UNIVERSIDADE DE LISBOA
FACULDADE DE CIÊNCIAS
DEPARTAMENTO DE ESTATÍSTICA E INVESTIGAÇÃO OPERACIONAL

Modeling and Optimization of Compactness on the Selection of Geographical Regions

Duarte Cambotas de Medeiros

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Professor Doutor Miguel Fragoso Constantino

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Resumo

Topologicamente, um conjunto A, definido num espaço Euclidiano, é compacto, se e somente se A é um conjunto fechado e limitado. No entanto, geometricamente, a compacidade de uma figura geométrica mede o grau de compacidade da mesma. Por outras palavras mede, o quão uma figura geométrica se assemelha a um círculo. Ao longo dos anos, vários autores propuseram diversas medidas de compacidade. Algumas das medidas propostas utilizam características da figura geométrica para calcular a compacidade, nomeadamente, a área, o perímetro ou o diâmetro (ou distância máxima). Outras, baseiam-se na comparação da figura geométrica com o círculo inscrito ou o círculo circunscrito. Para além disso, apesar de não serem tão comuns na literatura, algumas medidas de compacidade baseiam-se na comparação com uma figura padrão e na dispersão de elementos de área. Frequentemente, estuda-se a compacidade de dois tipos de figuras geométricas - figuras contínuas e figuras discretas. As figuras contínuas referem-se a figuras que podem ser desenhadas com o auxilio de lápis e papel, como por exemplo, limites administrativos. Figuras discretas, por outro lado, são figuras resultantes da união de pixeis, que podem tomar várias formas. Apesar de não existir uma definição formal para compacidade de uma figura geométrica, esta, tem propriedades interessantes para diversas aplicações, nomeadamente na seleção de distritos eleitorais. Uma das medidas de compacidade mais utilizadas neste contexto é o momento de inércia, também conhecido como o segundo momento de área.

Em Investigação Operacional a compacidade está presente no *Districting Problem* e é uma parte fulcral do mesmo. O *Districting Problem* é um problema NP-difícil que consiste na criação de distritos compactos através da seleção de unidades básicas, indivisíveis. Este problema tem diversas aplicações na vida real, nomeadamente em logística. Por exemplo, uma empresa com vários clientes precisa de dividir os mesmos por regiões geográficas compactas de forma a que um único trabalhador visite os clientes dessa região e, para além disso, a procura esperada e o tempo de trabalho seja divido homogeneamente pelos trabalhadores. Assim, dependente da aplicação, há critérios que devem ser tidos em consideração para além da compacidade, tais como a contiguidade ou o equilíbrio. A contiguidade refere-se há exigência das unidades básicas de cada distrito estarem interligadas e o equilíbrio refere-se à homogeneidade de uma ou mais características importantes.

Esta dissertação foca-se no desenvolvimento e avaliação de modelos, em Programação Inteira, para a seleção de distritos compactos. Os modelos utilizados baseiam-se em três medidas de compacidade comuns na literatura. Das várias medidas de compactidade propostas na literatura, as três medidas de compacidade seccionadas baseiam-se no momento de inércia, na distância máxima e no perímetro. Estas medidas são caracterizadas por serem a razão entre a área da figura geométrica e um dos parâmetros anteriormente referidos. Os modelos desenvolvidos tentam criar distritos compactos, fixando o numerador da fração, a área, num intervalo relativamente pequeno, e minimizando o denominador.

Considerando um conjunto de unidades básicas I. Sabe-se para qualquer unidade básica a área e o perímetro. Para além disso, também é sabido a distância máxima e o perímetro comum entre quaisquer duas unidades básicas de I. Os modelos utilizados constroem distritos com área mínima L e área máxima

U, minimizando algum dos critérios selecionados. Dois conjuntos de modelos foram criados, um para selecionar um único distrito compacto e outro para selecionar múltiplos distritos compactos. Neste último, todas as unidades básicas têm de pertencer a um e um só distrito. Para cada conjunto de modelos, criaram-se seis modelos com diferentes funções objetivo. As funções objetivo utilizadas baseia-se na minimização do primeiro e segundo momento de área, do primeiro e segundo momento de área multiplicado pela área da unidade básica (respetivamente, primeiro e segundo momento de área pesado), da distância máxima e do perímetro. Cada um dos modelos foi otimizado de forma a reduzir o número de variáveis e, consequentemente, reduzir o tempo de otimização. Os doze modelos propostos foram testados em cinco instâncias diferentes, tendo apresentado resultados distintos. Três instâncias referem-se ao mapa dos concelhos de Portugal Continental, enquanto que as outras duas correspondem a regiões geográficas divididas homogeneamente em quadrados ou em hexágonos. Desenvolveu-se também, um conjunto de algoritmos em *Python* para calcular todos os parâmetros dos modelos, assim como, desenhar algumas das instâncias utilizadas e as soluções obtidas. Os resultados apresentados ao longo deste trabalho correspondem a valores de L e de U respetivamente de 15% e 20%.

Devido à complexidade dos modelos, limitou-se o tempo máximo otimização em de 10000 segundos, menos de 3 horas. Os modelos de seleção de um único distrito obtiveram sempre solução ótima, independentemente da função objetivo, enquanto que alguns modelos para a seleção de múltiplos distritos obtiveram soluções sub-otimais. Reparou-se, ao selecionar múltiplos distritos, que minimizar a distância máxima e o perímetro é mais caro computacionalmente que minimizar os momentos de área ou os momentos de área pesados. Nos modelos apresentados, não foram adicionadas restrições de contiguidade, o que levaram a algumas soluções não-contíguas.

A compacidade é um conceito subjetivo que depende da perceção de cada individuo. Assim, para avaliar as soluções obtidas pelos modelos de forma objetiva escolhue-se um conjunto de medidas e mediu-se a compacidade de cada distrito para cada uma das medidas. As medidas de avaliação dos modelos correspondem às medidas de compacidade na qual os modelos se baseiam. Esta avaliação foi realizada para os modelos de seleção de um único distrito, como para os modelos de seleção de múltiplos distritos. Neste último, apresenta-se o valor mínimo, máximo e médio da compactidade para cada instância.

Os modelos que minimizam o diâmetro e o perímetro dos distritos obtiveram boas soluções na seleção de um único distrito. Nos modelos que criam um único distrito verificou-se que para a medida compacidade baseada no perímetro, o melhor modelo foi o que minimiza o perímetro do distrito. O mesmo acontece para a medida de compacidade baseada na distância máxima e o modelo baseado na distância máxima. Relativamente, à medida de compacidade baseada no momento de inércia, o modelo que obteve melhores resultados foi o diâmetro, para as instâncias contínuas, o perímetro para a instância com quadrados e os momentos de área para a instância com hexágonos. Os modelos baseados na distância máxima e no perímetro apresentaram sempre soluções contíguas, o que nem sempre aconteceu com os modelos baseados nos momentos de área e nos momentos de área pesados. Para além disso, por serem modelos de minimização, assim que a restrição de área do distrito é satisfeita, a otimização é terminada. Consequentemente, outras unidades básicas poderiam ser adicionadas não piorando a solução ótima, e em alguns casos, até melhorar a compacidade. Assim, modelos de Programação Inteira Fracionaria, potencialmente, podem formular mais adequadamente a compacidade. Em todo o caso, os modelos apresentados podem beneficiar de uma análise pós-otimização. As soluções destes modelos podem-se considerar soluções iniciais para uma heurística que maximiza a compacidade, e, dependente da aplicação, tem, também, em conta outros critérios como a contiguidade ou o balanço.

Para os modelos de seleção de múltiplos distritos verificou-se que os modelos baseados nos momen-

tos de área e momentos de área pesados, são os únicos capazes de encontrar uma solução ótima num intervalo de tempo, relativamente baixo. Assim, estes modelos podem ser mais adequados para este tipo de problema.

Por fim, mais testes computacionais devem ser realizados com o objetivo de criar distritos a compactos em Programação Inteira. Consequentemente, o desenvolvimento de outros modelos mais eficientes e/ou baseados noutras medidas de compacidade permitirão a comparação entre as diferentes abordagens para resolver o problema.

Palavras chave: Compactness, Moment of Inertia, Districting Problem, *p*-Compact Regions Problem

Abstract

Compactness is a natural characteristic of every geometric shape. Throughout the years, many authors have proposed different measures to quantify how circular a shape is. The most commonly used measures consist of the division of the area of the shape with another characteristic, including moment of inertia (or second moment of area), diameter, or perimeter. Although more recent works reflect on the moment of inertia compactness measure, there is no consensus on the best way to measure compactness.

This property of shapes is applied in Operations Research in the Districting Problem, where indivisible basic units are selected to create compact and contiguous districts for a real-world application. This problem has many applications including political districting or design sales territories. Our approach uses three compactness measures, using six different Integer Programming models, to test the best way to model compactness. We created models to select a single district or multiple districts.

Our models were tested using instances of the municipalities of Portugal, where each basic unit has a unique shape and area, and regions divided into uniform squares and hexagons. The models used focus on minimizing the first moment of area (multiplied by the area), the second moment of area (multiplied by the area), the diameter, and the perimeter, where each district has a fixed area interval. The solutions returned by the models had the compactness evaluated using the original compactness measures.

Our analysis reveals significant variations in the configuration of each district and compactness scores across different evaluation measures. The models presented do not maximize compactness directly, since the optimal solution is reached when the area of the district is feasible. Minimizing the diameter or the perimeter of the districts is expensive computationally for selecting multiple districts, even though returning satisfying results in selecting a single district.

Keywords: Compactness, Moment of Inertia, Districting Problem, *p*-Compact Regions Problem

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Chapter 1

Introduction

Compactness is a very subjective property of all geometric shapes. Although there is no precise definition of compactness, compactness measures (sometimes called circularity measures) quantify the degree to which a geometric shape is compact (Gillman, 2002). Informally, these measures evaluate how close to a circle a geometric shape is (Kalcsics and Roger Z. Ríos-Mercado, 2019; Roger Z. Ríos-Mercado, 2020). In the literature, many authors have proposed several compactness measures. Some derive from the perimeter or area of the shape and others come from parameters related to circles such as the inscribed and circumscribed circle. A small percentage of compactness measures are based on the direct comparison to a circle or even on the dispersion of elements of area (MacEachren, 1985). Moreover, some compactness measures are more suitable for specific geometric shapes. Most compactness measures proposed are for continuous geometric shapes - any geometric shape that can be drawn by an individual with a paper and pencil - but some are specially designed for discrete geometric shapes - geometric shapes that result from the union of homogeneous pixels.

Compactness is usually used in regionalization problems, such as in the design of political districts especially to avoid gerrymandering (S. W. Hess et al., 1965). The term gerrymandering, introduced in 1812, is the practice of redrawing the boundaries of an electoral district to benefit a particular political party. Gerrymandering can also be used to increase or decrease the voting power of racial minorities (Lublin, 1997). Young, 1988 discussed the use of one of the most popular compactness measures, the moment of inertia (or second moment of area), in political redistricting. The Districting Problem is a common problem in Operations Research with a lot of applications in logistics and political redistricting. The Districting Problem consists of the selection of indivisible basic units to create compact and connected districts for a specific application. The basic units can consist of administrative regions, like municipalities of a country, (for example: Almeida and Manquinho, 2022) or by dividing a geographical region into areas of uniform shape and size (for example: Önal et al., 2016; Önal and Briers, 2003).

The ongoing research focuses on applying the District Problem in a specific real-world application while taking into consideration many factors, one of which is maximizing compactness. In the literature, there are many ways to measure the compactness of a shape, making this characteristic hard to measure objectively (MacEachren, 1985; Young, 1988). The objective functions of the models in the literature minimize a characteristic of the district. Some objective functions commonly used include the minimization of the moment of inertia (for example: Önal et al., 2016), the moment of inertia multiplied by another characteristic (or weighted moment of inertia) (for example: S. W. Hess et al., 1965), the diameter (for example: Roger Z Ríos-Mercado and Fernández, 2009; Benzarti et al., 2013) or the perimeter (for example: (Önal and Briers, 2003; Almeida and Manquinho, 2022)).

With this dissertation, we aim to analyze the various ways to select compact districts in Mathematical

Programming. We created several Integer Programming models based on the multiple compactness measures. We present models for selecting one or multiple districts. We tested the models with different types of instances. Some instances consider administrative regions as basic units, while in others, the geographical region is divided into homogeneous basic units with a specific shape. Since there are many ways to measure compactness and we can not rely solely on human intuition, we evaluate the results of the models with the original compactness measures to get an objective way of comparison.

The second chapter of this dissertation focuses on compactness. We start with an introduction to compactness measures and desirable properties, followed by a list of relevant continuous and discrete compactness measures in the literature. Furthermore, we introduce the Districting Problem and briefly mention some applications noting the importance of compactness in each problem. Followed by a model presented in the literature review, in the third chapter we built models for selecting single and multiple districts. For each type of model, we created six different models based on some compactness measures mentioned earlier. With the models already created, the fourth chapter introduces the instances used in the computational experiments, followed by the methodology used to calculate the parameters of the models. Moreover, we discuss the solutions presented by the models, considering the human intuition of compactness and the evaluation measures used. We also review the efficiency of the models implemented relative to the optimization time and final gap. Concluding the thesis, the fifth chapter summarizes the key findings of the study and reflects on the research objectives. The chapter concludes with recommendations for future research and potential improvements to the models and different approaches.

Chapter 2

Compactness

In this chapter, we review the desirable properties of a compactness measure and some compactness measures common in the literature. We also do a literature review on districting problems mentioning some applications. We focus on the different ways to model compactness in the problem context.

2.1 Compactness Measures

Compactness is an intuitive notion, although it may be difficult to define. For example, a circle is more compact than a rectangle or another irregular shape. Compactness is, therefore, a natural characteristic of a geometric shape (Santiago-Montero et al., 2009). Throughout this dissertation, we use the word shape to refer to geometric shapes - the outline, and the interior of an object in two dimensions. Informally, compactness can be defined as how close a shape is to a circle (Kalcsics and Roger Z. Ríos-Mercado, 2019; Roger Z. Ríos-Mercado, 2020). Throughout the years, many authors have proposed very different ways to measure compactness, making this characteristic hard to measure objectively (MacEachren, 1985; Young, 1988). Some authors have proposed using other characteristics of a shape, such as the perimeter, the area, or the diameter (for example: Frolov, 1975; Ritter, 1852; Haggett, 1980), to measure the compactness. At the same time, other authors prefer using statistical analysis based on the central limit theorem (Kaufman et al., 2021). Hence, a measure of compactness is difficult to define mathematically. In general, there seems to exist more or less a consensus in the literature about what should be the properties of a "good" compactness measure. Ideally, a compactness measure should: be possible to measure in any shape; be dimensionless i.e. it entirely depends on the shape itself, and not depend on the scale or the orientation; be in the interval $[0, 1]$, where 1 represents the value of a perfectly compact shape; be possible to apply to one shape and a set of shapes; be following the human intuition.

The compactness measures of a shape can be divided into two large groups - continuous measures and discrete measures. Continuous measures refer to compactness measures that are commonly used in continuous shapes (i.e. any shape), while discrete measures refer to measures used in shapes, that consist of the union of pixels of the same size. Usually, discrete shapes result from the approximation of a continuous shape in a grid of pixels. The type of pixel may vary, for instance, a pixel can be a square, a hexagon, or a triangle. The most common type of pixel used is the square since, in a computer, images are represented by square pixels. Figure 2.1 shows a circle, a continuous shape, and an approximation of a circle using squared pixels, a discrete shape. Note that both shapes have different compactness values, although both represent a circle. Intuitively, the more pixels, the more detail in the shape, and the better the approximation to an actual circle.

Some measures of compactness can be more appropriate depending on the goal of the problem

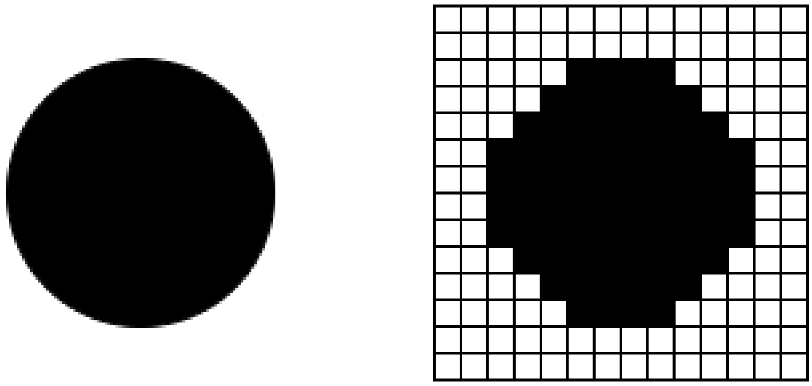


Figure 2.1: A continuous circle on the left and a continuous circle on the right

(MacEachren, 1985). In the literature, some authors use the measure of compactness C while others prefer to use the square of the measure, C^2. By taking the square of a measure, differences among relatively compact shapes will be more apparent, while differences among irregular shapes will be softened (MacEachren, 1985). Figure 2.2, we present the behavior of a compactness measure C and the behavior when squared, C^2. The figure demonstrates that the C^2 measure is stricter than the C for achieving compactness, for the same compactness output. For example, a shape to have a compactness of 0.4, in C, needs to be 0.64 compact in C^2.

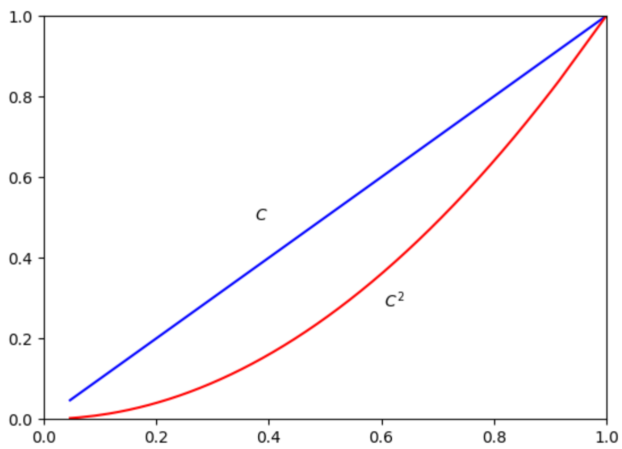


Figure 2.2: Comparison between measures C and C^2

Consider a shape S in which the compactness we want to study. Suppose that S can be approximated to a polygon without holes. The classic compactness measure used is the proportion between the area of S, A, and the perimeter, P, as seen in the following equation:

$$\gamma_1 = \frac{A}{P} \tag{2.1}$$

The compactness measure, usually called the Ritterian coefficient, given by (2.1) and proposed in Ritter, 1852, does not have most of the properties listed above, therefore it is not an adequate compactness measure. The compactness values are not in the interval $[0,1]$, and, the measure is dimensional (Frolov, 1975) - the same shape with different sizes has different compactness values. Consider, the example in figure 2.3. Suppose that each small square has an area of 1 and each side has a length 1. The square on the left has a compactness of 0.5, but the square on the right has a compactness of 0.75. We can easily verify that a square composed of 25 pixels has compactness above 1.

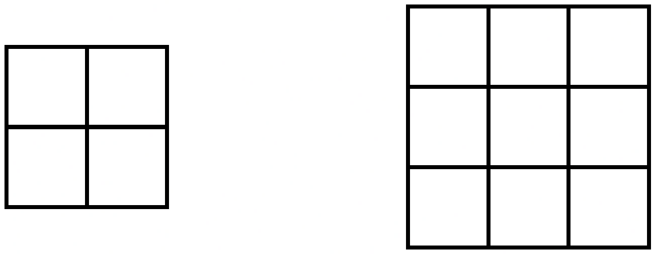


Figure 2.3: Basic example

2.1.1 Continuous Measures

Let S be a shape that can be well-approximated by a planar simple polygon, a polygon with no holes and does not intersect itself. Furthermore, S has m vertices oriented counterclockwise, in a Cartesian coordinate system. The set of vertices of S are given by $V = \{P_i = (x_i, y_i) : i = 1, ..., m\}$. For simplicity suppose that $P_1 = P_{m+1}$. It is possible to calculate the area, A, the perimeter, P, and the diameter, L, using the following equations.

$$A = \frac{1}{2}\sum_{i=1}^{m}(y_i + y_{i+1})(x_i - x_{i+1}) \tag{2.2}$$

$$P = \sum_{i=1}^{m} d(P_i, P_{i+1}) \tag{2.3}$$

$$L = \max_{i,j=1,\dots,m} d(P_i, P_j) \tag{2.4}$$

The formula in (2.2) is known as the shoelace formula or the Gauss's area formula (Braden, 1986), and returns the area of a simple polygon, using a sequence of trapezoids. In (2.3) and (2.4), respectively, the perimeter and the diameter of S are calculated based on the Euclidean distance. Below, we introduce some compactness measures that utilize these characteristics of the shape to deduce the compactness.

$$\gamma_2 = 4\pi\frac{A}{P^2} \tag{2.5}$$

The measure (2.5) also known as Polsby–Popper test (Polsby and Popper, 1991) is one of the most used measures of compactness. Although, many authors refer to Polsby and Popper, 1991 as the origin of this measure, Li et al., 2013 claims that it was first introduced by Osserman, 1978. This measure can also be seen as the ratio of the area of the shape, to the area of a circle with the same perimeter (Kaufman et al., 2021). This measure does not depend on the size of the shape, therefore it is adimensional. The compactness values for the Polsby–Popper test are in the interval $[0,1]$, where the circle, the most compact shape, has a compactness value of 1 (Li et al., 2013).

$$\gamma_3 = \frac{2\sqrt{\pi A}}{P} \tag{2.6}$$

Young, 1988 claims that the origin of (2.6), called Schwartzberg test, comes from Schwartzberg, 1966 however, Frolov, 1975 and MacEachren, 1985 references Nagel, 1835, where it is introduced a similar compactness measure called the Nagel's formula. Analyzing Nagel's equation further demonstrates that the Schwartzberg test is the inverse of Nagel's formula. The Schwartzberg test is the square root of the Polsby–Popper test, (2.5), thus it satisfies the same properties as the latter.

$$\gamma_4 = \frac{4A}{\pi L^2} \tag{2.7}$$

According to Frolov, 1975, Haggett, 1980 obtained the Haggett index, given by (2.7). Horton, 1932 also used the ratio of the area of a shape to the square of the diameter, making (2.7) the Horton's ratio multiplied by a constant. This index increases from zero to one with increasing compactness values and it is adimensional.

$$\gamma_5 = \frac{2\sqrt{A}}{\sqrt{\pi}L} \tag{2.8}$$

The Schumm index given by (2.8) is the square root of the Hagget index, mentioned earlier. Schumm, 1956 proposed this compactness measure using the ratio between the diameter of a circle, having the same area of the shape, and the length of the diameter of the shape (Frolov, 1975; MacEachren, 1985). (2.8) measure follows the desired properties of compactness measures. MacEachren, 1985 compared many measures of compactness and, among the easiest to calculate, this measure returned decent results.

Other compactness measures compare the shape S with a circle, and some authors like Bunge, 1967 propose comparing the shape with an octagon but this approach is not very common in the literature. Consider a radius r_1, the radius of the minimal enclosing circle, i.e. the smallest circle that contains S, and a radius r_2, the largest inscribing circle, i.e. the largest circle that can be drawn inside the shape. Calculating the exact value for r_1 and r_2 can be a very tedious task, especially if each polygon has a high number of vertices. In the literature, there are heuristic methods to calculate r_1 only considering the vertices of the polygon, the edges should be removed. Nielsen and Nock, 2008 proposed a generalization of an algorithm introduced by Welzl, 1991 to calculate the largest circle considering a number of points. Relatively to r_2, Huber, 2017 states that using the Voronoi diagram of a polygon (including polygons with holes) it is possible to calculate the maximum inscribed circle. The following measures utilize the minimal enclosing circle and the largest inscribed circle to calculate compactness.

$$\gamma_6 = \frac{A}{\pi r_1^2} \tag{2.9}$$

$$\gamma_7 = \frac{\sqrt{A}}{r_1\sqrt{\pi}} \tag{2.10}$$

$$\gamma_8 = \frac{r_2^2}{r_1^2} \tag{2.11}$$

$$\gamma_9 = \frac{r_2}{r_1} \tag{2.12}$$

$$\gamma_{10} = \frac{\pi r_2^2}{A} \tag{2.13}$$

$$\gamma_{11} = \frac{r_2\sqrt{\pi}}{\sqrt{A}} \tag{2.14}$$

Ehrenburg, 1892 proposed six different compactness measures using the radius of the circumscribing circle (r_1), the radius of the largest inscribing circle (r_2) and the radius of a circle having the same area as the shape in study (Frolov, 1975). For convenience the measures presented in (2.9), (2.10), (2.11), (2.12), (2.13) and (2.14) are written using the area and returning to a circle the value of one, and less compact shapes values below one. All six measures are adimensional, the compactness value does not depend on the orientation of the shape or the scale. The Reock degree of compactness measure, proposed by Reock, 1961 (Frolov, 1975; Young, 1988; Kaufman et al., 2021; Polsby and Popper, 1991), very common in the literature, is essentially the same compactness equation in (2.9). In general, there seems to exist a consensus in the literature stating the poor results of the Reock degree of compactness. Kim and Anderson, 1984 also proposed the inverse of (2.9) to calculate the compactness using discrete circles. The drawbacks of this measure are that it can not be applied to figures with holes and is not scale invariant (Li et al., 2013).

$$\gamma_{12} = 1 - \left[\frac{\sum |\frac{r_i}{\sum r_i} - \frac{1}{p}|}{2} \right] \tag{2.15}$$

Boyce and Clark, 1964 proposed what's called the Boyce-Clark test, (2.15), (Frolov, 1975; MacEachren, 1985; Young, 1988; Kaufman et al., 2021), a compactness measure based on p radials, separated by equal angular intervals. The $i-$th radial has length r_i, and is drawn from a central point to the border of the shape. The measure was rewritten to produce results on the interval $[0,1]$, with increasing compactness values (MacEachren, 1985). According to Young, 1988 the disadvantage of this measure is that it returns good results to spiral-like shapes.

$$\gamma_{13} = \frac{\max x_i - \min x_i}{\max y_i - \min y_i} \tag{2.16}$$

The Length-width test (2.16) introduced by Harris, 1964 (Kaufman et al., 2021; Frolov, 1975; Young, 1988) is the ratio of the length, to the width of the minimum bounding rectangle. The formula above, from Kaufman et al., 2021, calculates the compactness based on the vertices of the shape, where the coordinates of the $i-$th vertex of the shape are (x_i, y_i). This measure of compactness does not guarantee results in the interval desired and it is not orientation invariant.

$$\gamma_{14} = \frac{A^2}{2\pi I_g} \tag{2.17}$$

The Blair and Bliss index or moment of inertia (2.17) proposed by Blair and Biss, 1967 (MacEachren, 1985; Frolov, 1975) uses the dispersion of area around an axis to calculate the compactness of a shape. Each shape can be viewed as a composition of infinitesimal units of area da, and the moment of inertia of a shape is defined as the second moment of an area about a point, usually the centroid, g. It can be formulated as the following integral:

$$I_g = \int z^2 da \tag{2.18}$$

In (2.18), z represents the distance between the element of area da and the centroid g. The compactness index, based on the second moment of area, is calculated as the ratio between the moment of inertia of a circle and a shape with the same area. The moment of inertia of a circle, through its centroid, depends only on the radius, r, and it is given by $\frac{\pi r^4}{2}$.

Similarly, we can induce the first moment of area of a shape. The first moment of area of a shape relative to its centroid is represented by Q_g, and instead of using the square distance between the g and da, it is used only the absolute distance, as in (2.19).

$$Q_g = \int |z| da \tag{2.19}$$

The compactness measure (2.17) returns values ranging from 0 to 1, with larger values meaning more compact shapes. Although the moment of inertia, I_g, of two identical shapes may vary depending on the axis chosen, the moment of inertia compactness index is insensitive to scale or orientation. This measure is, also, additive, since the compactness of a shape S, can be calculated using the moment of inertia of smaller shapes that, together, form S (Li et al., 2013). Suppose shape S results from the union of shapes U and P, in which the intersection of the interiors is empty. In that case, the moment of inertia of S, is given by $I_S = I_P + I_U + A_P \cdot d^2(S,P) + A_U \cdot d^2(S,U)$, where I_P and I_U represent the moment of inertia, through the centroid, of P and U, d represents the Euclidian distance between the centroids of two shapes, and A_P and A_U are the area of the shapes P and U, respectively (Li et al., 2013).

Not all compactness measures require using elements of the shape, Kaufman et al., 2021 presented a statistical model to predict the compactness of electoral districts given by the perception of judges and public officials.

2.1.2 Discrete Measures

Discrete measures are the result of the union of pixels, usually squares (Bogaert et al., 2000). Given a digital shape, with area A, and perimeter P, some common compactness measures, in the literature, are the following:

$$\gamma_{15} = \frac{4\sqrt{A}}{P} \tag{2.20}$$

The compactness measure (2.20), based on (2.6), was modified to return perfect compact ratios for squares (Bogaert et al., 2000).

$$\gamma_{16} = \frac{\mu}{\sigma} \tag{2.21}$$

Haralick, 1974 did the first work on this measure (Santiago-Montero et al., 2009), the author calculated the centroid of the shape and measured all the Euclidean distances from the centroid and each boundary vertex. The μ and σ are the mean and standard deviation of those distances. The values of compactness increase if the shape is more similar to a circle.

$$\gamma_{17} = \frac{C_D - C_{D_{min}}}{C_{D_{max}} - C_{D_{min}}} \tag{2.22}$$

Bribiesca, 1997 was the first work using the Normalized Discrete Compactness (Santiago-Montero et al., 2009). Given a digital shape with n squared pixels, the idea is to count the number of cell sides that are shared between region cells, also known as the contact perimeter. In (2.22), C_D is the perimeter of contact given by $C_D = \frac{4n-P}{2}$, and $C_{D_{min}}$ and $C_{D_{max}}$ are the bottom and top limits of the contact perimeter, respectively, $C_{D_{min}} = n-1$ and $C_{D_{max}} = \frac{4n-4\sqrt{n}}{2}$. The more similar to a square, the higher the compactness values this measure returns. According to Santiago-Montero et al., 2009 this measure returns more accurate results. This measure also has a 3D version.

Many other compactness measures are utilized in the literature in Bogaert et al., 2000 and in Santiago-Montero et al., 2009. Bogaert et al., 2000 focuses on compactness measures for shapes composed of squares, while Santiago-Montero et al., 2009 presents a comparison of a wide range of discrete compactness measures.

2.2 Models and Applications

Districting consists of the selection of basic units, small geographical areas, usually polygons, into larger groups called districts. Depending on the context of the problem, different criteria may be utilized to select the districts. Some popular criteria include balance, compactness, and contiguity. Balance describes the wish to have districts with similar sizes according to a possible measure (e.g. area and population). A district is contiguous if, and only if, one can travel between any two basic units without leaving the district, while compact districts are the ones with "somewhat rounded shapes" according to Kalcsics and Roger Z. Ríos-Mercado, 2019, hopefully without holes. Usually, compact and contiguous districts have important properties in the context of the problem, for example, when clustering clients in the vehicle Routing Problem by travel time between clients. Unfortunately, there is no formal mathematical definition of compactness, which makes it difficult to model. In the literature, there is a wide range of problems where the Districting Problem is applied, those applications go from political districting to service or distribution districting and designing sales territory.

The Districting Problem is an NP-hard problem (Kalcsics and Roger Z. Ríos-Mercado, 2019), so exact solutions through formulations are limited to relatively small instances. Many authors have proposed different heuristics to solve the problem, a popular approach are GRASP algorithms - Greedy Randomized Adaptive Search Procedure. GRASP algorithms can successfully handle connectivity constraints when constructing solutions from scratch. The first GRASP implementation to the districting problem was by Roger Z Ríos-Mercado and Fernández, 2009 (Kalcsics and Roger Z. Ríos-Mercado, 2019). More recently, in Duque et al., 2011, it was introduced the p-Regions Problem, similar to a districting problem, where it aggregates n spacial units into p contiguous groups while minimizing the dissimilarity intragroup. Li et al., 2014 proposed a heuristic approach called MERGE (memory-based randomized greedy and edge reassignment) to solve a version of this problem with a focus on compactness (and contiguity) of the regions, called the p-Compact-Regions Problem. Other metaheuristics were also proposed including, tabu search, simulated annealing, and genetic algorithms (Kalcsics and Roger

Z. Ríos-Mercado, 2019).

2.2.1 Political Districting

The redrawing of electoral maps is made through the selection of basic units to form clusters, i.e., electoral districts. To be approved, those clusters are evaluated in several criteria. Ideally, each elected member should represent the same amount of people. For instance, consider two electoral districts A and B, if the population of district A is twice the population of district B, then district A elects the twice amount of members compared to district B. Other criteria are compactness and contiguity, electoral districts should be contiguous and compact to prevent odd shapes from existing, that is, gerrymandering. One of the problems with formulations that focus on compactness is to ensure the contiguity of the districts.

One of the first formulations for electoral districting was made by S. W. Hess et al., 1965, who proposed, an Integer Programming model, based on the weighted moment of inertia (the moment of inertia multiplied by a basic unit measure). This model was proposed in the context of political redistricting where population homogeneity (balance), among districts, is intended. Consider the set I of basic units. The model selects p districts, assigning each basic unit i with population P_i, to a district, with a total population between L and U. The decision variable x_{ki}, is a binary decision variable, that takes the value 1, if the basic unit i belongs to the district centered in k, and 0, otherwise.

$$\min: \sum_{k\in I}\sum_{i\in I} d_{ki}^2 P_i x_{ki} \tag{2.23}$$

$$\text{s.t.} \sum_{i\in I} x_{ki} = 1 \qquad \forall k \in I \tag{2.24}$$

$$\sum_{k\in I} x_{kk} = p \tag{2.25}$$

$$\sum_{i\in I} P_i x_{ki} \geq L x_{kk} \qquad \forall k \in I \tag{2.26}$$

$$\sum_{i\in I} P_i x_{ki} \leq U x_{kk} \qquad \forall k \in I \tag{2.27}$$

$$x_{ki} \in \{0,1\} \qquad \forall k,i \in I \tag{2.28}$$

The objective function (2.23) considers the Euclidean distance between the centers of the basic units and the respective population, the weighted moment of inertia. With this objective function, basic units further away from the center and with bigger populations are penalized. Constraint (2.24) ensures that every basic unit must belong to one and only one district, while constraint (2.25) guarantees that p districts are selected. Constraints (2.26) and (2.27) ensure the balance of the districts, by creating a lower and upper limit for the population in each district. Finally, (2.28) is a domain constraint that state that the decision variable is binary.

The model proposed by Hess does not guarantee the contiguity of the districts for some instances and Validi et al., 2022 proposed contiguity constraints for Hess's model. Almeida and Manquinho, 2022 proposed a new compact boolean formulation model, focusing on drawing new electoral districts for Portugal. The model presented differs from Hess's and Validi's model due to the focus on the border length as a measure of the shape of the electoral district, instead of the moment of inertia. Although the results seem reasonable, not every author agrees with compactness as a good measure for creating legislative districts, for instance, Young, 1988 concludes, after reviewing several compactness measures, that compactness is a "hazy and ill-defined" concept, that is very difficult to apply rigorously in the law.

2.2.2 Sales Territory Design

Another application of compactness is in sales territory design. This problem consists of subdividing the market area into regions that are attended by one or multiple sales representatives of a company, by assigning each salesperson to a set of (potential) customers. Each customer should be visited only by one sales representative and the income opportunity, travel time and workload should be divided by each salesperson homogeneously (Zoltners and Sinha, 2005). The problem becomes more complex when customers have time windows to be visited or when customers have different needs (e.g. one customer may need weekly visits while others semesterly visits are enough).

This problem requires the balancing of multiple attributes, but compactness and contiguity are also important - compact and contiguous districts result in smaller travel times than non-compact and/or contiguous districts. Sidney W. Hess and Samuels, 1971 introduced a model where it minimizes the squared distance between basic units multiplied by an attribute of the basic unit - the weighted moment of inertia - with balancing constraints. Zoltners and Sinha, 1983 presented a general approach with a focus on an adjacency tree graph. Roger Z Ríos-Mercado and Fernández, 2009 focuses on minimizing the maximum distance of each district and adds connectivity constraints, proposed by Drexl and Haase, 1999. This constraint is similar to connectivity constraints used in routing problems.

2.2.3 Service Districting

According to Benzarti et al., 2013 the first work done on the home health care problem was done by Busby and Carter, 2006, who created a decision tool for a community care center. Benzarti et al., 2013 developed two models for districting based on the approach made by Bergey et al., 2003, a home health care problem multi-objective. In this problem, it is important to consider districting and the equity of the workload, where compactness has an important job. In one model the compactness is a hard constraint and the workload is an objective function, while in the other, compactness is the objective function and the workload is a hard constraint. One tries to minimize the maximum total workload difference between each district and the average, while the other focuses on compactness and minimizes the maximum distance between any two basic units assigned to the same district. In the first model, compactness is a constraint stating that the distance between two basic units assigned to the same district should not be bigger than the maximum distance predefined. In the second model, the care workload of each district should not deviate from the average care workload by more than a percentage τ, defined *a priori*.

2.2.4 Distribution Districting

The Districting Problem has also applications in logistics and appears in the literature as distribution districting. When there is uncertainty in the demand of customers in the Vehicle Routing Problem some authors propose a two-phase approach (Kalcsics and Roger Z. Ríos-Mercado, 2019). Haugland et al., 2007 assumes that in the first stage, the districts are created with stochastic demand, and later, when the demand is revealed the routing is done. In this problem, the districts should be compact and contiguous, but also the workload should be divided homogeneously among drivers (for example: Galvão et al., 2006) The compactness of the districts helps in the homogeneity of expected the workload since each district should have about the same demand. Moreover, the travel time between customers is reduced.

2.2.5 Reserve Network Design

Other applications of compactness include authors like Önal et al., 2016, who have proposed a model for modeling compact reserves for rare and endangered species within the boundaries of military installations. Önal et al., 2016 refers to several criteria in site selection but focuses on compactness and contiguity. By connecting reserves, animals gain access to more resources across the entire network, boosting their survival rates and chances of populating new habitats. The problem was addressed by partitioning the military area into disjoint spatial units (squares), and then each spatial unit was selected to be part of a reserve in the network or left out. In his problem, each basic unit has a habitat quality, where each reserve must provide the minimum habitat quality, and the reserves together must also have a habitat quality above the desired. These constraints cause contiguity issues in the model, which were solved by adding a contiguity constraint and by changing the definition of distance between any two spatial units. Other compactness measures were utilized in the selection of reserve networks as minimizing the perimeter in Önal and Briers, 2003.

Chapter 3

Modeling Compactness

In this chapter, we present models in Integer Programming to select a single compact district and models to select multiple compact districts. As discussed in the previous chapter, there is no consensus on the definition of compactness. Most authors seem to agree on the use of the moment of inertia as a compactness measure, but depending on the context of the problem, another compactness measure might be more appropriate (MacEachren, 1985). Therefore, we intend to study various models, based on different compactness measures and evaluate the results according to the contiguity of the districts with a special focus on compactness. In the models presented, no contiguous constraints were added also to evaluate the contiguity of each objective function.

In this dissertation, we present models based on the first and second moments of area, weighted first and second moments of area, diameter, and perimeter. We opted to use these measures due to the simplicity of calculation of the parameters and implementation of the models. Furthermore, the models implemented have been used in the literature in the context of the Districting Problem in various applications.

Given a set I with n basic units, we can build a model to select compact districts based on Hess's work. We assume that each basic unit can be approximated to a planar simple polygon. Consider the decision variable x_{ki}, it is 1 if basic unit i belongs to the district centered in k, and equals 0 otherwise. Consider, as well, an interval L and U, respectively, the lower and upper bounds of the area of each district. The most common parameters used in compactness are the area of the shape A, the perimeter P, and the maximum distance or diameter L. All those parameters can be calculated, for each basic unit, using the formulas in (2.2), (2.3), (2.4), respectively. However, to calculate the perimeter of a shape, that results in the union of shapes, the contact perimeter also needs to be calculated. The contact perimeter of any two shapes can be calculated by verifying if an edge of a shape belongs to the adjacent shape. If it belongs, then we sum the length of that edge, if not, we move to the next edge. Thus, let A_i be the area of a basic unit i, CP_{ij} the contact perimeter of basic units i and j and D_{ij} the maximum distance between any two vertices of basic units i and j. Note that CP_{ii} is the perimeter of basic unit i. Additionally, since the moment of inertia's compactness measure seems to return good results (Li et al., 2013; MacEachren, 1985), consider d_{ki} the distance between the centroids of basic units k and i.

3.1 Single District Model

As seen above, there are many ways to measure the compactness of a shape. Most of the compactness measures are introduced as fractions using some characteristics of a shape, generally using the area in the numerator. One possible approach to model compactness in Integer Programming is by fixing the area of a district in a close interval and minimizing the denominator of the fraction. By considering different objective functions, using the same constraints, it is possible to analyze which objective function produces more compact districts. A general model for selecting a single district can be seen below, using the x_{ki} decision variable defined earlier. The x_{ki} is a binary decision variable that takes the value 1 if basic unit i belongs to the district centered in k, and takes the value 0 otherwise.

$$\text{max: Compactness} \tag{3.1}$$

$$\text{s. t.:} \sum_{k \in I} x_{kk} = 1 \tag{3.2}$$

$$\sum_{k \in I} x_{ki} \leq 1 \qquad \forall i \in I \tag{3.3}$$

$$L x_{kk} \leq \sum_{i \in I} A_i x_{ki} \leq U x_{kk} \qquad \forall k \in I \tag{3.4}$$

$$x_{ki} \in \{0,1\} \qquad \forall k,i \in I \tag{3.5}$$

In this model, we create one district with an area in the interval $[L,U]$ by selecting a set of basic units. The objective function (3.1) maximizes the compactness of the district. Constraints (3.2) ensures that only one district is selected, while (3.3) ensures that each basic unit belongs, at most, to one district. (3.4) implies that the district has an area bigger than L and smaller than U. Finally, (3.5) guarantees that the decision variables are binary.

Since the area of the district is restricted, the moment of area can be translated into an objective function by minimizing the sum of the distance between a central basic unit and the others. That distance between any two basic units is, usually, given by the distance between the centroids of each basic unit. Thus, the objective functions (3.6) and (3.7) can be added to the model, respectively, the first and second moments of area. Note that these objective functions are an estimate of the true moment of area. It is expected that, in general, the model with the objective function (3.7) returns no worse results than (3.6), since by squaring the distance, farther basic units are more penalized. The objective functions (3.6) and (3.7) prioritize bigger basic units, and lastly, the smaller ones. By multiplying the distance squared by the area of each basic unit, similar to Hess's model, bigger basic units have a bigger weight to the objective function. The weighted first and second moments of area, are as objective functions in (3.8) and (3.9), respectively.

$$\min : \sum_{i \in I} \sum_{k \in I} d_{ki} x_{ki} \tag{3.6}$$

$$\min : \sum_{i \in I} \sum_{k \in I} d_{ki}^2 x_{ki} \tag{3.7}$$

$$\min : \sum_{i \in I} \sum_{k \in I} d_{ki} A_i x_{ki} \tag{3.8}$$

$$\min : \sum_{i \in I} \sum_{k \in I} d_{ki}^2 A_i x_{ki} \tag{3.9}$$

In the compactness measures mentioned in section 2.1 the diameter and the perimeter are, also, commonly used in the literature. The diameter of a district is the maximum of maximum distance between any two basic units, while the perimeter of a district can be expressed as the sum of the perimeter of its basic units minus the length of the intersection of its basic units. Note that we must consider the intersection between basic units i and j and between j and i, to calculate the perimeter correctly. Since we are selecting only one district and we don't need to identify the center of the district, we can, therefore, optimize the model, by using the x_i decision variable, instead of the x_{ki}. x_i is binary and takes the value 1 if basic unit i is selected to belong in the district, and 0 otherwise. In (3.10) and (3.11) it is represented these objective functions in Integer Programming, respectively, as well as, the new constraints. Notice that using x_i decision variable we get a model with fewer constraints.

$$\min: \max_{i,j \in I}\{D_{ij}x_i x_j\} \tag{3.10}$$

$$\min: \sum_{i \in I} CP_{ii}x_i - \sum_{i \in I}\sum_{\substack{j \in I \\ j \neq i}} CP_{ij}x_i x_j \tag{3.11}$$

$$\text{s.t.: } L \leq \sum_{i \in I} A_i x_i \leq U \tag{3.12}$$

$$x_i \in \{0,1\} \qquad \forall i \in I \tag{3.13}$$

If we look closer at objective function (3.10) we notice that it is not linear due to the product of two decision variables and we have a min-max problem. The min-max problem is linearized by introducing v, a continuous decision variable that represents the diameter of the district. We get a model minimizing v and we add constraints stating that v is greater or equal to $D_{ij}x_i x_j$ and that v is greater or equal to 0. In (3.10), notice that $x_i x_j$ takes the value 1, if and only if both basic units i and j belong to the district, and 0 otherwise. There, we can replace the product of these two variables by $x_i + x_j - 1$. If both basic units i and j belong to the district we get $v \geq D_{ij}$. If only i or j don't belong to the district the constraint $v \geq 0$ guarantees that v is non-negative. The model can be seen below. The objective function (3.14) minimizes v, the diameter of the district. Constraints (3.15) make sure that v is the diameter of the district. (3.16) is the district area constraint, and (3.17) and (3.18) are the domain constraints for the decision variables.

$$\min: v \tag{3.14}$$

$$\text{s.t.: } v \geq D_{ij}(x_i + x_j - 1) \qquad \forall i,j \in I \tag{3.15}$$

$$L \leq \sum_{i \in I} A_i x_i \leq U \tag{3.16}$$

$$x_i \in \{0,1\} \qquad \forall i \in I \tag{3.17}$$

$$v \geq 0 \tag{3.18}$$

Similarly to the (3.10), the (3.11) objective function is not linear. In this case, we should introduce a new decision variable y_{ij} that takes the value 1 if basic units i and j belong to the district and 0 otherwise. To not overload the problem with variables we can restrict the number of y_{ij} variables created. The value of $CP_{ij} = 0$ for most combinations of i and j, and $CP_{ij} \neq 0$ for adjacent basic units. Then, if we consider the function $\mathcal{N}(i)$, which returns the indices of the basic units adjacent to i, we create y_{ij} for all $i \in I$ and for all $j \in \mathcal{N}(i)$. Thus, the number of variables is reduced. Furthermore, the contact perimeter between basic units i and j is the same as the contact perimeter between j and i. Therefore, we can reduce even further the number of variables in the problem by not creating y_{ji}, if y_{ij} exists. Note that the formula for the perimeter should be adapted accordingly since we have fewer variables. By introducing the new variable to the model we need constraints to guarantee that $y_{ij} = 1$, if and only if, $x_i = 1$ and $x_j = 1$.

Constraints (3.21) and (3.22) state that if $y_{ij} = 1$, then $x_i = 1$ and $x_j = 1$. We have a minimizing objective function with negative coefficients in the y_{ij} variables, thus if the problem has no constraints we would get $y_{ij} = 1\ \forall i, j \in I$ and $x_i = 0\ \forall i \in I$. Then, constraints (3.21) and (3.22) guarantee that $y_{ij} = 0$, if $x_i = 0$ or $x_j = 0$. For simplicity writing the model consider the set $J(i) = \{j, j \in I : i < j, j \in \mathcal{N}(i)\}$. The model can be seen below.

$$\min: \sum_{i\in I} CP_{ii}x_i - 2\sum_{i\in I}\sum_{j\in J(i)} CP_{ij}y_{ij} \tag{3.19}$$

$$\text{s.t.: } L \leq \sum_{i\in I} A_i x_i \leq U \tag{3.20}$$

$$x_i \geq y_{ij} \qquad \forall i \in I, \forall j \in J(i) \tag{3.21}$$

$$x_j \geq y_{ij} \qquad \forall i \in I, \forall j \in J(i) \tag{3.22}$$

$$x_i \in \{0,1\} \qquad \forall i \in I \tag{3.23}$$

$$y_{ij} \in \{0,1\} \qquad \forall i \in I, \forall j \in J(i) \tag{3.24}$$

Note that in the models presented we decided not to add contiguity constraints. It is easy to create examples, for each objective function, obtaining non-contiguous solutions. In figure 3.1, if basic units A, B and C are in the solution and the lower bound area is not reached, another basic unit must be added. If by adding basic units E or F, the solution is still infeasible (because the upper area limit is surpassed), then the only solution for the problem is by adding basic unit D. Note that, for the same model but with contiguity constraints, the instance would be infeasible.

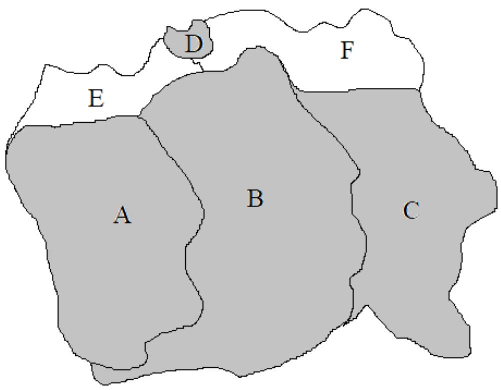


Figure 3.1: Contiguity

In summary, we have the following six single districting models:

- First moment of area - Objective function (3.6); Constraints: (3.2), (3.3), (3.4), (3.5);
- Second moment of area - Objective function (3.7); Constraints: (3.2), (3.3), (3.4), (3.5);
- First weighted moment of area - Objective function (3.8); Constraints: (3.2), (3.3), (3.4), (3.5);
- Second moment weighted of area - Objective function (3.9); Constraints: (3.2), (3.3), (3.4), (3.5);
- Diameter - Objective function: (3.14); Constraints: (3.15), (3.16), (3.17), (3.18);
- Perimeter - Objective function: (3.19); Constraints: (3.20), (3.21), (3.22), (3.23); (3.24);

3.2 Multiple of Districts Model

Similarly to the previous subsection, a general model to select multiple compact districts in a geographical region can be seen below. We choose to use a partition model since in many applications all basic units need to belong in a district.

$$\max: \sum_{k\in I} \text{Compactness}_k \tag{3.25}$$

$$\text{s. t.:} \sum_{k\in I} x_{ki} = 1 \qquad \forall i \in I \tag{3.26}$$

$$Lx_{kk} \le \sum_{i\in I} A_i x_{ki} \le U x_{kk} \qquad \forall k \in I \tag{3.27}$$

$$x_{ki} \in \{0,1\} \qquad \forall k,i \in I \tag{3.28}$$

In this model, a partition of a geographical region is created, and it is maximized the average compactness of the districts. Here we use the x_{ki} decision variable defined earlier. The objective function (3.25) maximizes the average compactness of the districts. Constraint (3.26) ensures that every basic unit belongs to one district and constraint (3.27) implies that each district has an area in the interval $[L,U]$. Finally, (3.28) ensures that the variable is binary.

Similarly, to the single-district model the objective functions (3.6), (3.7), (3.8), (3.9) can be considered without any change. For the diameter and perimeter objective functions, we can use the x_{ki} decision variable, resulting in (3.29) and (3.30), respectively. Although the formulation presented is general and works for every objective function, it is not adequate for the diameter and perimeter objective functions. In both objective functions, we have more decision variables than we need. By reducing the number of decision variables we manage to get optimal solutions quicker.

$$\min: \sum_{k\in I} \max_{i,j\in I} \{D_{ij} x_{ki} x_{kj}\} \tag{3.29}$$

$$\min: \sum_{k\in I}\sum_{i\in I} CP_{ii} x_i - \sum_{i\in I}\sum_{\substack{j\in I\\ j\neq i}} CP_{ij} x_{ki} x_{kj} \tag{3.30}$$

Similar to the single district model a similar linearization can be done. Notice, that in this model we don't need the center of the basic unit, we just need a basic unit that identifies the district. In the example, in the figure 3.2 the solution is the same, but the center (in white) is different. By reducing the number of variables, the number of optimal solutions is also reduced. Hence, instead of considering all x_{ki} decision variables, we can only create the decision variables, where $k \le i$. Thus, the basic units will be assigned to the basic units with the smallest index in the district. Making this change changes the definition of the x_{ki} decision variable, it takes the value 1 if the basic unit i belongs to the district identified by k, and 0 otherwise. For simplicity writing the model consider the set $K(i) = \{k, k \in I : k \le i\}$. The model for the diameter is below. The objective function (3.31) minimizes the average district diameter and constraint (3.32) guarantees that the diameter is the maximum distance between any two basic units in the district. (3.33) guarantees that each basic unit is assigned to one district. Equations (3.35) and (3.36) are domain constraints of the variables.

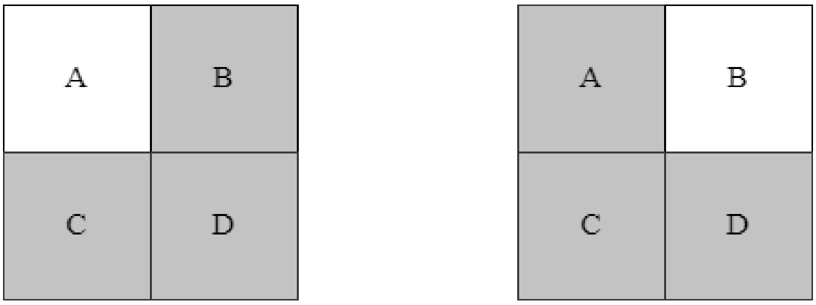


Figure 3.2: Multiple centers

$$\min: \sum_{k \in K(i)} v_k \tag{3.31}$$

$$\text{s.t.: } v_k \geq D_{ij}(x_{ki} + x_{kj} - 1) \qquad \forall k \in K(i) \tag{3.32}$$

$$\sum_{k \in K(i)} x_{ki} = 1 \qquad \forall i \in I \tag{3.33}$$

$$Lx_{kk} \leq \sum_{i \in I} A_i x_{ki} \leq U x_{kk} \qquad \forall k \in K(i) \tag{3.34}$$

$$x_{ki} \in \{0,1\} \qquad \forall i \in I, \forall k \in K(i) \tag{3.35}$$

$$v_k \geq 0 \qquad \forall k \in K(i) \tag{3.36}$$

The formulation for the perimeter follows the same principle as the diameter, not all x_{ki} variables need to be created and a similar linearization to the single district with the perimeter objective function can be done. We may only create variables x_{ki} with $k \leq i$ and we create the y_{kij} binary decision variables. y_{kij} that take the value 1 if basic units i and j belong to the district identified by k. Similarly, to the single district model for the perimeter, we only create the y_{kij} decision variables for $k, i, j \in I$ where $j \in \mathcal{N}(i)$ and $k \leq i < j$. For simplicity writing the model consider the set $K(i) = \{k, k \in I : k \leq i\}$ and the set $J(i) = \{j, j \in I : i < j, j \in \mathcal{N}(i)\}$.The model can be seen below. The objective function (3.37) minimizes the average perimeter of the districts. Equation (3.38) guarantees that every basic unit is assigned to a district. (3.40) and (3.41) guarantee that $y_{kij} = 1$, if and only if $x_{ki} = x_{kj} = 1$. Finally, (3.42) and (3.43) are domain constraints that state that the decision variables are binary.

$$\min: \sum_{k \in K(i)} \left(\sum_{i \in I} CP_{ii} x_{ki} - 2 \sum_{i \in I} \sum_{j \in J(i)} CP_{ij} y_{kij} \right) \tag{3.37}$$

$$\text{s.t.: } \sum_{k \in K(i)} x_{ki} = 1 \qquad \forall i \in I \tag{3.38}$$

$$Lx_{kk} \leq \sum_{i \in I} A_i x_{ki} \leq U x_{kk} \qquad \forall k \in K(i) \tag{3.39}$$

$$x_{ki} \geq y_{kij} \qquad \forall i \in I, \forall k \in K(i), \forall j \in J(i) \tag{3.40}$$

$$x_{kj} \geq y_{kij} \qquad \forall i \in I, \forall k \in K(i), \forall j \in J(i) \tag{3.41}$$

$$x_{ki} \in \{0,1\} \qquad \forall i \in I, \forall k \in K(i) \tag{3.42}$$

$$y_{kij} \in \{0,1\} \qquad \forall i \in I, \forall k \in K(i), \forall j \in J(i) \tag{3.43}$$

In summary, we have the following six multiple districting models:

- First moment of area - Objective function (3.6); Constraints: (3.26), (3.27), (3.28);
- Second moment of area - Objective function (3.7); Constraints: (3.26), (3.27), (3.28);
- First weighted moment of area - Objective function (3.8); Constraints: (3.26), (3.27), (3.28);
- Second weighted moment of area - Objective function (3.9); Constraints: (3.26), (3.27), (3.28);
- Diameter - Objective function: (3.31); Constraints: (3.32), (3.33), (3.34), (3.35), (3.36);
- Perimeter - Objective function: (3.37); Constraints: (3.38), (3.39), (3.40), (3.41); (3.42), (3.43);

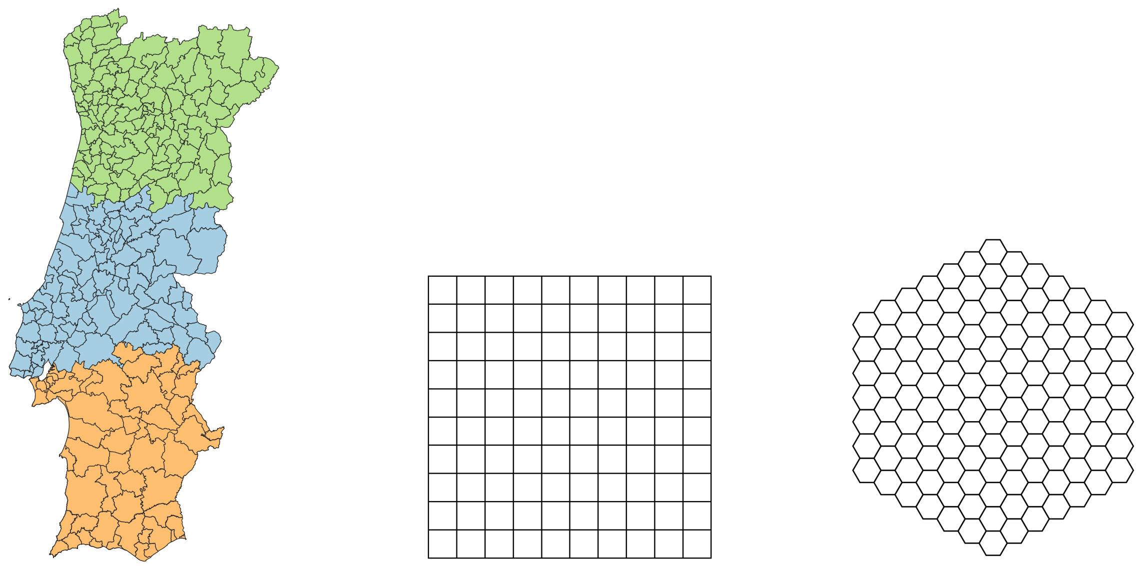
Chapter 4

Computational Experiments

In this chapter, we conduct computational experiments to evaluate the compactness of each solution and compare solutions for the models presented in chapter 3.

4.1 Test Instances

We decided to experiment with these models using different types of instances with continuous and discrete basic units, as seen in the literature (for example: Önal et al., 2016 and Li et al., 2013). We chose to use the municipalities of Portugal's mainland ([SNIG], 2024) for the continuous instances and for the discrete instances, we created a map composed by squares and a map composed of hexagons. Due to the number of municipalities in Portugal, we had to divide the problem into three instances with similar total areas - North, Center, and South. In figure 4.1 it is possible to visualize the instances used, in particular in 4.1a the division of the municipalities of Portugal into three instances. In green, we have the North instance, in blue the Center instance, and in orange the South instance. In 4.1b and 4.1c the two discrete instances.



(a) North, Center, and South instances

(b) Square instance

(c) Hexagon instance

Figure 4.1: Instances tested

The North instance has 125 basic units with a total area of 31508 km^2, while the center instance has

96 basic units with a total area of 29689 km^2. The South instance has a total area of 27906 km^2 and a total of 57 basic units. Visually, the North instance has more and smaller basic units compared to the Center and the South instances, which may help in selecting more compact districts. Conversely, the South instance has fewer and bigger basic units, which may lead to less compact results. Naturally, the average area for the North instance is smaller than the Center which is smaller than the South instance. Relatively to the discrete instances, we have 100 basic units in the Square instance, each square with area 1, while the Hexagon instance has 127 basic units composed of regular hexagons with side length 1. In these instances, since every basic unit has the same area, it is expected to have contiguous solutions.

The solutions presented in the appendix are relative to the models presented above with L and U equal to 15% and 20% of the total area of the instance, respectively. Hence, in the multiple district models, five or six districts are going to be selected. The computational results were made in a Windows Pro machine with an AMD Ryzen 7 7730U processor with 16 GB of memory RAM. The computational time in the tables presented refers to the time took to read the model, load the data, and find the optimal solutions. We establish a limit of 10000 seconds of optimization (less than 3 hours), thus, some solutions presented are not optimal. For those, the value of the objective functions is highlighted. We used the Fico Xpress IVE solver in the version 9.1.

In figure A.1 it is possible to observe the solutions for the six models when it is selected only one district, while in figure A.2 the solutions are for multiple districts. Note that the solution in figures A.2e and A.2f is not optimal, since the computational time was reached. Similarly, figures A.3 and A.5 can describe the solutions for the discrete instances selecting only one district. And in figures A.4 and A.6 the solutions for multiple districts. As in the continuous instances, the solutions for the diameter and perimeter objective functions with multiple districts are not optimal (figures A.4c, A.4d, A.6c, and A.6d), since the computational time limit was reached. Moreover, some of the models have multiple optimal solutions, we present the first optimal solution given by the solver.

In tables B.1 and B.2, in appendix, it is possible to observe the computational results for the single district model, and in tables B.3 and B.4 the computational results for the multiple district model. On those tables, it is described the value of the objective function, the linear relaxation, the respective computational times, and the gaps (the linear relaxation gap and final gap).

Additionally, the compactness of the solutions obtained was evaluated using the original compactness measures. In tables C.1 and C.2 we present the results obtained for the single district models and the continuous and discrete instances, respectively. Similarly, in C.3 and C.4 we showcase the results for the multiple district models.

4.2 Methodology

All the parameters in the models were calculated using the Python programming language. We got access to the data of the municipalities of Portugal and extracted the vertices to an Excel, using QGIS desktop. We implemented a polygon class called MyPolygon in Python to get full control of each basic unit. Each polygon is defined by an ID set on the creation and its vertices. A map, a set of polygons, is defined by a list of polygons, in the MyMap class. For each map, we computed the centroids for each basic unit, the distance between every two centroids, the maximum distance of any two polygons, and the contact perimeter of any two adjacent polygons. We also implemented a software to create personalized maps, with square or hexagon basic units. In this software, the user can add or remove polygons as intended to create customized maps, under the class MyMapEditor. For example, it is possible to create holes in the map or non-contiguous maps. In figure 4.2 we present a possible design of the classes mentioned. The calculation of the parameters for the models took a high number of hours specifically for the perimeter and diameter, since every polygon, from the municipalities of Portugal, has more than 1000 vertices.

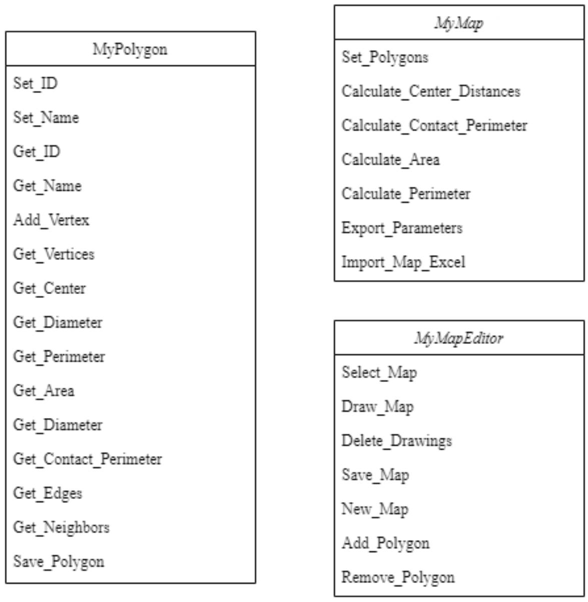
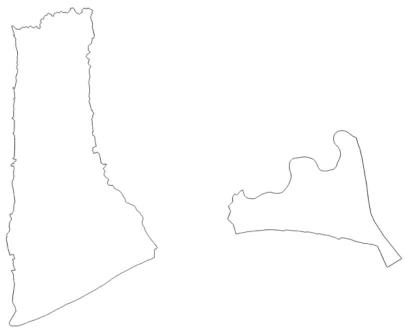
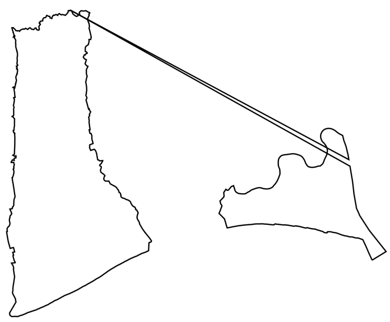


Figure 4.2: Design of classes implemented

Although values for the perimeter and area seem to be reasonable for most basic units, there are a few exceptions. Mainly, in the South instance, we have non-contiguous municipalities, for example, Vila Real de Santo António (see figure 4.3a). This basic unit is composed of two non-contiguous parts so, the values obtained for the perimeter and area do not correspond to the real values of area and perimeter of the municipality of Vila Real de Santo António. As shown in figure 4.3b we present the polygon used to calculate the perimeter and area. The polygons are not the exact same making these values estimates.



(a) Municipalitie of Vila Real de Santo António



(b) Polygon of Vila Real de Santo António

Figure 4.3: Municipalities and Polygons

After having the solutions for all models a program was created to calculate the moment of inertia of each solution (see: Li et al., 2013).

4.3 Discussion

4.3.1 Solution

In A.1 and in A.3 and A.5 we display the solutions obtained from the software, for the single district set of models.

Visually, for the continuous instances, the model whose solution resembles more like the circle is the diameter. However, the districts' perimeter appears higher than other solutions. Eventually, an evaluation compactness measure based on the perimeter might return less compact results. Notice that sometimes there are multiple optimal solutions. For example, other computational results for the diameter model would have the district, in the Center instance, with a small hole. That hole could be added to the district without violating the area constraint.

For models whose objective function is based on the moment of area and weighted moment of area, the solutions do not look very deformed. Nonetheless, extra basic units could have been added to the district making the solution appear more compact. These models do not add extra basic units to the district when the lower area limit is met. Furthermore, in the weighted first moment of area, the solution is non-contiguous, for the Center and South instance.

The districts that aim to minimize the perimeter seem to be reasonably compact. Perhaps other basic units could also be added to the district without reducing compactness. The solution returned by the model is contiguous despite the model does not guarantee contiguity.

Regarding the discrete instances, the solutions are all contiguous and all seem relatively compact. The number of optimal solutions in these instances is vast, for example, in the Square instance, the first and second moments of area solution are still optimal for the diameter model. In the perimeter solution, a basic unit is missing to have a perfect squared district. That basic unit could be added to the solution making the district more compact and without changing the optimal value or violating any constraint.

Similarly, the diameter solution in the Hexagon instance is very irregular with a large perimeter. Probably, basic units could be added making the solution appear more compact, but the value of the objective function would increase. Notice that in the Square and Hexagon instances, the solutions obtained by the models have the least number of basic units to make the solution feasible. We salient that the moments of area models, for each instance, returned the same optimal solution.

Equivalently, to the single district models, in figures A.2 and in A.4 and A.6 we display the solutions obtained for the multiple district models, for the continuous and discrete instances, respectively.

When dealing with multiple districts the compactness of the individual district might decrease since every basic unit must belong to a district with a certain area. We obtained sub-optimal solutions for the most diameter and perimeter models, so, naturally, the districts do not seem very compact in those instances. We opted not to analyze those solutions.

In the continuous instances, the solutions obtained using the first and second moment of area are very similar. For both, there is one non-contiguous district and three with odd shapes.

In the weighted moment of area models, the solutions are also similar. Still, the solution in the weighted second moment of area seems more compact since there are more non-contiguous districts in the weighted first moment of area and fewer districts with odd shapes. It is challenging to visually compare the solutions obtained by the moment of area and weighted moment of area models, and state which ones are the best.

Similarly to the single districts, the discrete instances have multiple optimal solutions. In the square instance, the moment of area models have the same solution, it is only organized differently. While, in the Hexagon instance, we have similar solutions for the moment of area models.

4.3.2 Computational Results

In tables B.1 and B.2 we present the computational results for the different models presented in chapter 3, when modeling a single compact district, for the continuous and discrete instances, respectively. The computational results include the value of the objective function after optimization (Opt.), the time of optimization (Time), the linear relaxation (LR), the time optimizing the linear relaxation (Time), the linear gap (Linear Gap), and the final gap (Final Gap), both in percentage. The computational time for those and the following tables was measured in seconds.

In the continuous instances, the results show that the solution for these models is optimal since the final gap for all the instances and models is 0.0 %. The models based on any of the moments of area took less time to find the optimal solution when compared to the weighted moments of area, the slowest models. Although not significant, the diameter model took more time to optimize than the perimeter model.

The linear gap is greater in the diameter model, of 100 %, followed by the weighted second moment of area. Conversely, the best linear gaps come from the first moment of area and perimeter models. The perimeter model is the slowest to optimize the linear relaxations, although the time compared to the other models is not significant.

In the discrete instances, the diameter is the slowest model to find an optimal solution, and the linear gap is 100%. The perimeter model is the fastest to return the optimal solution, but the linear gap is still high. The best linear gap was returned by the first moment of the area model.

Similarly, in tables B.3 and B.4 we display the best value found (Opt.), the computational time (Time), as well as, the linear Relaxation (LR), the corresponding time (Time), and the linear and final gaps (Linear Gap, Final Gap), for the multiple districts model, for all instances. For simplicity, we decided to highlight the sub-optimal solutions, for the remainder of this dissertation.

In this set of tests we have sub-optimal solutions, mainly in the diameter and perimeter models, except for the South instance with the perimeter model. Notice that for any of the highlighted instances, the optimization time is over 10000 seconds. The optimization time is higher for the weighted moments of area than in the first and second moments of area models.

For the continuous instances, we get that the perimeter models have lower linear relaxation gaps. Conversely, in the discrete solutions, we have the first moment of area model getting better linear gaps, followed by the perimeter. In both continuous and discrete instances, we get that the diameter has the higher relations gaps, about 100%, followed by the second moment of area.

The computational time, optimizing the linear relaxations is, higher in the diameter and perimeter models. For the continuous instances, we get a computational time lower than 20 seconds, while for the discrete instances, the time increases to 40 seconds.

4.3.3 Evaluation

As discussed in the previous section, it is difficult to identify a single best model using only intuition, mainly because the perception of the compactness of two individuals may vary. We choose to analyze the solutions obtained, using the compactness measures that originated the models presented. We utilized, to calculate the performances, the equations presented in (2.5), for the perimeter, (2.7) for the diameter, and (2.17) for the moment of inertia.

4. COMPUTATIONAL EXPERIMENTS

In tables C.1 and C.2 we present the compactness value of the solution given by the multiple objective functions of the single districts models, for each evaluation measure and each instance.

An analysis of the results shows that, for the continuous instances, the models based on the diameter and perimeter tend to have better results for all the evaluation measures. There is one exception, for the North instance in the diameter model and the perimeter evaluation measure, which has a compactness of 0.24, while the models for the moments of area and weighted moments of area have a compactness of 0.30.

The perimeter evaluation measure rates better for the perimeter model, as well as, the diameter evaluation measure returns better values in the diameter model. Regarding, the moment of inertia evaluation measure we have the highest results for the district with the diameter model.

Generally speaking, the diameter and the perimeter objective functions seem to perform better than the moment of inertia models, despite taking more computational time.

The discrete instances had similar results. In the Square instance, the solutions returned by the perimeter model perform better in two out of three evaluation measures and only perform better in one evaluation measure in the Hexagon instance. In both instances, the perimeter evaluation measure performs better in the perimeter model. Similarly, the diameter evaluation measure returns better compactness values for the diameter model. The moment of inertia evaluation measure returns better compactness values for the perimeter model, in the Square instance. In the Hexagon instance, the models based on the moments of area perform better when using the moment of inertia evaluation measure.

The evaluation measures chosen to get compactness ratings are very different, from each other. The moment of inertia compactness measure rates the solutions of the models with compactness above 0.73, while the diameter compactness measure rates the solutions presented between 0.24 and 0.83. Finally, the perimeter compactness measure returns a compactness values ranging from 0.19 to 0.74.

Similarly to the previous tables, in C.3 and C.4 we can see the performance, for each evaluation measure, for the solution obtained in each multiple district model. Note that, in the last tables, because of the multiple districts, we present the minimum, maximum, and average compactness values for each evaluation measure. The solution is not optimal for most of the diameter and perimeter objective functions, for that reason, those were excluded from the analysis.

There is no consensus on whether weighted first and second moments of area have better average performances than the models based on the moment of area. For the North instance, the evaluation seems better for the weighted moments of area models. The weighted second moment of area model performs better than the other models, in the Center instance, although we get very similar average performances. In the South instance, the first and second moment of area models perform better than the other models. It appears that depending on the type of instance different models might return better results.

In the North and Center instances, there seems to be more of an equilibrium among the compactness of the districts. The minimum and maximum compactness are not very distant from the average. However, in the South instance, there is an unbalance between the compactness of the districts. We have for example an average compactness of 0.82, using the moment of inertia evaluation measure, and a minimum compactness of 0.66.

In the discrete instances, the average compactness for each evaluation measure is the same and there is not much difference between the minimum and maximum compactness values for each model. Therefore, the discrete set of tests is unquestionably similar.

Chapter 5

Conclusion and Future Work

This chapter, the last of this dissertation, mentioned the relevant conclusions of this work. The chapter ends with a brief discussion of possible future work.

5.1 Conclusion

We used several Integer Programming and Mixed-Integer Programming models to select compact districts in continuous and discrete instances. The models developed aim to compare and study the different ways to create compact districts. We presented two models based on the moment of inertia. The models minimize an estimate for the first and second moment of area of the districts. We also adapted these two models, using the weighted moments of area, similar to Hess's model in political redistricting. We also created other models influenced by other compactness measures and the easiness of modeling them in Integer Programming. Namely, we tried to minimize the diameter and the perimeter of the districts. In total, we created twelve models, six for creating a single district and another six for creating multiple districts.

Our computational results were based on continuous and discrete instances, as some applications in the literature (for example: S. W. Hess et al., 1965 for continuous instances and Önal et al., 2016 for squared-like instances). The continuous instances utilized were the maps of the municipalities of Portugal. Due to the elevated number of basic units, we had to divide the map into three instances, the North, the Center, and the South. Each instance has a different number and size of basic units. The discrete instances correspond to two large areas, one composed of squares and another of hexagons. The solutions obtained were evaluated and compared among themselves, using original compactness measures.

The work presented in this dissertation makes us conclude that more computational results need to be made, especially with extra computational time and contiguity constraints and different restrictions on the area for each district. The results obtained show that there is no best way to model compactness and depending on the instance in hand and its characteristics different models might have better results. Also, depending on the application, different models might be preferable.

We also conclude that the models based on the moment of the area tend to return solutions quicker but sometimes with worse performances. The diameter and perimeter models did not get an optimal solution in a reasonable time for the multiple districts. Yet these models might be a good option for selecting a single district. Generally, although no contiguity constraints were added, the models based on the moment of area returned more often non-contiguous solutions.

The models presented do not maximize compactness directly. They often stop optimizing once the area limit is reached, even though more units could fit without exceeding the upper area limit. These basic units could lead to an increase in the compactness of the district, but the models do not consider them because they would increase the value of the objective function. Therefore, some of the models presented provide a good initial solution that satisfies the area constraint, it might not be the most compact option. The initial solutions might be a good starting point for heuristic methods that maximize compactness and/or other criteria.

5.2 Future Work

One of the evident limitations of this dissertation was the computational time and the complexity of the models. The instances used, have many basic units which increase the number of variables and constraints exponentially (mainly in the diameter and perimeter multiple district models). We made efforts to reduce the number of variables in the models, but we still got non-optimal solutions with less than three hours of optimization. Future work could explore other techniques and methods to reduce the number of variables and constraints, developing more efficient models. Additionally, investigating alternative optimization solvers or parallel computing approaches might improve the ability to tackle larger and more complex models.

The models presented in this dissertation do not guarantee contiguous districts. Thus, future work could involve adding contiguity constraints and analyzing and comparing the solutions obtained. Potentially, with contiguity constraints, the solutions returned by the models could improve the overall compactness while reducing the computational time. Furthermore, the development of post-optimization heuristics, based on a compactness measure, could improve the solutions returned by these models, such as MERGE proposed by Li et al., 2014. Depending on the context of the problem, these algorithms can consider multiple criteria not only compactness and contiguity.

This dissertation relies on the evaluation of the solutions returned by the models using the original compactness measures. However, using such measures could lead to bias. Future work could explore the usage of alternative compactness measures for a more impartial evaluation of each solution. Moreover, we tried to apply the statistical model presented by Kaufman et al., 2021, which evaluates compactness using human intuition, but computational limitations prevented its full implementation. Further research with this model and more powerful computing resources could provide valuable insights into evaluating compact districts.

This study focused on specific optimization models and compactness measures. Future research could explore developing and testing new models based on different compactness measures. Additionally, directly optimizing for a chosen compactness measure through Linear Fractional Programming is a possible path with potential promising outcomes.

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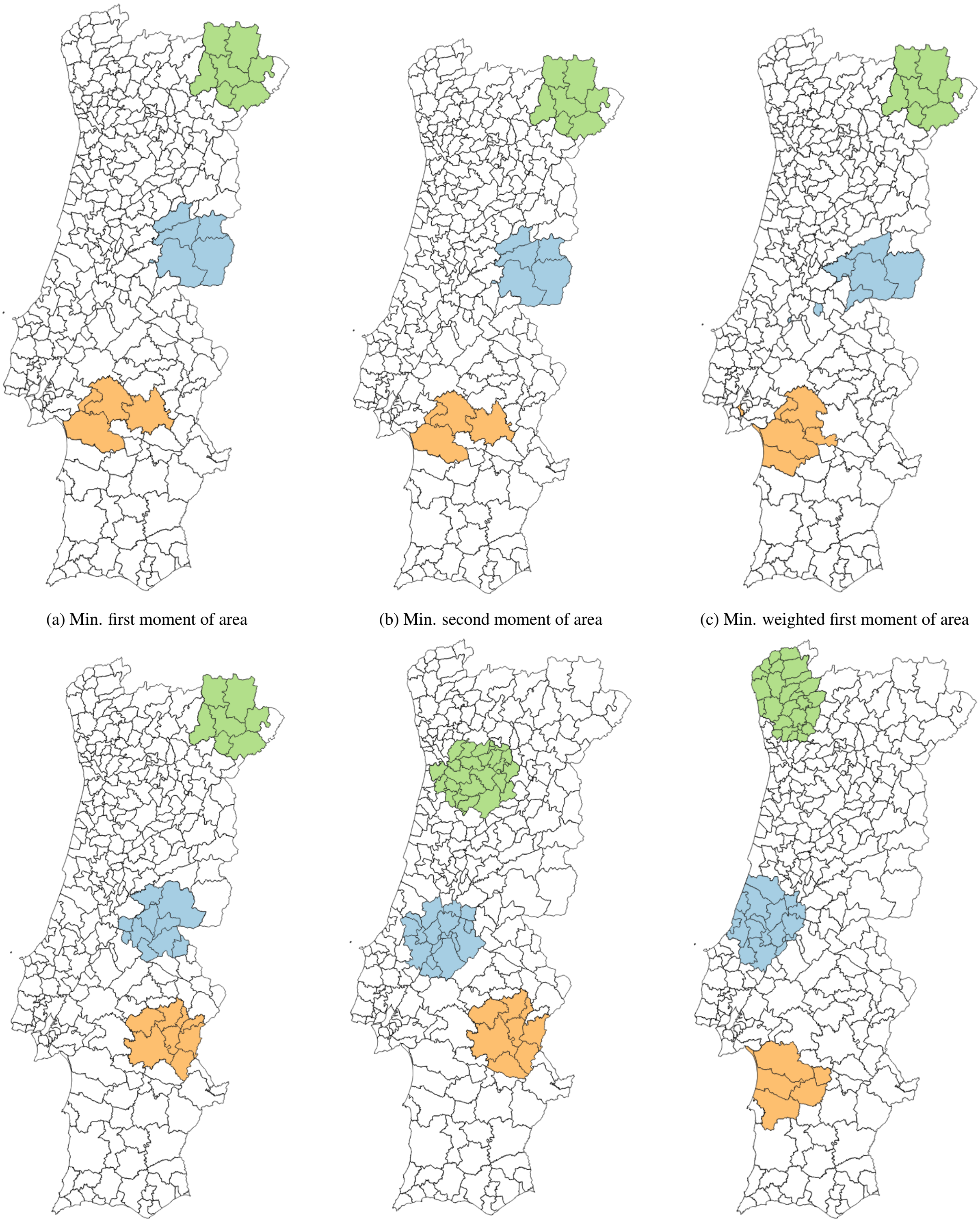
Appendix A

Solutions

A. SOLUTIONS



(a) Min. first moment of area

(b) Min. second moment of area

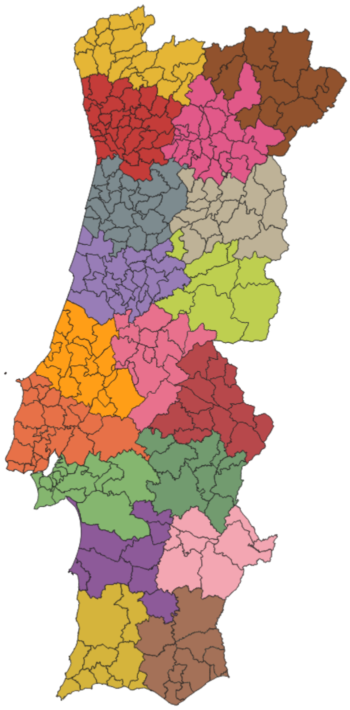
(c) Min. weighted first moment of area

(d) Min. weighted second moment of area

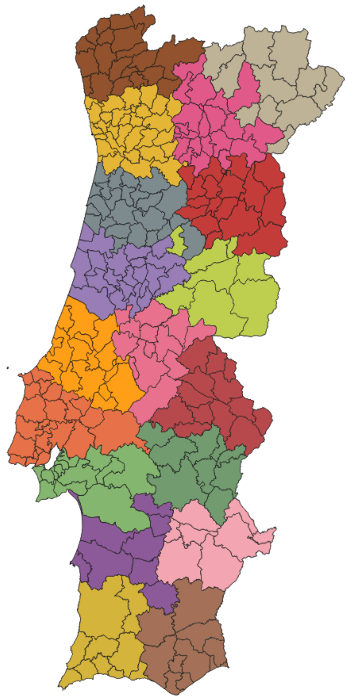
(e) Min. diameter

(f) Min. perimeter

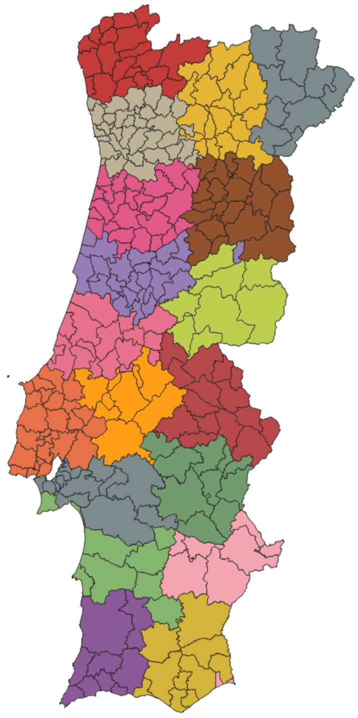
Figure A.1: Districts obtained for the single district models, for the North, Center, and South instances, with $L = 15\%$ and $U = 20\%$ of the total area of the instance.



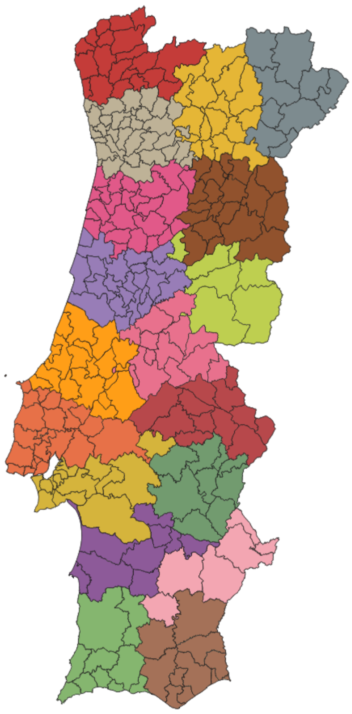
(a) Min. first moment of area



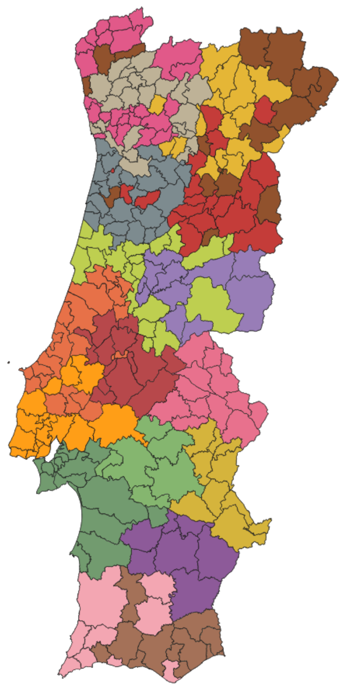
(b) Min. second moment of area



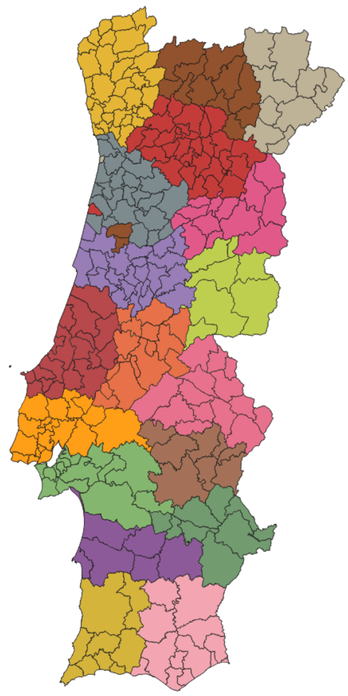
(c) Min. weighted first moment of area



(d) Min. weighted second moment of area

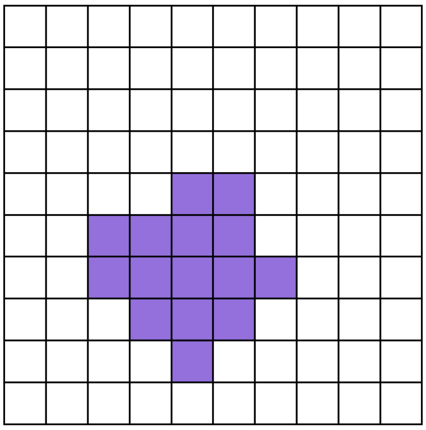


(e) Min. diameter (71.4%, 55.8%, 39.4%)

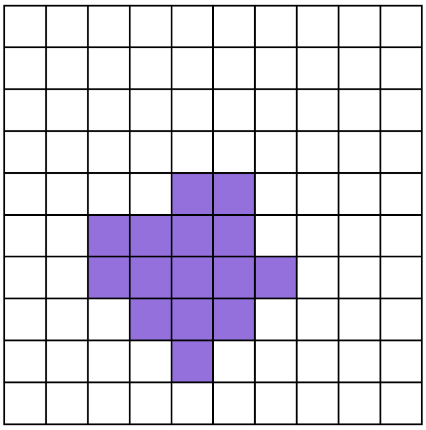


(f) Min. perimeter (38.7%, 21.3%, 0.0%)

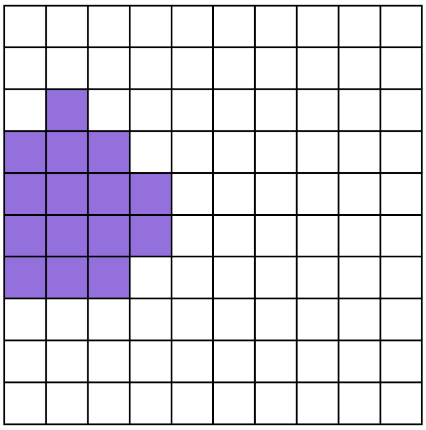
Figure A.2: Districts obtained for the multiple district models, for the North, Center, and South instances, with $L = 15\%$ and $U = 20\%$ of the total area of the instance.



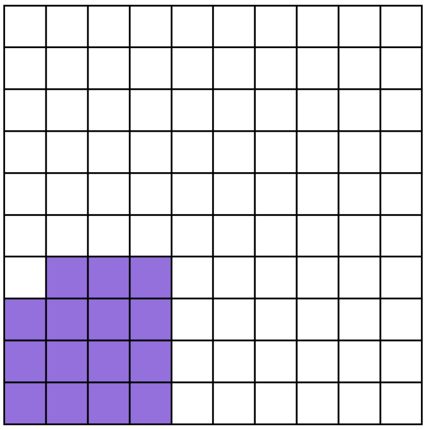
(a) Min. first moment of area



(b) Min. second moment of area

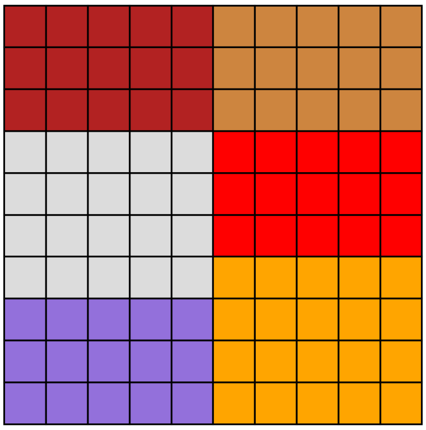


(c) Min. diameter

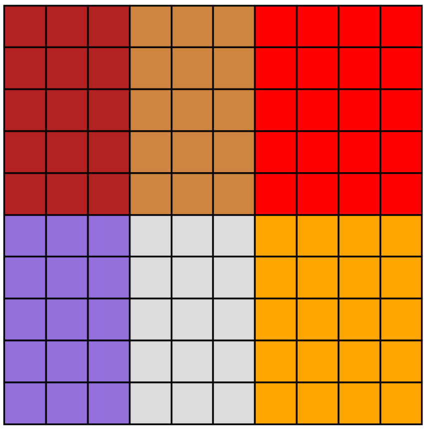


(d) Min. perimeter

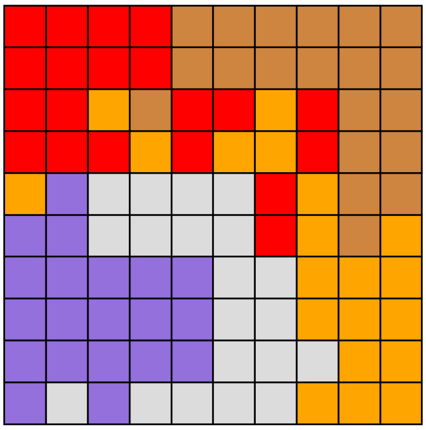
Figure A.3: Districts obtained for the single district models, for the Square instance, with $L = 15\%$ and $U = 20\%$ of the total area of the instance.



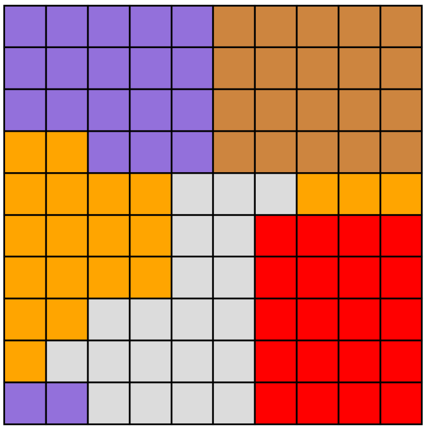
(a) Min. first moment of area



(b) Min. second moment of area

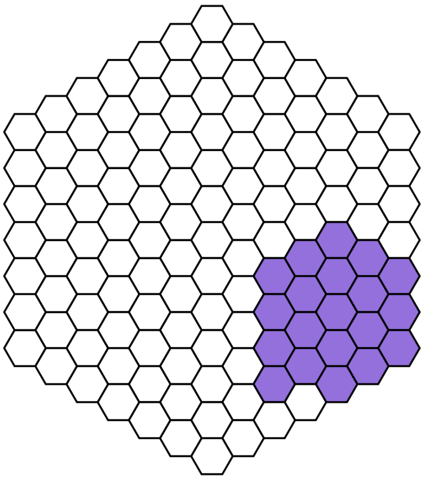


(c) Min. diameter (63.6%)

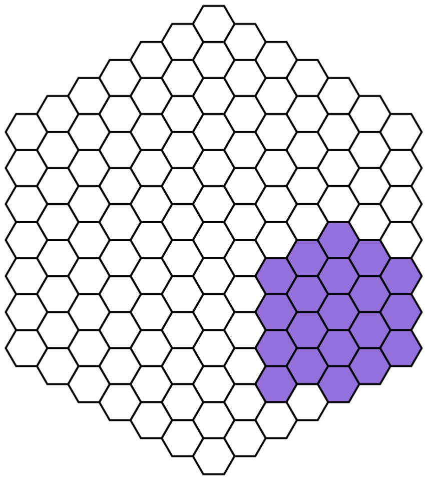


(d) Min. perimeter (43.4%)

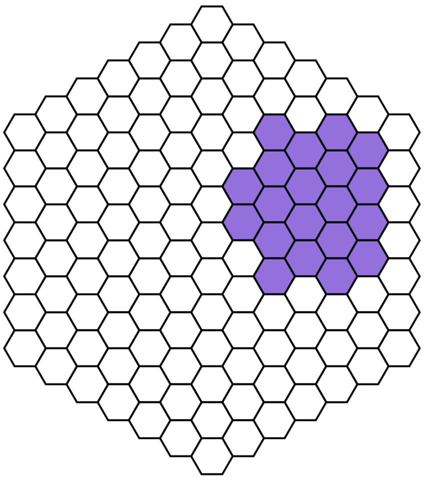
Figure A.4: Districts obtained for the multiple district models, for the Square instance, with $L = 15\%$ and $U = 20\%$ of the total area of the instance.



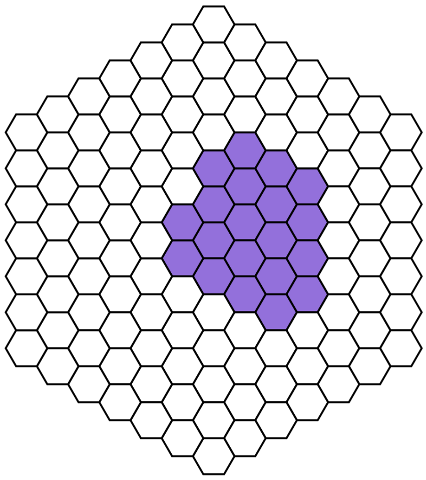
(a) Min. first moment of area



(b) Min. second moment of area

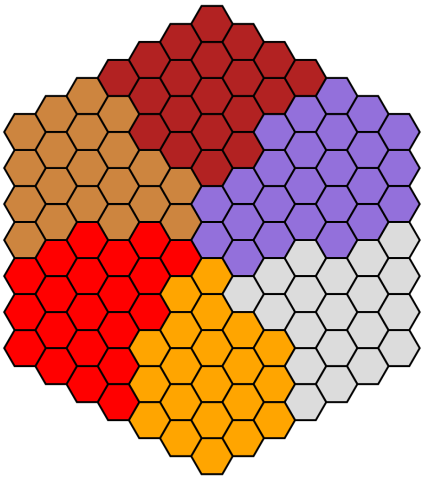


(c) Min. diameter

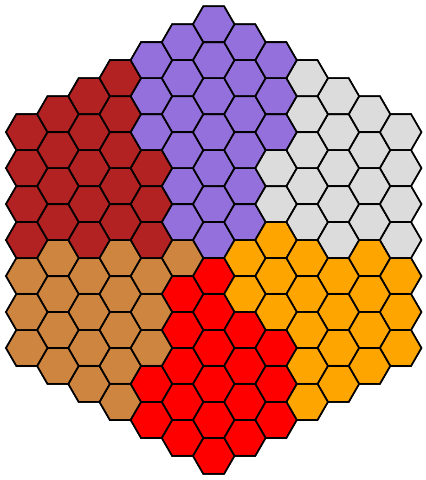


(d) Min. perimeter

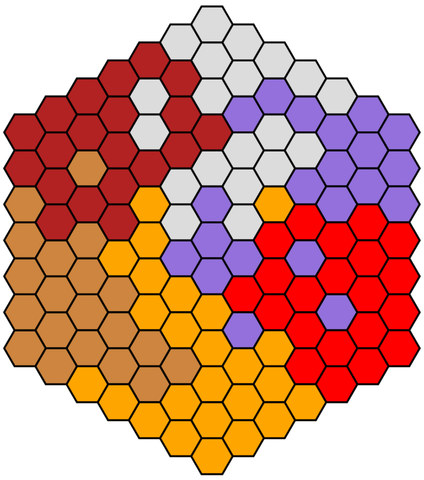
Figure A.5: Districts obtained for the single district models, for the Hexagon instance, with $L = 15\%$ and $U = 20\%$ of the total area of the instance.



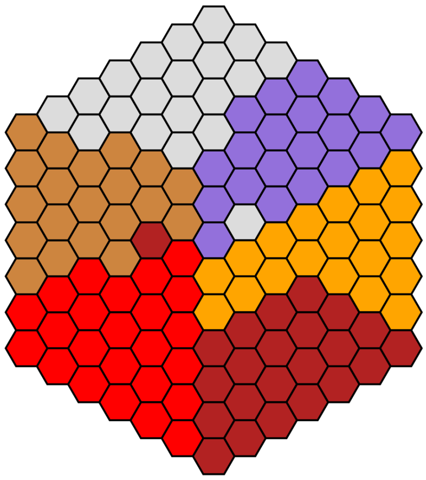
(a) Min. first moment of area



(b) Min. second moment of area



(c) Min. diameter (72.2%)



(d) Min. perimeter (48.0%)

Figure A.6: Districts obtained for the multiple district models, for the Hexagon instance, with $L = 15\%$ and $U = 20\%$ of the total area of the instance.

Appendix B

Computational Results

B. COMPUTATIONAL RESULTS

Table B.1: Computational results obtained for the single district models, for the North, Center, and South instances, with $L = 15\%$ and $U = 20\%$ of the total area of the instance.

Instance		Obj. Func.	Integer Model		Linear Relaxation		Gap	
			Opt.	Time	LR	Time	Linear	Final
North	125	First M.	164.0	1.38	84.0	0.17	48.2	0.0
		Second M.	4544.0	1.15	1049.8	0.18	76.9	0.0
		W. First M.	113464.0	13.41	32287.4	0.17	71.5	0.0
		W. Second M.	3170968.0	6.02	230115.0	0.17	92.7	0.0
		Diameter	85.0	10.49	0.0	0.15	100.0	0.0
		Perimeter	402.0	1.88	185.4	0.27	53.7	0.0
Center	96	First M.	104.0	0.36	64.9	0.10	37.5	0.0
		Second M.	2794.0	0.34	970.5	0.10	65.2	0.0
		W. First M.	90417.0	3.36	32596.8	0.12	64.0	0.0
		W. Second M.	2619617.0	2.45	254281.8	0.12	90.3	0.0
		Diameter	89.0	1.96	0.0	0.09	100.0	0.0
		Perimeter	372.0	0.58	206.2	0.19	44.4	0.0
South	57	First M.	90.0	0.16	55.0	0.07	38.9	0.0
		Second M.	2826.0	0.18	906.6	0.06	67.9	0.0
		W. First M.	88315.0	0.69	43940.2	0.06	50.2	0.0
		W. Second M.	2638863.0	0.47	506094.4	0.07	80.8	0.0
		Diameter	91.0	0.38	0.0	0.07	100.0	0.0
		Perimeter	404.0	0.44	188.5	0.08	53.2	0.0

Table B.2: Computational results obtained for the single district models, for the Square, and Hexagon instances, with $L = 15\%$ and $U = 20\%$ of the total area of the instance.

Instance		Obj. Func.	Integer Model		Linear Relaxation		Gap	
			Opt.	Time	LR	Time	Linear	Final
Square	100	First M.	22.1	3.06	14.0	0.12	36.7	0.0
		Second M.	38.0	3.09	14.0	0.12	63.1	0.0
		Diameter	5.4	4.29	0.0	0.07	100.0	0.0
		Perimeter	16.0	0.44	6.0	0.19	62.5	0.0
Hexagon	127	First M.	53.7	5.30	31.2	0.16	40.4	0.0
		Second M.	164.8	3.60	54.0	0.16	66.6	0.0
		Diameter	9.5	8.53	0.0	0.11	100.0	0.0
		Perimeter	32.0	2.88	11.7	0.62	62.5	0.0

Table B.3: Computational results obtained for the multiple district models, for the North, Center, and South instances, with $L = 15\%$ and $U = 20\%$ of the total area of the instance.

Instance		Obj. Func.	Integer Model		Linear Relaxation		Gap	
			Opt.	Time	LR	Time	Linear	Final
North	125	First M.	3300.0	605.94	1314.6	0.15	55.1	0.0
		Second M.	104959.0	307.30	16289.1	0.15	79.2	0.0
		W. First M.	870920.0	3817.63	391110.9	0.15	60.1	0.0
		W. Second M.	27808352.0	524.99	5793232.3	0.16	84.5	0.0
		Diameter	831.0	10019.71	27.0	29.81	96.8	71.4
		Perimeter	3100.0	10002.58	1732.9	7.48	44.1	38.7
Center	96	First M.	2448.0	39.57	1080.8	0.08	48.1	0.0
		Second M.	76366.0	11.84	14961.9	0.09	73.4	0.0
		W. First M.	794386.0	164.27	412619.1	0.10	55.8	0.0
		W. Second M.	26240029.0	77.41	6980683.9	0.11	80.4	0.0
		Diameter	748.0	10006.24	43.0	5.76	94.2	55.8
		Perimeter	3137.0	10000.56	2179.7	12.33	30.5	21.3
South	57	First M.	1474.0	1.48	841.7	0.06	33.6	0.0
		Second M.	50023.0	0.68	15847.8	0.07	56.7	0.0
		W. First M.	727457.0	4.23	482874.4	0.06	42.9	0.0
		W. Second M.	24713806.0	1.99	10711423.0	0.07	68.3	0.0
		Diameter	695.0	10001.44	39.0	1.24	94.4	39.4
		Perimeter	3610.0	1778.36	2695.7	0.55	25.3	0.0

Table B.4: Computational results obtained for the multiple district models, for the Square, and Hexagon instances, with $L = 15\%$ and $U = 20\%$ of the total area of the instance.

Instance		Obj. Func.	Integer Model		Linear Relaxation		Gap	
			Opt.	Time	LR	Time	Linear	Final
Square	100	First M.	159.9	9768.14	93.3	0.36	41.2	0.0
		Second M.	300.2	306.06	93.3	0.11	68.7	0.0
		Diameter	42.9	10008.11	1.4	6.41	95.3	63.6
		Perimeter	112.0	10004.82	56.3	3.34	49.1	43.4
Hexagon	127	First M.	366.4	93.67	208.2	0.17	43.0	0.0
		Second M.	1237.8	428.21	360.1	0.36	70.8	0.0
		Diameter	73.8	10014.33	2.0	13.64	97.3	72.2
		Perimeter	238.0	10001.14	113.6	43.38	52.1	48.0

Appendix C

Evaluation Results

C. EVALUATION RESULTS

Table C.1: Evaluation of the solutions obtained by the single district models, for the North, Center, and South instances, using γ_2, γ_4, and γ_{14} compactness measures.

Instance	Obj. Func.	Evaluation Measures		
		γ_2 (Perimeter)	γ_4 (Diameter)	γ_{14} (Mom. of Inertia)
North	First M.	0.30	0.64	0.92
	Second M.	0.30	0.64	0.92
	W. First M.	0.30	0.64	0.92
	W. Second M.	0.30	0.64	0.92
	Diameter	0.24	0.83	0.96
	Perimeter	0.37	0.66	0.92
Center	First M.	0.29	0.73	0.95
	Second M.	0.29	0.73	0.95
	W. First M.	0.19	0.24	0.76
	W. Second M.	0.24	0.67	0.90
	Diameter	0.29	0.73	0.97
	Perimeter	0.43	0.61	0.93
South	First M.	0.19	0.44	0.73
	Second M.	0.19	0.44	0.73
	W. First M.	0.21	0.44	0.85
	W. Second M.	0.24	0.65	0.92
	Diameter	0.24	0.71	0.94
	Perimeter	0.33	0.62	0.89

Table C.2: Evaluation of the solutions obtained by the single district models, for the Square, and Hexagon instances, using γ_2, γ_4, and γ_{14} compactness measures.

Instance	Obj. Func.	Evaluation Measures		
		γ_2 (Perimeter)	γ_4 (Diameter)	γ_{14} (Mom. of Inertia)
Square	First M.	0.47	0.66	0.90
	Second M.	0.47	0.66	0.90
	Diameter	0.58	0.66	0.92
	Perimeter	0.74	0.60	0.95
Hexagon	First M.	0.64	0.68	0.96
	Second M.	0.64	0.68	0.96
	Diameter	0.57	0.73	0.95
	Perimeter	0.64	0.68	0.95

Table C.3: Evaluation of the solutions obtained by the multiple district models, for the North, Center, and South instances, using γ_2, γ_4, and γ_{14} compactness measures.

Instance	Obj. Func.	γ_2 (Perimeter)			γ_4 (Diameter)			γ_{14} (Mom. of Inertia)		
		Min	Max	Avg.	Min	Max	Avg.	Min	Max	Avg.
North	First M.	0.17	0.27	0.22	0.50	0.63	0.57	0.70	0.91	0.83
	Second M.	0.14	0.25	0.20	0.47	0.59	0.53	0.78	0.92	0.83
	W. First M.	0.20	0.31	0.24	0.53	0.64	0.58	0.79	0.93	0.86
	W. Second M.	0.20	0.31	0.24	0.53	0.64	0.58	0.79	0.93	0.86
	Diameter	0.05	0.10	0.07	0.17	0.63	0.40	0.19	0.73	0.47
	Perimeter	0.20	0.34	0.25	0.14	0.57	0.36	0.34	0.86	0.72
Center	First M.	0.17	0.33	0.22	0.34	0.57	0.49	0.72	0.87	0.80
	Second M.	0.17	0.33	0.22	0.34	0.57	0.49	0.72	0.87	0.80
	W. First M.	0.14	0.28	0.20	0.28	0.62	0.48	0.66	0.92	0.81
	W. Second M.	0.16	0.31	0.22	0.33	0.60	0.50	0.72	0.89	0.82
	Diameter	0.05	0.23	0.12	0.29	0.57	0.43	0.38	0.82	0.58
	Perimeter	0.15	0.29	0.24	0.40	0.54	0.47	0.50	0.96	0.74
South	First M.	0.09	0.26	0.17	0.31	0.63	0.49	0.66	0.90	0.82
	Second M.	0.09	0.26	0.17	0.31	0.63	0.49	0.66	0.90	0.82
	W. First M.	0.09	0.24	0.16	0.22	0.55	0.40	0.60	0.89	0.78
	W. Second M.	0.08	0.29	0.17	0.33	0.74	0.48	0.59	0.95	0.77
	Diameter	0.04	0.25	0.13	0.35	0.62	0.46	0.53	0.90	0.67
	Perimeter	0.10	0.24	0.17	0.31	0.60	0.46	0.65	0.93	0.81

Table C.4: Evaluation of the solutions obtained by the multiple district models, for the Square and Hexagon instances, using γ_2, γ_4, and γ_{14} compactness measures.

Instance	Obj. Func.	Evaluation Measures								
		γ_2 (Perimeter)			γ_4 (Diameter)			γ_{14} (Mom. of Inertia)		
		Min	Max	Avg.	Min	Max	Avg.	Min	Max	Avg.
Square	First M.	0.74	0.78	0.75	0.56	0.62	0.58	0.84	0.93	0.87
	Second M.	0.74	0.78	0.75	0.56	0.62	0.58	0.84	0.93	0.87
	Diameter	0.14	0.44	0.26	0.19	0.62	0.39	0.28	0.84	0.52
	Perimeter	0.32	0.78	0.55	0.20	0.62	0.41	0.35	0.93	0.65
Hexagon	First M.	0.50	0.59	0.54	0.55	0.64	0.59	0.84	0.93	0.88
	Second M.	0.50	0.57	0.54	0.55	0.64	0.59	0.84	0.91	0.88
	Diameter	0.13	0.31	0.22	0.35	0.72	0.48	0.41	0.81	0.61
	Perimeter	0.37	0.54	0.44	0.34	0.61	0.46	0.62	0.86	0.73