ldots, x_m : \Box\psi_m.
$$

If for all x_i, there is a z_i such that $x_i R z_i$ and $z_i : \psi$ is already in the succedent of some formula along this branch, skip this Stage. Otherwise, let $x_{i_1}, \ldots, x_{i_k}$ denote the x_i such that there is no z_i

with $x_i R z_i$ and $z_i : \psi$ is already in the succedent of some formula along this branch. Furthermore, let $y_{i_1}, \ldots, y_{i_k}$ be variables which have not appeared in the reduction tree up to this Stage, that is, variables that satisfy the variable condition in $\Box_{\mathsf{R}}$. Write on top of $\Gamma' \Rightarrow \Delta'$ the following sequent:

$$x_{i_1} R y_{i_1}, \ldots, x_{i_k} R y_{i_k}, \Gamma' \Rightarrow \Delta'', y_{i_1} : \psi_{i_1}, \ldots, y_{i_k} : \psi_{i_k}.$$

$\Diamond'_{\mathsf{L}}$: This step might be applied to each topmost sequent $\Gamma' \Rightarrow \Delta'$, containing in the antecedent formulas having $\Diamond$ as their outermost logical connective. Each such sequent will have the following form, for some $m > 0$, and where we explicitly display all the $\Diamond$-formulas in the antecedent:

$$x_1 : \Diamond \psi_1, \ldots, x_m : \Diamond \psi_m, \Gamma'' \Rightarrow \Delta'.$$

If for all x_i, there is a z_i such that $x_i R z_i$ and $z_i : \psi$ is already in the antecedent of some formula along this branch, skip this step. Otherwise, let $x_{i_1}, \ldots, x_{i_k}$ denote the x_i such that there is no z_i with $x_i R z_i$ and $z_i : \psi$ is already in the antecedent of some formula along this branch. Furthermore, let $y_{i_1}, \ldots, y_{i_k}$ be variables which have not appeared in the reduction tree up to this Stage, that is, variables that satisfy the variable condition in $\Diamond_{\mathsf{L}}$. Write on top of $\Gamma' \Rightarrow \Delta'$ the following sequent:

$$x_{i_1} R y_{i_1}, \ldots, x_{i_k} R y_{i_k}, y_{i_1} : \psi_{i_1}, \ldots, y_{i_k} : \psi_{i_k}, \Gamma'' \Rightarrow \Delta'.$$

With these changes taken into account, the proof-search from Example 4.9 would terminate when we reached the sequent $xRx, xRy, yRy, x : \Box\Diamond p, y : p \Rightarrow x : \bot$. However, these modifications are not enough as there is a second, more complex, issue that needs to be addressed: $\Box_{\mathsf{R}}$ and $\Diamond_{\mathsf{L}}$ may be repeatedly applied to formulas which are only distinct in the labels. Before continuing to the next example that illustrates this problem, consider the following definition.

Definition 4.10. Let Σ be a set of labelled formulas and x a label. We define $fm(x, \Sigma)$ as the set of formulas in $\mathcal{L}^{\Box,\Diamond}$ which are labelled by x in Σ. Furthermore, if $\Gamma \Rightarrow \Delta$ is a sequent in a branch of a tree and x, y are labels such that $fm(x, \downarrow\Gamma) = fm(y, \downarrow\Gamma)$ and $fm(x, \downarrow\Delta) = fm(y, \downarrow\Delta)$, we say that x and y are *duplicate* (with respect to that branch).

Example 4.11. Consider now the slightly more complex formula $\Box\Diamond\Diamond p \to \bot$. Again, we can easily see that this formula is not valid since, for instance, it is not satisfiable in the following model:

$$\begin{tikzcd} x \arrow[loop left] \arrow[rr] & & y, \boldsymbol{p} \arrow[loop right] \end{tikzcd}$$

Let's analyse the tree produced by our algorithm:

$$
\begin{array}{cl}
\vdots & \\ \cline{1-1}
xRx, xRy, yRy, yRz, zRz, xRz, zRw, wRw, xRw, yRw, x : \Box\Diamond\Diamond\, p, z : p, y : \Diamond\Diamond\, p, w : \Diamond\, p, w : \Diamond\Diamond\, p \Rightarrow x : \bot & \\ \cline{1-1}
xRx, xRy, yRy, yRz, zRz, xRz, zRw, wRw, xRw, yRw, x : \Box\Diamond\Diamond\, p, z : p, y : \Diamond\Diamond\, p, w : \Diamond\, p \Rightarrow x : \bot & \Box_{\mathsf{L}} \\ \cline{1-1}
xRx, xRy, yRy, yRz, zRz, xRz, zRw, x : \Box\Diamond\Diamond\, p, z : p, y : \Diamond\Diamond\, p, w : \Diamond\, p \Rightarrow x : \bot & \textit{Ref} + \textit{Trans} \\ \cline{1-1}
xRx, xRy, yRy, yRz, zRz, xRz, x : \Box\Diamond\Diamond\, p, z : p, y : \Diamond\Diamond\, p, z : \Diamond\Diamond\, p \Rightarrow x : \bot & \Diamond_{\mathsf{L}} \\ \cline{1-1}
xRx, xRy, yRy, yRz, zRz, xRz, x : \Box\Diamond\Diamond\, p, z : p, y : \Diamond\Diamond\, p \Rightarrow x : \bot & \Box_{\mathsf{L}} \\ \cline{1-1}
xRx, xRy, yRy, yRz, x : \Box\Diamond\Diamond\, p, z : p, y : \Diamond\Diamond\, p \Rightarrow x : \bot & \textit{Ref} + \textit{Trans} \\ \cline{1-1}
xRx, xRy, yRy, x : \Box\Diamond\Diamond\, p, y : \Diamond\, p, y : \Diamond\Diamond\, p \Rightarrow x : \bot & \Diamond_{\mathsf{L}} \\ \cline{1-1}
xRx, xRy, yRy, x : \Box\Diamond\Diamond\, p, y : \Diamond\, p \Rightarrow x : \bot & \Box_{\mathsf{L}} \\ \cline{1-1}
xRx, xRy, x : \Box\Diamond\Diamond\, p, y : \Diamond\, p \Rightarrow x : \bot & \textit{Ref} \\ \cline{1-1}
xRx, x : \Box\Diamond\Diamond\, p, x : \Diamond\Diamond\, p \Rightarrow x : \bot & \Diamond_{\mathsf{L}} \\ \cline{1-1}
xRx, x : \Box\Diamond\Diamond\, p \Rightarrow x : \bot & \Box_{\mathsf{L}} \\ \cline{1-1}
x : \Box\Diamond\Diamond\, p \Rightarrow x : \bot & \textit{Ref}
\end{array}
$$

Clearly, even with the changes that were applied to Stages 9 and 10, the construction will not be finite. One can easily verify that y and w are duplicate (we denote the top sequent of this derivation branch by $\Gamma \Rightarrow \Delta$):

- $fm(y, \downarrow\Gamma) = fm(w, \downarrow\Gamma) = \{\Diamond\, p, \Diamond\Diamond\, p\}$
- $fm(y, \downarrow\Delta) = fm(w, \downarrow\Delta) = \emptyset$

Therefore, intuitively, it is redundant to apply $\Diamond_{\mathsf{L}}$ to the formula $w : \Diamond\, p$, creating a new label. In a sense, we have already performed this step when we applied that rule to $y : \Diamond\, p$, which generated z. As we can see by continuing the derivation, if we were to apply this rule to w we would simply be creating a duplicate of z. In the following we collapsed the non-principal and non-active relational atoms into sets $\mathcal{R}$, $\mathcal{R}'$, etc.:

$$
\begin{array}{cl}
\vdots & \\ \cline{1-1}
\mathcal{R}''', wRv, vRv, yRv, xRv, x : \Box\Diamond\Diamond\, p, z : p, y : \Diamond\Diamond\, p, v : p, w : \Diamond\Diamond\, p, v : \Diamond\Diamond\, p \Rightarrow x : \bot & \\ \cline{1-1}
\mathcal{R}'', xRv, x : \Box\Diamond\Diamond\, p, z : p, y : \Diamond\Diamond\, p, v : p, w : \Diamond\Diamond\, p \Rightarrow x : \bot & \Box_{\mathsf{L}} \\ \cline{1-1}
\mathcal{R}', xRw, yRw, wRv, x : \Box\Diamond\Diamond\, p, z : p, y : \Diamond\Diamond\, p, v : p, w : \Diamond\Diamond\, p \Rightarrow x : \bot & \textit{Ref} + \textit{Trans} \\ \cline{1-1}
\mathcal{R}, x : \Box\Diamond\Diamond\, p, z : p, y : \Diamond\Diamond\, p, w : \Diamond\, p, w : \Diamond\Diamond\, p \Rightarrow x : \bot & \Diamond_{\mathsf{L}}
\end{array}
$$

This example hints at the fact that another modification that must be made to $\Box_{\mathsf{R}}$ and $\Diamond_{\mathsf{L}}$ in order to prevent the creation of new labels which, in a sense, already were created along the proof search. That is, we also do not apply $\Box_{\mathsf{R}}$ (resp. $\Diamond_{\mathsf{L}}$) to a formula $x : \Box\,\phi$ in the antecedent (resp. $x : \Diamond\,\phi$ in the succedent), if we have already applied that same rule to another label y which is a duplicate of x and such that yRx. We modify step $\Box'_{\mathsf{R}}$ and $\Diamond'_{\mathsf{L}}$ resulting in their final version, giving rise to Stages $9''$ and $10''$. The formulation of $\Box''_{\mathsf{R}}$ is as follows:

$\Box''_{\mathsf{R}}$: This step might be applied to each topmost sequent $\Gamma' \Rightarrow \Delta'$, containing in the succedent formulas having $\Box$ as their outermost logical connective. Each such sequent will have the following form, for some $m > 0$, and where we explicitly display all the $\Box$-formulas in the succedent:

$$\Gamma' \Rightarrow \Delta'', x_1 : \Box\,\psi_1, \ldots, x_m : \Box\,\psi_m.$$

If for all x_i,

1. there is a z_i such that $x_i R z_i$ and $z_i : \psi$ are already in the succedent of some formula along this branch or,
2. there is a z_i duplicate of x_i such that $z_i R x_i$[2],

skip this step. Otherwise, let $x_{i_1}, \ldots, x_{i_k}$ denote the x_i which do not meet either of the two conditions above. Furthermore, let $y_{i_1}, \ldots, y_{i_k}$ be variables which have not appeared in the reduction tree up to this Stage, that is, variables that satisfy the variable condition in $\Diamond_{\mathsf{L}}$. Write on top of $\Gamma' \Rightarrow \Delta'$ the following sequent:

$$x_{i_1} R y_{i_1}, \ldots, x_{i_k} R y_{i_k}, \Gamma' \Rightarrow \Delta'', y_{i_1} : \psi_{i_1}, \ldots, y_{i_k} : \psi_{i_k}.$$

The modification on Stage $10'$ is analogous. We also should update the saturated conditions to guarantee that when we stop proof search we have a saturated branch. We modify the conditions $(\Box_{\mathsf{R}})$ and $(\Diamond_{\mathsf{L}})$ as follows:

$(\Box_{\mathsf{R}})$ If $x : \Box\phi$ is in $\downarrow\Delta$, then for some y, we have that xRy is in $\downarrow\Gamma$ and $y : \phi$ is in $\downarrow\Delta$ *or* there is y such that yRx is in $\downarrow\Gamma$ and y is a duplicate of x;

$(\Diamond_{\mathsf{L}})$ If $x : \Diamond\phi$ is in $\downarrow\Gamma$, then for some y, we have that yRx and $y : \phi$ is in $\downarrow\Gamma$ *or* there is y such that yRx is in $\downarrow\Gamma$ and y is a duplicate of y;

To accommodate for the existence of duplicates we should guarantee that we only move on to the application of $\Box_{\mathsf{R}}$ and $\Diamond_{\mathsf{L}}$ after all the other rules have been exhaustively applied. This avoids a situation where we unnecessarily generate a label z from y, but later in the tree we verify that y is a duplicate of some other label x (meaning that the application of the rule which generated z should have been blocked by the duplicate condition).

We are now ready to define the algorithm from $\mathbf{LabS4}$.

Modified construction of the reduction tree for $\mathbf{\Gamma \Rightarrow \Delta}$.
Begin by placing $\Gamma \Rightarrow \Delta$ at the root of the tree. If this sequent it initial, the construction ends. Otherwise, the construction will continue by writing on top of every non-initial top sequent the result of performing a step corresponding to at least one (possibly several) root first application of a rule of $\mathbf{LabS4}$. The order is detailed in the algorithm from Figure 3.2, where steps *Ref,Trans*, $\Box_{\mathsf{L}}, \Diamond_{\mathsf{R}}, \Box_{\mathsf{R}}$ and $\Diamond_{\mathsf{L}}$ have been replaced by *Ref*$'$, *Trans*$'$, $\Box'_{\mathsf{L}}, \Diamond'_{\mathsf{R}}, \Box''_{\mathsf{R}}$ and $\Diamond''_{\mathsf{L}}$, respectively. Move to steps $\Box''_{\mathsf{R}}$ and $\Diamond''_{\mathsf{L}}$ only if every other step has been applied exhaustively. If every resulting top sequent is initial the construction stops and the resulting tree gives rise to a derivation for the sequent at the root. If there is a top sequent which is not initial but none of the steps applied, the construction stops and the search for a proof of $\Gamma \Rightarrow \Delta$ has *failed*. Furthermore, the branch going from the root of the tree up to this sequent is saturated, and will be used to construct the countermodel. Otherwise, perform the steps again, starting in $\wedge_{\mathsf{L}}$. The resulting algorithm is summarized in Figure 4.2. We go back to Example 4.11 to understand the construction of the countermodel.

[2]Note that, we may only skip an application of $\Box_{\mathsf{R}}$/ $\Diamond_{\mathsf{L}}$ due to the existence of duplicate labels x and y if we guarantee that xRy. Essentially, this guarantees that proof search did not stop too soon. The reason why this relational atom is needed will become clear in the proof for Lemma 4.14.

input: sequent $\Gamma \Rightarrow \Delta$;

Place $\Gamma \Rightarrow \Delta$ at the root;
If this sequent is initial, stop;

Stages $1 - 6$:
For each leaf sequent and $\star \in \{\wedge_{\mathsf{L}}, \wedge_{\mathsf{R}}, \vee_{\mathsf{L}}, \vee_{\mathsf{R}}, \rightarrow_{\mathsf{L}}, \rightarrow_{\mathsf{R}}\}$ apply step $\star$;

Stage $7' - 8'$:
For each leaf sequent apply step *Ref*$'$, *Trans*$'$;

Stage $9' - 10'$:
Apply steps $\Box'_{\mathsf{L}}$, $\Diamond'_{\mathsf{R}}$;

For each topsequent, if none of the above stages has been applied, go to Stage $11''$.
Else, go to Stage1.

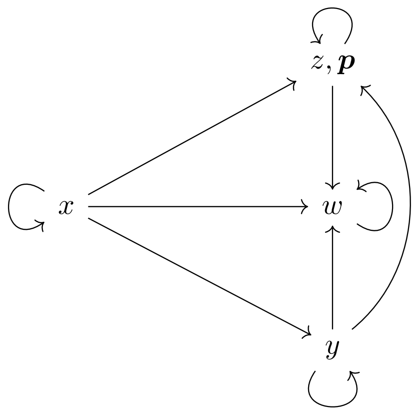
Stage $11'' - 12''$:
Apply steps $\Box''_{\mathsf{R}}$, $\Diamond''_{\mathsf{L}}$;

If every leaf is an initial sequent, stop;
If there is a non initial leaf sequent to which none of the stages were applied, stop;
Else, go to Stage 1;

output: a finite tree with $\Gamma \Rightarrow \Delta$ at the root.

Figure 4.2: Algorithm for the construction of the reduction tree for **LabS4**- terminating.

Example 4.12. According to our algorithm, the construction of the derivation tree should terminate with the topsequent $xRx, xRy, yRy, yRz, zRz, xRz, zRw, wRw, xRw, yRw, x : \Box\Diamond\Diamond p, z : p, y : \Diamond\Diamond p, w : \Diamond p, w : \Diamond\Diamond p \Rightarrow x : \bot$. The following structure is the Kripke-model extracted from the tree, according to the construction in Section 4.1.



Note, however, that this is not a countermodel for $\Box\Diamond\Diamond p \to \bot$. In fact, let $v \in \{x, y, w, z\}$ be some world in this model, we have that vRw and $\mathcal{M}^\times, w \not\Vdash \Diamond\Diamond p$, hence $\mathcal{M}^\times, v \not\Vdash \Box\Diamond\Diamond p$. Therefore, $\mathcal{M}^\times, v \Vdash \Box\Diamond\Diamond p \to \bot$, that is, the formula is satisfiable in $\mathcal{M}^\times$. It is clear that the construction of the countermodel as detailed in Section 4.1 is no longer suitable. In fact, in order to make this model a countermodel, there should be an arrow from w to some world from which z is accessible.

Constructing the countermodel.
Consider some sequent $\Gamma \Rightarrow \Delta$ for which the algorithm has produced a finite reduction tree which is not a derivation. Consider a saturated branch $\mathcal{B}^\times = (\Gamma_i \Rightarrow \Delta_i)_{0 \leq i \leq k}$ where $\Gamma \Rightarrow \Delta \equiv \Gamma_0 \Rightarrow \Delta_0$ in the reduction tree constructed for this sequent, according to the algorithm in Figure 4.2. We use $\mathcal{B}^\times$ to construct the countermodel. Let $\mathcal{M}^\times = (W^\times, R^\times, [\![\cdot]\!]^\times)$ where the interpretation $\rho^\times$, set of worlds $W^\times$ and valuation $[\![\cdot]\!]^\times$ are constructed as in Section 3.3. Furthermore, the accessibility relation $R^\times$ is defined as follows:

- If $xRy \in \mathbf{\Gamma}$ then $(\rho^\times(x), \rho^\times(y)) \in R^\times$;
- Whenever rule $\Box_{\mathsf{R}}$ (resp. $\Diamond_{\mathsf{L}}$) was not applied to a formula $x : \Box\psi \in \mathbf{\Delta}$ (resp. $x : \Diamond\psi \in \mathbf{\Gamma}$) because there was a label z, which is a duplicate of x and such that zRx occurs in $\mathbf{\Gamma}$, then additionally:
$$(\rho^\times(x), \rho^\times(z)) \in R^\times$$
- Close $R^\times$ under transitivity;

It is essential to proof that the structure being constructed is, in fact, a countermodel and that the underlying accessibility relation is both reflexive and transitive.

Lemma 4.13. Let $\mathcal{M}^\times$ be the model constructed as described above. Then $R^\times$ is reflexive and transitive.

Proof. Let $xRy \in \mathbf{\Gamma}$. Let i be such that the first time we try to apply *Ref* to xRy is at sequent $\Gamma_i \Rightarrow \Delta_i$. Then, either yRx was already in $\downarrow\Gamma_i$, or *Ref* is applied to this formula, whence $yRx \in \Gamma_{i+1}$. In both cases we get that $yRx \in \mathbf{\Gamma}$, hence $R^\times$ is reflexive. The proof that $R^\times$ is transitive proceeds analogously. Note that for the new $R-$edges that are added in the countermodel construction, transitivity is guaranteed by the fact that we close under transitivity. QED.

Lemma 4.14 (Truth Lemma)**.** Let $\mathcal{M}^{\times}$ be the model and $\rho^{\times}$ the interpretation constructed as described above. The following hold:

$$\text{For every } x : \phi, yRz \in \mathbf{\Gamma} \text{ we have that } \mathcal{M}^{\times}, \rho^{\times} \Vdash x : \phi \text{ and } \mathcal{M}^{\times}, \rho^{\times} \Vdash yRz; \tag{4.3}$$

$$\text{For every } x : \phi \in \mathbf{\Delta} \text{ we have that } \mathcal{M}^{\times}, \rho^{\times} \nVdash x : \phi. \tag{4.4}$$

Proof. The condition on the relational atoms follows by the construction of the countermodel. For the other conditions, the proof is by induction on the weight w of ϕ. The base case $w = 0$ follows as in the proof for Lemma 3.24. For $w \geq 1$ it is only relevant to consider the cases where $\phi \equiv \Box \psi$ with $x : \Box \psi \in \mathbf{\Delta}$ and $\phi \equiv \Diamond \psi$ with $x : \Diamond \psi \in \mathbf{\Gamma}$. We show the details for the first case, as the proofs are analogous. Let $x : \Box \psi \in \mathbf{\Delta}$.

- If rule $\Box_{\mathsf{R}}$ was applied to $x : \Box \psi$ in the construction of the reduction tree, we get that $y : \psi \in \mathbf{\Delta}$ and $xRy \in \mathbf{\Gamma}$ for some y. By the inductive hypothesis, $\mathcal{M}^{\times}, y \nVdash \psi$ hence $\mathcal{M}^{\times}, \rho^{\times} \nVdash x : \Box \psi$.
- Otherwise, consider the sequent $\Gamma_i \Rightarrow \Delta_i$, with $x : \Box \psi \in \Delta_i$, to which we did not apply $\Box_{\mathsf{R}}$. We distinguish between two cases:
  - $\Box_{\mathsf{R}}$ was not applied because there is $xRy \in \mathbf{\Gamma}$ and $y : \psi \in \Delta_j$ for some $j \leq i$. In particular, we have that $y : \psi \in \mathbf{\Delta}$. By the inductive hypothesis, $\mathcal{M}^{\times}, y \nVdash \psi$ hence $\mathcal{M}^{\times}, \rho^{\times} \nVdash x : \Box \psi$.
  - $\Box_{\mathsf{R}}$ was not applied because there is z such that z is a duplicate of x and zRx. Since we have that zRx we can guarantee that x and y have not been generated in the same step of the construction of the reduction tree, hence, $z : \Box \psi \in \Delta_j$ for some $j < i$.
    If $\Box_{\mathsf{R}}$ was applied to this formula, we have that for some w and $k > j$, $zRw \in \mathbf{\Gamma}$ and $w : \psi \in \Delta_k$, whence $w : \psi \in \mathbf{\Delta}$. Furthermore, by construction, we have that xRz and that xRw. By the inductive hypothesis, $\mathcal{M}^{\times}, w \nVdash \psi$ hence $\mathcal{M}^{\times}, \rho^{\times} \nVdash x : \Box \psi$.
    Otherwise, go down the derivation and find the first label v such that v is only a duplicate of labels appearing after it was generated. We can do this since there is a finite number of sequents from the root of the tree up to $\Gamma_j \Rightarrow \Delta_j$. Then, $\Box_{\mathsf{R}}$ must have been applied to $v : \Box\psi$ and, by transitivity, we have that vRx. By proceding analogously to the case above we conclude that $\mathcal{M}^{\times}, \rho^{\times} \nVdash x : \Box \psi$.

QED.

It remains to prove the following result:

Proposition 4.15. The algorithm presented in Figure 4.2 always terminates.

Proof. In order to establish our result we must prove that any branch of a tree outputted by the algorithm from Figure 4.2 is finite. We provide only a sketch of this result.
By the same argument we used in the proof of Theorem 4.5, there are only finitely many formulas (modulo labels) occurring in a branch. However, it is not straightforward to prove that we only have finitely many labels occurring in a branch[3]. The key is that, contrary to what happens in $\mathbf{LabK}$, the presence of the rules for R allows for formulas to be carried from one label to another. Hence, even though rules $\Diamond_{\mathsf{L}}/\Box_{\mathsf{R}}$ (rules who create new labels) can only be applied once to a labelled formula, formulas of the

[3]Note that the algorithm discussed at the begining of Section 4.2 produced trees which did not satisfy this condition.

shape $\Diamond_\mathsf{L}/\Box_\mathsf{R}$ could be carried to another label, and the rule $\Diamond_\mathsf{L}/\Box_\mathsf{R}$ could be re-applied[4]. We claim that for each label x occurring in a branch, there are only finitely many z occurring in the branch, such that the relational atom xRz occurs in the branch. To prove this, observe that:

- If x labels only propositional formulas in the branch, then we are done;
- If x labels some modal formulas, we might be in the situation discussed in Example 4.11. In this case, since we only have finitely many formulas (modulo labels) in a branch, and such formulas can be combined in finitely many ways, there are only finitely many sets of formulas that can be labelled by a label along a branch. Hence, along every R-path of labels, that is, a (possibly infinite) set of labels $(x_i)_{i>0}$ such that for each i, $x_i R x_{i+1}$ occurs in the branch, we are bound to find duplicated labels. Because of the modified steps in the algorithm, this will eventually stop the application of the labels-generating rules along a branch. Thus, every R-path of labels is finite.

QED.

The proof of the following result is analogous to the proof of Theorem 4.6.

Theorem 4.16. **S4** *is decidable.*

Corollary 4.17. **S4** has the finite model property.

[4]In Example 4.11 we can see $\Diamond_\mathsf{L}$ being applied indefinitely to formulas $x : \Diamond\Diamond p$ (modulo labels). Note that the algorithm has been modified to prevent this.

Conclusion

We conclude this work by reflecting on the main results we proved, along with the methodologies employed to achieve these outcomes.

Overview and main results

The purpose of this thesis was to present a proof theoretical analysis of modal logics. In order to attain this goal, we began by introducing in Chapter 1, modal logics, their syntax, semantics and relevant properties such as soundness and completeness. Furthermore, we presented a Gentzen-style calculus for classical propositional logic, which serves as a starting point for the rest of the work. To extend the study of sequent calculi to modal logics, we opted to focus on the labelled approach. In Chapter 3, we presented labelled systems for every logic in the $\mathbf{S5}$-cube, that is, the family of logics built from the basic modal logic $\mathbf{K}$ by considering certain combination of frame conditions, namely reflexivity, symmetry and transitivity. These systems possess remarkable properties such as admissibility of weakening and contraction, and invertibility of the logical rules. The proofs for these results, along with a proof of the notable cut-admissibility theorem can be found in Chapter 3. The strategies used to establish these results are standard and very similar to the ones used in Chapter 2, when we studied the classical case. We paid special attention to proving completeness of the logics in the cube with respect to the labelled systems, using solely semantical methods (thus avoiding the usage of cut to establish this result). We did this by presenting a proof-search algorithm which, given a formula which is not valid, produces an infinite reduction tree that can be used to build an infinite countermodel. Finally, in Chapter 4, we focused on a proof-theoretical approach to decidability, taking $\mathbf{K}$ and $\mathbf{S4}$ as case studies. This was attained by taking the non-terminating algorithms used to prove semantical completeness in [30] and adapting them to get terminating proof-search mechanisms. As an important corollary to proof-search termination, we also obtained the finite model property. Even though the results obtained in Chapter 4 concerning the decidability of $\mathbf{K}$ and $\mathbf{S4}$ have been established in the literature, the approach taken here, in particular the use of labelled sequent calculi, constitutes an original contribution of this work.

Future work

The methods implemented in Chapter 4 allow us to provide purely syntactical proofs of important results. Although we focused on the logics $\mathbf{K}$ and $\mathbf{S4}$, it is of interest to analyse how our work could be adapted for other systems within the modal cube. For instance, we could be interested in studying $\mathbf{K4}$ to which a very similar method to the one we have provided in Section 4.2 should apply. Furthermore, future work could include developing a general strategy to deal with a bigger family of modal logics, outside of the $\mathbf{S5}$-cube.

Bibliography

[1] A. Avron. The method of hypersequents in the proof theory of propositional non-classical logics. In W. Hodges, editor, *Logic: Foundations to Applications*, pages 1–32. 1996.

[2] D. Basin, S. Matthews, and L. Viganò. Labelled Propositional Modal Logics: Theory and Practice. *Journal of Logic and Computation*, 7(6):685–717, 12 1997.

[3] D. Basin, S. Matthews, and L. Viganò. Natural deduction for non-classical logics. *Studia Logica: An International Journal for Symbolic Logic*, 60(1):119–160, 1998.

[4] E. W. Beth. *Semantic Entailment and Formal Derivability*. Noord-Hollandsche, 1955.

[5] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2002.

[6] K. Brünnler. Deep sequent systems for modal logic. *Archive for Mathematical Logic*, 48(6):551–577, 2009.

[7] B. F. Chellas. *Modal Logic: An Introduction*. Cambridge University Press, 1980.

[8] A. Church. A note on the entscheidungsproblem. *Journal of Symbolic Logic*, 1(1):40–41, 1936.

[9] R. Dyckhoff and S. Negri. Geometrisation of first-order logic. *The Bulletin of Symbolic Logic*, 21(2):123–163, 2015.

[10] M. Fitting. *Proof Methods for Modal and Intuitionistic Logics*. Synthese Library Volume. Springer Netherlands, 1983.

[11] J. Garson. Modal Logic. In E. N. Zalta and U. Nodelman, editors, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, Spring 2024 edition, 2024.

[12] G. Gentzen. Untersuchungen über das logische schließen i. *Mathematische Zeitschrift*, 39:176–210, 1935.

[13] R. Goldblatt. Mathematical modal logic: A view of its evolution. *Journal of Applied Logic*, 1(5):309–392, 2003.

[14] G. Governatori. Labelled tableaux for non-normal modal logics. 08 2002.

[15] G. Governatori. Labelled modal tableaux. In M. Kracht, M. de Rijke, H. Wansing, and M. Zakharyaschev, editors, *Advances in Modal Logic*, pages 87–110. CSLI Publications, 2008.

[16] G. E. Hughes and M. Cresswell. *A Companion to Modal Logic*. University paperbacks. Methuen, 1984.

[17] S. Jaśkowski. On the rules of suppositions in formal logic. In S. Jaśkowski, editor, *? Itepmc-call1967*. Oxford at the Clarendon Press, 1934.

[18] S. Kanger. *Provability in Logic*. Acta Universitatis Stockholmiensis. Stockholm, 1957.

[19] W. Kneale and M. Kneale. *The Development of Logic*. Clarendon Press, Oxford, England, 1962.

[20] S. Krantz. The history and concept of mathematical proof. 2007.

[21] S. Kripke. Semantical considerations on modal logic. *Acta Philosophica Fennica*, 16:83–94, 1963.

[22] S. Kripke. *Naming and Necessity*. Harvard University Press, 1980.

[23] S. A. Kripke. A completeness theorem in modal logic. *The Journal of Symbolic Logic*, 24(1):1–14, 1959.

[24] S. A. Kripke. Semantical analysis of modal logic i. normal propositional calculi. *Zeitschrift fur mathematische Logik und Grundlagen der Mathematik*, 9(5-6):67–96, 1963.

[25] S. A. Kripke. Semantical analysis of intuitionistic logic i. In J. Crossley and M. Dummett, editors, *Formal Systems and Recursive Functions*, volume 40 of *Studies in Logic and the Foundations of Mathematics*, pages 92–130. Elsevier, 1965.

[26] R. E. Ladner. The computational complexity of provability in systems of modal propositional logic. *SIAM J. Comput.*, 6:467–480, 1977.

[27] C. Lewis and C. Langford. *Symbolic Logic*. Century philosophy series. Century Company, 1932.

[28] P. Mateus, A. Sernadas, C. Sernadas, and L. Vigano. Modal sequent calculi labelled with truth values: Completeness, duality and analyticity. *Logic Journal of the IGPL*, 12:227–274, 05 2004.

[29] S. Negri. Proof analysis in modal logic. *Journal of Philosophical Logic*, 34(5/6):507–544, 2005.

[30] S. Negri. Kripke completeness revisited. In G. Primiero, editor, *Acts of Knowledge: History, Philosophy and Logic*, pages 233–266. College Publications, 2009.

[31] S. Negri. Proofs and countermodels in non-classical logics. *Logica Universalis*, 8(1):25–60, Mar 2014.

[32] R. Smith. Prior analytics book 1. *Ancient Philosophy*, 31(2):417–424, 2011.

[33] A. S. Troelstra and H. Schwichtenberg. *Basic Proof Theory*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2000.

[34] J. van Benthem. *Modal Logic for Open Minds*. CSLI Publications. Center for the Study of Language and Information, 2010.

[35] J. van Benthem. Rolling down the river: Saul kripke and the course of modal logic. In J.-Y. Beziau, editor, *Universal Logic: An Anthology*, pages 187–197. Birkhäuser Basel, 2012.

[36] L. Viganò. *Labelled Non-Classical Logics*. Kluwer Academic Publishers, Boston, 2000.