FROM QUASI-VARIABLE THINKING TO ALGEBRAIC THINKING: A STUDY WITH GRADE 4 STUDENTS[1]

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Abstract

This communication presents a study that is part of a broader teaching experiment research that focus on the development students' algebraic thinking in one grade 4 class. The particular goal of this communication is to analyze students' algebraic thinking when they explore numerical equalities with two unknown quantities. Data collection focuses on the students' work on one task in the classroom, and is based on participant observation and on the analyses of students' worksheets. We conclude that students are starting to evidence the emergence of algebraic thinking by expressing the generalization of the numerical relationships in different representations.

Introduction

In line with recent international trends that consider that the introduction to algebraic thinking should begin in the first years of school, and that it should be understood as a way of thinking that brings meaning, depth and coherence to other subjects learning (National Council of Teachers of Mathematics, 2000), the new national mathematics curriculum in Portugal (Ministério da Educação, 2007) considers the development of algebraic thinking from the first years of schooling, one of the four fundamental axes of teaching-learning in mathematics. It is then of great importance to understand how to develop students' algebraic thinking from the early years on.

The present communication aims to analyze students' algebraic thinking when they explore numerical equalities with two unknown quantities. Namely, we seek to understand: (i) How do students recognize the relationship between two unknown quantities in numerical equalities?; (ii) What kind of representations do they use to express the generalization of the involved numerical relationships?

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Theoretical Background

Algebraic thinking can be looked at "as a process in which students generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways" (Blanton & Kaput, 2005, p. 413). One of the possible approaches for the development of algebraic thinking is based on the potentially algebraic character of arithmetic, in other words, the generalized arithmetic. That implies the construction of the generalization through numerical relationships and arithmetic operations and their properties and, also, includes the notion of equivalence related to the equal sign (=). Carpenter, Franke and Levi (2003) acknowledge those ideas as relational thinking, which means, the ability to look at expressions or equations in this broader perspective, revealing the existing relationships in those expressions or equations.

The students' general explanations about the reason for the veracity of a numerical expression such as 78 - 49 + 49 = 78, and their ability to use specific examples of what later will be seen as a general relationship ($a - b + b = a$), have been described as quasi-variable thinking (Fujii & Stephens, 2008). The expression quasi-variable means a "number sentence or group of number sentences that indicate an underlying mathematical relationship which remains true whatever the numbers used are" (Fujii, 2003, p. 59). Within this perspective, students can use generalizable numerical expressions, focusing their attention in the expressions structure, and identifying and discussing the algebraic generalization before the introduction of formal algebraic symbology.

The use of the potentially algebraic nature of arithmetics through generalisable numerical sentences to represent quasi-variables can provide an important bridge between arithmetic and algebraic thinking and, also, a gateway to the concept of variable (Fujii, 2003). Britt and Irwin (2011) agree that a pathway for algebraic thinking should provide opportunities for all students to work with several layers of awareness of generalization, in such a way that "students use three semiotic systems to express that generalization: first they should work with numbers as quasi-variables, then with words and finally with the literal symbols of algebra" (p. 154).

Methodology

The results presented in this communication are part of a broader study that centers on the implementation of a teaching experiment (Gravemeijer & Cobb, 2006) which aims to promote the development of algebraic thinking of grade 4 students. This communication focuses on evidence from the work done by students around one of the mathematical tasks proposed in the classroom that involves numerical equalities with two unknown quantities. For the data collection we videotaped the class in which the students performed the task and we analyzed, in particular, the collective discussion with the class after they completed their work on the task. We also used the students' worksheets for analyses.

Description of the teaching experiment

The teaching experiment took place in the school year of 2010/11 and the mathematical tasks proposed to the class drew on the mathematical topics defined by the annual plan made by the school teacher. However, these tasks were innovative considering the usual teacher's practice as they accommodate the prospect of conceiving the algebraic thinking as guiding the syllabus (NCTM, 2000), through a logic of curricular integration. These tasks focused on the exploration of numerical relations and operations properties, in a number sense development perspective (MacIntosh, Reys & Reys, 1992), and had as goals the identification of regularities and the expression of the generalization through natural language, and the beginning of a way towards mathematical symbolization. The use of some informal symbology started to be introduced, particularly, in four tasks (10^{th}, 12^{th}, 14^{th} and 17^{th}). For instance, in the tenth task it was proposed by the teacher-researcher the use of the symbol "?" to express "what is the number" in expressions like "?x5=100". The other three tasks explored computation strategies and its generalization in both natural and mathematical language.

The twentieth task, "Ana and Bruno's stickers" (appendix I), was inspired in the study by Stephens and Wang (2008), and it was the first one to introduce numerical equalities with two unknown numbers, corresponding to interrelated quantities. This task presented a modeling context for an arithmetic compensation situation, involving addition and subtraction operations. In the resolution of the task, students clearly reported the arithmetical compensation and expressed the generalization of the numerical relation in natural language. Spontaneously, one of the students suggested that the generalization could also be written in mathematical language, proposing the expression "B-2=A". This proposal was collectively discussed, and the class arrived to the correct expression "A=B+2". In the collective discussion, the teacher-researcher explored with the students several representations like tables, diagrams and the scale model.

Results

The task discussed in this communication, "Find A and B" (Fig. 1), equally inspired by the study by Stephens and Wang (2008), in continuity with the twentieth task previously referred, also presented an arithmetic compensation situation, but now involving the multiplication and the division. The task was solved by the students in pairs and had moments of collective discussion with the class.

1. Think about the following mathematical sentence:

6 × Ⓐ = 12 × Ⓑ

a) Put numbers in box Ⓐ and box Ⓑ to make three correct sentences like the one above.

b) What is the relationship between the numbers in box Ⓐ and box Ⓑ?

c) If the mathematical sentence is the following one, what is the relationship between numbers in box Ⓐ and box Ⓑ?

15 × Ⓐ = 5 × Ⓑ

Figure 1 –Task "Find A and B".

In the resolution of the task, the students did not manifest any difficulties in the first two points. Concerning the first one, students used different values for unknown quantities for box A and box B. Moreover, students were able to identify the relation between the numbers in boxes A and B, referring in natural language, as in the following example, that "The relationship that exists is that the numbers are always the double". These two students (António and Fábio) arrived at this relationship by dividing 12 by 6. The table they present shows that they understand the direction of the variation between the unknown numbers represented by box A and box B, in the case of the first two pairs of values. Nonetheless and possibly suggested by the additive nature of the compensation in the previous task, they concretize mistakenly the third pair of values, assuming that "the difference is 2". These students have generalized the relation between the two involved unknown quantities, since they are able to symbolically represent, through two different mathematical expressions, the relation between the values of A and B (like double or half).

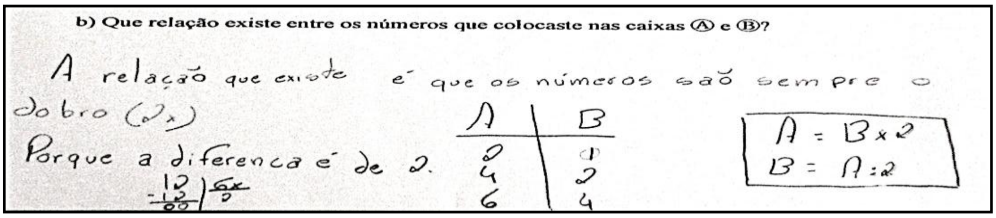

Figure 2 – Question b) resolution, made by the pair António and Fábio.

In the collective discussion with the class, other students show that they go beyond capturing the numerical relation between the unknown values, as they have reached a generalization of the values that satisfy the present equality. For instance, João says that "the A box will always be the double of the B box. The numbers in the A box will always be double from what is in the B box". Matilde adds that "The A box can be any number, but it has to always be the double of the B box". This last student also symbolically represent the involved relations: "A=2xB" and "B=A:2".

These are evidences that students are interpreting the values for A and B not as some fixed numbers that they have to figure out, but as quantities in one predetermined relationship.

In question c), also involving division and multiplication, but that presents more complex relations of triple and one third, all students were able to express the relationship between the values for A and B in a more or less specific way, using different representations, completing the relationship explanation in natural language as another way of representing it, like a table or an arrows diagram. For instance, in the next students' resolution we see the double use of a two column proto-table and of a structured table, even though in both cases with just the same two pairs of values. The students can express, in natural language, clearly, the relation between the values given to the A and B boxes: "The relationship is that the box B is the triple of the [box] A and box A is one third of the [box] B".

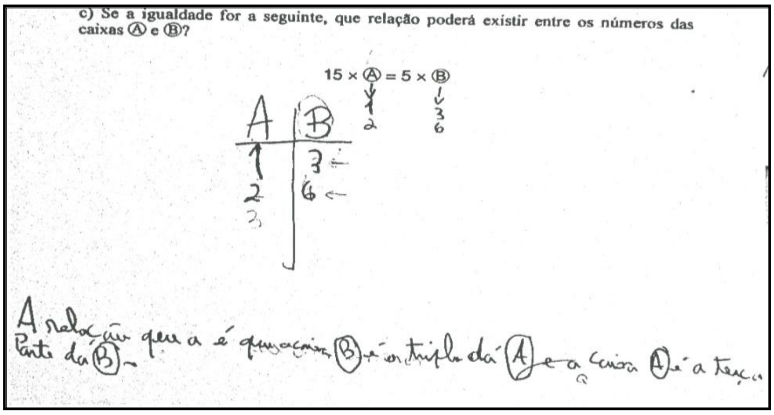

Figure 3 - Question c) resolution, made by the pair João and Marco.

Another pair of students uses different ways to express the relationship between the numbers in the A and B boxes. The arrows diagram shows the type of relationships between A and B values and also the dependency relation between 15 and 5, giving strong evidence of the generalization students made about this situation of dependence between two unknown quantities. They also use the scale representation to illustrate an example of that equality.

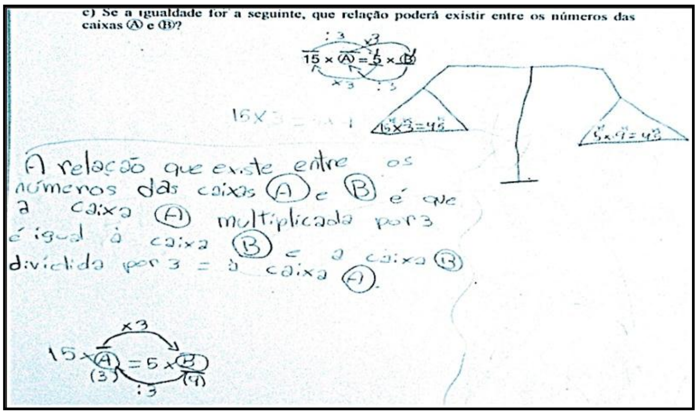

Figure 4 - Question c) resolution, made by the pair Gonçalo and Joana.

In the collective discussion moment, we observed that the majority of students not only understood the numerical relations portrayed by this question, but could also explain them in symbolic language through the mathematical expressions: “B=3xA” and “A=B:3”.

Final considerations and conclusions

Despite being in an early stage, the work developed by the students so far manifests evidences of the recognition of the relationships between two unknown quantities in numerical equalities. Students identify clearly the existing relationship between the numbers placed in A and B boxes (as double or half and triple or one third), referring the arithmetical compensation values and direction. Also, students describe the condition to which any number could be used in A and B boxes, keeping the initial equality.

The initial use of a significant modeling context in the twentieth task, “Ana and Bruno’s stickers”, seems to have contributed to the attribution of meaning to A and B, as unknown quantities in a certain relationship. Therefore, besides being able to use different values for A and B, students were also able to generalize that the equality would be possible for any given number, as long as it obeys the identified dependency relations. Possibly, suggested by the representation in a table form, some students started using the co-variation notion between the quantities, but still in a very incipient way and with some mistakes arising from the attempt to apply to this second task, the same additive structure from the twentieth task. According to Fujii (2003), the use of quasi-variables can provide a gateway to the concept of variable and, although this notion is still in an embryonic state, this shows promising for the development of this concept.

Relatively to the forms of representation presented in the students’ resolutions, we can find that they used natural language, tables, arrow diagrams and were even able to present symbolically the A values regarding B and vice versa. Even though this process is still in an early stage, we can find that the students started to use symbolic language to mathematically express what they translate in natural language. One student’s attempt to find a way to represent B regarding A, in the twentieth task, without being asked to do that, shows the acknowledgement of that representation utility and how the students gave it value and meaning.

In conclusion, we can refer that students start to manifest evidence of algebraic thinking by mobilizing generalization of the relationships involved in numerical equalities with two unknown quantities. The clear expression of the relationships involved in the numerical equality, the identification of the variation taking into account its value and direction and the use of particular cases for the generalization construction, indicate that students used arithmetic in an algebraic way. The use of several representations to express the relations involved in the numerical equality also reveals the apprehension of a way of thinking that goes beyond arithmetics’ basic procedures and which takes into account the structural nature of the involved numerical expressions. These are promising aspects concerning these students' emergence of algebraic thinking, which allows us to consider the potentialities of the teaching experiment that was put into practice.

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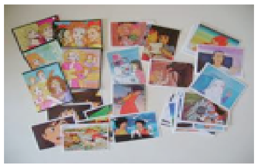
Appendix I

"Ana and Bruno's stickers"



Ana and Bruno are doing a stickers collection. On the last Sunday, their grandmother gave them the same amount of stickers to paste on their books of stickers. Ana pasted 18 stickers on her book and kept the remaining on the box A. Bruno pasted 20 stickers on his book and kept the remaining on the box B.

We can represent the **amount of Ana's stickers** in this way:

18+ Ⓐ

Number of stickers pasted on Ana's book — Number of stickers kept on box A

We can also represent the **amount of Bruno's stickers** in this way:

20+ Ⓑ

Number of stickers pasted on Bruno's book — Number of stickers kept on box B

As the two kids have the same amount of stickers, we can make this equality:

18+ Ⓐ = 20 + Ⓑ

a) How many stickers might Ana have in box Ⓐ and how many stickers might Bruno have in box Ⓑ?

b) Find if there are other values for the number of stickers in boxes Ⓐ and Ⓑ, in way that the total amount of stickers of the two kids is the same.

c) What is the relationship between the numbers you used in box Ⓐ and box Ⓑ?