Universidade de Lisboa
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Option Pricing in Illiquid Markets with Jumps

José Manuel Teixeira dos Santos Cruz

Supervisors:
Daniel Ševčovič
Maria do Rosário Lourenço Grossinho

PHD thesis elaborated to obtain the degree of Doctor in Applied Mathematics in Economics and Management by the University of Lisbon.

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Contents

Chapter 1

Introduction

We know that the Black-Scholes model is the most used in financial markets because one of the main reasons for using it is the existence of an analytical formula to price European options. However, evidence from the stock market suggests that this model is not the most realistic one, since it assumes that the market is liquid, complete and without transaction costs.

It is well known that the sample paths of a Brownian motion are continuous, but the stock price of a typical company suffers sudden jumps on an intraday scale, making the price trajectories discontinuous. In the classical Black-Scholes model the logarithm of the price process has normal distribution. However the empirical distribution of stock returns exhibits fat tails. Finally, when we calibrate the theoretical prices to the market prices, we realize that the implied volatility is not constant as a function of strike neither as a function of time to maturity, contradicting this way the prediction of the Black-Scholes model. Several alternatives have been proposed in the literature for the generalization of this model. The models with jumps can, at least in part, solve the problems inherent to the Black-Scholes model. The jump models have also an important role in the options market. While in the Black-Scholes model the market is complete, implying that every payoff can be exactly replicated, in jump models there is no perfect hedge and this way the options are not redundant.

We also relax the assumption of liquid market. Investors and risk managers have realized that financial models based upon on the assumption that an investor can trade large amounts of an asset without affecting its price is no longer true in markets that are not liquid. Market Illiquidity has been studied in the literature in [49], [36], [70], [78], [37], [73]. The first major contribution was done by Robert Jarrow, in 1994 who studied the market manipulation strategies that may arise in illiquid markets. This paper also studies option pricing theory, in discrete time, when there is a large trader. The pricing argument used here was a condition to ensure that no market manipulation strategy is used by the large trader and the large trader's optimality conditions, thus replacing the usual free-arbitrage argument.

Then, Frey in 1998 extended Jarrow's analysis to the continuous time case. In this paper is shown a result of existence and uniqueness of solution of a nonlinear partial differential equation satisfied by the large trader's hedging strategy. In the same year, Platten and Schweizer proposed an explanation for the smile and skewness for the implied volatilities and show that hedging strategies followed by large traders can lead to option price bias.

In 1998, Sircar and Papanicolaou present a model in which the derivative security price is characterized by a nonlinear partial differential equation that becomes the Black-Scholes equation when there is no feedback. When the programme traders are a small fraction of the economy the nonlinear partial differential equation is analysed by peturbation methods using numerical and analytical methods. This equation is derived using an argument similar to the one used in the derivation of the classical Black-Scholes equation. The findings are that this model also predict increased implied volatilities as in Platten and Schweizer.

In 2000, Schonbucher and Willmott analyse also the feedback effects from the presence of hedging strategies. Also a nonlinear partial differential equation is derived for an option replication strategy and these effects are studied for the case of a Put option. The effects are more pronounced in markets with low liquidity and can induce discontinuities in the price process.

So this leads us to consider also a jump process into the models already used in the literature.

Jump models have been studied for example in [60], [55], [14], [21], [31], [26], [65] and [21] but none of these papers takes into account the market's illiquidity. So it seems to be a good approach to extend the models that study market's illiquidity to the case where a jump process is considered. This way not only it is assumed that trading strategies affects the stock price but also the possibility to account for sudden jumps that might occur when the market is under stress.

The objective of this thesis is to study under which conditions we can obtain the function that represents the option price as a solution of a certain partial integro-differential equation. Moreover, we will discuss some examples where the price function is not regular enough in order to be a classical solution of this partial integro-differential equation.

The prices of options such as European options and barrier options can be characterized in terms of solutions of a partial integro differential equation with some boundary conditions depending on the type of option considered. Conversely, if we have a solution of a certain partial integro-differential equation (PIDE) satisfying some conditions, then it is possible to arrive at a stochastic representation of the Feynman-Kač kind, analogous to the Black-Scholes case. The main difference between a model with jumps and the Black-Scholes case is a non-local term that appears in the equation, because now the price process possesses jumps, and the option price can be discontinuous. This non-local term makes PIDEs less easy to solve than partial differential equations. However, one of the numerical schemes used in the literature is presented to solve such equations. In analytical terms, if the price is not a classical (smooth) solution of the PIDE, the notion of viscosity solution can be used. In this thesis, we also analyze existence and uniqueness of solutions to the partial integro-differential equation (PIDE) in the framework of Bessel potential spaces. As a model we consider a model for pricing vanilla call and put options on underlying assets following Lévy stochastic processes. Using the theory of abstract semilinear parabolic equations we prove existence and uniqueness of solutions in the Bessel potential space representing a fractional power space of the space of Lebesgue p-integrable functions with respect to the second order Laplace differential operator. We generalize known existence results for a wider class of Lévy measures including those having strong singular kernel. We also prove existence and uniqueness of solutions to the penalized PIDE representing approximation of the linear complementarity problem arising in pricing American style of options.

In a fixed income market, practitioners usually use the price of standard (plain vanilla) products corrected by an adjustment called the convexity adjustment. Convexity adjustments are used by practitioners to value non standard products using information on plain vanilla products.

In this thesis we explicitly compute the interbank convexity adjustment of FRAs (Forward Rate Agreements), combining the classical affine term structure (ATS) framework with a shot-noise process that is able to capture the counter-party risk of interbank contracts.

In Section 2.1.1 we recall basic notions related to option pricing models with underlying assets following Lévy stochastic processes. In Section 2.1.2 we introduce the Lévy exponential models for financial assets. In Section 2.2 we give some examples of financial models and we introduce a notion of an admissible activity Lévy measure and we show that this class of Lévy measures includes jump-diffusion finite activity measures present in e.g. Merton's or Kou's double exponential models as well as infinite activity Lévy measures appearing in e.g. Variance Gamma, Normal inverse Gaussian or the so-called CGMY models. In Section 3.1 we present the definition of a price of an European option as a discounted expected value of the terminal payoff and a simple derivation of the integro-differential equation whose solution is the discounted expected value of the terminal payoff. Moreover, we present the partial integro-differential extension of the classical Black-Scholes equation for pricing vanilla options on underlyings following a Lévy stochastic process. In Section 3.2 we present a result shown by Nualart and Schoutens [65] that allows a probabilistic representation of solutions of PIDE's through the use of a Feynman-Kač formula. Section 3.3 is dedicated to present in detail the relation between the price of European options (Subsection 3.3.1) and barrier options (Subsection 3.3.2), and the solutions of the associated integro-differential equations. Also, in Subsection 3.3.2 some continuity results are presented for barrier options. Section 3.4 shows how to perform option pricing using Fourier techniques. Section 3.5 shows a model which allows to model feedback effects in a Lévy Model. We show that under some conditions the price representing a security's price satisfies a certain partial integro-differential equation (PIDE). Section 3.6 is devoted to the problem of existence and uniqueness of solution to the governing linear PIDE in the framework of the Bessel potential spaces. We follow the methodology of abstract semilinear parabolic equations developed by Henry in [46]. First, we provide sufficient conditions guaranteeing existence and uniqueness of a solution to the linear PIDE in Bessel potential spaces in 3.6.1. Next in 3.6.2 we deal with existence and uniqueness of a solution to a nonlinear extension of PIDE representing the penalty method for pricing American style of put options. Section 4.1 is dedicated to present numerical schemes involving finite difference methods to solve PIDEs. In subsection 4.1.2 we present an implicit-explicit scheme to solve a non-linear PIDE. Then in subsection

4.1.5 we give some numerical results when considering the Variance-Gamma process using a Finite difference scheme. Section (4.2) presents a Radial Basis Interpolation scheme to solve numerically a classical PIDE and then its extension to the nonlinear case.

Section 4.3 studies the nonlinear case when the influence of the large trader is considered to be small. Section 4.4 deals with the consistency, stability and monotony of the numerical scheme developed in Section 4.1. Chapter 5 deals with the pricing of Interest rate derivatives, in particular the pricing of the Forward Rate Agreement contract. Section 5.2 presents problem formulation, Section 5.3 presents the basic assumption concerning the risk-free interest rate, the pricing of non-defaultable bonds in 5.3.1 and defaultable bonds in 5.3.2. Here we compute the convexity adjustment for the case whithout jump processes in 5.3.2.1 and then in 5.3.2.2 for the case of a shot-noise process.

In the appendix we present the proofs of Propositions 3.3.1, 3.3.2 and 3.3.5.

Chapter 2

Financial models using Lévy Processes

2.1 Background

Over recent decades, the Black–Scholes model and its generalizations become widely used in financial markets because of its simplicity and existence of the analytic formula for pricing European style options. According to the classical theory developed by Black, Scholes and Merton, the price $V(t,S)$ of an option in a stylized financial market at time $t \in [0,T]$ and depending on the underlying asset price S can be computed as a solution to the linear Black–Scholes parabolic equation:

$$\frac{\partial V}{\partial t}(t,S) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}(t,S) + rS\frac{\partial V}{\partial S}(t,S) - rV(t,S) = 0, \ t \in [0,T), S > 0. \tag{2.1}$$

Here $\sigma > 0$ is the historical volatility of the underlying asset driven by the geometric Brownian motion, $r > 0$ is the risk-free interest rate of zero-coupon bond. A solution is subject to the terminal pay-off condition $V(T,S) = \Phi(S)$ at maturity $t = T$.

The models used in the literature postulate an economy with two traded assets, a risky asset, usually a stock with price at time t denoted by S_t, and also a riskless asset, typically a bond with price at time t denoted by B_t. The bond is taken as a numeraire and the bond market is assumed to be perfectly elastic since the bond market is more liquid than the stock market. We consider the Black-Scholes model

$$\begin{cases} \mathrm{d}S_t = \alpha S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W(t), \\ \mathrm{d}B_t = rB_t \,\mathrm{d}t. \end{cases}$$

In the Black-Scholes model we allow for short sales and assume a frictionless market, i.e there are no transaction costs and the market is liquid. We assume that the price trajectories are continuous, the stock's volatility is constant, r is deterministic and constant. We also assume that the today's observed prices contain all the information of the stock. Finally, we assume that for every derivative there is a replicating portfolio.

Given these assumptions we construct a self-financing portfolio based on the price of a derivative and its underlying, Π and S,

$$\mathrm{d}P_t = h_t^S S_t + h_t^\Pi \Pi_t. \tag{2.2}$$

Since this portfolio is self-financing

$$P_t = h_t^S \,\mathrm{d}S_t + h_t^\Pi \,\mathrm{d}\Pi_t, \tag{2.3}$$

or in terms of the relative weights $u_t^S = \frac{h_t^S S_t}{V_t}$, $u_t^\Pi = \frac{h_t^\Pi \Pi_t}{V_t}$

$$\mathrm{d}P_t = u_t^S \frac{\mathrm{d}S_t}{S_t} + u_t^\Pi \frac{\mathrm{d}\Pi_t}{\Pi_t} \tag{2.4}$$

Given the assumptions and since P_t depends on S, we will have $\Pi_t = V(t, S_t)$.

Then applying Ito's lemma to $V \in C^{1,2}$ we get

$$\mathrm{d}V = \alpha_V V \,\mathrm{d}t + \sigma_V V \,\mathrm{d}W_t,$$

where

$$\begin{aligned} \alpha_V &= \frac{V_t + \alpha S V_S + \frac{1}{2}\sigma^2 S^2 V_{SS}}{V}, \\ \sigma_V &= \frac{\sigma S_t V_S}{V}. \end{aligned} \tag{2.5}$$

So far we have

$$\begin{cases} \mathrm{d}P = u_t^S \frac{\mathrm{d}S_t}{S_t} + u_t^\Pi \frac{\mathrm{d}\Pi_t}{\Pi_t}, \\ \mathrm{d}V = \alpha_V V \,\mathrm{d}t + \sigma_V V \,\mathrm{d}W_t. \end{cases}$$

Using the dynamics for S and Π we can further simplify and obtain

$$\begin{cases} \mathrm{d}V = V\left(u^S\alpha + u^\Pi \alpha_V\right)\,\mathrm{d}t + V\left(u^S\sigma + u^\Pi\sigma_V\right)\,\mathrm{d}W_t, \\ \mathrm{d}V = \alpha_V V \,\mathrm{d}t + \sigma_V V \,\mathrm{d}W_t. \end{cases}$$

In order to have a risk-free arbitrage portfolio we must make sure that the stochastic part is zero and the drift term is equal to the risk-free short rate

$$\begin{cases} u^S\alpha + u^\Pi\alpha_V = r, \\ u^S\sigma + u^\Pi\sigma_V = 0, \\ u^S + u^\Pi = 1. \end{cases}$$

Then in order for the system to have a unique solution we must have

$$-\sigma - \frac{\sigma_V - \sigma}{\alpha_V - \alpha}(r - \alpha) = 0. \tag{2.6}$$

Then, plugging in (2.5) into (2.6) and after some algebra we obtain the classical Black-Scholes PDE (2.1).

Evidence from stock markets observations indicates that this model is not the most realistic one, since it assumes that the market is liquid, complete, frictionless and without transaction costs. We also recall that the linear Black–Scholes equation provides a solution corresponding to a perfectly replicated portfolio which need not be a desirable property. Indeed forming now a self-financed portfolio P consisting of the stock and bond S and B we obtain

$$\mathrm{d}P_t = P_t \left(u_t^S \frac{\mathrm{d}S_t}{S_t} + u_t^B \frac{\mathrm{d}B_t}{B_t} \right),$$

where $u_t^S = \frac{h_t^S S_t}{V_t}, u_t^B = \frac{h_t^B B_t}{V_t}$, are the weights to be invested in the stock and bond respectively. We have, after using Ito's Lemma ([17]), the following

$$\begin{cases} \mathrm{d}P = P \left(u^S \alpha + u^B r \right) \mathrm{d}t + P u^S \sigma \, \mathrm{d}W, \\ \mathrm{d}V = \alpha_V V \, \mathrm{d}t + \sigma_V V \, \mathrm{d}W_t, \end{cases}$$

where

$$\begin{cases} \alpha_V = \frac{V_t + \alpha S V_S + \frac{1}{2}\sigma^2 S^2 V_{SS}}{V}, \\ \sigma_V = \frac{\sigma S_t V_S}{V}. \end{cases}$$

Then since we want to make sure that $P = V$ we must have

$$\begin{cases} u^S \sigma = \frac{\sigma S_t V_S}{V}, \\ u^B + u^S = 1, \end{cases}$$

which gives the relative weights

$$\begin{cases} u^S = \frac{S_t V_S}{V}, \\ u^B = 1 - \frac{S_t V_S}{V}. \end{cases}$$

In the last two decades some of these assumptions have been relaxed in order to model, for instance, the presence of transaction costs (see e.g. Kwok [51] and Avellaneda and Paras [9]), feedback and illiquid market effects due to large traders choosing given stock-trading strategies (Schönbucher and Willmott [74], Frey and Patie [39], Frey and Stremme [38]), risk from the unprotected portfolio (Jandačka and Ševčovič [48]). In all aforementioned generalizations of the linear Black–Scholes equation (2.1) the constant volatility σ is replaced by a nonlinear function $\tilde{\sigma}(S\partial_S^2 V)$ depending on the second derivative $\partial_S^2 V$ of the option price itself. In the class of generalized Black–Scholes equation with such a nonlinear diffusion function, an important role is played by the nonlinear Black–Scholes model derived by Frey and Stremme in [48] (see also [39],[36]). In this model the asset dynamics takes into account the presence of feedback effects due to a large trader choosing his/her stock-trading strategy (see also [74]). The diffusion coefficient is again non-constant:

$$\tilde{\sigma}(S\partial_S^2 V)^2 = \sigma^2 \left(1 - \varrho S \partial_S^2 V \right)^{-2}, \tag{2.7}$$

where $\sigma, \varrho > 0$ are constants.

Now we relax the liquid market assumption. In this economy there are two types of traders, the reference traders and the program traders. The program traders are also referred to as portfolio insurers since they use dynamic hedging strategies to insure against movements in stock's price. They can be a single trader or a group of traders who act together. It is assumed that their trades influence the equilibrium stock price. The reference traders can be thought of as a representative trader of many small agents and therefore it is assumed that they act as price takers. Usually it is assumed that $\tilde{D}(t, Y_t, S_t)$ is the reference trader's demand function which depends on the income process or some other fundamental state variable that influences the reference trader's demand. The aggregate demand of the program traders is denoted by $\phi(t, S_t) = \xi\Phi(t, S_t)$, where ξ is the number of identical written securities that the program traders are trying to hedge and $\Phi(t, S_t)$ is the demand per security being hedged. We assume for simplicity that ξ is the same for every program trader. The general case where different securities are considered can be seen for example in [78]. Assume the supply of the stock $\tilde{S_0}$ is constant and define $D(t, y, s) = \frac{\tilde{D}(t,y,s)}{\tilde{S_0}}$ as the quantity demanded of the reference trader per unit of supply. Then the total demand relative to the supply at time t is given by $G(t, y, s) = D(t, y, s) + \rho\Phi(t, s)$, where $\rho = \frac{\xi}{\tilde{S_0}}$ and $\rho\Phi(t, s)$ the proportion of the total supply of the stock that is being traded by the programme traders . So, in order to get market equilibrium we must have $G(t, y, s) = 1$. Assume that G is monotonic on the last two arguments and sufficiently smooth in s and y. Then we can invert $G(t, y, s) = 1$ to obtain $S_t = \psi(t, Y_t)$ where ψ is sufficiently smooth. For example, in [78] it is assumed that Y_t has the following dynamics

$$\mathrm{d}Y_t = \mu(t, Y_t)\,\mathrm{d}t + \eta(t, Y_t)\,\mathrm{d}W_t.$$

Then the authors obtain a generalization of the Black-Scholes pricing partial differential equation

$$\begin{aligned}
&\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial^2 S} S_{t-}^2 \left(\frac{\sigma}{1 - \rho S_{t-} \frac{\partial \phi}{\partial S}(t, S_{t-})} \right)^2 + S_{t-} r \frac{\partial V}{\partial S} - rV = 0, (S,t) \in]0, \infty[\times]0, T], \\
&V(S,T) = \phi(S), 0 < S < \infty. \qquad (2.8)
\end{aligned}$$

The derivation of this equation is done in the spirit of the original argument used in the derivation of the original Black-Scholes equation. We suppose that the price of a derivative security is a smooth function given by $P_t = V(t, S_t)$ and consider a self-financing replicating strategy (α_t, β_t) consisting of a bond and a risky asset:

$$\mathrm{d}P_t = \alpha_t\,\mathrm{d}S_t + \beta_t\,\mathrm{d}B_t, \qquad (2.9)$$

where $\alpha_t = \Phi(t, S_t)$. Then applying Ito's Lemma to $V(t, S_t)$ we get

$$\mathrm{d}V(t, S_t) = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial^2 S} S_{t-}^2 v(t, S_t)^2 \right) \mathrm{d}t + \frac{\partial V}{\partial S}\,\mathrm{d}S_t. \qquad (2.10)$$

So by comparing (2.9) with (2.10) we see that

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial^2 S} S_{t^-}^2 v(t, S_t)^2 = \beta_t r B_t, \tag{2.11}$$

$$\alpha_t = \frac{\partial V}{\partial S}. \tag{2.12}$$

Then since $\beta_t = \frac{V(t,S_t)-\alpha_t S_t}{B_t}$ the equation turns out to be

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial^2 S} S_{t^-}^2 v(t, S_t)^2 = r\left(V(t, S_t) - \frac{\partial V}{\partial S} S_t\right). \tag{2.13}$$

To obtain the adjusted volatility simply take into account the dynamics of the income process and apply Ito's lemma to $\psi(t, Y_t)$

$$\mathrm{d}S_t = \left(\frac{\partial \psi}{\partial t} + \mu(t, Y_t)\frac{\partial \psi}{\partial y} + \frac{1}{2}\frac{\partial^2 \psi}{\partial^2 y} S_{t^-}^2 \eta(t, Y_t)^2\right) \mathrm{d}t + \eta(t, Y_t)\frac{\partial \psi}{\partial y}\,\mathrm{d}W_t. \tag{2.14}$$

Thus

$$v(t, S_t) = \eta(t, Y_t)\frac{\partial \psi}{\partial y} = -\eta(t, Y_t)\frac{D_y(t, Y_t, S_t)}{D_s(t, Y_t, S_t) + \rho\frac{\partial \Phi}{\partial S}}, \tag{2.15}$$

because $G(t, y, s) = 1$ and since $\frac{\partial G}{\partial S} \neq 0$ we can differentiate with respect to y to obtain $\frac{\partial \psi}{\partial y}$.

In this thesis we follow Frey's approach which is to begin by proposing a dynamics for the stock price instead of deriving it using the market equilibrium and assuming a certain dynamic for the income process as done for example in [78]. This way Frey obtains the same stock price dynamics as in [78] corresponding to a situation where the the demand function is of logarithm type and does not depend on t and when the income process follows a Geometric Brownian motion, $D(y, s) = \ln(\frac{y^\gamma}{s})$ where $\gamma = \frac{\sigma}{\eta}$. In fact

$$D_y(y, s) = \gamma\frac{1}{y}, D_s(y, s) = -\frac{1}{s},\ \mathrm{d}Y_t = \mu Y_t\,\mathrm{d}t + \eta Y_t\,\mathrm{d}W_t, \tag{2.16}$$

$$v(t, s) = -\eta y\frac{\gamma\frac{1}{y}}{-\frac{1}{s} + \rho\frac{\partial \Phi}{\partial S}} = \frac{\sigma s}{1 - \rho s\frac{\partial \Phi}{\partial S}}. \tag{2.17}$$

The goal is to extend the dynamics used by Frey to a Lévy process. So this model can be thought of as a deviation to the standard jump-diffusion model instead of the Geometric Brownian Motion. The level of deviation to the jump-diffusion model is measured by a parameter ρ which is the market liquidity parameter.

Another important direction in generalizing the original Black–Scholes equation arise from the fact that the sample paths of a Brownian motion are continuous, but the realized stock price of a typical company exhibits random jumps over the intraday scale, making the price trajectories discontinuous. In the classical Black–Scholes model the underlying asset price process is assumed to follow a geometric Brownian motion. However, the empirical distribution of stock returns exhibits fat tails. Several alternatives have been proposed in the literature for the generalization of this model. The models with jumps and diffusion can, at least in part, solve the problems inherent to the linear Black–Scholes model and they have also an important role in the options market. While in the Black–Scholes model the market is complete, implying that every pay-off can be perfectly replicated, in jump–diffusion models there is no perfect hedge and this way the options are not redundant. It turns out that the option price can be computed from the solution $V(t,S)$ of the following partial integro-differential (PIDE) Black–Scholes equation ([24]):

$$\begin{aligned}&\frac{\partial V}{\partial t}(t,S)+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}(t,S)+rS\frac{\partial V}{\partial S}(t,S)-rV(t,S)\\&+\int_{\mathbb{R}}V(t,S+H(z,S))-V(t,S)-H(z,S)\frac{\partial V}{\partial S}(t,S)\nu(\,\mathrm{d}z)=0,\end{aligned}\tag{2.18}$$

where $H(z,S)=S(e^z-1)$ and ν is the so-called Lévy measure characterizing the underlying asset process with random jumps in time and space. Note that, if $\nu=0$ then (2.18) reduces to the classical linear Black–Scholes equation (2.1).

The novelty and main purpose of this thesis is to take into account both directions of generalizations of the Black–Scholes equation. The assumption that an investor can trade large amounts of the underlying asset without affecting its price is no longer true, especially in illiquid markets. Therefore, we will derive, analyze, and perform numerical computation of the model. We relax the assumption of liquid market following the Frey–Stremme model under the assumption that the underlying asset price follows a Lévy stochastic process with jumps. We will show that the corresponding PIDE nonlinear equation has the form:

$$\begin{aligned}&\frac{\partial V}{\partial t}+\frac{1}{2}\frac{\sigma^2}{(1-\varrho S\partial_S\phi)^2}S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV\\&+\int_{\mathbb{R}}V(t,S+H(t,z,S))-V(t,S)-H(t,z,S)\frac{\partial V}{\partial S}\nu(\,\mathrm{d}z)=0,\end{aligned}\tag{2.19}$$

where the function $H(t,z,S)$ may depend e.g. on the large trader strategy function $\phi=\phi(t,S)$. This function may depend on the delta $\partial_S V$ of the price V, if $\varrho>0$.

2.1.1 Lévy Processes: definitions

Let us start with the definition of a Lévy process.

Definition 2.1.1 *Consider a fixed probability space* $(\Omega,\mathcal{F},\mathbb{Q})$. *A stochastic process* X_t *such that* $X_0=0$ *is called a Lévy process if:*

- X_t *has independent increments: for every* $t_0<t_1<..<t_n$, *the random variables* $X_{t_0},X_{t_1}-X_{t_0},X_{t_2}-X_{t_1},...,X_{t_n}-X_{t_{n-1}}$ *are independent.*
- X_t *has stationary increments: the law of* $X_{t+h}-X_t$ *does not depend on* t*;*

- X_t *is stochastically continuous ,i.e. for all* $a > 0$ *and* $s > 0$*:*

$$\lim_{t \to s} \mathbb{P}[|X_t - X_s| > a] = 0.$$

If we drop the stationary increments condition we say that the process is an additive process.

Definition 2.1.2 *Consider a fixed probability space* $(\Omega, \mathcal{F}, \mathbb{Q})$*. A stochastic process* X_t *such that* $X_0 = 0$ *is called an additive process if:*

- X_t *has independent increments: for every* $t_0 < t_1 < .. < t_n$*, the random variables* $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}$ *are independent.*
- X_t *is stochastically continuous ,i.e. for all* $a > 0$ *and* $s > 0$*:*

$$\lim_{t \to s} \mathbb{P}[|X_t - X_s| > a] = 0.$$

We give now the definition of infinite divisible distributions

Definition 2.1.3 *A given random variable* X *taking values in* $\mathbb{R}^n$ *with probability law* p_x *is infinitely divisible if* $\forall\ n \in \mathbb{N}$ *there exists independent and identically distributed random variables* $X_1^{(n)}, X_2^{(n)}, .., X_n^{(n)}$*such that*

$$X \stackrel{d}{=} X_1^{(n)} + X_1^{(n)} + ... + X_n^{(n)}.$$

The next theorem is an important result which gives us the link between additive and Lévy processes and infinitely divisible distributions. The proof can be found in theorems 8.1, 9.1 and 9.8 in [72].

Theorem 2.1.4 *If* $X = \{X_t, t \in [0, T]\}$ *is an additive process with increasing and positive measure* p_t *on* $\mathbb{R}\backslash \{0\}$ *such that* $p_s(B) \to p_t(B)$ *as* $s \to t$ *for all measurable sets* $B \subset \{x : |x| \geq \varepsilon\}$*, for some* $\varepsilon > 0$*, and for all* $t \in [0, T]$ *satisfying*

$$\int_{\mathbb{R}} \left(1 \wedge x^2\right) p_t(\mathrm{d}x) < \infty. \tag{2.20}$$

Then the distribution of the random variable X_t *is infinitely divisible for each* $t \in [0, T]$ *and the characteristic function of* X_t *is given by the Lévy-Khintchine representation formula*

$$\mathbb{E}\left[\exp\left(izX_t\right)\right] = \exp\left(\psi_t\left(z\right)\right),$$

where the characteristic exponent is defined by

$$\psi_t\left(z\right) := i\gamma_t z - \frac{A_t}{2}z^2 + \int_{\mathbb{R}} \left(\exp\left(izx\right) - 1 - izx\mathbf{1}_{\{|x|<1\}}\right) p_t\left(\mathrm{d}x\right),$$

and where A_t *is a nonnegative and increasing continuous function and* γ_t *is a continuous function.*

We only consider a right continuous with limits to the left (cádlag) version of X_t and will denote $\Delta X_t = X_t - X_{t^-}$, the jump of X at time t.

If we consider a Lévy process X_t then the characteristic function has the following Lévy-Khintchine representation ([72],[24],[6]):

$$\mathbb{E}\left[e^{izX_t}\right] = e^{t\phi(z)}, \phi(z) = -\frac{\sigma^2 z^2}{2} + i\gamma z + \int_{-\infty}^{+\infty} \left(e^{izx} - 1 - izx1_{|x|\leq 1}\right) \nu(dx).$$

where $\sigma \geq 0$ and $\gamma \in \mathbb{R}$ and ν is a positive Radon measure on $\mathbb{R} \setminus \{0\}$ verifying:

$$\int_{-1}^{1} x^2 \nu(dx) < \infty. \tag{2.21}$$

and

$$\int_{|x|>1} \nu(dx) < \infty. \tag{2.22}$$

The measure ν is defined by:

$$\nu(A) = \mathbb{E}\left[\#\{t \in [0,1] : \Delta X_t \in A\}\right] = \frac{1}{T}\mathbb{E}\left[\#\{t \in [0,T] : \Delta X_t \in A\}\right], A \in \mathcal{B}(\mathbb{R}), \tag{2.23}$$

and is called the Lévy measure of X. It gives the mean number, per unit of time, of jumps whose amplitude belongs to A.

The Lévy-Itô decomposition gives a representation where X is interpreted as a combination of a Brownian motion with drift and a infinite sum of independent compensated Poisson processes with several jump sizes x (see [24])

$$X_t = \gamma t + \sigma W_t + \int_0^t \int_{|x|\geq 1} x J_X(ds, dx) + \int_0^t \int_{|x|<1} x \widetilde{J}_X(ds, dx), \tag{2.24}$$

where J_X is the Poisson random measure defined in the following way:

$$J_X([0,t] \times A) = \#\{s \in [0,t] : \Delta X_s \in A\}. \tag{2.25}$$

The compensated Poisson measure is defined by:

$$\widetilde{J}_X([0,t] \times A) = J_X([0,t] \times A) - t\nu(A). \tag{2.26}$$

A Lévy process is a strong Markov process, the associated semigroup is a convolution semigroup and its infinitesimal generator $L : f \to Lf$ is an integro-differential operator given by (see [6]):

$$Lf(x) = \lim_{t\to 0} \frac{\mathbb{E}\left[f(x + X_t)\right] - f(x)}{t} \tag{2.27}$$

$$= \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} + \gamma\frac{\partial f}{\partial x} + \int_{\mathbb{R}} \left[f(x+y) - f(x) - y1_{|y|\leq 1}\frac{\partial f}{\partial x}(x)\right] \nu(\mathrm{d}y), \tag{2.28}$$

which is well defined for $f \in C^2(\mathbb{R})$ with compact support.

We now define the concept of structure-preserving equivalent measures in a given market.

Definition 2.1.5 *If $\mathbb{Q}$ is equivalent to the original probability measure $\mathbb{P}$ and the Lévy process X is a $\mathbb{Q}$-Lévy process we say that a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}, \mathbb{F})$ is a $\mathbb{P}$-structure-preserving equivalent measure, in the market model M.*

We now proceed by presenting a characterization result for structure preserving equivalent measures. A proof for the following theorem can be found in [72, Theorem 33.1].

Theorem 2.1.6 *Consider the Lévy process $X = \{X_t, t \in [0,T]\}$ with Lévy triplet $[\gamma, \sigma^2, \nu(\mathrm{d}x)]$ and with Lévy Ito decomposition under the probability measure $\mathbb{P}$. Then there is $\mathbb{P}$-structure-preserving equivalent measure $\mathbb{Q}$, such that X is a $\mathbb{Q}$-Lévy process with triplet $[\tilde{\gamma}, \tilde{\sigma}^2, \tilde{\nu}(dx)]$ if and only if*

(i) $\tilde{\nu}(dx) = h(x)\nu(dx)$ *for some Borel function* $h : \mathbb{R} \to (0, \infty)$.

(ii) $\tilde{\gamma} = \gamma + c\sigma + \int_{\mathbb{R}} x\mathbf{1}_{\{|x|<1\}}(h(x) - 1)\nu(dx)$ *for some* $c \in \mathbb{R}$.

(iii) $\tilde{\sigma} = \sigma$.

(iv) $\int_{\mathbb{R}}(1 - \sqrt{h(x)})^2\nu(dx) < \infty$.

Furthermore the density process $\{\frac{\mathrm{d}\mathbb{Q}_t}{\mathrm{d}\mathbb{P}_t} = L_t, t \in [0,T]\}$ *converges uniformly in t for every bounded interval, $\mathbb{P}$-a.s. and is given by*

$$L_t = \exp\left(cW_t - \frac{1}{2}c^2 t + \int_0^t \int_{|x|>\varepsilon} \log h(x) N(\mathrm{d}s, \mathrm{d}x) - t\int_{|x|>\varepsilon} (h(x) - 1)\nu(\mathrm{d}x)\right) \quad (2.29)$$

for small ϵ and with $\mathbb{E}_{\mathbb{P}}[L_t] = 1$, for every $t \in [0,T]$.

2.1.2 Exponential Lévy models

Let $\{S_t, t \geq 0\}$ be a stochastic process representing the price of a financial asset under a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. The filtration $\{\mathcal{F}_t\}$ represents the price history up to time t. If the market is arbitrage-free, then there is a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which the discounted prices of all traded financial assets are $\mathbb{Q}-$ martingales. This result is known as the fundamental theorem of asset pricing (see [24]). The measure $\mathbb{Q}$ is also known as the risk neutral measure. We consider here the exponential Lévy model in which the risk-neutral dynamics of S_t under $\mathbb{Q}$ is given by $S_t = e^{rt+X_t}$, where X_t is a Lévy process under $\mathbb{Q}$ with characteristic triplet (σ, γ, ν). Then the arbitrage-free market hypothesis imposes that $\widehat{S}_t = S_t e^{-rt} = e^{X_t}$ is a martingale, which is equivalent to the following conditions imposed on the triplet (σ, γ, ν)

$$\int_{|y|>1} e^y \nu(\mathrm{d}y) < \infty, \gamma = -\frac{\sigma^2}{2} - \int_{-\infty}^{+\infty} \left(e^y - 1 - y1_{|y|\leq 1}\right) \nu(dy). \quad (2.30)$$

Then the infinitesimal generator (2.28) becomes

$$Lf(x) = -\frac{\sigma^2}{2}\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} + \int_{\mathbb{R}} \left[f(x+y) - f(x) - (e^y - 1)\frac{\partial f}{\partial x}\right] \nu(\mathrm{d}y). \quad (2.31)$$

The risk-neutral dynamics of S_t under $\mathbb{Q}$ is given by

$$S_t = S_0 + \int_0^t rS_{u^-}\, \mathrm{d}u + \int_0^t \sigma S_{u^-}\, \mathrm{d}W_u + \int_0^t \int_{\mathbb{R}} (e^x - 1)\, S_{u^-} \widetilde{J}_X\,(du, dx)\,. \tag{2.32}$$

The price process S_t is also a Markov process with state space $(0, \infty)$ and infinitesimal generator (see [24])

$$L^S f(x) = \lim_{h\to 0} \frac{\mathbb{E}[f(xe^{X_h})] - f(x)}{h} \tag{2.33}$$

$$= rx\frac{\partial f}{\partial x} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 f}{\partial x^2} + \int_{\mathbb{R}} \left[f\,(xe^y) - f\,(x) - x\,(e^y - 1)\,\frac{\partial f}{\partial x}\right] \nu\,(\,\mathrm{d}y)\,. \tag{2.34}$$

A Lévy process Y_t is called a Lévy type stochastic integral if

$$\mathrm{d}Y_t = \gamma\, \mathrm{d}t + \sigma\, \mathrm{d}W_t + \int_{|y|<1} H(t,y)\tilde{J}_Y(\,\mathrm{d}t,\,\mathrm{d}y) + \int_{|y|>1} K(t,y)J_Y(\,\mathrm{d}t,\,\mathrm{d}y).$$

An important result that will be needed later is the Ito's lemma which can be found in [6]

Result 2.1.7 *Let $f \in C^{1,2}([0,T]\times\mathbb{R})$ and let Y_t be Lévy type stochastic integral. Then*

$$\begin{aligned}
\mathrm{d}f(t,Y_t) &= \frac{\partial f}{\partial t}\,\mathrm{d}t + \frac{\partial f}{\partial y}\,\mathrm{d}Y_t^c + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}\,\mathrm{d}[Y_t^c, Y_t^c] \\
&+ \int_{|y|>1} f(t, Y_t + K(t,y)) - f(t,Y_t)J_Y(\,\mathrm{d}t,\,\mathrm{d}y) \\
&+ \int_{|y|<1} f(t, Y_t + H(t,y)) - f(t,Y_t)\tilde{J}_Y(\,\mathrm{d}t,\,\mathrm{d}y) \\
&- \int_{|y|<1} f(t, Y_t + H(t,y)) - f(t,Y_t) - H(t,y)\frac{\partial f}{\partial y}\nu(\,\mathrm{d}y)\,\mathrm{d}t.
\end{aligned} \tag{2.35}$$

An example of a Lévy Process is the jump-diffusion model first introduced by Merton in [60]. This model assumes the following dynamics for the stock's price logarithm

$$\mathrm{d}X_t = \left(b + \int_{|x|<1} x\nu(\,\mathrm{d}x)\right)\mathrm{d}t + \sigma\,\mathrm{d}W_t + \int_{|x|<1} x\tilde{J}_X(\,\mathrm{d}t,\,\mathrm{d}x) + \int_{|x|>1} xJ_X(\,\mathrm{d}t,\,\mathrm{d}x).$$

Then one obtains the dynamics for the stock price applying Ito's lemma to $S_t = e^{X_t}$

$$\mathrm{d}S_t = S_{t^-}\left(b + \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma S_{t^-}\,\mathrm{d}W_t + S_{t^-}\int_{\mathbb{R}} (e^x - 1)\,J_X(\,\mathrm{d}t,\,\mathrm{d}x),$$

or in terms of jumps of S_t

$$\mathrm{d}S_t = S_{t^-}\left(b + \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma S_{t^-}\,\mathrm{d}W_t + \int_0^\infty yJ_S(\,\mathrm{d}t,\,\mathrm{d}y).$$

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2.2 Examples of Lévy processes in finance

The exponential Lévy models considered in the financial literature are of two types. The first type of models are called jump-diffusion models where we represent the log-price as a Lévy process with a non zero diffusion part ($\sigma > 0$) and with a jump process with finite activity (i.e $\nu(\mathbb{R}) < \infty$). The second type of models are called infinite activity pure jump models in which case there is no diffusion part and only a jump process with infinite activity (i.e $\nu(\mathbb{R}) = \infty$).

Definition 2.2.1 *A Lévy measure ν is called an admissible activity Lévy measure if*

$$0 \leq \frac{\nu(\,\mathrm{d}z)}{\mathrm{d}z} \leq h(z) \equiv C|z|^{-\alpha}\left(e^{D^- z}1_{z\geq 0} + e^{D^+ z}1_{z<0}\right)e^{-\mu z^2}, \tag{2.36}$$

for any $z \in \mathbb{R}$ and the shape parameters $\alpha \geq 0$, $D^\pm \in \mathbb{R}$ and $\mu \geq 0$.

Remark 1 *The conditions $\int_{\mathbb{R}} \min(z^2, 1)\nu(\,\mathrm{d}z) < \infty$ (see (2.22)) and $\int_{|z|>1} e^z \nu(\,\mathrm{d}z) < \infty$ (see (2.30)) are satisfied provided that ν is an admissible Lévy measure with shape parameters $\alpha < 3$, and, either $\mu > 0, D^\pm \in \mathbb{R}$, or $\mu = 0$ and $D^- + 1 < 0 < D^+$.*

There is a wide class of exponential Lévy models proposed in the financial modelling literature that differ from each other only in the choice of the Lévy measure. In this section we present some examples of such models.

2.2.1 Jump-Diffusion models

A Lévy process of jump-diffusion type is of the following form:

$$X_t = \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$$

where $\sigma > 0, N_t$ is a Poisson process with intensity λ that counts the jumps of X_t and $Y_i, i = 1, 2, 3...$ are independent and identically distributed random variables with distribution given by μ. The Lévy measure ν is given by $\lambda\mu$ and the drift γ is equal to

$$-\frac{\sigma^2}{2} - \int_{\mathbb{R}} \left(e^y - 1 - y1_{|y|\leq 1}\right)\nu\left(\,\mathrm{d}y\right).$$

2.2.1.1 Merton's model

This model was introduced by Merton [60] and was the first jump-diffusion model proposed in the financial literature. The random variables $Y_i, i = 1, 2, 3...$ are normally distributed with mean m and variance δ. Its Lévy density is given by:

$$\nu(x) = \lambda \frac{1}{\delta\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\delta^2}} \tag{2.37}$$

Then it's possible to obtain the probability density of X_t as a series that converges rapidly (see [24]):

$$p_t(x) = \sum_{j=0}^{\infty} e^{-\lambda t}(\lambda t)^j \frac{e^{-\frac{(x-\gamma t-jm)^2}{2(\sigma^2 t+j\delta^2)}}}{j!\sqrt{2\pi(\sigma^2 t + j\delta^2)}}. \tag{2.38}$$

Thus, we can express the price of an European call option as a weighted sum of Black-Scholes prices:

$$C_{Merton}(S_0, K, T, \sigma, r) = e^{-rT} \sum_{j=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!} e^{r_j T} C_{BS}(S_0 e^{\frac{j\delta^2}{T}}, K, T, \sigma_j, r_j), \tag{2.39}$$

where $r_j = r - \lambda(e^{m+\frac{\delta^2}{2}} - 1) + \frac{jm}{T}$, $\sigma_j = \sqrt{\sigma^2 + \frac{j\delta^2}{T}}$ and $C_{BS}(S, K, T, \sigma, r)$ is the well known Black-Scholes formula.

For the Merton model the Lévy measure satisfies $\nu(\mathbb{R}) < \infty$. Moreover, ν is the admissible activity measure with the shape parameters $\mu = 1/(2\delta^2) > 0$, $\alpha = 0$ and any $D^{\pm}$.

2.2.1.2 Kou's double exponential model

Another popular and frequently used model is the so-called double exponential model which was introduced by Kou in [50] and it also referred to as Kou's model. In this model the distribution of jumps have a density of the form:

$$\nu(\,dx) = \lambda \left(\theta \lambda^+ e^{-\lambda^+ x} 1_{x>0} + (1-\theta)\lambda^- e^{\lambda^- x} 1_{x<0}\right) dx, \tag{2.40}$$

where λ is the intensity of jumps, θ is the probability of having a positive jump and $\lambda^{\pm} > 0$ correspond to the level of decay of the distribution of positive and negative jumps. This implies that the distribution of jumps is asymmetric and the tails of the distribution of returns are semi-heavy. In Kou's double exponential model for the Lévy measure we again have $\nu(\mathbb{R}) < \infty$, and, ν is an admissible activity Lévy measure with the shape parameters $\mu = 0, \alpha = 0$, and $D^+ = \lambda^- > 0, D^- = -\lambda^+ < 0$.

2.2.2 Infinite activity pure jump models

Examples of infinite activity Lévy processes are the Variance Gamma (see [55]) and Normal Inverse Gaussian (NIG) processes (see [14]). They are constructed by means of subordination of a Brownian motion and a tempered α-stable process: the Variance Gamma process corresponds to $\alpha = 0$ and the NIG process corresponds to $\alpha = 1/2$. These models are popular in the literature because the probability density function of the subordinator is known in a closed form for those values of α (see [24]). The Variance Gamma and NIG processes are special cases of the Generalized Hyperbolic model which is a process of infinite variation without a Gaussian part. Other examples of these kind of processes are the Meixner process and CGMY model which consist of more general processes and more complicated Lévy measures.

2.2.2.1 Variance Gamma Process

The Variance Gamma process is a pure discontinuous process of infinite activity and finite variation ($\int_{|x|\leq 1}|x|\nu(\,\mathrm{d}x) < \infty$) that is widely used in the financial modelling. Its Lévy measure is given by

$$\nu(x) = \frac{1}{\kappa|x|}e^{Ax-B|x|} \text{ with } A = \frac{\theta}{\sigma^2} \text{ and } B = \frac{\sqrt{\theta^2+2\frac{\sigma^2}{\kappa}}}{\sigma^2},$$

where σ and θ are parameters related with the volatility and drift of the Brownian motion with drift and κ is the parameter related with the variance of the subordinator, in this case the Gamma process (see [24]). The probability density is given by

$$p_t(x) = Ce^{Ax}|x|^{\frac{t}{k}}K_{\frac{t}{k}-\frac{1}{2}}(|x|),$$

where K is the modified Bessel Function of second kind.

The characteristic function of $X_t + \gamma t$ is equal to :

$$\Phi_t(u) = e^{itu\gamma}\phi_t(u) = e^{itu\gamma}\left(1+\frac{\sigma^2u^2\kappa}{2} - i\theta\kappa u\right)^{-t/\kappa},$$

where γ is determined by the martingale condition and $\phi_t(u)$ is the characteristic function of X_t. In fact, we must have

$$\mathbb{E}[e^{-rT}S_T|\mathcal{F}_t] = e^{-rt}S_t, \tag{2.41}$$

where

$$S_t = S_0e^{rt+\gamma t+X_t} \tag{2.42}$$

is the risk-neutral process introduced in [55,56]. Therefore, $\gamma = \frac{1}{\kappa}log(1-\frac{\sigma^2\kappa}{2}-\theta\kappa)$.

For the Variance Gamma process we have $\nu(\mathbb{R}) = \infty$. Moreover, ν is an admissible activity Lévy measure with shape parameters $\mu = 0$, $D^+ = A+B > 0$, $D^- = A-B < 0$, and $\alpha = 1$. The condition $D^- + 1 < 0 < D^+$ is satisfied provided that $\kappa(2\theta+\sigma^2) < 2$.

2.2.2.2 Normal Inverse Gaussian model

The NIG process is a process of infinite activity and infinite variation without any Brownian component. Its Lévy measure is given by (see [24])

$$\nu(x) = \frac{C}{|x|}e^{Ax}K_1(B|x|)$$

and

$$C = \frac{\sqrt{\theta^2+\frac{\sigma^2}{\kappa}}}{2\pi\sigma\sqrt{\kappa}}, A = \frac{\theta}{\sigma^2}, B = \frac{\sqrt{\theta^2+\frac{\sigma^2}{\kappa}}}{\sigma^2},$$

where θ,σ and κ have the same meaning as in the Variance Gamma process. The probability density is:

$$p_t(x) = Ce^{Ax}\frac{K_1(B\sqrt{x^2+\frac{t^2\sigma^2}{\kappa}})}{\sqrt{x^2+\frac{t^2\sigma^2}{\kappa}}},$$

where K is the modified Bessel Function of second kind. The characteristic function is given by

$$\Phi_t(u) = e^{\frac{t}{\kappa}-\frac{t}{\kappa}\sqrt{1+u^2\sigma^2\kappa-2iu\theta\kappa}}. \tag{2.43}$$

2.2.2.3 Generalized Hyperbolic model

The Generalized Hyperbolic model is a process of infinite variation without gaussian part. Its characteristic function is given by (see [24])

$$\phi_t(u) = e^{i\mu u}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+iu)^2}\right)^{\frac{t}{2\kappa}}\frac{K_{\frac{t}{\kappa}}(\delta\sqrt{\lambda^2-(\beta+iu)^2})}{K_{\frac{t}{\kappa}}(\delta\sqrt{\alpha^2-\beta^2})}, \tag{2.44}$$

where δ is a scale parameter, μ is the shift parameter and κ has the same meaning that in the Variance Gamma process. The parameters λ, α and β determine the shape of the distribution. The density function

$$p_t(x) = C(\sqrt{\delta^2+(x-\mu)^2})^{\frac{t}{k}-\frac{1}{2}}K_{\frac{t}{\kappa}-\frac{1}{2}}(\alpha\sqrt{\delta^2-(x-\mu)^2})e^{\beta(x-\mu)},$$

where K is the modified Bessel function and

$$C = \frac{(\sqrt{\alpha^2-\beta^2})^{\frac{t}{k}}}{\sqrt{2\pi}\alpha^{\frac{t}{\kappa}-\frac{1}{2}}\delta^{\frac{t}{\kappa}}K_{\frac{t}{\kappa}}(\delta\sqrt{\alpha^2-\beta^2})}.$$

The Variance Gamma process is obtained for $\mu = 0$ and $\delta = 0$. The Normal Inverse Gaussian process corresponds to $\lambda = -\frac{1}{2}$. Here θ,σ and κ have the same meaning as in the Variance Gamma process and K_1 is the modified Bessel function of the second kind (see [24]).

For the Lévy measure of the NIG process we have $\nu(\mathbb{R}) = \infty$. Recall that the modified Bessel function K_1 of the second kind satisfies the following asymptotic behavior:

$$K_1(x) = e^{-x}\sqrt{\frac{\pi}{2}}x^{-\frac{1}{2}}(1+O(\frac{1}{x})),\ \text{as } x\to\infty, \quad K_1(x) \sim \frac{1}{2}\left(\frac{x}{2}\right)^{-1} \text{ as } x\to 0\,,$$

(see [1]). Thus ν is an admissible activity Lévy measure with the shape parameters $\mu = 0$, $D^+ = A+B > 0$, $D^- = A-B < 0$, $\alpha = 2$. The condition $D^- + 1 < 0 < D^+$ is satisfied provided that $\kappa(2\theta+\sigma^2) < 1$.

2.2.2.4 CGMY process

The so called CGMY distribution process introduced by Carr *et al.* in [20],[21] has four parameters C, G, M and Y with the characteristic function given by
$\Phi_t(u) = \exp(\Gamma(-Y)Ct\left((M-iu)^Y - M^Y + (G+iu)^Y - G^Y\right))$, and the Lévy measure given by

$$\nu(\mathrm{d}x) = \left(Ce^{Gx}|x|^{-1-Y}1_{x<0} + Ce^{-Mx}|x|^{-1-Y}1_{x>0}\right) \mathrm{d}x, \tag{2.45}$$

where $C, G, M > 0$ and $Y < 2$. The parameter C measures the overall level of activity. The parameters G and M are the left and right tail decay parameters, respectively. When $G = M$ the distribution is symmetric. Low values of Y yield a finite activity process. The process has infinite activity and finite variation when $Y \in (0, 1)$. For higher values of $Y \in [1, 2)$ the process has infinite activity and infinite variation. ν is an admissible activity Lévy measure with the shape parameters $\mu = 0, \alpha = 1 + Y$, and $D^+ = G > 0, D^- = -M < 0$.

Chapter 3

Integro-differential equations for option pricing

3.1 Definitions

The value of a European option is defined as the discounted conditional expectation of the terminal payoff $H(S_T)$ under the risk neutral probability $\mathbb{Q}$:

$$\begin{aligned} C(t, S_t) &= \mathbb{E}\left[e^{-r(T-t)} H(S_T) \,|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[e^{-r(T-t)} H(S_T) \,|S_t = S\right] \\ &= e^{-r(T-t)} \mathbb{E}[H(S e^{r(T-t)+X_{T-t}})], \end{aligned}$$

because of the Markov property and the fact that X_t is a Lévy process.

If H is in the domain of the infinitesimal generator L^S, then if we differentiate $C(t, S_t)$ with respect to t ,we obtain the following integro-differential equation :

$$\frac{\partial C}{\partial t}(t, S) + L^S C(t, S) - rC(t, S) = 0; C(T, S) = H(S), \tag{3.1}$$

where L^S is defined by (2.34).

Defining $\tau = T - t, x = ln\left(\frac{S}{S_0}\right), h(x) = H(S_0 e^x)$ and $f(\tau, x) = e^{r\tau} C(T - t, S_0 e^x)$ we get

$$f(\tau, x) = \mathbb{E}\left[H\left(S e^{r\tau + X_\tau}\right)\right] = \mathbb{E}\left[H\left(S_0 e^{x + r\tau + X_\tau}\right)\right] = \mathbb{E}\left[h(x + r\tau + X_\tau)\right]. \tag{3.2}$$

The associated infinitesimal generator is given by (2.31). Then, similarly to the previous case, differentiating (3.2) with respect to τ we obtain the integro-differential equation

$$\frac{\partial f}{\partial \tau} = Lf + r\frac{\partial f}{\partial x}, (\tau, x) \in (0, T] \times \mathbb{R}; \tag{3.3}$$

$$f(0, x) = h(x), x \in \mathbb{R}. \tag{3.4}$$

Indeed, by the definition of the associated infinitesimal generator we get

$$\begin{aligned}
Lf(x) &= \lim_{k\to 0} \frac{\mathbb{E}[f(\tau, x+X_k)] - f(\tau,x)}{k} \\
&= \lim_{k\to 0} \frac{\mathbb{E}[h(x+r\tau+X_{k+\tau})] - \mathbb{E}[h(x+r\tau+X_{\tau})]}{k} \\
&= \lim_{k\to 0} \frac{\mathbb{E}[h(x+r(\tau+k)+X_{k+\tau})] - \mathbb{E}[h(x+r\tau+X_{\tau})]}{k} \\
&\quad - \lim_{k\to 0} \frac{\mathbb{E}[h(x+r(\tau+k)+X_{k+\tau})] - \mathbb{E}[h(x+r\tau+X_{k+\tau})]}{k} \\
&= \frac{\partial f}{\partial \tau} - \lim_{k\to 0} \frac{\mathbb{E}[h(x+r(\tau+k)+X_{k+\tau})] - \mathbb{E}[h(x+r\tau+X_{k+\tau})]}{k} \\
&= \frac{\partial f}{\partial \tau} - \lim_{z\to 0} r \frac{\mathbb{E}[h(x+z+r\tau+X_{\frac{z}{r}+\tau})] - \mathbb{E}[h(x+r\tau+X_{\frac{z}{r}+\tau})]}{z} = \frac{\partial f}{\partial \tau} - r\frac{\partial f}{\partial x}.
\end{aligned}$$

3.2 Feynman-Kač formula for PIDEs

For $t \geq 0$, let $\mathcal{F}_t$ denote the $\sigma-$algebra generated by the random variables $\{X_s, 0 \leq s \leq t\}$ and

$$H_T^2 = \left\{ \phi_t, t \in [0,T] : \|\phi\|^2 = \mathbb{E}[\int_0^T |\phi_t|^2 \,\mathrm{d}t] < \infty \right\}$$

M_T^2 is the subspace of H_T^2 that contains predictable processes. Let $H_T^2(l^2)$ and $M_T^2(l^2)$ denote the corresponding spaces of l^2-valued processes equipped with the norm:

$$\|\phi\|^2 = \mathbb{E}[\int_0^T \sum_{i=1}^{\infty} |\phi_t^{(i)}|^2 \,\mathrm{d}t].$$

Finally set $\mathcal{H}_T^2 = H_T^2 \times M_T^2(l^2)$.

Following Nualart and Schoutens [64], they define the power-jump processes for every $i = 1, 2, 3, ...$, $\left\{ X_t^{(i)}, t \geq 0 \right\}$ and the compensated power-jump processes or Teugel martingales $\left\{ Y_t^i = X_t^i - \mathbb{E}\left[X_t^{(i)} \right], t \geq 0 \right\}$, in the following way:

$$X_t^{(1)} = X_t,$$

$$X_t^{(i)} = \sum_{0<s\leq t} (\Delta X_s)^i, i = 2, 3, 4, ...,$$

$$Y_t^{(i)} = X_t^{(i)} - t\mathbb{E}\left[X_t^{(i)} \right], i \geq 1,$$

Then applying a orthonormalization procedure to the martingales $Y^{(i)}$ we obtain a set of pairwise strongly orthonormal martingales $\left\{ H^{(i)}, t \geq 0 \right\}, i = 1, 2, ...$ such that each $H^{(i)}$ is a linear combination of the $Y^{(j)}, j = 1, 2, ...i$:

$$H^{(i)} = c_{i,i} Y^{(i)} + ... + c_{i,1} Y^{(1)},$$

where

$$c_{1,1} = [\int_{\mathbb{R}} y^2 \nu(dy)]^{-1/2}$$

and

$$\mathbb{E}[X_1] = a + \int_{|x|\geq 1} x\nu(dx).$$

The constants $c_{i,j}$ are the orthonormalization coefficients of the polynomials $\{1, x, x^2, x^3, ...\}$ with respect to the measure $\mu(dx) = x^2\nu(dx) + \sigma^2\delta_0(dx)$ and the polynomials we want to find are of the form

$$q_{i-1}(x) = c_{i,1} + c_{i,2}x + c_{i,3}x^2 + ... + c_{i,i-1}x^{i-2} + c_{i,i}x^{i-1}, i = 1, 2, 3....$$

Then, we just have to multiply by x to get the desired pairwise strongly orthonormal martingales:

$$p_i(x) = c_{i,1}x + c_{i,2}x^2 + c_{i,3}x^3 + ... + c_{i,i-1}x^{i-1} + c_{i,i}x^i, i = 1, 2, 3....$$

We now see that:

$$H_t^{(i)} = p_i(Y^{(i)}).$$

An important result in Nualart and Schoutens [64] is the predictable representation property:

Theorem 3.2.1 *Let $F \in L^2(\Omega, \mathbb{F}_T, \mathbb{P})$. Then F has a representation of the form:*

$$F = \mathbb{E}[F] + \sum_{j=1}^{\infty} \int_0^T \phi_t^{(j)} \, \mathrm{d}H_t^{(j)} \text{ where } \phi_t^{(j)} \tag{3.5}$$

are predictable processes such that

$$\mathbb{E}[\int_0^T \sum_{j=1}^{\infty} |\phi_t^{(j)}|^2 \, \mathrm{d}t] < \infty. \tag{3.6}$$

Consider the Backward Stochastic Differential Equation (BSDE):

$$-dY_t = b(t, Y_{t^-}, Z_t) \, \mathrm{d}t - \sum_{i=0}^{\infty} Z_t^{(i)} dH_t^{(i)}, Y_T = \xi, \tag{3.7}$$

where $H_t^{(i)}$ is the orthonormalized Teugel martingale of order i associated with the Lévy process X, $b : \Omega \times [0, T] \times \mathbb{R} \times M_T^2(l^2) \to \mathbb{R}$ is a measurable function and uniformly Lipschitz in the first two components and $\xi \in L_T^2$.

Consider the particular case of a BSDE:

$$dY_t = \sum_{i=0}^{\infty} Z_t^{(i)} dH_t^{(i)}, Y_T = h\left(X_T\right), \tag{3.8}$$

Let $f(\tau, x)$ be the solution of the following PIDE:

$$\frac{\partial f}{\partial \tau} = c\frac{\partial f}{\partial x} + \int_{\mathbb{R}} \left[f(\tau, x+y) - f(\tau, x) - y\frac{\partial f}{\partial x} \right] \nu(\mathrm{d}y),$$

$$f(0, x) = h(x), \tag{3.9}$$

where $c = r + \gamma + \int_{|y|\geq 1} y\nu(\mathrm{d}y) = a + \int_{|y|\geq 1} y\nu(\mathrm{d}y)$.

Defining $g(t, x) = f(\tau, x)$, where $\tau = T - t$, we obtain from (3.9):

$$\frac{\partial g}{\partial t} + c\frac{\partial g}{\partial x} + \int_{\mathbb{R}} \left[g(t, x+y) - g(t, x) - y\frac{\partial g}{\partial x} \right] \nu(\mathrm{d}y) = 0,$$

$$g(T, x) = h(x) \tag{3.10}$$

If g is sufficiently smooth, then by applying the Itô formula to $g(t, X_t)$ we obtain the following probabilistic representation for the case of a Lévy process given by $X_t = (r + \gamma)t + J_t = a + J_t$,where J_t is a pure jump process. For a detailed proof of this proposition see [65].

Proposition 3.2.1 *Assume $\sigma = 0$ and $\exists \lambda > 0$ such that*

$$\int_{|x|>1} e^{\lambda|x|}\nu(dx) < \infty.$$

If $g \in C^{1,2}$ is a classical solution of (3.10) and $\frac{\partial g}{\partial x}$ and $\frac{\partial^2 g}{\partial x^2}$ are bounded by a polynomial function of x, uniformly in t, then the unique adapted solution of (3.8) is given by

$$Y_t = g(t, X_t),$$

where

$$Z_t^1 = \int_{\mathbb{R}} \left[g(t, X_{t^-} + y) - g(t, X_{t^-}) - y\frac{\partial g}{\partial x}(t, X_{t^-}) \right] p_1(y)\nu(\mathrm{d}y) + \frac{\partial g}{\partial x}(t, X_{t^-}) \left(\int_{\mathbb{R}} y^2 \nu(\mathrm{d}y) \right)^{1/2}, \tag{3.11}$$

$$Z_t^i = \int_{\mathbb{R}} \left[g(t, X_{t^-} + y) - g(t, X_{t^-}) - y\frac{\partial g}{\partial x}(t, X_{t^-}) \right] p_i(y)\nu(\mathrm{d}y), i \geq 2$$

and $g(t, x) = \mathbb{E}[h(X_T)|X_t = x]$.

The probabilistic representation

$$g(t, x) = \mathbb{E}[h(X_T)|X_t = x] \tag{3.12}$$

obtained in the previous proposition is a Feynman-Kač formula for the solution of the PIDE (3.10).

Sketch of the proof. We can apply Itô's formula for processes with jumps, presented in Proposition 8.19 of [24], to $g(s, X_s)$ from $s = t$ to $s = T$:

$$g(T, X_T) = g(t, X_t) + \int_t^T \frac{\partial g}{\partial t}(s, X_{s^-})\,\mathrm{d}s + \int_t^T \frac{\partial g}{\partial x}(s, X_{s^-})\,\mathrm{d}X_s + \sum_{t<s\leq T}[g(s, X_s) - g(s, X_{s^-}) - \frac{\partial g}{\partial x}(s, X_{s^-})\Delta X_s]. \tag{3.13}$$

Making use of Lemma 5 in [64] and applying it to $h(s, y) = g(s, X_s) - g(s, X_{s^-} + y) - \frac{\partial g}{\partial x}(s, X_{s^-})y$ we get,

$$\begin{aligned} g(T, X_T) &= g(t, X_t) + \int_t^T \frac{\partial g}{\partial t}(s, X_{s^-})\,\mathrm{d}s + \int_t^T \frac{\partial g}{\partial x}(s, X_{s^-})\,\mathrm{d}X_s \\ &+ \sum_{i=1}^{\infty} \int_t^T \int_{\mathbb{R}} (g(s, X_s) - g(s, X_{s^-} + y) - \frac{\partial g}{\partial x}(s, X_{s^-})y)p_i(y)\nu(\,\mathrm{d}y)\,\mathrm{d}H_s^{(i)} \\ &+ \int_t^T (\int_{\mathbb{R}} g(s, X_s) - g(s, X_{s^-} + y) - \frac{\partial g}{\partial x}(s, X_{s^-})y\nu(\,\mathrm{d}y))\,\mathrm{d}s. \end{aligned} \tag{3.14}$$

But $X_t = Y_t^{(1)} + t\mathbb{E}[X_1] = (\int_{\mathbb{R}} y^2\nu(\,\mathrm{d}y))^{1/2}H_t^{(1)} + t(a + \int_{|y|\geq 1} y\nu(\,\mathrm{d}y))$, so

$$\begin{aligned} h(X_T) &= g(t, X_t) + \int_t^T [\frac{\partial g}{\partial t}(s, X_{s^-}) + \int_{\mathbb{R}} (g(s, X_s) - g(s, X_{s^-} + y) - \frac{\partial g}{\partial x}(s, X_{s^-})y)\nu(\,\mathrm{d}y) \\ &+ (a + \int_{|y|\geq 1} y\nu(\,\mathrm{d}y))\frac{\partial g}{\partial x}(s, X_{s^-})]\,\mathrm{d}s + \int_t^T \frac{\partial g}{\partial x}(s, X_{s^-})(\int_{\mathbb{R}} y^2\nu(\,\mathrm{d}y))^{1/2}\,\mathrm{d}H_s^{(1)} \\ &+ \sum_{i=1}^{\infty} \int_t^T (\int_{\mathbb{R}} (g(s, X_s) - g(s, X_{s^-} + y) - \frac{\partial g}{\partial x}(s, X_{s^-})y)p_i(y)\nu(\,\mathrm{d}y))\,\mathrm{d}H_s^{(i)}. \end{aligned} \tag{3.15}$$

Then because $g(t, x)$ solves (3.9) and taking expectations in (3.15), we get:

$$g\,(t, x) = \mathbb{E}\left[h\,(X_T)\,|X_t = x\right].$$

■

The next example shows how to perform the orthonormalization procedure described above and presents the Feynman-Kač formula for a pure jump process.

Example 3.2.2 *Consider the case where we have the sum of two compensated Poisson processes,* $X_t = N_t^1 + N_t^2$ *where* $N_t^1 = N_t - \lambda_1 t$ *and* $N_t^2 = N_t - \lambda_2 t$*, with Lévy measure* $\nu(dx) = (\lambda_1 + \lambda_2)\delta_1(x)\,\mathrm{d}x$*. Then performing a orthonormalization procedure we get*

$$\begin{aligned} &\psi_0 = 1 \Rightarrow q_0 = \frac{1}{(\int_{\mathbb{R}} x^2\nu(dx))^{1/2}} = \frac{1}{(\int_{\mathbb{R}} x^2(\lambda_1 + \lambda_2)\delta_1(x)\,\mathrm{d}x)^{1/2}} = \frac{1}{(\lambda_1 + \lambda_2)^{1/2}} \\ &\psi_1 = x + a_{1,0}q_0 \Rightarrow \psi_1 = x - \langle x, q_0\rangle\, q_0 = x - \int_{\mathbb{R}} x\frac{1}{\sqrt{\lambda_1 + \lambda_2}}(\lambda_1 + \lambda_2)\delta_1(x)\frac{1}{\sqrt{\lambda_1 + \lambda_2}}\,\mathrm{d}x \\ &\quad = x - 1 \Rightarrow \psi_1 = 0 \Rightarrow q_1 = 0. \end{aligned}$$

By recurrence we get that $q_i = 0, i = 1, 2, 3, ...$. *Then in terms of previous notation* $p_1(x) = \frac{x}{\sqrt{\lambda_1+\lambda_2}}$ *and* $p_i(x) = 0, i = 2, 3, ...$, *which implies*

$$H_t^{(1)} = \frac{1}{\sqrt{\lambda_1 + \lambda_2}} X_t \text{ and } H_t^{(i)} = 0, i = 2, 3, ... \tag{3.16}$$

Then, by Proposition (3.2.1)

$$Y_t = h\left(X_T\right) - \int_t^T Z_s^{(1)} dH_s^{(1)},$$

where

$$\begin{aligned} Z_t^{(1)} &= [g(t, x+1) - g(t, x) - \frac{\partial g}{\partial x}]\sqrt{\lambda_1 + \lambda_2} + \frac{\partial g}{\partial x}\sqrt{\lambda_1 + \lambda_2} \\ &= [g(t, x+1) - g(t, x)]\sqrt{\lambda_1 + \lambda_2}. \end{aligned}$$

Then,

$$\begin{aligned} Y_t &= h\left(X_T\right) - \int_t^T [g(t, X_{s^-} + 1) - g(t, X_{s^-})]\sqrt{\lambda_1 + \lambda_2} \frac{1}{\sqrt{\lambda_1 + \lambda_2}} \, \mathrm{d}X_s \Leftrightarrow \\ Y_t &= h\left(X_T\right) - \int_t^T [g(t, X_{s^-} + 1) - g(t, X_{s^-})] \, \mathrm{d}X_s. \end{aligned}$$

Moreover,

$$g(t, x) = \mathbb{E}[h\left(X_T\right) | X_t = x]. \tag{3.17}$$

Notice that in Proposition 3.2.1, g is assumed to be smooth and its derivatives have to be bounded by a polynomial function of x, uniformly in t. However, these conditions are rarely satisfied in applications.

Example 3.2.3 *Consider an European call option with payoff function* $H(x) = (x-1)^+$ *and strike price* $K = 1$. *We see that the first derivative of the payoff function has a discontinuity at* $x = 1$*:*

$$H^{'}(x) = \begin{cases} 1 & \text{if } x > 1, \\ 0 & \text{if } x < 1. \end{cases}$$

Then, we see that the second derivative diverges at $x = 1$. *So, when* t *tends to* T *and if the option is at the money* $(S = K)$ *the second derivative of the price function tends to the second derivative of the payoff function that diverges when* $S = K$. *This means that the gamma of the call option is not uniformly bounded in time.*

3.3 Option prices as classical solutions of PIDEs

3.3.1 European Options

Consider an European option with maturity T and payoff $H\left(S_T\right)$. Assume that the payoff function is a Lipschitz function

$$\left|H\left(x\right) - H\left(y\right)\right| \leq c\left|x - y\right| \tag{3.18}$$

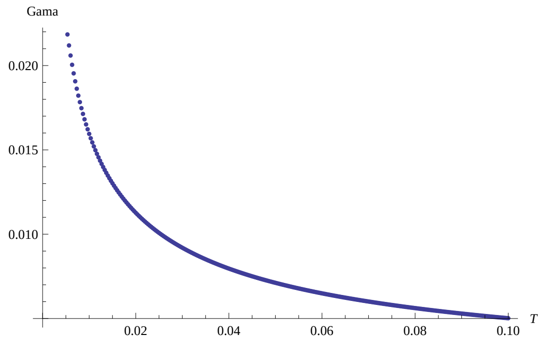

Figure 3.1: As T tends to zero the gamma of the option tends to infinity.

for some $c > 0$. As we already know, the value of that option at time t is :

$$C(t, S_t) = \mathbb{E}\left[e^{-r(T-t)} H(S_T) \,|S_t = S\right] = e^{-r(T-t)} \mathbb{E}\left[H\left(S e^{r(T-t)+X_{T-t}}\right)\right]. \tag{3.19}$$

We will assume that $\widehat{S}_t = e^{X_t}$ is a square integrable martingale

$$\int_{|x|>1} e^{2y} \nu(dy) < \infty. \tag{3.20}$$

Then the dynamics of $\widehat{S}_t$ is given by:

$$\frac{\mathrm{d}\widehat{S}_t}{\widehat{S}_{t-}} = \sigma \,\mathrm{d}W_t + \int_{\mathbb{R}} (e^x - 1)\, \widetilde{J}_X(dt, dx), \sup_{t\in[0,T]} \mathbb{E}\left[\widehat{S}_t^2\right] < \infty. \tag{3.21}$$

The proofs of the following propositions are presented in [83] and are shown in greater detail in the appendix. These propositions will be needed to prove the Proposition 3.3.3.

Proposition 3.3.1 *Let the payoff function H satisfy the Lipschitz condition* (3.18). *Then the forward value of an European option defined by* (3.2), $f(\tau, x) = \mathbb{E}[H(Se^{x+r\tau+X_\tau})]$, *is continuous on* $[0, T] \times \mathbb{R}$.

Proposition 3.3.2 *Let h be a measurable function with polynomial growth at infinity:* $\exists p > 0, |h(x)| \le C(1 + x^p)$. *If*

$$\sigma > 0 \quad or \quad \exists \beta \in (0, 2) \quad such \quad that \quad \liminf_{\epsilon\to 0} \frac{1}{\epsilon^{2-\beta}} \int_{-\epsilon}^{\epsilon} |x|^2 \nu(\mathrm{d}x) > 0 \tag{3.22}$$

and

$$\forall n \ge 0, \int_{|y|>1} |y|^n \nu(\mathrm{d}y) < \infty, \tag{3.23}$$

Then, $f(\tau, x) = \mathbb{E}[h(x + r\tau + X_\tau)]$ *belongs to* $C^\infty((0, T] \times \mathbb{R})$.

The proof of this proposition, following the proof given in Voltchkova [26], is presented here in greater detail.

Proposition 3.3.3 *Consider the exponential Lévy model* $S_t = S_0 e^{rt+X_t}$ *where the Lévy process* X *verifies* (3.20) *and* (3.22). *Then the value of a European option with terminal payoff* $H(S_T)$ (*satisfying* (3.18)) *given by*

$$C : [0,T] \times (0,\infty) \to \mathbb{R}, (t,S) \to C(t,S) = \mathbb{E}\left[e^{-r(T-t)} H(S_T) \,|S_t = S\right] \tag{3.24}$$

is continuous on $[0,T] \times (0,\infty)$, C^∞ *on* $(0,T) \times (0,\infty)$ *and satisfies the integro-differential equation:*

$$\begin{aligned} &\frac{\partial C}{\partial t}(t,S) + rS\frac{\partial C}{\partial S}(t,S) + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}(t,S) - rC(t,S) + \\ &+ \int \left[C(t,Se^y) - C(t,S) - S(e^y - 1)\frac{\partial C}{\partial S}(t,S)\right]\nu(\,\mathrm{d}y) = 0; \end{aligned} \tag{3.25}$$

on $[0,T) \times (0,\infty)$ *with the terminal condition:*

$$C(T,S) = H(S) \quad, \forall S > 0 \tag{3.26}$$

Proof. By Proposition 3.3.1 we know that $C(t,S) = e^{r\tau} f(\tau,x)$ is continuous on $[0,T] \times \mathbb{R}$ and by Proposition 3.3.2, $C(t,S_t) \in C^\infty((0,T) \times (0,\infty))$.

It remains to prove that $C(t,S)$ satisfies (3.25).

The risk neutral dynamics of S_t under $\mathbb{Q}$ is given by

$$\mathrm{d}S_t = rS_{t^-}\,\mathrm{d}t + \sigma S_{t^-}\,\mathrm{d}W_t + \int_{\mathbb{R}} (e^x - 1) S_{t^-} \widetilde{J}_X(dt,dx).$$

Applying the Itô formula to $\widehat{C}_t = e^{-rt} C(t,S_t)$, where $S_t = e^{rt+X_t}$ we get (see Proposition 8.18 of [24]),

$$\begin{aligned} \mathrm{d}(e^{-rt}C(t,S_t)) = e^{-rt}(&-rC(t,S_{t^-})\,\mathrm{d}t + \frac{\partial C}{\partial t}(t,S_{t^-})\,\mathrm{d}t + \frac{\sigma^2}{2} S_{t^-}^2 \frac{\partial^2 C}{\partial S^2}(t,S_{t^-})\,\mathrm{d}t \\ &+ \frac{\partial C}{\partial S}(t,S_{t^-})\,\mathrm{d}S_t + \int_{\mathbb{R}} (C(t,y+S_{t^-}) - C(t,S_{t^-}) - y\frac{\partial C}{\partial S}(t,S_{t^-}))\widetilde{J}_S(dt,dy)). \end{aligned}$$

Simplifying and plugging in the dynamics for S_t we get:

$$\begin{aligned} \mathrm{d}\widehat{C}_t &= e^{-rt}\frac{\partial C}{\partial S}(t,S_{t^-})\sigma S_{t^-}\,\mathrm{d}W_t + e^{-rt}\int_{\mathbb{R}} (C(t,S_{t^-}e^x) - C(t,S_{t^-}))\widetilde{J}_X(dt,dx) \\ &\quad + e^{-rt}(-rC(t,S_{t^-}) + \frac{\partial C}{\partial t}(t,S_{t^-}) + \frac{\sigma^2}{2}e^{-rt}S_{t^-}^2\frac{\partial^2 C}{\partial S^2}(t,S_{t^-}) + rS_{t^-}\frac{\partial C}{\partial S}(t,S_{t^-}) \\ &\quad + \int_{\mathbb{R}} (C(t,S_{t^-}e^x) - C(t,S_{t^-}) - S_{t^-}(e^x - 1)\frac{\partial C}{\partial S}(t,S_{t^-}))\nu(\,\mathrm{d}x))\;\mathrm{d}t \\ &= b(t)\,\mathrm{d}t + \,\mathrm{d}M_t, \end{aligned}$$

where

$$\begin{aligned} b(t) &= -rC(t,S_{t^-}) + \frac{\partial C}{\partial t}(t,S_{t^-}) + \frac{\sigma^2}{2}e^{-rt}S_{t^-}^2\frac{\partial^2 C}{\partial S^2}(t,S_{t^-}) + rS_{t^-}\frac{\partial C}{\partial S}(t,S_{t^-}) \\ &\quad + \int_{\mathbb{R}} (C(t,S_{t^-}e^x) - C(t,S_{t^-}) - S_{t^-}(e^x - 1)\frac{\partial C}{\partial S}(t,S_{t^-}))\nu(\,\mathrm{d}x), \\ M_t &= \int_0^T e^{-rt}\frac{\partial C}{\partial S}(t,S_{t^-})\sigma S_{t^-}\,\mathrm{d}W_t + \int_0^T e^{-rt}\int_{\mathbb{R}} (C(t,S_{t^-}e^x) - C(t,S_{t^-}))\widetilde{J}_X(dt,dx). \end{aligned}$$

It remains to prove that M_t is a martingale, because by proposition 8.9 of [24], if $\widehat{C}_t - M_t = \int_0^t b(s)\,\mathrm{d}s$ is a continuous martingale with finite variation paths then $\int_0^t b(s)\,\mathrm{d}s = X_0$ a.s, which implies that b(t)=0 a.s.

In order for $\int_0^t e^{-rt}\int_{\mathbb{R}}(C(t,S_{t^-}e^x) - C(t,S_{t^-}))\widetilde{J}_X\,(dt,dx)$ to be a martingale we have to show that:

$$\mathbb{E}[\int_0^T e^{-2rt}\int_{\mathbb{R}}(C(t,S_{t^-}e^x) - C(t,S_{t^-}))^2\nu(\,\mathrm{d}x)\,\mathrm{d}t] < \infty.$$

Then, by the Lipschitz condition

$$\mathbb{E}[\int_0^T e^{-2rt}\int_{\mathbb{R}}(C(t,S_{t^-}e^x) - C(t,S_{t^-}))^2\nu(\,\mathrm{d}x)\,\mathrm{d}t] \le \mathbb{E}[\int_0^T e^{-2rt}\int_{\mathbb{R}} c^2 S_{t^-}^2(e^x-1)^2\nu(\,\mathrm{d}x)\,\mathrm{d}t].$$

Moreover,

$$\begin{aligned} c^2\int_{\mathbb{R}}(e^x-1)^2\nu(\,\mathrm{d}x) &= c^2\int_{|x|\le 1}(e^x-1)^2\nu(\,\mathrm{d}x) + c^2\int_{|x|>1}(e^x-1)^2\nu(\,\mathrm{d}x)\\ &\le \widetilde{k}^2\int_{|x|\le 1}|x|^2\nu(\,\mathrm{d}x) + c^2\int_{|x|>1}(e^x-1)^2\nu(\,\mathrm{d}x)\\ &= \widetilde{k}^2\int_{|x|\le 1}|x|^2\nu(\,\mathrm{d}x) + c^2\int_{|x|>1}(e^{2x}+1-2e^x)\nu(\,\mathrm{d}x)\\ &\le \widetilde{k}^2\int_{|x|\le 1}|x|^2\nu(\,\mathrm{d}x) + \widetilde{k}^2\int_{|x|>1}(e^{2x}+1)\nu(\,\mathrm{d}x), \end{aligned}$$

for some $\widetilde{k}$ sufficiently big.

Then,

$$\begin{aligned} &\mathbb{E}[\int_0^T e^{-2rt}\int_{\mathbb{R}} c^2 S_{t^-}^2\,(e^x-1)^2\,\nu(\,\mathrm{d}x)\,\mathrm{d}t]\\ &\le \mathbb{E}[\int_0^T S_{t^-}^2 e^{-2rt}\left(\widetilde{k}^2\int_{|x|\le 1}|x|^2\nu(\,\mathrm{d}x) + \widetilde{k}^2\int_{|x|>1}(e^{2x}+1)\nu(\,\mathrm{d}x)\right)\,\mathrm{d}t]\\ &= \widetilde{k}^2\left(\int_{|x|\le 1}|x|^2\nu(\,\mathrm{d}x) + \int_{|x|>1}(e^{2x}+1)\nu(\,\mathrm{d}x)\right)\mathbb{E}[\int_0^T S_{t^-}^2 e^{-2rt}\,\mathrm{d}t]\\ &= \widetilde{k}^2\left(\int_{\mathbb{R}} 1\wedge|x|^2\nu(\,\mathrm{d}x) + \int_{|x|>1} e^{2x}\nu(\,\mathrm{d}x)\right)\int_0^T \mathbb{E}[S_{t^-}^2]e^{-2rt}\,\mathrm{d}t < \infty. \end{aligned}$$

Then $\int_0^t e^{-rt}\int_{\mathbb{R}}(C(t,S_{t^-}e^x) - C(t,S_{t^-}))\widetilde{J}_X\,(dt,dx)$ is a square integrable martingale.

It remains to prove that $\int_0^T e^{-rt}\frac{\partial C}{\partial S}(t,S_{t^-})\sigma S_{t^-}\,\mathrm{d}W_t$ is also a martingale, such that M_t is a martingale.

$$\begin{aligned} \mathbb{E}[\int_0^T e^{-2rt}(\frac{\partial C}{\partial S}(t,S_{t^-})\sigma S_{t^-})^2\,\mathrm{d}t] &\le \mathbb{E}[\int_0^T e^{-2rt}\left\|\frac{\partial C}{\partial S}(t,S_{t^-})\right\|_{L^\infty}^2\sigma^2 S_{t^-}^2\,\mathrm{d}t]\\ &\le c^2\sigma^2\int_0^T e^{-2rt}\mathbb{E}[S_{t^-}^2]\,\mathrm{d}t < \infty, \end{aligned}$$

because if C is Lipschitz, then $\frac{\partial C}{\partial S}(t, S_{t^-}) \in L^\infty$. ■

The condition (3.22) holds for all jump-diffusion models with Brownian component or for processes with Lévy densities with behavior near zero as $\nu(x) \sim \frac{c}{x^{1+\beta}}$ with $\beta > 0$. This condition is not satisfied for the Generalized Hyperbolic model or in particular for the Variance Gamma model. Note also that when $\nu = 0$ then we obtain the well known Black-Scholes PDE. The next example shows that if we do not impose any conditions on a given Lévy triplet, then the function that represents the price of a binary option is not smooth.

Example 3.3.1 *Consider the Generalized Hyperbolic model and for simplicity assume $\delta = 0$. Then the density function becomes*

$$p_t(x) = C|x-\mu|^{\frac{t}{\kappa}-\frac{1}{2}} K_{|\frac{t}{\kappa}-\frac{1}{2}|}(\alpha|x-\mu|)e^{\beta(x-\mu)}.$$

Notice that, when

$$z \to 0 \ , \ K_v(z) \sim \frac{1}{2}\Gamma(v)(\frac{z}{2})^{-v} \Rightarrow$$

$$\begin{aligned}\lim_{x\to\mu} p(t,x) &= \lim_{x\to\mu} C|x-\mu|^{\frac{t}{\kappa}-\frac{1}{2}} K_{|\frac{t}{k}-\frac{1}{2}|}(\alpha|x-\mu|)e^{\beta(x-\mu)} \\ &= \lim_{x\to\mu} C|x-\mu|^{\frac{t}{\kappa}-\frac{1}{2}-|\frac{t}{\kappa}-\frac{1}{2}|}\frac{1}{2}\Gamma(\frac{t}{\kappa}-\frac{1}{2})e^{\beta(x-\mu)} = \infty\end{aligned}$$

if and only if

$$\frac{t}{\kappa}-\frac{1}{2}-|\frac{t}{\kappa}-\frac{1}{2}| < 0 \Leftrightarrow 2(\frac{t}{\kappa}-\frac{1}{2}) < 0.$$

Then we conclude that $p(t,x)$ is locally unbounded at $x = \mu$ if $t < \frac{\kappa}{2}$.

If $0 < \frac{t}{\kappa}-\frac{1}{2}-|\frac{t}{\kappa}-\frac{1}{2}| < 1$, *then* $p(t,x) \in C^0$ *but not in* C^1.

If $0 < \frac{t}{\kappa}-\frac{1}{2}-|\frac{t}{\kappa}-\frac{1}{2}| < 1$ *or* $1 < 2\frac{t}{\kappa} < 2$, *then* $p(t,x) \in C^0$ *but not in* C^1.

If $1 < \frac{t}{\kappa}-\frac{1}{2}-|\frac{t}{\kappa}-\frac{1}{2}| < 2$ *or* $2 < 2\frac{t}{\kappa} < 3$, *then* $p(t,x) \in C^1$ *but not in* C^2.

So by recurrence we conclude that

$$\textit{if } p-1 < \frac{t}{\kappa}-\frac{1}{2}-|\frac{t}{\kappa}-\frac{1}{2}| < p \ \textit{ or } p < 2\frac{t}{\kappa} < p+1 \ , \textit{ then } p(t,x) \in C^{p-1} \textit{ but not in } C^p.$$

So if $t \in (p\frac{\kappa}{2}, (p+1)\frac{\kappa}{2})$ then $p(t,x)$ belongs to $C^{p-1}(\mathbb{R})$ but not in $C^p(\mathbb{R})$ and for $t < \frac{\kappa}{2}$, $p(t,.)$ is locally unbounded.

Consider a binary option whose payoff function is given by $h(x) = 1_{x\geq l_0}$. Its price is given by

$$\begin{aligned}C(t,S) &= e^{-r(T-t)}E[H(S_T)|S_t = S] = e^{-r(T-t)}E[h(x + r(T-t) + X_{T-t})] \\ &= e^{-r(T-t)}E[1_{x+r(T-t)+X_{T-t}\geq l_0}] = e^{-r(T-t)}\mathbb{Q}[x + r(T-t) + X_{T-t} \geq l_0] \\ &= e^{-r(T-t)}\mathbb{Q}[X_{T-t} \geq l_0 - r(T-t) - x] = \int_d^\infty p(t,x)\,\mathrm{d}x,\end{aligned}$$

where $d = l_0 - r(T-t) - x$. Then for $t < \frac{k}{2}$ the binary option is continuous but is not differentiable.

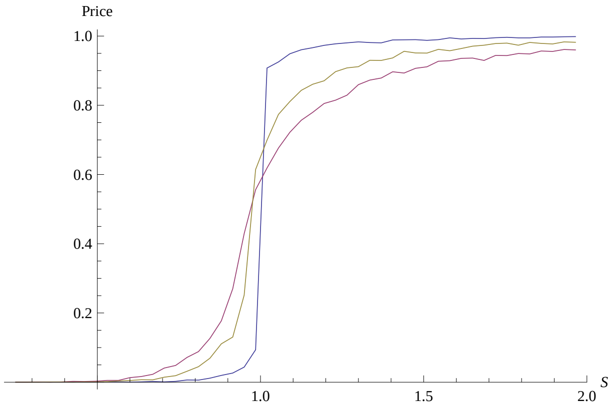

Figure 3.2: The price of a binary option is not differentiable at the money, using the Monte Carlo method, when $\mu = 0$ with $\kappa = 2, r = 0, \sigma = 0.25, \theta = -0.1$. Blue: $T = 0.1$, red: $T = 0.5$, yellow: $T = 1$.

3.3.2 Barrier Options

We now present the result without proof, analogous to Proposition 3.3.3. It tells us that the price function of a barrier option is smooth enough if and only if it satisfies a PIDE. For a full detailed proof of this proposition, see [83].

Proposition 3.3.4 *Consider* $S_t = S_0 e^{rt+X_t}$ *where the Lévy process* X *verifies* (3.20). *Let* $\theta_t = \inf\{s \geq t | S_t \notin (L, U)\}$ *where* $0 \leq L < U \leq \infty$ *and suppose that* $H \geq 0$ *and* $\exists N > 0 : H(S) \leq N(1 + S)$. *Define*

$$C_b(t, S) = e^{-r(T-t)} \mathbb{E}[H(S_T) 1_{T<\theta_t} | S_t = S], \tag{3.27}$$

as the value of a knock-out option, where $C_b(t, S) \in C^{1,2}([0, T) \times (L, U))$. *Then it satisfies the integro-differential equation:*

$$\frac{\partial C_b}{\partial t}(t, S) + rS \frac{\partial C_b}{\partial S}(t, S) + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C_b}{\partial S^2}(t, S) - rC_b(t, S) +$$

$$+ \int \left[C_b(t, Se^y) - C_b(t, S) - S(e^y - 1) \frac{\partial C_b}{\partial S}(t, S) \right] \nu(\mathrm{d}y) = 0; \tag{3.28}$$

on $[0, T) \times (L, U)$ *with the conditions:*

$$C_b(T, S) = H(S) \ \forall S \in (L, U), \tag{3.29}$$

$$C_b(t, S) = 0 \ \forall S \notin (L, U). \tag{3.30}$$

Conversely, every solution of (3.28) – (3.30) *belonging to* $C^{1,2}([0, T) \times (L, U))$ *has the stochastic representation given by* (3.27).

Before we study the continuity of barrier option prices we will need the definition of first passage process: Let $\{Y_t\}$ be a Lévy process defined by $Y_t = rt + X_t$. Finally set $M_t = \sup_{0\leq s\leq t} Y_s$. Following the notation of Sato [72], we define

$$R_x = \inf\{s \geq 0 | Y_s > x\}, R_x^- = \inf\{s \geq 0 | -Y_s > x\},$$

$$R_x^{''} = \inf\{s \geq 0 | Y_s \vee Y_{s^-} \geq x\}.$$

We know that $\{R_x, x \geq 0\}$ is a process with non-decreasing paths, so we can define $R_{x^-}(\omega) = \lim_{\epsilon\to 0} R_{x-\epsilon}(\omega)$. As for the right continuity, since Y_t is right-continuous, R_x is also right-continuous. Following the terminology of Voltchkova:

Definition 3.3.2 *Consider a Lévy process Y_t with triplet (σ, γ, ν).*
If $\sigma = 0$ and $\nu(\mathbb{R}) < \infty$, then Y_t is of type A (Compound Poisson).
If $\sigma = 0$, $\nu(\mathbb{R}) = \infty$ and $\int_{|x|\leq 1} |x|\nu(\,\mathrm{d}x) < \infty$, then Y_t is of type B (finite variation,infinite activity).
If $\sigma > 0$ or $\int_{|x|\leq 1} |x|\nu(\,\mathrm{d}x) = \infty$, then Y_t is of type C (infinite variation) .

In order to prove the continuity of barrier option prices we need some properties of the process $\{R_x\}$.

The first lemma is an extension of the Lemma 3.5.3 presented in [83], in the sense that also applies to Lévy processes of type A. The second and third lemmas are presented in [83].

Lemma 3.3.3 *If $\{Y_t\}$ is of type B or C or A with $\gamma \neq 0$ then:*

$$\forall x > 0, \mathbb{Q}[R_{x^-} = R_x] = 1. \tag{3.31}$$

Proof.
Introducing

$$\Omega_1 = \{\omega \in \Omega : R_{x^-} < R_x\}, \Omega_2 = \left\{\omega \in \Omega : R_x^{''} = R_x\right\}.$$

Define $R_{x'} = \inf\{s \geq 0 | Y_s \geq x\}$. By Lemma 49.6 of [72] we have that, for any $x > 0, \mathbb{P}[R_x = R_{x'} = R_x^{''}] = 1$, because Y_t is non zero and is not Compound Poisson process, which means that is of type B or C or A with $\gamma \neq 0$. Then $\mathbb{Q}[\Omega_2] = 1$. In order that, $\mathbb{Q}[R_{x^-} = R_x] = 1$ we must have $\mathbb{Q}[\Omega_1] = 0$, because we always have $R_{x^-} \leq R_x$. So we have to prove that $\mathbb{Q}[R_{x^-} < R_x] = \mathbb{Q}[R_{x^-} < R_x^{''}] = \mathbb{Q}[\Omega_1 \cap \Omega_2] = 0$.

By contradiction, suppose that $\exists \omega \in \Omega_1 \cap \Omega_2 \Rightarrow \omega \in \Omega_1, \omega \in \Omega_2$.Then,

$$\exists u \geq 0, R_{x^-} = u, \tag{3.32}$$
$$\exists u < t, t = R_x^{''}. \tag{3.33}$$

By definition of $R_{x^-} = \lim_{\epsilon\to 0}, R_{x-\epsilon}$ and because $R_{x^-} = u$ we get

$$\forall_{\delta>0}, \exists_{\eta>0} : |\epsilon| < \eta \Rightarrow u - \delta < R_{x-\epsilon} < u + \delta.$$

$$\forall_{\epsilon>0}, \forall_{\delta>0}, \exists s < u+\delta : Y_s > x-\epsilon.$$

Now, choose $\epsilon_n = \delta_n = \frac{1}{n} \to 0$.Then,

$$\exists s_n \forall n : s_n < u + \frac{1}{n}, Y_{s_n} > x - \frac{1}{n}.$$

Because $\{s_n\}$ is bounded, there is a convergent subsequence $s_{n_k} \uparrow s_0$ with $s_0 \leq u < t$. This means that

$$Y_{s_n} > x - \frac{1}{n} \Rightarrow Y_{s_0^-} \geq x,$$

and if $s_{n_k} \downarrow s_0$ with $s_0 \leq u < t$, then

$$Y_{s_n} > x - \frac{1}{n} \Rightarrow Y_{s_0} \geq x.$$

Then, $Y_{s_0^-} \vee Y_{s_0} \geq x$. But this contradicts (3.33) because it implies that $\forall s < t$, $Y_{s^-} \vee Y_s < x$. Then $\Omega_1 \cap \Omega_2 = \emptyset$. ■

Lemma 3.3.4 *If $\{Y_t\}$ is of type B with $R_0 = 0$ a.s or of type C, then:*

$$\forall x > 0, \forall t \geq 0, \mathbb{Q}[R_x = t] = 0. \tag{3.34}$$

Lemma 3.3.5 *If $\{Y_t\}$ is of type B or C, then $\forall x > 0, \forall t \geq 0$:*

$$\mathbb{Q}[R_x \leq t < R_{x+\epsilon}] \to 0, \tag{3.35}$$

$$\mathbb{Q}[R_{x-\epsilon} \leq t < R_x] \to 0, \tag{3.36}$$

when $\epsilon \to 0$.If we have also $R_0 = 0$ a.s, then (3.35) is satisfied for $x = 0, t > 0$.

The next proposition shows that the up-and-out option is continuous. The sketched proof of this proposition is shown in [83] and a more detailed version is shown in the appendix.

Proposition 3.3.5 *Let Y_t be of type B or C with $R_0 = 0$ a.s. Suppose that $H : (0, U) \to [0, \infty)$ is Lipschitz:*

$$\forall S_1, S_2 \in (0, U), |H(S_1) - H(S_2)| \leq k|S_1 - S_2|, \tag{3.37}$$

for some $k > 0$ and let $u = log(\frac{U}{S_0})$. Then the function $f_U(\tau, x)$ defined by

$$f_U(\tau, x) = \begin{cases} E\left[H\left(S_0 e^{x+Y_\tau}\right) 1_{\tau < R_{u-x}}\right] & \text{if } x < u, \\ 0 & \text{if } x \geq u, \end{cases} \tag{3.38}$$

is continuous on $(0, T] \times \mathbb{R}$.

The following proposition gives the continuity result for the case of a down-and-out option. The proof of this proposition, similar to the previous one, can be found in [83].

Proposition 3.3.6 *Let Y_t be of type B or C with $R_{0^-} = 0$ a.s. Suppose that $H : (L, \infty) \to [0, \infty)$ is Lipschitz:*

$$\forall S_1, S_2 \in (0, U), |H(S_1) - H(S_2)| \leq k|S_1 - S_2|, \tag{3.39}$$

with $L < S_0$ and let $l = log(\frac{L}{S_0})$. Then the function $f_L(\tau, x)$ defined by

$$f_L(\tau, x) = \begin{cases} E\left[H\left(S_0 e^{x+Y_\tau}\right) 1_{\tau < R^-_{x-l}}\right] & \text{if } x > l, \\ 0 & \text{if } x \leq l, \end{cases} \tag{3.40}$$

is continuous on $(0, T] \times \mathbb{R}$.

Finally the continuity result of a double-barrier option with payoff $H(S_T)1_{T<inf\{t\geq 0, S_t \in (L,U)\}}$, where $L < S_0 < U$, $u = log(\frac{U}{S_0})$ and $l = log(\frac{L}{S_0})$ is presented here without proof and can be found in [83].

Proposition 3.3.7 *Let Y_t be of type B or C with $R_{0^-} = 0$ and $R_0 = 0$ a.s. Suppose that $H : (L, \infty) \to [0, \infty)$ is Lipschitz:*

$$\forall S_1, S_2 \in (0, U), |H(S_1) - H(S_2)| \leq k|S_1 - S_2|, \tag{3.41}$$

with $k > 0$. Then the function $f_D(\tau, x)$ defined by

$$f_D(\tau, x) = \begin{cases} E\left[H\left(S_0 e^{x+Y_\tau}\right) 1_{\tau < R^-_{x-l} \cap R_{u-x}}\right] & \text{if } x \in (l, u), \\ 0 & \text{if } x \notin (l, u), \end{cases} \tag{3.42}$$

is continuous on $(0, T] \times \mathbb{R}$.

The results for the continuity of a up-and-out option and a down-and-out option are proven here when the Lévy process is of type A.

Proposition 3.3.8 *Suppose $\{Y_t\}$ is a Lévy process of type A with $\gamma \neq 0$, $R_0 = 0$ a.s and $\mathbb{Q}[R_x = t] = 0, \forall x \geq 0, t \geq 0, (t, x) \neq (0, 0)$. Suppose that H:$(0, U) \to (0, \infty)$ is Lipschitz. Then for every τ sufficiently small $f_u(\tau, x)$ defined by*

$$f_u(\tau, x) = \begin{cases} E\left[H\left(S_0 e^{x+Y_\tau}\right) 1_{\tau < R_{u-x}}\right] & \text{if } x < u, \\ 0 & \text{if } x \geq u, \end{cases}$$

is continuous.

Proof. Considering $\tau > 0$ and $x = u$, we have by definition,

$$|f_u(\tau, u - \epsilon) - f_u(\tau, u)| = E\left[H\left(S_0 e^{u-\epsilon+Y_\tau}\right) 1_{\tau < R_\epsilon}\right] \leq ME\left[1_{\tau < R_\epsilon}\right] = M\mathbb{Q}\left[\tau < R_\epsilon\right].$$

Let $\{\epsilon_n\} \to 0$ and $\Omega_n = \{\omega \in \Omega : \tau < R_{\epsilon_n}\}$, then $\{\Omega_n\}$ is a decreasing sequence: $\Omega_1 \supset \Omega_2 \supset ...\Omega_n \supset ...$, because R_x is an increasing process. Then

$$\lim_{n\to\infty} \mathbb{Q}\left[\tau < R_{\epsilon_n}\right] = \mathbb{Q}\left[\cap_{n=1}^{\infty} \Omega_n\right] = \mathbb{Q}\left[\tau < R_0\right] = 0,$$

because $R_0 = 0$ a.s.

For $\tau > 0$ and $x < u$:

$$\begin{aligned}|f_u(\tau, x+\epsilon) - f_u(\tau, x)| &= |E\left[H\left(S_0 e^{x+\epsilon+Y_\tau}\right)1_{\tau<R_{u-x-\epsilon}}\right] - \mathbb{E}[H\left(S_0 e^{x+Y_\tau}\right)1_{\tau<R_{u-x}}\right]| \\ &= |E[H(S_0 e^{x+\epsilon+Y_\tau}) - H(S_0 e^{x+Y_\tau}))1_{\tau<R_{u-x-\epsilon}} \\ &\quad + H(S_0 e^{x+Y_\tau})(1_{\tau<R_{u-x-\epsilon}} - 1_{\tau<R_{u-x}})]| \\ &\leq cS_0 e^{x+r\tau}\mathbb{E}[e^{Y_\tau}]|e^\epsilon - 1| + M\mathbb{Q}[R_{u-x-\epsilon} \leq \tau < R_{u-x}]\end{aligned}$$

But, $|e^\epsilon - 1| \to 0$ as $\epsilon \to 0$ and

$$\mathbb{Q}[R_{u-x-\epsilon} \leq \tau < R_{u-x}] \to \mathbb{Q}[R_{(u-x)^-} \leq \tau < R_{u-x}] \leq \mathbb{Q}[R_{(u-x)^-} \neq R_{u-x}] = 0, \quad (3.43)$$

because of lemma (3.3.3). In a similar way:

$$\begin{aligned}|f_u(\tau, x-\epsilon) - f_u(\tau, x)| &= |E\left[H\left(S_0 e^{x-\epsilon+Y_\tau}\right)1_{\tau<R_{u-x+\epsilon}}\right] - \mathbb{E}[H\left(S_0 e^{x+Y_\tau}\right)1_{\tau<R_{u-x}}\right]| \\ &\leq cS_0 e^{x+r\tau}|1 - e^{-\epsilon}| + M\mathbb{Q}[R_{u-x} \leq \tau < R_{u-x+\epsilon}] \to 0,\end{aligned}$$

because $|1 - e^{-\epsilon}| \to 0$ as $\epsilon \to 0$, and

$$\mathbb{Q}[R_{u-x} \leq \tau < R_{u-x+\epsilon}] \to \mathbb{Q}[R_{u-x} = \tau] = 0. \quad (3.44)$$

The continuity in time is proven in the same way as in the proof of Proposition 3.3.5 (see the Appendix). Finally, using the triangular inequality we can prove continuity for all $(\tau, x) \in [0, T] \times (-\infty, u)$. ■

Proposition 3.3.9 *Suppose $\{Y_t\}$ is a Lévy process of type A with $\gamma \neq 0$, $R_0^- = 0$ a.s and $\mathbb{Q}[R_x^- = t] = 0, \forall x \geq 0, t \geq 0, (t, x) \neq (0, 0)$.Suppose that H:$(L, \infty) \to (0, \infty)$ is Lipschitz. Then, for every τ sufficiently small $f_l(\tau, x)$ defined by*

$$f_l(\tau, x) = \begin{cases} E\left[H\left(S_0 e^{x+Y_\tau}\right)1_{\tau<R_{x-l}^-}\right] & \text{if } x > l, \\ 0 & \text{if } x \leq l, \end{cases}$$

is continuous.

Proof. Considering $\tau > 0$ and by definition of the price of a down-and-out option,

$$\begin{aligned}|f_l(\tau, l+\epsilon) - f_l(\tau, l)| &= E\left[H\left(S_0 e^{l+\epsilon+Y_\tau}\right)1_{\tau<R_\epsilon^-}\right] \\ &\leq CE\left[\left(1 + S_0 e^{l+\epsilon+Y_\tau}\right)1_{\tau<R_\epsilon^-}\right] \\ &= C\mathbb{Q}\left[\tau < R_\epsilon^-\right] + CS_0 e^{l+\epsilon}E\left[e^{Y_\tau}1_{\tau<R_\epsilon^-}\right].\end{aligned}$$

Let $\{\epsilon_n\} \to 0$ and $\Omega_n = \left\{\omega \in \Omega : \tau < R_{\epsilon_n}^-\right\}$, then $\{\Omega_n\}$ is a decreasing sequence: $\Omega_1 \supset \Omega_2 \supset ...\Omega_n \supset ...$, because R_x^- is an increasing process. Then,

$$\lim_{n\to\infty} \mathbb{Q}\left[\tau < R_{\epsilon_n}^-\right] = \mathbb{Q}\left[\cap_{n=1}^\infty \Omega_n\right] = \mathbb{Q}\left[\tau < R_0^-\right] = 0$$

because $R_0^- = 0$ a.s. Then, $\mathbb{Q}\left[\tau < R_\epsilon^-\right] \to 0$ as $\epsilon \to 0$.
The quantity $e^{Y_\tau} 1_{\tau < R_\epsilon^-}$ is bounded by an integrable variable e^{Y_τ} that is, $e^{Y_\tau} 1_{\tau < R_\epsilon^-} \leq e^{Y_\tau}$ and converges in probability to zero because, $\forall\ \delta > 0$,

$$\mathbb{Q}\left[e^{Y_\tau} 1_{\tau < R_\epsilon^-} > \delta\right] \leq \mathbb{Q}\left[\tau < R_\epsilon^-\right].$$

Then, by dominated convergence theorem,

$$\lim_{\epsilon \to 0} E\left[e^{Y_\tau} 1_{\tau < R_\epsilon^-}\right] = \lim_{\epsilon \to 0} \int_\Omega e^{Y_\tau} 1_{\tau < R_\epsilon^-}\, \mathrm{d}\mathbb{Q} = 0. \tag{3.45}$$

Then, $E\left[e^{Y_\tau} 1_{\tau < R_\epsilon^-}\right] \to 0$ as $\epsilon \to 0$. This means that $f_l(\tau, x)$ is continuous in $x = l$. The proof for all (τ, x), follows the same steps of the proof of Proposition 3.5.9 in [83]. ■

The following set of propositions show the discontinuity of up-and-out option and down-and-out option when the Lévy process is of type A.

Proposition 3.3.10 *Let $\{Y_t\}$ be a Lévy process of type A with $R_0 > 0$ a.s. Suppose that H:$(0, U) \to (0, \infty)$ is Lipschitz.Then for every τ sufficiently small $f_u(\tau, x)$ defined by*

$$f_u(\tau, x) = \begin{cases} E\left[H\left(S_0 e^{x+Y_\tau}\right) 1_{\tau < R_{u-x}}\right] & \text{if } x < u, \\ 0 & \text{if } x \geq u, \end{cases}$$

is discontinuous in x=u.

Proof. Considering $\tau > 0$, we have by definition

$$|f_u(\tau, u - \epsilon) - f_u(\tau, u)| = E\left[H\left(S_0 e^{u-\epsilon+Y_\tau}\right) 1_{\tau < R_\epsilon}\right] \leq M E\left[1_{\tau < R_\epsilon}\right] = M\mathbb{Q}\left[\tau < R_\epsilon\right].$$

Let $\{\epsilon_n\} \to 0$ and $\Omega_n = \{\omega \in \Omega : \tau < R_{\epsilon_n}\}$, then $\{\Omega_n\}$ is a decreasing sequence, i.e $\Omega_1 \supset \Omega_2 \supset \ldots\Omega_n \supset \ldots$, because R_x is an increasing process. Then,

$$\lim_{n \to \infty} \mathbb{Q}\left[\tau < R_{\epsilon_n}\right] = \mathbb{Q}\left[\cap_{n=1}^{\infty} \Omega_n\right] = \mathbb{Q}\left[\tau < R_0\right] = \mathbb{Q}\left[Y_\tau \leq 0\right] > 0,$$

because $\{Y_t\}$ is a pure jump Lévy process. For $\tau = 0$

$$\begin{aligned} |f_u(0, u - \epsilon) - f_u(0, u)| &= E\left[H\left(S_0 e^{u-\epsilon+Y_0}\right) 1_{0 < R_\epsilon}\right] \leq M E\left[1_{0 < R_\epsilon}\right] && (3.46) \\ &= M\mathbb{Q}\left[0 < R_\epsilon\right] \to M\mathbb{Q}\left[0 < R_0\right] = M > 0, && (3.47) \end{aligned}$$

as $\epsilon \to 0$, because $R_0 > 0$ a.s. Then $f_u(\tau, x)$ is discontinuous at the barrier. ■

Proposition 3.3.11 *Let $\{Y_t\}$ be a Lévy process of type A and $R_0^- > 0$ almost surely. Suppose that H:$(L, \infty) \to (0, \infty)$ is Lipschitz. Then for every τ sufficiently small $f_l(\tau, x)$ defined by*

$$f_l(\tau, x) = \begin{cases} E\left[H\left(S_0 e^{x+Y_\tau}\right) 1_{\tau < R_{x-l}^-}\right] & \text{if } x > l, \\ 0 & \text{if } x \leq l, \end{cases}$$

is discontinuous in x=l.

Proof. Considering $\tau > 0$, we have by definition

$$\begin{aligned}|f_l(\tau, l+\epsilon) - f_l(\tau, l)| &= E\left[H\left(S_0 e^{l+\epsilon+Y_\tau}\right) 1_{\tau < R_\epsilon^-}\right] \leq CE\left[\left(1 + S_0 e^{l+\epsilon+Y_\tau}\right) 1_{\tau < R_\epsilon^-}\right] \\ &= C\mathbb{Q}\left[\tau < R_\epsilon^-\right] + CS_0 e^{l+\epsilon} E\left[e^{Y_\tau} 1_{\tau < R_\epsilon^-}\right].\end{aligned}$$

Let $\{\epsilon_n\} \to 0$ and $\Omega_n = \left\{\omega \in \Omega : \tau < R_{\epsilon_n}^-\right\}$, then $\{\Omega_n\}$ is a decreasing sequence ,that is $\Omega_1 \supset \Omega_2 \supset ...\Omega_n \supset ...$, because R_x is an increasing process. Then

$$\lim_{n\to\infty} \mathbb{Q}\left[\tau < R_{\epsilon_n}^-\right] = \mathbb{Q}\left[\cap_{n=1}^\infty \Omega_n\right] = \mathbb{Q}\left[\tau < R_0^-\right] > 0,$$

because by theorem 46.2 of Sato $\{R_x\}$ is a pure jump lévy process. The quantity $e^{Y_\tau} 1_{\tau < R_\epsilon^-}$ is bounded by an integrable variable e^{Y_τ} that is, $e^{Y_\tau} 1_{\tau < R_\epsilon^-} \leq e^{Y_\tau}$ but doesn't converge in probability to zero because $\forall\ \delta > 0$,

$$\mathbb{Q}\left[e^{Y_\tau} 1_{\tau < R_\epsilon^-} > \delta\right] \leq \mathbb{Q}\left[\tau < R_\epsilon^-\right] > 0.$$

Then, by dominated convergence theorem,

$$\lim_{\epsilon\to 0} E\left[e^{Y_\tau} 1_{\tau < R_\epsilon^-}\right] = \lim_{\epsilon \to 0} \int_\Omega e^{Y_\tau} 1_{\tau < R_\epsilon^-}\, \mathrm{d}\mathbb{Q} \tag{3.48}$$

$$= \int_\Omega \lim_{\epsilon\to 0} e^{Y_\tau} 1_{\tau < R_\epsilon^-}\, \mathrm{d}\mathbb{Q} = \int_\Omega 0\, \mathrm{d}\mathbb{Q} > 0. \tag{3.49}$$

Then,

$$|f_l(\tau, l+\epsilon) - f_l(\tau, l)| \to \mathbb{Q}\left[\tau < R_\epsilon^-\right] + CS_0 e^{l+\epsilon} E\left[e^{Y_\tau} 1_{\tau < R_0^-}\right] > 0$$

as $\epsilon \to 0$. Then $f_l(\tau, x)$ is discontinuous at the barrier. ■

The next two examples show that if we do not impose any restriction on the Lévy process, then the value of a knock-out option is discontinuous for $t = 0$:

Example 3.3.6 *Let us consider the following lévy process $X_t = N_t^1 - N_t^2$ where N_t^1 and N_t^2 are independent Poisson processes with jump intensities λ_1 and λ_2.*

Assuming $r = 0$, we have $\lambda_2 = e\lambda_1$ in order for $S_t = S_0 e^{X_t}$ to be a martingale. Consider now a up and out option with payoff function $H : (0, U) \to (0, \infty)$ defined by $H_T = 1_{T<\theta(S_0)}$, and $\theta(S) = \inf\left\{t \geq 0 : S_0 e^{X_t} \geq U\right\}$ is the first exit time if the process starts from S. We will show that the initial option value $C(0, S) = \mathbb{E}\left[H(S_T) 1_{T<\theta(S_0)} | S_0 = S\right] = \mathbb{E}\left[H\left(Se^{X_T}\right) 1_{T<\theta(S)}\right]$ is not continuous at $S^* = U/e$.

$$|C(0, S^*+\epsilon) - C(0, S^*-\epsilon)| = \left|\mathbb{E}\left[1_{\theta(S^*-\epsilon)\leq T<\theta(S^*+\epsilon)}\right]\right| = \mathbb{Q}\left[\theta(S^*-\epsilon) \leq T < \theta(S^*+\epsilon)\right].$$

Consider the following event $\{N_T^1 = 1, N_T^2 = 0\}$ of non-zero probability. Then, if S_t starts from $S^* - \epsilon$, then $S_T = (S^* - \epsilon) e^1 = \left(\frac{U}{e} - \epsilon\right) e^1 = U - \epsilon e^1 < U$, which means that $T \geq \theta(S^* - \epsilon)$. On the other hand, if S_t starts from $S^* + \epsilon$, then $S_T = (S^* + \epsilon) e^{-1} = \left(\frac{U}{e} + \epsilon\right) e^1 = U + \epsilon e^1 > U$, implying $T < \theta(S^* + \epsilon)$. Because $\{N_T^1 = 1, N_T^2 = 0\}$ is a possible realization of the trajectory of X_t, we have:

$\{\omega \in \Omega : N_T^1 = 0, N_T^2 = 1\} \subset \{\omega \in \Omega : \theta(S^* - \epsilon) \leq T < \theta(S^* + \epsilon)\}$. *Then,*

$$\begin{aligned}|C(0, S^* + \epsilon) - C(0, S^* - \epsilon)| &= \mathbb{Q}[\theta(S^* - \epsilon) \leq T < \theta(S^* + \epsilon)] \\ &\geq \mathbb{Q}\left[N_T^1 = 1, N_T^2 = 0\right] = e^{-T\lambda_1} T \lambda_1 e^{-T\lambda_2} > 0\end{aligned}$$

Thus, $C(0, S)$ *is discontinuous at* $S = S^*$.

Example 3.3.7 *Let us consider the following Lévy process* $X_t = N_t^1 - N_t^2$, *where* N_t^1 *and* N_t^2 *are independent Poisson processes with jump intensities* λ_1 *and* λ_2. *Assuming* $r = 0$, *we have* $\lambda_2 = e\lambda_1$ *in order for* $S_t = S_0 e^{X_t}$ *to be a martingale. Consider now a knock-out option with a payoff function defined by* $H_T = 1_{T<\theta(S_0)}$, *where* $\theta(S) = \inf\left\{t \geq 0 : S_0 e^{X_t} \leq L\right\}$ *is the first exit time if the process starts from* S. *We will show that the initial option value* $C(0, S) = \mathbb{E}\left[1_{T<\theta(S_0)} | S_0 = S\right] = \mathbb{E}\left[1_{T<\theta(S)}\right]$ *is not continuous at* $S^* = Le$. *Let* $0 < \epsilon < S^* - L$, *so that* $L = L - S^* + S^* < S^* - \epsilon < S^* < S^* + \epsilon$.

$$|C(0, S^* + \epsilon) - C(0, S^* - \epsilon)| = \left|\mathbb{E}\left[1_{\theta(S^*-\epsilon) \leq T < \theta(S^*+\epsilon)}\right]\right| = \mathbb{Q}[\theta(S^* - \epsilon) \leq T < \theta(S^* + \epsilon)].$$

Consider the following event $\{N_T^1 = 0, N_T^2 = 1\}$ *of non-zero probability. Then, if* S_t *starts from* $S^* - \epsilon$, *then* $S_T = (S^* - \epsilon) e^{-1} = (Le - \epsilon) e^{-1} = L - \epsilon e^{-1} < L$, *which means that* $T \geq \theta(S^* - \epsilon)$. *On the other hand, if* S_t *starts from* $S^* + \epsilon$, *then* $S_T = (S^* + \epsilon) e^{-1} = (Le + \epsilon) e^{-1} = L + \epsilon e^{-1} > L$, *implying* $T < \theta(S^* + \epsilon)$. *Because* $\{N_T^1 = 0, N_T^2 = 1\}$ *is a possible realization of the trajectory of* X_t, *we have:*
$\{\omega \in \Omega : N_T^1 = 0, N_T^2 = 1\} \subset \{\omega \in \Omega : \theta(S^* - \epsilon) \leq T < \theta(S^* + \epsilon)\}$. *Then,*

$$\begin{aligned}|C(0, S^* + \epsilon) - C(0, S^* - \epsilon)| &= \mathbb{Q}[\theta(S^* - \epsilon) \leq T < \theta(S^* + \epsilon)] \\ &\geq \mathbb{Q}\left[N_T^1 = 0, N_T^2 = 1\right] = e^{-T\lambda_1} T \lambda_2 e^{-T\lambda_2} \\ &= e^{-T\lambda_1(1+e)} T \lambda_2 > 0.\end{aligned}$$

Thus, $C(0, S)$ *is discontinuous at* $S = S^*$.

3.4 Option Pricing using Fourier Transform methods

An European call option on some asset S_t with a given maturity T and exercise price K, gives its holder the right but not the obligation to buy the asset at date T for a fixed price. Since the holder of the option will only exercise if the actual price of the asset is higher we know that the holder receives then the difference between the fixed amount paid K and the amount received for selling immediately the asset , i.e $(S_T - K)^+$. Similarly the holder of a Put option receives $(K - S_T)^+$.

We can consider a more general contract, the European option, which gives the holder the right but not the obligation to receive $\Phi(S_T)$ at maturity T. It is well known that (see [24]) if Φ is convex, Φ' is its left derivative and κ the second derivative in the sense of distributions, then

$$\Phi(S_T) = \Phi(0) + \Phi'(0) S_T + \int_0^\infty (S_T - K)^+ \kappa(\mathrm{d}K). \tag{3.50}$$

This result shows that we can express any European option's payoff in terms of call option payoffs and also shows that we can value more complex European options by knowing call options payoffs. This is true for example for butterfly spreads and straddle options which are linear combinations of a finite number of call or put options.

We already know that the price of an European call option can be represented as the risk-neutral conditional expectation of a certain payoff:

$$\begin{aligned} V(t,S_t) &= e^{-r(T-t)}\mathbb{E}\left[(S_T-K)^+ \,|\mathcal{F}_t\right] \\ &= e^{-r(T-t)}\mathbb{E}\left[\left(S_t e^{r(T-t)+X_{T-t}}-K\right)^+ |S_t=S\right] \\ &= e^{-r\tau}\mathbb{E}\left[\left(Se^{r\tau+X_\tau}-K\right)^+\right] \\ &= e^{-r\tau}\mathbb{E}\left[\left(Ke^{r\tau+w+X_\tau}-K\right)^+\right] \\ &= e^{-r\tau}K\mathbb{E}\left[\left(e^{r\tau+w+X_\tau}-1\right)^+\right], \end{aligned} \tag{3.51}$$

since a Lévy process has independent and stationary increments, due to the Markov property and making the transformation $w=\ln(\frac{S}{K})$.

Finally we can define the forward relative call option's price in terms of log-moneyness x and the time until maturity τ

$$u(\tau,x) = e^{r\tau}\frac{V(t,S_t)}{K} \tag{3.52}$$

$$= \mathbb{E}\left[\left(e^{r\tau+w+X_\tau}-1\right)^+\right], \tag{3.53}$$

where $V(t,S_t)$ is given by (3.51). We will now see how to price an option using Fourier transform methods in an exponential Lévy model. Remember the definition of Fourier transform of a function f:

$$\mathcal{F}f(u) = \int_{\mathbb{R}} e^{ixu} f(x)\,\mathrm{d}x. \tag{3.54}$$

We can also define the inverse Fourier transform by

$$\mathcal{F}^{-1}f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ixu} f(u)\,\mathrm{d}u. \tag{3.55}$$

Note that for $f\in L_2(\mathbb{R})$, $\mathcal{F}^{-1}\mathcal{F}f=f$. We will follow the method of Carr and Madan because it is easier to implement than Lewis's method but it has lower convergence. As it was done in [25], we set $S_0=1$, $\kappa=\ln(K)$. It is assumed that the stock price has moments of order $1+\alpha$ for some $\alpha>0$

$$\exists\alpha>0: \int_{|y|>1} e^{(1+\alpha)y}\nu(\,\mathrm{d}y) < \infty. \tag{3.56}$$

Now we would like to compute

$$V(0,S_0)= e^{-rT}\mathbb{E}\left[\left(e^{rT+X_T}-e^{\kappa}\right)^+\right], \tag{3.57}$$

and so we define it as a function of κ

$$V(\kappa)= e^{-rT}\mathbb{E}\left[\left(e^{rT+X_T}-e^{\kappa}\right)^+\right] \tag{3.58}$$

$$= e^{-rT}\int_{\kappa-rT}^{\infty} e^{rT+x}-e^{\kappa}\nu(\,\mathrm{d}x). \tag{3.59}$$

A natural idea seems to compute its Fourier Transform

$$\mathcal{F}V(u)= \int_{\mathbb{R}} e^{i\kappa u}V(\kappa)\,\mathrm{d}\kappa. \tag{3.60}$$

However, we see from (3.59) that $V(\kappa)$ tends to a constant and so (3.60) is not integrable. So the idea is to define the time-value of the option

$$z(\kappa)= e^{-rT}\mathbb{E}\left[\left(e^{rT+X_T}-e^{\kappa}\right)^+\right]-\left(1-e^{\kappa-rT}\right)^+ \tag{3.61}$$

and compute its Fourier transform

$$\zeta(u)=\mathcal{F}z(u)= \int_{\mathbb{R}} e^{i\kappa u}z(\kappa)\,\mathrm{d}\kappa. \tag{3.62}$$

So note that

$$z(\kappa)= e^{-rT}\mathbb{E}\left[\left(e^{rT+X_T}-e^{\kappa}\right)^+\right]-\left(1-e^{\kappa-rT}\right)^+ \tag{3.63}$$

$$= e^{-rT}\int_{-\infty}^{\infty}\left(e^{rT+x}-e^{\kappa}\right)\left(1_{\kappa\le rT+x}-1_{\kappa\le rT}\right)p(\,\mathrm{d}x). \tag{3.64}$$

Then, since now the integrand is integrable we can change the order of integration and get

$$\begin{aligned}
\zeta(u)=& \int_{\mathbb{R}} e^{i\kappa u}z(\kappa)\,\mathrm{d}\kappa\\
=& \int_{\mathbb{R}} e^{i\kappa u}e^{-rT}\int_{-\infty}^{\infty}\left(e^{rT+x}-e^{\kappa}\right)\left(1_{\kappa\le rT+x}-1_{\kappa\le rT}\right)p(x)\,\mathrm{d}x\,\mathrm{d}\kappa\\
=& \int_{\mathbb{R}} p(x)\int_{-\infty}^{\infty} e^{i\kappa u}e^{-rT}\left(e^{rT+x}-e^{\kappa}\right)\left(1_{\kappa\le rT+x}-1_{\kappa\le rT}\right)\,\mathrm{d}\kappa\,\mathrm{d}x\\
=& \int_{\mathbb{R}} p(x)\int_{rT}^{rT+x} e^{i\kappa u}e^{-rT}\left(e^{rT+x}-e^{\kappa}\right)\,\mathrm{d}\kappa\,\mathrm{d}x.
\end{aligned}$$

Observe that

$$\begin{aligned}\int_{rT}^{rT+x} e^{i\kappa u}e^{-rT}\left(e^{rT+x}-e^{\kappa}\right)\mathrm{d}\kappa &= \int_{rT}^{rT+x} e^{i\kappa u+x}-e^{i\kappa+\kappa-rT}\,\mathrm{d}\kappa\\ &= \left[\frac{e^{i\kappa u+x}}{iu}-\frac{e^{i\kappa u+\kappa-rT}}{iu+1}\right]_{rT}^{rT+x}\\ &= \frac{e^{iurT}\left(1-e^{x}\right)}{iu+1}-\frac{e^{iurT+x}}{iu(iu+1)}+\frac{e^{iurT+ixu+x}}{iu(iu+1)}.\end{aligned}$$

Now notice that since $E\left[e^{X_\tau}\right]=1$, we have

$$\begin{aligned}\zeta(u) &= \int_{\mathbb{R}} p(x)\left(-\frac{e^{iurT+x}}{iu(iu+1)}+\frac{e^{iurT+ixu+x}}{iu(iu+1)}\right)\mathrm{d}x\\ &= \frac{e^{iurT}}{iu(iu+1)}\int_{\mathbb{R}} p(x)e^{x}\left(e^{ixu}-1\right)\mathrm{d}x\\ &= \frac{e^{iurT}}{iu(iu+1)}\left(\int_{\mathbb{R}} p(x)e^{x(1+iu)}\,\mathrm{d}x-1\right)\\ &= \frac{e^{iurT}}{iu(iu+1)}\left(\int_{\mathbb{R}} p(x)e^{x(-i^2+iu)}\,\mathrm{d}x-1\right)\\ &= \frac{e^{iurT}}{iu(iu+1)}\left(\int_{\mathbb{R}} p(x)e^{xi(u-i)}\,\mathrm{d}x-1\right)\\ &= \frac{e^{iurT}}{iu(iu+1)}\left(\Psi(u-i)-1\right), \qquad (3.65)\end{aligned}$$

where $\Psi(u)$ is the characteristic function of X_t.

If $u\to 0$ then we see that since when $u=0$, $\Psi(u-i)=1$ by the martingale condition. Also we see that by (3.56), $\Psi(u-i)$ is analytic then

$$\lim_{u\to 0}\int_{\mathbb{R}} p(x)e^{xi(u-i)}\,\mathrm{d}x = \int_{\mathbb{R}} p(x)e^{x}\,\mathrm{d}x = 1, \qquad (3.66)$$

$$\lim_{u\to 0}\int_{\mathbb{R}} p(x)xie^{xi(u-i)}\,\mathrm{d}x = ai, \qquad (3.67)$$

for some $a\in\mathbb{R}$.

We conclude that

$$\begin{aligned}\lim_{u\to 0}\frac{e^{iurT}}{iu(iu+1)}\left(\Psi(u-i)-1\right) &= \lim_{u\to 0}\frac{irTe^{iurT}\left(\Psi(u-i)-1\right)+e^{iurT}\Psi'(u-i)}{-2u+i}=k\\ &= \lim_{u\to 0}\frac{irTe^{iurT}\left(\Psi(u-i)-1\right)+e^{iurT}ai}{-2u+i}=a<\infty.\end{aligned}$$

Then we can find the option price by inverting the Fourier Transform $\zeta(u)$:

$$z(\kappa)=\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\kappa u}\zeta(u)\,\mathrm{d}u. \qquad (3.68)$$

In order to compute numerically this integral we resort to the discrete inverse Fourier transform which implies discretizing and truncating the integral in the following manner

$$z(\kappa) \approx \frac{1}{2\pi} \int_{-\frac{A}{2}}^{\frac{A}{2}} e^{-i\kappa u} \zeta(u) \,\mathrm{d}u \approx \frac{A}{2\pi N} \sum_{k=0}^{N-1} \omega_k e^{-i\kappa x_k} \zeta(x_k), \tag{3.69}$$

where $x_k = -\frac{A}{2} + kh, h = \frac{A}{N-1}$ is the discretization step and $\omega_k = \frac{k}{h}$.

3.5 Modelling the feedback effects with jump process

This section is devoted to derivation of the novel option pricing model taking into account feedback effects of a large trader on the underlying asset following a jump-diffusion Lévy process. We show that the price of an option can be computed from a solution to a fully nonlinear partial integro-differential equation (PIDE) (2.19). We also derive a formula for the trading strategy function ϕ which minimizes the variance of the tracking error. Let us suppose that a large trader uses a stock-holding strategy α_t and S_t is a cadlag process (right continuous with limits to the left). Henceforth, we shall identify S_t with S_{t^-}. We assume S_t has the following dynamics:

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t + \rho S_t \,\mathrm{d}\alpha_t + \int_{\mathbb{R}} S_t (e^x - 1) J_X(\,\mathrm{d}t, \,\mathrm{d}x). \tag{3.70}$$

It can be viewed as a perturbation of the classical jump-diffusion model. Indeed, if a large trader does not trade then $\alpha_t = 0$ or the market liquidity parameter ρ is set to zero then the stock price S_t follows the classical jump-diffusion model.

In what follows, we will assume the following structural hypothesis:

Assumption 3.5.1 *Assume the trading strategy $\alpha_t = \phi(t, S_t)$ and the parameter $\rho \geq 0$ satisfy $\rho L < 1$, where $L = \sup_{S>0} |S \frac{\partial \phi}{\partial S}|$.*

Next we show an explicit formula for the dynamics of S_t satisfying (3.70) under certain regularity assumptions made on the stock-holding function $\phi(t, S)$.

Proposition 3.5.1 *Suppose that the stock-holding strategy $\alpha_t = \phi(t, S_t)$ satisfies Assumption 3.5.1 where $\phi \in C^{1,2}([0, T] \times \mathbb{R}^+)$. If the process $S_t, t \geq 0$, satisfies the implicit stochastic equation (3.70) then the process S_t satisfies the following SDE:*

$$\mathrm{d}S_t = b(t, S_t) S_t \,\mathrm{d}t + v(t, S_t) S_t \,\mathrm{d}W_t + \int_{\mathbb{R}} H(t, x, S_t) J_X(\,\mathrm{d}t, \,\mathrm{d}x), \tag{3.71}$$

where

$$b(t, S) = \frac{1}{1 - \rho S \frac{\partial \phi}{\partial S}(t, S)} \left(\mu + \rho \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} v(t, S)^2 S^2 \frac{\partial^2 \phi}{\partial S^2} \right) \right), \tag{3.72}$$

$$v(t, S) = \frac{\sigma}{1 - \rho S \frac{\partial \phi}{\partial S}(t, S)}, \tag{3.73}$$

$$H(t, x, S) = S(e^x - 1) + \rho S \left[\phi(t, S + H(t, x, S)) - \phi(t, S) \right]. \tag{3.74}$$

Proof. We can rewrite the SDE (3.71) for S_t, in the following way:

$$\begin{aligned} dS_t &= \left(b(t,S_t)S_t + \int_{|x|<1} H(t,x,S_t)\nu(\,dx)\right) dt + v(t,S_t)S_t\,dW_t \\ &\quad + \int_{|x|\geq 1} H(t,x,S_t)J_X(\,dt,\,dx) + \int_{|x|<1} H(t,x,S_t)\tilde{J}_X(\,dt,\,dx). \end{aligned}$$

Since $\phi(t,S)$ is assumed to be a smooth function then, by applying Itô formula (2.35) to the process $\phi(t,S_t)$, we obtain

$$\begin{aligned} d\alpha_t &= \left(\frac{\partial\phi}{\partial t} + \frac{1}{2}v(t,S_t)^2S_t^2\frac{\partial^2\phi}{\partial S^2}\right) dt + \frac{\partial\phi}{\partial S}\,dS_t \qquad (3.75) \\ &\quad + \int_{\mathbb{R}} \phi(t,S_t+H(t,x,S_t)) - \phi(t,S_t) - H(t,x,S_t)\frac{\partial\phi}{\partial S}(t,S_t)J_X(\,dt,\,dx). \end{aligned}$$

Now, inserting the differential $d\alpha_t$ into (3.70), we obtain

$$\begin{aligned} dS_t &= \mu S_t\,dt + \sigma S_t\,dW_t + \int_{\mathbb{R}} S_t(e^x-1)J_X(\,dt,\,dx) + \rho S_t\frac{\partial\phi}{\partial S}\,dS_t \\ &\quad + \rho S_t\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}v(t,S_t)^2S_t^2\frac{\partial^2\phi}{\partial S^2}\right) dt \qquad (3.76) \\ &\quad + \rho S_t\int_{\mathbb{R}} \phi(t,S_t+H(t,x,S_t)) - \phi(t,S_t) - H(t,x,S_t)\frac{\partial\phi}{\partial S}(t,S_t)J_X(\,dt,\,dx). \end{aligned}$$

Rearranging terms in (3.76) we conclude

$$\begin{aligned} &(1-\rho S_t\frac{\partial\phi}{\partial S}(t,S_t))\,dS_t = (\mu S_t + \rho S_t(\frac{\partial\phi}{\partial t} + \frac{1}{2}v(t,S_t)^2S_t^2\frac{\partial^2\phi}{\partial S^2}))\,dt \\ &+\sigma S_t\,dW_t - \rho S_t\int_{\mathbb{R}} H(t,x,S_t)\frac{\partial\phi}{\partial S}(t,S_t)J_X(\,dt,\,dx) \qquad (3.77) \\ &+\int_{\mathbb{R}} S_t(e^x-1) + \rho S_t\left(\phi(t,S_t+H(t,x,S_t)) - \phi(t,S_t)\right) J_X(\,dt,\,dx). \end{aligned}$$

Comparing terms in (3.71) and (3.77) we end up with expressions (3.72), (3.73), and the implicit equation for the function H:

$$\begin{aligned} H(t,x,S) &= \frac{1}{1-\rho S\frac{\partial\phi}{\partial S}(t,S)}\left(S(e^x-1) + \rho S\left(\phi(t,S+H(t,x,S)) - \phi(t,S)\right)\right) \\ &\quad - \frac{1}{1-\rho S\frac{\partial\phi}{\partial S}(t,S)}\rho S\frac{\partial\phi}{\partial S}(t,S)H(t,x,S). \qquad (3.78) \end{aligned}$$

Simplifying this expression for H we conclude (3.74), as claimed. ■

The function H is given implicitly by equation (3.74). If we expand its solution H in terms of a small parameter ρ, i.e. $H(t,x,S) = H^0(t,x,S) + \rho H^1(t,x,S) + O(\rho^2)$ as $\rho \to 0$, we conclude the following proposition:

Proposition 3.5.2 *Assume ρ is small. Then the first order approximation of the function $H(t,x,S)$ reads as follows:*

$$H(t,x,S) = S(e^x - 1) + \rho S\left(\phi(t, Se^x) - \phi(t,S)\right) + O(\rho^2) \quad \text{as } \rho \to 0. \quad (3.79)$$

Proposition 3.5.3 *Assume that the asset price process $S_t = e^{X_t + rt}$ fulfills SDE (3.71) where the Lévy measure ν is such that $\int_{|x|\geq 1} e^{2x}\nu\,(\mathrm{d}x) < \infty$. Denote by $V(t,S)$ the price of a derivative security given by*

$$V(t,S) = \mathbb{E}\left[e^{-r(T-t)}\Phi(S_T)|S_t = S\right] = e^{-r(T-t)}\mathbb{E}\left[\Phi(Se^{r(T-t)+X_{T-t}})\right]. \quad (3.80)$$

Assume that the pay-off function Φ is a Lipschitz continuous function and the function ϕ has a bounded derivative. Then $V(t,S)$ is a solution to the PIDE:

$$\begin{aligned}&\frac{\partial V}{\partial t} + \frac{1}{2}v(t,S)^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \\ &\quad + \int_{\mathbb{R}} V(t, S + H(t,x,S)) - V(t,S) - H(t,x,S)\frac{\partial V}{\partial S}(t,S)\nu(\,\mathrm{d}x) = 0, \quad (3.81)\end{aligned}$$

where $v(t,S)$ and $H(t,x,S)$ are given by (3.73) and (3.74), respectively.

Proof. The asset price dynamics of S_t under the $\mathbb{Q}$ measure is given by

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + v(t,S_t)S_t\,\mathrm{d}W_t + \int_{\mathbb{R}} H(t,x,S_t)\tilde{J}_X(\,\mathrm{d}t,\,\mathrm{d}x). \quad (3.82)$$

If we apply Itô's lemma to $V(t,S_t)$ we obtain $\mathrm{d}(V(t,S_t)e^{-rt}) = a(t)\,\mathrm{d}t + \mathrm{d}M_t$ where

$$\begin{aligned}a(t) &= \frac{\partial V}{\partial t} + \frac{1}{2}v(t,S_t)^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + rS_t\frac{\partial V}{\partial S} - rV \\ &\quad + \int_{\mathbb{R}} V(t, S_t + H(t,x,S_t)) - V(t,S_t) - H(t,x,S_t)\frac{\partial V}{\partial S}(t,S_t)\nu(\,\mathrm{d}x), \\ \mathrm{d}M_t &= e^{-rt}S_t v(t,S_t)\frac{\partial V}{\partial S}\,\mathrm{d}W_t + e^{-rt}\int_{\mathbb{R}} V(t,S_t + H(t,x,S_t)) - V(t,S_t)\tilde{J}_X(\,\mathrm{d}t,\,\mathrm{d}x).\end{aligned}$$

Our goal is to show that M_t is a martingale. Consequently, we have $a \equiv 0$ a.s., and V is a solution to (3.81) (see Proposition 8.9 of [24]). To prove the term $\int_0^T e^{-rt}\int_{\mathbb{R}} V(t, S_t + H(t,x,S_t)) - V(t,S_t)\tilde{J}_S(\,\mathrm{d}t,\,\mathrm{d}y)$ is a martingale it is sufficient to show that

$$\mathbb{E}\left[\int_0^T e^{-2rt}\left(\int_{\mathbb{R}} V(t,S_t + H(t,x,S_t)) - V(t,S_t)\nu(\,\mathrm{d}x)\right)^2\mathrm{d}t\right] < \infty. \quad (3.83)$$

Since $\sup_{0\leq t\leq T}\mathbb{E}\left[e^{X_{T-t}}\right] < \infty$ and the pay-off function Φ is Lipschitz continuous, $V(t,S)$ is Lipschitz continuous as well with some Lipschitz constant $C > 0$. As the function $\phi(t,S)$ has bounded derivatives we obtain

$$S\left|\phi(t, S + H(t,x,S)) - \phi(t,S)\right| \leq S\left|\frac{\partial\phi}{\partial S}\right||H(t,x,S)| \leq L|H(t,x,S)|$$

(see Assumption 3.5.1). Since $H(t,x,S) = S(e^x-1) + \rho S(\phi(t, S + H(t,x,S)) - \phi(t,S))$ we obtain $|H(t,x,S)|^2 \le S^2(e^x-1)^2/(1-\rho L)^2$. As V is Lipschitz continuous with the Lipschitz constant $C>0$ we have

$$\begin{aligned}&\mathbb{E}\left[\int_0^T e^{-2rt}\left(\int_{\mathbb{R}} V(t,S_t+H(t,x,S_t)) - V(t,S_t)\nu(\,\mathrm{d}x)\right)^2 \mathrm{d}t\right]\\ &\le \frac{C^2}{(1-\rho L)^2}\mathbb{E}\left[\int_0^T\int_{\mathbb{R}} e^{-2rt}|S_t|^2(e^x-1)^2\nu(\,\mathrm{d}x)\,\mathrm{d}t\right] < \infty,\end{aligned}$$

because $\sup_{t\in[0,T]}\mathbb{E}\left[S_t^2\right]<\infty$. Here $C_0 = \int_{\mathbb{R}}(e^x-1)^2\nu(\,\mathrm{d}x)<\infty$ due to the assumptions made on the measure ν. It remains to prove that $\int_0^T e^{-rt}S_t v(t,S_t)\frac{\partial V}{\partial S}(t,S_t)\,\mathrm{d}W_t$ is a martingale. Since $S\frac{\partial\phi}{\partial S}(t,S)$ is assumed to be bounded we obtain

$$0 < v(t,S) = \frac{\sigma}{1-\rho S\frac{\partial\phi}{\partial S}(t,S)} \le \frac{\sigma}{1-\rho L} \equiv C_1 < \infty.$$

Therefore $\mathbb{E}[\int_0^T e^{-2rt}(\frac{\partial V}{\partial S}(t,S_t)v(t,S_t)S_t)^2\,\mathrm{d}t] \le C^2C_1^2\int_0^T e^{-2rt}\mathbb{E}[S_t^2]\,\mathrm{d}t<\infty$ because S_t is a martingale. Hence M_t is a martingale as well. As a consequence, $a\equiv 0$ and so V is a solution to PIDE (3.81), as claimed. ■

Remark 2 *If $\rho=0$ then $H(t,x,S)=S(e^x-1)$ and equation (3.81) reduces to:*

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV + \int_{\mathbb{R}} V(t,Se^x) - V(t,S) - S(e^x-1)\frac{\partial V}{\partial S}(t,S)\nu(\,\mathrm{d}x) = 0, \tag{3.84}$$

which is the well-known classical PIDE. If there are no jumps ($\nu=0$) and a trader follows the delta hedging strategy, i.e. $\phi(t,S)=\partial_S V(t,S)$, then equation (3.81) reduces to the Frey–Stremme option pricing model:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\sigma^2}{(1-\varrho S\partial_S^2 V)^2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{3.85}$$

(cf. [36]). Finally, if $\rho=0$ and $\nu=0$ equation (3.81) reduces to the classical linear Black–Scholes equation.

For simplicity, we assume the interest rate is zero, $r=0$. Then the function $V(t,S)$ is a solution to the PIDE:

$$\begin{aligned}\frac{\partial V}{\partial t} &+ \frac{1}{2}v(t,S)^2S^2\frac{\partial^2 V}{\partial S^2}\\ &+ \int_{\mathbb{R}} V(t,S+H(t,x,S)) - V(t,S) - H(t,x,S)\frac{\partial V}{\partial S}(t,S)\nu(\,\mathrm{d}x) = 0.\end{aligned} \tag{3.86}$$

Let us define the tracking error of a trading strategy $\alpha_t = \phi(t,S_t)$ as follows: $e_T^M := \Phi(S_T) - V_0 = V(T,S_T) - V_0 - \int_0^T \alpha_t\,\mathrm{d}S_t$.

By applying Itô's formula to $V(t, S_t)$ and using (3.86) we obtain

$$
\begin{aligned}
& V(T,S_T) - V_0 = V(T,S_T) - V(0,S_0) = \int_0^T \mathrm{d}V(t,S_t) \\
&= \int_0^T \frac{\partial V}{\partial S}\,\mathrm{d}S_t + \int_0^T \frac{\partial V}{\partial t} + \frac{1}{2}v(t,S_t)^2 S_t^2 \frac{\partial^2 V}{\partial^2 S}\,\mathrm{d}t \\
&\quad + \int_0^T \int_{\mathbb{R}} V(t, S_t + H(t,x,S_t)) - V(t,S_t) - H(t,x,S_t)\frac{\partial V}{\partial S} J_X(\,\mathrm{d}t,\,\mathrm{d}x) \\
&= \int_0^T \frac{\partial V}{\partial S}\,\mathrm{d}S_t - \int_0^T \int_{\mathbb{R}} V(t, S_t + H(t,x,S_t)) - V(t,S_t) - H(t,x,S_t)\frac{\partial V}{\partial S}\nu(\,\mathrm{d}x)\,\mathrm{d}t \\
&\quad + \int_0^T \int_{\mathbb{R}} V(t, S_t + H(t,x,S_t)) - V(t,S_t) - H(t,x,S_t)\frac{\partial V}{\partial S} J_X(\,\mathrm{d}t,\,\mathrm{d}x) \\
&= \int_0^T \frac{\partial V}{\partial S}\,\mathrm{d}S_t + \int_0^T \int_{\mathbb{R}} V(t, S_t + H(t,x,S_t)) - V(t,S_t) - H(t,x,S_t)\frac{\partial V}{\partial S} \tilde{J}_X(\,\mathrm{d}t,\,\mathrm{d}x).
\end{aligned}
$$

Using expression (3.82) for the dynamics of the asset price S_t (with $r = 0$), the tracking error e_T^M can be expressed as follows:

$$
\begin{aligned}
e_T^M &= V(T,S_T) - V_0 - \int_0^T \alpha_t\,\mathrm{d}S_t = \int_0^T \left(\frac{\partial V}{\partial S}(t,S_t) - \alpha_t\right)\mathrm{d}S_t \\
&\quad + \int_0^T \int_{\mathbb{R}} V(t, S_t + H(t,x,S_t)) - V(t,S_t) - H(t,x,S_t)\frac{\partial V}{\partial S} \tilde{J}_X(\,\mathrm{d}t,\,\mathrm{d}x) \\
&= \int_0^T v(t,S_t) S_t \left(\frac{\partial V}{\partial S} - \alpha_t\right)\mathrm{d}W_t \qquad (3.87) \\
&\quad + \int_0^T \int_{\mathbb{R}} V(t, S_t + H(t,x,S_t)) - V(t,S_t) - \alpha_t H(t,x,S_t) \tilde{J}_X(\,\mathrm{d}t,\,\mathrm{d}x).
\end{aligned}
$$

Remark 3 *For the delta hedging strategy $\alpha_t = \phi(t,S_t) = \frac{\partial V}{\partial S}(t,S_t)$ the tracking error function e_T^M can be expressed as follows:*

$$
e_T^M = \int_0^T \int_{\mathbb{R}} V(t, S_t + H(t,x,S_t)) - V(t,S_t) - H(t,x,S_t)\frac{\partial V}{\partial S}(t,S_t) \tilde{J}_X(\,\mathrm{d}t,\,\mathrm{d}x).
$$

Clearly, the tracking error for the delta hedging strategy need not be zero for $\nu \not\equiv 0$.

Next, we propose a criterion that can be used to find the optimal hedging strategy.

Proposition 3.5.4 *The trading strategy $\alpha_t = \phi(t,S_t)$ of a large trader minimizing the variance $\mathbb{E}\left[(\epsilon_T^M)^2\right]$ of the tracking error is given by the implicit equation:*

$$
\begin{aligned}
\phi(t,S_t) &= \beta^\rho(t,S_t)\big[v(t,S_t)^2 S_t^2 \frac{\partial V}{\partial S}(t,S_t) \\
&\quad + \int_{\mathbb{R}} (V(t, S_t + H(t,x,S_t)) - V(t,S_t)) H(t,x,S_t)\,\nu(\,\mathrm{d}x)\big], \qquad (3.88)
\end{aligned}
$$

where $\beta^\rho(t,S_t) = 1/[v(t,S_t)^2 S_t^2 + \int_{\mathbb{R}} H(t,x,S_t)^2 \nu(\,\mathrm{d}x)]$ and $H(t,x,S) = S(e^x - 1) + \rho S[\phi(t, S + H(t,x,S)) - \phi(t,S)]$.

Proof. Using expression (3.87) for the tracking error ϵ_T^M and Itô's isometry we obtain

$$\begin{aligned}\mathbb{E}\left[(\epsilon_T^M)^2\right] &= \mathbb{E}\left[\int_0^T v(t,S_t)^2 S_t^2 \left(\frac{\partial V}{\partial S}(t,S_t)-\alpha_t\right)^2 \mathrm{d}t\right] \\ &+\mathbb{E}\left[\int_0^T\int_{\mathbb{R}} \left(V(t,S_t+H(t,x,S_t))-V(t,S_t)-\alpha_t H(t,x,S_t)\right)^2 \nu(\,\mathrm{d}x)\,\mathrm{d}t\right].\end{aligned}$$

The minimizer α_t of the above convex quadratic minimization problem satisfies the first order necessary conditions $\mathrm{d}(\mathbb{E}\left[\epsilon_T^2\right],\alpha_t)=0$, that is,

$$\begin{aligned} 0 &= -2\mathbb{E}\left[\int_0^T \left(v(t,S_t)^2 S_t^2\left(\frac{\partial V}{\partial S}(t,S_t)-\alpha_t\right)\right.\right. \\ &\left.\left. +\int_{\mathbb{R}} H(t,x,S_t)\big(V(t,S_t+H(t,x,S_t))-V(t,S_t)-\alpha_t H(t,x,S_t)\big)\nu(\,\mathrm{d}x)\right)\omega_t\,\mathrm{d}t\right]\end{aligned}$$

for any variation ω_t. Thus the tracking error minimizing strategy α_t is given by (3.88). ■

Remark 4 *The optimal trading strategy minimizing the variance of the tracking error need not satisfy the structural Assumption 3.5.1. For instance, if $\nu=0$ then the tracking error minimizer is just the delta hedging strategy $\phi=\partial_S V$. In the case of a call or put option its gamma, i.e. $\partial_S^2 V(t,S)$ becomes infinite as $t\to T$ and $S=K$. Given a level $L>0$ we can however minimize the tracking error $\mathbb{E}\left[\epsilon_T^2\right]$ under the additional constraint $\sup_{S>0}|S\frac{\partial\phi}{\partial S}(t,S)|\le L$. That is we can solve the following convex constrained nonlinear optimization problem*

$$\min_{\phi}\quad \mathbb{E}\left[\epsilon_T^2\right]\quad s.t.\quad |S\partial_S\phi|\le L$$

instead of the unconstrained minimization problem proposed in Proposition 3.5.4.

Remark 5 *Notice that, if $\nu=0$ and $\rho\ge 0$, the trading strategy α_t reduces to the Black–Scholes delta hedging strategy, i.e. $\alpha_t=\frac{\partial V}{\partial S}(t,S_t)$. If $\nu\not\equiv 0$ and $\rho=0$, then the optimal trading strategy becomes $\alpha_t=\phi^0(t,S_t)$ where*

$$\phi^0(t,S_t)=\beta^0(t,S_t)\left(\sigma^2 S_t^2\frac{\partial V}{\partial S}(t,S_t)+\int_{\mathbb{R}} S_t(e^x-1)\left(V(t,S_te^x)-V(t,S_t)\right)\nu(\,\mathrm{d}x)\right),$$

where $\beta^0(t,S_t)=1/[\sigma^2S_t^2+\int_{\mathbb{R}}S_t^2(e^x-1)^2\nu(\,\mathrm{d}x)]$.

We conclude this section by the following proposition providing the first order approximation of the tracking error minimizing trading strategy for the case when the parameter $\rho\ll 1$ is small. In what follows, we derive the first order approximation of $\phi^\rho(t,S_t)$ in the form $\phi^\rho(t,S_t)=\phi^0(t,S_t)+\rho\phi^1(t,S_t)+O(\rho^2)$ as $\rho\to 0$.

Clearly, the first order Taylor expansion for the volatility function $v(t,S)$ has the form:

$$v(t,S)^2=\frac{\sigma^2}{(1-\rho S\partial_S\phi)^2}=\sigma^2+2\rho\sigma^2 S\frac{\partial\phi^0}{\partial S}(t,S)+O(\rho^2),\quad \text{as } \rho\to 0.$$

With regard to Proposition 3.5.2 (see (3.79)) we have $H(t,x,S) = H^0(t,x,S)+\rho H^1(t,x,S)+O(\rho^2)$, where

$$H^0(t,x,S) = S(e^x-1), \qquad H^1(t,x,S) = S[\phi^0(t,Se^x)-\phi^0(t,S)]. \tag{3.89}$$

The function β^ρ can be expanded as follows: $\beta^\rho(t,S) = \beta^0(t,S)+\rho\beta^{(1)}(t,S)+O(\rho^2)$,

$$\begin{aligned}\beta^0(t,S) &= 1/[\sigma^2S^2+S^2\int_{\mathbb{R}}(e^x-1)^2\nu(\,\mathrm{d}x)], \qquad (3.90)\\ \beta^{(1)}(t,S) &= -(\beta^0(t,S))^2\left[2\sigma^2S^3\frac{\partial\phi^0}{\partial S}(t,S)+2S^2\int_{\mathbb{R}}(e^x-1)[\phi^0(t,Se^x)-\phi^0(t,S)]\nu(\,\mathrm{d}x)\right].\end{aligned}$$

Using the first order expansions of the functions v^2, β^ρ and H we obtain the following results.

Proposition 3.5.5 *For small values of the parameter $\rho \ll 1$, the tracking error variance minimizing strategy $\alpha_t = \phi^\rho(t,S_t)$ is given by*

$$\phi^\rho(t,S_t) = \phi^0(t,S_t)+\rho\phi^{(1)}(t,S_t)+O(\rho^2), \quad \text{as } \rho\to 0, \tag{3.91}$$

where

$$\begin{aligned}\phi^{(1)}(t,S) &= \beta^0(t,S)\left[2\sigma^2S^3\frac{\partial V}{\partial S}(t,S)\frac{\partial\phi^0}{\partial S}(t,S)\right.\\ &\quad \left.+\int_{\mathbb{R}}\left(V(t,Se^x)-V(t,S)+\frac{\partial V}{\partial S}(t,Se^x)H^0(t,x,S)\right)H^1(t,x,S)\nu(\,\mathrm{d}x)\right]\\ &\quad +\beta^{(1)}(t,S)\left[\sigma^2S^2\frac{\partial V}{\partial S}(t,S)+\int_{\mathbb{R}}\left(V(t,Se^x)-V(t,S)\right)H^0(t,x,S)\nu(\,\mathrm{d}x)\right]\end{aligned}$$

and the functions H^0, H^1, β^0 and $\beta^{(1)}$ are defined as in (3.89) and (3.90).

In this section we investigated a novel nonlinear option pricing model generalizing the Frey–Stremme model under the assumption that the underlying asset price follows a Lévy stochastic process. We derived the fully-nonlinear PIDE for pricing options under influence of a large trader. We also proposed the hedging strategy minimizing the variance of the tracking error.

3.6 Existence of solutions in Bessel potential spaces

Using the theory of abstract semilinear parabolic equations we prove existence and uniqueness of solutions in the Bessel potential space. Our aim is to generalize known existence results for a wide class of Lévy measures including those having strong singular kernel. We also prove existence and uniqueness of solutions to the penalized PIDE representing approximation of the linear complementarity problem arising in pricing American style of options under Lévy stochastic process. In the past years, existence results of PIDE has been studied in the literature. In [15] A. Bensoussan and J.-L. Lions (see Theorem 3.3 and Theorem 8.1) and also M. G. Garroni and J. L. Menaldi in [40] investigated the existence and uniqueness of classical solutions for the case $\sigma > 0$. In [61] Mikulevicius and Pragarauskas extended these results for the case $\sigma = 0$. Furthermore, in [62],[63] they investigated existence and uniqueness of classical solutions in Hölder and Sobolev spaces of the Cauchy problem to the partial-integro-differential equation of the order of kernel singularity $\alpha \in (0, 2)$. Qualitative results using the notion of viscosity solutions were provided by M. Crandall and P.-L. Lions in [27]. They were generalized to PIDE by Awatif [11] and Soner [79] for the first order operators and by Alvarez and Tourin [5], Barles *et al.* [12], and Pham [69] for the second order operators. In [57],[58] Mariani and SenGupta proved existence of weak solutions of a generalized integro-differential equation using the Schaefer fixed point theorem first for bounded domains and then for unbounded domains. On other hand, in [77], Amster *et al.* proved the existence of solutions using the method of upper and lower solutions in a general domain in the case of several assets and for the regime-switching jump-diffusion model in [34]. In [8],[7] Arregui et al. applied the theory of abstract parabolic equations in Banach spaces (cf. [46]) for the proof of existence and uniqueness of solutions of a system of nonlinear PDEs for pricing of XVA derivatives. In a recent paper, Cruz and Ševčovič [28] investigated a nonlinear extension of the option pricing PIDE model (2.18) from numerical point of view. We consider a model for pricing vanilla call and put options on underlying assets following Lévy stochastic processes. Using the theory of abstract semilinear parabolic equations we prove existence and uniqueness of solutions in the Bessel potential space representing a fractional power space of the space of Lebesgue p-integrable functions with respect to the second order Laplace differential operator. We generalize known existence results for a wider class of Lévy measures including those having strong singular kernel with the third order of singularity. We also prove existence and uniqueness of solutions to the penalized PIDE representing approximation of the linear complementarity problem for a PIDE arising in pricing American style of options.

The goal of this section is to prove main results regarding existence and uniqueness of solution to the linear and nonlinear PIDE for pricing vanilla options on the underlying asset following a Lévy stochastic process for a wide class of admissible activity Lévy measures.

3.6.1 Existence results for the linear PIDE

In this section, we analyze solutions to the semilinear parabolic partial integro-differential equation (PIDE):

$$\begin{aligned} \frac{\partial u}{\partial \tau} &= \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + \omega \frac{\partial u}{\partial x} + g(\tau, u) \\ &\quad + \int_{\mathbb{R}} \left[u(\tau, x+z) - u(\tau, x) - (e^z - 1)\frac{\partial u}{\partial x}(\tau, x) \right] \nu(\,\mathrm{d}z), \qquad (3.92) \\ &\quad u(0, x) = u_0(x), \end{aligned}$$

$x \in \mathbb{R}, \tau \in (0,T)$, where g is Hölder continuous in the τ variable and it is Lipschitz continuous in the u variable. Here ν is a positive Radon measure on $\mathbb{R}$ such that $\int_{\mathbb{R}} \min(z^2, 1)\nu(\,dz) < \infty$. Our goal is to prove existence and uniqueness of a solution to (3.92) in the framework of Bessel potential spaces. These functional spaces represent a nested scale $\{X^\gamma\}_{\gamma\geq 0}$ of Banach spaces such that

$$X^1 \equiv D(A) \hookrightarrow X^{\gamma_1} \hookrightarrow X^{\gamma_2} \hookrightarrow X^0 \equiv X,$$

for any $0 \leq \gamma_2 \leq \gamma_1 \leq 1$ where A is a sectorial operator in the Banach space X with a dense domain $D(A) \subset X$. For example, if $A = -\Delta$ is the Laplacian operator in $\mathbb{R}^n$ with the domain $D(A) \equiv W^{2,p}(\mathbb{R}^n) \subset X \equiv L^p(\mathbb{R}^n)$ then X^γ is embedded in the Sobolev-Slobodecki space $W^{2\gamma,p}(\mathbb{R}^n)$ consisting of all functions having 2γ-fractional derivative belonging to the Lebesgue space $L^p(\mathbb{R}^n)$ of p-integrable functions (cf. [46]). In this paper, our goal is to prove existence and uniqueness of solutions to (3.92) for a general class of the so-called admissible activity Lévy measures ν satisfying suitable growth conditions at $\pm\infty$ and the origin. We can rewrite the PIDE (3.92) in the abstract form as follows:

$$\begin{aligned} &\frac{\partial u}{\partial \tau} + Au = \omega\frac{\partial u}{\partial x} + f[u] + g(\tau, u), \quad x \in \mathbb{R}, \tau \in (0,T), \\ &u(0,x) = u_0(x), x \in \mathbb{R}, \end{aligned} \tag{3.93}$$

where the linear operators A and f are defined by:

$$Au = -\frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}, \tag{3.94}$$

$$f[u](x) = \int_{\mathbb{R}} \left[u(x+z) - u(x) - (e^z - 1)\frac{\partial u}{\partial x}(x)\right] \nu(\,dz), \tag{3.95}$$

and g is a Hölder continuous mapping in the τ variable and it is Lipschitz continuous in the u variable. In order to prove existence, continuation and uniqueness of a solution to the problem (3.93) we follow the qualitative theory of semilinear abstract parabolic equations developed by Henry in [46]. First, we recall the concept of an analytic semigroup of linear operators and a sectorial operator in a Banach space.

Definition 3.6.1 *[46] A family of bounded linear operators $\{S(t), t \geq 0\}$ in a Banach space X is called an analytic semigroup if it satisfies the following conditions:*

i) $S(0) = I, S(t)S(s) = S(s)S(t) = S(t+s)$, for all $t, s \geq 0$;

ii) $S(t)x \to x$ when $t \to 0^+$ for all $x \in X$;

iii) $t \to S(t)x$ is a real analytic function on $0 < t < \infty$ for each $x \in X$.

The associated infinitesimal generator A is defined as follows: $Ax = \lim_{t\to 0^+} \frac{1}{t}(S(t)x - x)$ and its domain $D(A) \subseteq X$ consists of those x for which the limit exists in X.

Definition 3.6.2 *[46] Let $S_{a,\phi} = \{\lambda \in \mathbb{C} : \phi \le \arg(\lambda - a) \le 2\pi - \phi\}$ be a sector of complex numbers. A close densely defined linear operator $A : D(A) \subset X \to X$ which is called sectorial if there exists a constant $M \ge 0$ such that $\|(A-\lambda)^{-1}\| \le M/|\lambda - a|$ for all $\lambda \in S_{a,\phi} \subset \mathbb{C} \setminus \sigma(A)$.*

It is well known that that if A is a sectorial operator then $-A$ is an infinitesimal generator of an analytic semigroup $S(t) = \left\{e^{-At}, t \ge 0\right\}$ (cf. [46]). If X is a Banach space then we can define the scale of fractional power spaces $\{X^\gamma\}_{\gamma \ge 0}$ in the following way:

$$X^\gamma = D(A^\gamma) = Range(A^{-\gamma}) = \left\{u \in X : \ \exists \varphi \in X, u = A^{-\gamma}\varphi\right\},$$

where, for any $\gamma > 0$, the operator $A^{-\gamma}$ is defined by virtue of the Gamma function, i.e. $A^{-\gamma} = \frac{1}{\Gamma(\gamma)} \int_0^\infty \xi^{\gamma-1} e^{-A\xi} \,\mathrm{d}\xi$. The norm is defined as $\|u\|_{X^\gamma} = \|A^\gamma u\|_X = \|\varphi\|_X$. Note that $X^0 = X$, $X^1 = D(A)$, and $X^1 \equiv D(A) \hookrightarrow X^{\gamma_1} \hookrightarrow X^{\gamma_2} \hookrightarrow X^0 \equiv X$, for any $0 \le \gamma_2 \le \gamma_1 \le 1$.

In what follows, by $G * \varphi$ we shall denote the convolution operator defined by $(G * \varphi)(x) = \int_{\mathbb{R}^n} G(x-y)\varphi(y)\,\mathrm{d}y$.

Lemma 3.6.3 *[46, Section 1.6], [81, Chapter 5] The Laplace operator $-\Delta$ is sectorial in the Banach space $X = L^p(\mathbb{R}^n)$ of Lebesgue p-integrable functions for any $p \ge 1$ and $n \ge 1$. Its domain $D(A)$ is embedded into the Sobolev space $W^{2,p}(\mathbb{R}^n)$. The fractional power space $X^\gamma, \gamma > 0$, is the space of Bessel potentials: $X^\gamma = \mathscr{L}^p_{2\gamma}(\mathbb{R}^n) := \{G_{2\gamma} * \varphi, \ \varphi \in L^p(\mathbb{R}^n)\}$ where*

$$G_{2\gamma}(x) = \frac{(4\pi)^{-n/2}}{\Gamma(\gamma)} \int_0^\infty \xi^{-1+(2\gamma-n)/2} e^{-(\xi + \|x\|^2/(4\xi))} \,\mathrm{d}\xi$$

is the Bessel potential function. The norm of $u = G_{2\gamma} * \varphi$ is given by $\|u\|_{X^\gamma} = \|\varphi\|_{L^p}$. The fractional power space X^γ is continuously embedded into the fractional Sobolev-Slobodeckii space $W^{2\gamma,p}(\mathbb{R}^n)$.

Lemma 3.6.4 *Assume ν is an admissible activity Lévy measure with shape parameters $\alpha, D^\pm$ and μ where $\alpha < 3$ and either $\mu > 0, D^\pm \in \mathbb{R}$, or $\mu = 0, D^- + 1 < 0 < D^+$. Suppose that $\gamma \ge 1/2$ and $\gamma > (\alpha - 1)/2$. Then, for the mapping f defined by (3.95), there exists a constant $C > 0$ such that for any u satisfying $\partial_x u \in X^{\gamma - 1/2}$ the following estimate holds:*

$$\|f[u]\|_{L^p} \le C \|\partial_x u\|_{X^{\gamma-1/2}}.$$

In particular, if $u \in X^\gamma$ we have $\|f[u]\|_{L^p} \le C\|u\|_{X^\gamma}$ and the mapping f is a bounded linear operator from the fractional power space X^γ into $X = L^p(\mathbb{R})$.

Proof. The mapping f can be split as follows: $f[u] = \tilde{f}[u] + \tilde{\omega}\partial_x u$ where

$$\tilde{f}[u](x) \ = \ \int_{\mathbb{R}} \left(u(x+z) - u(x) - z\frac{\partial u}{\partial x}(x) \right) \nu(\,\mathrm{d}z).$$

and $\tilde{\omega} = \int_{\mathbb{R}} (z - e^z + 1)\,\nu(\,\mathrm{d}z)$. Since $z - e^z + 1 = O(z^2)$ as $z \to 0$, and

$$0 \le \nu(\,\mathrm{d}z) = h(z)\,\mathrm{d}z \le |z|^{-\alpha}\tilde{h}(z)\,\mathrm{d}z, \quad \text{where } \ \tilde{h}(z) = C_0 e^{-\mu z^2}\left(e^{D^- z} 1_{z \ge 0} + e^{D^+ z} 1_{z<0}\right),$$

we have $\tilde{\omega} \in \mathbb{R}$ provided that $0 \le \alpha < 3$ and either $\mu > 0, D^{\pm} \in \mathbb{R}$, or $\mu = 0$ and $D^- + 1 < 0 < D^+$.

First, we consider the case when $\gamma > 1/2$. We shall prove boundedness of the second linear operator $\tilde{f}$. If u is such that $\partial_x u \in X^{\gamma-1/2}$ then there exists $\varphi \in X = L^p(\mathbb{R})$ such that $\partial_x u = A^{-(2\gamma-1)/2}\varphi = G_{2\gamma-1} * \varphi$ and

$$\|\partial_x u\|_{X^{\gamma-1/2}} = \|\varphi\|_X = \|\varphi\|_{L^p}.$$

Hence, for any x, θ and z we have

$$\frac{\partial u}{\partial x}(x+\theta z) - \frac{\partial u}{\partial x}(x) = (G_{2\gamma-1}(x+\theta z - \cdot) - G_{2\gamma-1}(x - \cdot)) * \varphi(\cdot).$$

Recall the following inequality for the convolution operator:

$$\|G * \varphi\|_{L^p} \le \|G\|_{L^q} \|\varphi\|_{L^r},$$

where $p, q, r \ge 1$ and $1/p + 1 = 1/q + 1/r$ (see [46, Section 1.6]). In the special case when $q = 1$ we have $\|G * \varphi\|_{L^p} \le \|G\|_{L^1}\|\varphi\|_{L^p}$. According to [81, Chapter 5.4, Proposition 7] we know that the modulus of continuity of the Bessel kernel function $G_{2\gamma-1}$ satisfies the estimate:

$$\|G_{2\gamma-1}(\cdot + h) - G_{2\gamma-1}(\cdot)\|_{L^1} \le C_1 |h|^{2\gamma-1},$$

for any h where $C_1 > 0$ is a constant. Therefore, for any $\theta, z \in \mathbb{R}$ we have

$$\begin{aligned} &\int_{\mathbb{R}} \left| \frac{\partial u}{\partial x}(x+\theta z) - \frac{\partial u}{\partial x}(x) \right|^p \mathrm{d}x = \|\left(G_{2\gamma-1}(\cdot + \theta z) - G_{2\gamma-1}(\cdot)\right) * \varphi\|_{L^p}^p \\ &\le \|G_{2\gamma-1}(\cdot + \theta z) - G_{2\gamma-1}(\cdot)\|_{L^1}^p \|\varphi\|_{L^p}^p \le C_1^p |\theta z|^{(2\gamma-1)p} \|\partial_x u\|_{X^{\gamma-1/2}}^p. \end{aligned}$$

The latter inequality formally holds true also for the case $\gamma = 1/2$ because

$$\int_{\mathbb{R}} \left| \frac{\partial u}{\partial x}(x+\theta z) - \frac{\partial u}{\partial x}(x) \right|^p \mathrm{d}x \le 2^p \|\partial_x u\|_{L^p}^p = 2^p \|\partial_x u\|_{X^0}^p.$$

The rest of the proof of boundedness of the mapping f holds for $\gamma > 1/2$ as well as $\gamma = 1/2$. Now, as $u(x+z) - u(x) - z\frac{\partial u}{\partial x}(x) = z\int_0^1 \frac{\partial u}{\partial x}(x+\theta z) - \frac{\partial u}{\partial x}(x)\,\mathrm{d}\theta$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}} |u(x+z) - u(x) - z\frac{\partial u}{\partial x}(x)|^p \,\mathrm{d}x = |z|^p \int_{\mathbb{R}} \left| \int_0^1 \frac{\partial u}{\partial x}(x+\theta z) - \frac{\partial u}{\partial x}(x)\,\mathrm{d}\theta \right|^p \mathrm{d}x \\ &\le |z|^p \int_0^1 \int_{\mathbb{R}} \left| \frac{\partial u}{\partial x}(x+\theta z) - \frac{\partial u}{\partial x}(x) \right|^p \mathrm{d}x\,\mathrm{d}\theta \le C_1^p |z|^{2\gamma p} \|\partial_x u\|_{X^{\gamma-1/2}}^p. \end{aligned}$$

Now, as $0 \le \nu(\mathrm{d}z) = h(z)\,\mathrm{d}z \le |z|^{-\alpha}\tilde{h}(z)\,\mathrm{d}z = (|z|^{-\beta}\tilde{h}(z)^{\frac{1}{2}}) \cdot (|z|^{\beta-\alpha}\tilde{h}(z)^{\frac{1}{2}})\,\mathrm{d}z$, using the

Hölder inequality with exponents p, q such that $1/p + 1/q = 1$ we obtain

$$
\begin{aligned}
\|\tilde{f}[u]\|_{L^p}^p &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} u(x+z) - u(x) - z\frac{\partial u}{\partial x}(x)\nu(\mathrm{d}z) \right|^p \mathrm{d}x \\
&\leq \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \left| u(x+z) - u(x) - z\frac{\partial u}{\partial x}(x) \right| h(z)\,\mathrm{d}z \right|^p \mathrm{d}x \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} \left| u(x+z) - u(x) - z\frac{\partial u}{\partial x}(x) \right|^p |z|^{-\beta p}\tilde{h}(z)^{p/2}\,\mathrm{d}z \\
&\quad \times \left(\int_{\mathbb{R}} |z|^{(\beta-\alpha)q}\tilde{h}(z)^{q/2}\,\mathrm{d}z \right)^{p/q} \mathrm{d}x \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| u(x+z) - u(x) - z\frac{\partial u}{\partial x}(x) \right|^p \mathrm{d}x \right) |z|^{-\beta p}\tilde{h}(z)^{p/2}\,\mathrm{d}z \\
&\quad \times \left(\int_{\mathbb{R}} |z|^{(\beta-\alpha)q}\tilde{h}(z)^{q/2}\,\mathrm{d}z \right)^{p/q} \\
&\leq C_1^p \|\partial_x u\|_{X^{\gamma-1/2}}^p \int_{\mathbb{R}} |z|^{(2\gamma-\beta)p}\tilde{h}(z)^{p/2}\,\mathrm{d}z \left(\int_{\mathbb{R}} |z|^{(\beta-\alpha)q}\tilde{h}(z)^{q/2}\,\mathrm{d}z \right)^{p/q}.
\end{aligned}
$$

The integrals $C_2 = \int_{\mathbb{R}} |z|^{(2\gamma-\beta)p}\tilde{h}(z)^{p/2}\,\mathrm{d}z$ and $C_3 = \int_{\mathbb{R}} |z|^{(\beta-\alpha)q}\tilde{h}(z)^{q/2}\,\mathrm{d}z$ are finite provided that

$$(2\gamma - \beta)p > -1, \qquad (\beta - \alpha)q = (\beta - \alpha)\frac{p}{p-1} > -1$$

and $\mu > 0, D^\pm \in \mathbb{R}$, or $\mu = 0$ and $D^- < 0 < D^+$. The later inequalities are satisfied if there exists a parameter β such that

$$\alpha - 1 + 1/p < \beta < 2\gamma + 1/p.$$

Such a choice of β is possible because we have assumed $\gamma > (\alpha - 1)/2$. Hence there exists a constant $C > 0$ such that $\|\tilde{f}[u]\|_{L^p} \leq C\|\partial_x u\|_{X^{\gamma-1/2}}$ for any u satisfying $\partial_x u \in X^{\gamma-1/2}$, as claimed. Due to the continuity of the embedding $X^{\gamma-1/2} \hookrightarrow X$ we have $\|f[u]\|_{L^p} = \|\tilde{f}[u] + \tilde{\omega}\partial_x u\|_{L^p} \leq C\|\partial_x u\|_{X^{\gamma-1/2}} = C\|u\|_{X^\gamma}$ for any $u \in X^\gamma$ and f is a bounded linear operator from X^γ into $X = L^p$. $\diamondsuit$

Let us denote by $C([0,T], X^\gamma)$ the Banach space of all continuous functions from the interval $[0,T]$ to X^γ with the maximum norm $\|U(\cdot)\|_{C([0,T],X^\gamma)} = \sup_{\tau\in[0,T]} \|U(\tau)\|_{X^\gamma}$. We recall the well known result on existence and uniqueness of a solution to abstract parabolic equations in Banach spaces due to Henry [46].

Proposition 3.6.1 *[46, Chapter 3] Suppose that a densely defined closed linear operator $-A$ is a generator of an analytic semigroup $\{e^{-At}, t \geq 0\}$ in a Banach space X, $U_0 \in X^\gamma$ where $0 \leq \gamma < 1$. Assume $F : [0,T] \times X^\gamma \to X$ and $h : (0,T] \to X$ are Hölder continuous mappings in the τ variable, $\int_0^T \|h(\tau)\|_X \,\mathrm{d}x < \infty$, and F is a Lipschitz continuous mapping in the U variable. Then, there exists a unique solution $U \in C([0,T], X^\gamma)$ of the following abstract semilinear evolution equation:*

$$\frac{\partial U}{\partial \tau} + AU = F(\tau, U) + h(\tau), \qquad U(0) = U_0. \tag{3.96}$$

Moreover, $\partial_\tau U(\tau) \in X, U(\tau) \in D(A)$ *for any* $\tau \in (0,T)$.

Remark 6 *By a solution to (3.96) we mean a function* $U \in C([0,T], X^\gamma)$ *satisfying (3.96) in the integral (mild) sense, i.e.*

$$U(\tau) = e^{-A\tau}U_0 + \int_0^\tau e^{-A(\tau-s)}(F(s,U(s)) + h(s))\,\mathrm{d}s \text{ for any } \tau \in [0,T].$$

Recall that the key idea of the proof of Proposition 3.6.1 is based on the Banach fixed point argument combined with the decay estimate $\|e^{-At}\|_{X^\gamma} = \|A^\gamma e^{-At}\|_X \le M t^{-\gamma} e^{-at}$ *of the norm of the semigroup* e^{-At} *for any* $t > 0$.

As a direct consequence of Proposition 3.6.1 and Lemma 3.6.4 we deduce the following result:

Theorem 3.6.5 *Assume* ν *is an admissible activity Lévy measure with the shape parameters* $\alpha, D^\pm$ *and* μ *where* $\alpha < 3$ *and either* $\mu > 0, D^\pm \in \mathbb{R}$, *or* $\mu = 0, D^- + 1 < 0 < D^+$. *Assume* $\gamma \ge 1/2$ *and* $\gamma > (\alpha - 1)/2$. *Suppose that the function* $g(\tau, u)$ *is Hölder continuous in the* τ *variable and Lipschitz continuous in the* u *variable. Then for any* $u_0 \in X^\gamma$ *and* $T > 0$ *there exists a unique solution* $u \in C([0,T], X^\gamma)$ *to the PIDE (3.92).*

3.6.1.1 The Black-Scholes PIDE model

In this section, our purpose is to investigate properties of solutions to a PIDE generalizing the Black-Scholes model. An important definition concerning this generalization is the definition of a Lévy measure of a given process X_t . The measure $\nu(A)$ of a Borel set $A \subseteq \mathbb{R}$ defined in (2.23) gives, as we know, the mean number, per unit of time, of jumps of $X_t, t \ge 0$, whose amplitude belongs to the set A.

For the underlying asset price dynamics we will suppose that $S_t, t \ge 0$,follows the geometric Lévy proces, i.e. $S_t = e^{X_t}$ where $X_t, t \ge 0$, is a Lévy process. Then it is well known (cf. [24],[28]) that the price of a contingent claim in the presence of jumps is given by a solution $V(t,S)$ of the following partial integro-differential equation:

$$\begin{aligned} \frac{\partial V}{\partial t} &+ \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \\ &+ \int_{\mathbb{R}} \left[V(t, Se^z) - V(t,S) - (e^z - 1) S \frac{\partial V}{\partial S}(t,S) \right] \nu(\mathrm{d}z) = 0, \\ & V(T,S) = \Phi(S), \quad S > 0, t \in [0,T). \end{aligned} \tag{3.97}$$

Here Φ is the pay-off diagram of a plain vanilla option. For example, $\Phi(S) = (S-K)^+$ for a call option, or $\Phi(S) = (K-S)^+$ for a put option where $K > 0$ is the strike price. Here and after we shall denote by $a^+ = \max(a,0)$ and $a^- = \min(a,0)$ the positive and negative parts of a real number a, respectively.

If we consider the following change of variables $V(t,S) = e^{-r\tau}u(\tau,x)$ where $\tau = T - t$, $x = \ln(\frac{S}{K})$ then we obtain the following PIDE for the function $u(\tau, x)$:

$$\begin{aligned}\frac{\partial u}{\partial \tau} &= \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial x} \\ &\quad + \int_{\mathbb{R}} \left[u(\tau, x+z) - u(\tau, x) - (e^z - 1)\frac{\partial u}{\partial x}(\tau, x)\right] \nu(\,\mathrm{d}z), \\ u(0,x) &= \Phi(Ke^x), \quad x \in \mathbb{R}, \tau \in (0,T).\end{aligned} \tag{3.98}$$

Unfortunately, the initial condition $u(0,x) = \Phi(Ke^x)$ does not belong to the Banach space X for both call and put option pay-off diagrams Φ, i.e. $\Phi(S) = (S-K)^+$ and $\Phi(S) = (K-S)^+$. The idea how to formulate existence and uniqueness of a solution to the PIDE (3.98) is based on the idea of shifting the solution u by u_{BS} where the function $u_{BS}(\tau, x) = e^{r\tau} V_{BS}(T-\tau, Ke^x)$ corresponds to transformation of the classical solution V_{BS} to the linear Black-Scholes equation without PIDE part, i.e.

$$\begin{aligned}&\frac{\partial V_{BS}}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V_{BS}}{\partial S^2} + rS\frac{\partial V_{BS}}{\partial S} - rV_{BS} = 0, \\ &V_{BS}(T,S) = \Phi(S).\end{aligned}$$

Recall that the solution V_{BS} for a call or put option can be expressed explicitly:

$$\begin{aligned}V_{BS}^{call}(t,S) &= SN(d_1) - Ke^{-r(T-t)}N(d_2), \\ V_{BS}^{put}(t,S) &= Ke^{-r(T-t)}N(-d_2) - SN(-d_1),\end{aligned}$$

where

$$d_{1,2} = \frac{\ln(S/K) + (r \pm \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad \text{and} \quad N(d) = \int_{-\infty}^{d} \frac{e^{-\xi^2/2}}{\sqrt{2\pi}}\,\mathrm{d}\xi$$

is the cumulative distribution function of the normal distribution (cf. [51]). Furthermore, the transformed function u_{BS} is a solution to the linear parabolic PDE:

$$\begin{aligned}&\frac{\partial u_{BS}}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 u_{BS}}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial u_{BS}}{\partial x}, \\ &u_{BS}(0,x) = \Phi(Ke^x), \quad \tau \in (0,T), x \in \mathbb{R},\end{aligned} \tag{3.99}$$

where $\Phi(Ke^x) = K(e^x - 1)^+$ for the call option and $\Phi(Ke^x) = K(1 - e^x)^+$ for the put option.

In what follows, we shall provide important estimates for the function $f[u_{BS}]$.

Lemma 3.6.6 *Suppose that ν is an admissible activity Lévy measure ν with the shape parameters $\alpha, D^\pm$ and μ where $\alpha < 3$ and either $\mu > 0, D^\pm \in \mathbb{R}$, or $\mu = 0, D^- + 1 < 0 < D^+$. Suppose that $\frac{1}{2} \le \gamma < 1$ and $\frac{\alpha-1}{2} < \gamma < \frac{p+1}{2p} \le 1$. Then there exists a constant $C_0 > 0$ depending on the parameters p, σ, r, T, K only, and such that the function $f[u_{BS}(\tau,\cdot)]$ satisfies the following estimates:*

$$\begin{aligned}&\|f[u_{BS}(\tau,\cdot)]\|_{L^p} \le C_0 \tau^{-(2\gamma-1)\left(\frac{1}{2}-\frac{1}{2p}\right)}, \qquad 0 < \tau \le T, \\ &\|f[\partial_\tau u_{BS}(\tau,\cdot)]\|_{L^p} \le C_0 \tau^{-\gamma-\frac{1}{2}+\frac{1}{2p}}, \qquad 0 < \tau \le T, \\ &\|f[u_{BS}(\tau_1,\cdot)] - f[u_{BS}(\tau_2,\cdot)]\|_{L^p} \le C_0 |\tau_1 - \tau_2|^{-\gamma+\frac{p+1}{2p}}, \qquad 0 < \tau_1, \tau_2 \le T.\end{aligned}$$

Proof. First, we consider the case of a call option, i.e. $u_{BS} = u_{BS}^{call}$ with $u_{BS}(0,x) = \Phi(Ke^x) = K(e^x - 1)^+$. It is important to emphasize that $f[e^x] = 0$. Hence

$$f[u_{BS}] = f[u_{BS} - Ke^{r\tau+x}], \quad \text{and} \quad \partial_\tau f[u_{BS}] = f[\partial_\tau(u_{BS} - Ke^{r\tau+x})].$$

In what follows, we shall denote by C_0 any generic positive constant depending on the parameters p, σ, r, T, K only. With regard to Lemma 3.6.4 we shall estimate the $X^{\gamma-1/2}$ norm of the function v:

$$v(\tau,x) = \partial_x \left(u_{BS}(\tau,x) - Ke^{r\tau+x}\right) = Ke^{r\tau+x}(N(d_1(\tau,x)) - 1), \tag{3.100}$$

where $d_1(\tau,x) = (x + (r+\sigma^2/2)\tau)/(\sigma\sqrt{\tau})$. In the case of a put option we have

$$\partial_x u_{BS}^{put}(\tau,x) = -Ke^{r\tau+x}N(-d_1(\tau,x)) = -Ke^{r\tau+x}(1 - N(d_1(\tau,x))) = v(\tau,x).$$

Hence the proof of the statement of lemma for the case of a put option is essentially the same as the following argument for a call option.

Now, using integration by parts and substitution $\xi = d_1(\tau,x)$, we obtain

$$\begin{aligned} \|v(\tau,\cdot)\|_{L^p}^p &= K^p e^{pr\tau}\int_{-\infty}^{\infty} e^{px}(1-N(d_1))^p \,\mathrm{d}x \\ &\le K^p e^{pr\tau}\int_{-\infty}^{\infty} e^{px}(1-N(d_1)) \,\mathrm{d}x = K^p e^{pr\tau}\int_{-\infty}^{\infty} \frac{e^{px}}{p}\frac{e^{-d_1^2/2}}{\sqrt{2\pi}}\frac{1}{\sigma\sqrt{\tau}}\,\mathrm{d}x \\ &= K^p e^{pr\tau}\int_{-\infty}^{\infty} \frac{e^{p\sigma\sqrt{\tau}\xi - p(r+\sigma^2/2)\tau}}{p}\frac{e^{-\xi^2/2}}{\sqrt{2\pi}}\,\mathrm{d}\xi = \frac{1}{p}K^p e^{p(p-1)\tau\sigma^2/2}. \end{aligned}$$

Thus $\|v(\tau,\cdot)\|_{L^p} \le p^{-1/p}Ke^{(p-1)T\sigma^2/2} \equiv C_0$ for any $0 < \tau \le T$.

Now, as $\partial_x v = v + w$ where

$$w = Ke^{r\tau+x}N'(d_1)\frac{1}{\sigma\sqrt{\tau}} = Ke^{r\tau+x}\frac{e^{-d_1^2/2}}{\sigma\sqrt{2\pi\tau}}.$$

we obtain

$$\begin{aligned} \|w(\tau,\cdot)d_1(\tau,\cdot)^k\|_{L^p}^p &= \frac{K^p e^{pr\tau}}{(\sigma\sqrt{2\pi\tau})^{p-1}}\int_{-\infty}^{\infty} e^{px}\frac{e^{-pd_1^2/2}|d_1|^{pk}}{\sigma\sqrt{2\pi\tau}}\,\mathrm{d}x \\ &= \frac{K^p e^{pr\tau}}{(\sigma\sqrt{2\pi\tau})^{p-1}}\int_{-\infty}^{\infty} e^{p\sigma\sqrt{\tau}\xi - p(r+\sigma^2/2)\tau}\frac{e^{-\xi^2/2}|\xi|^{pk}}{\sqrt{2\pi}}\,\mathrm{d}\xi \\ &\le C_0^p\tau^{-\frac{p-1}{2}} \end{aligned} \tag{3.101}$$

for $k = 0,1,2$. Applying (3.101) with $k=0$ we obtain $\|w(\tau,\cdot)\|_{L^p} \le C_0\tau^{-\frac{1}{2}+\frac{1}{2p}}$. As a consequence, $\|v(\tau,\cdot)\|_{W^{1,p}} \le C_0\tau^{-\frac{1}{2}+\frac{1}{2p}}$. Since the Bessel potential space $\mathscr{L}_{2\gamma-1}^p$ is an interpolation space between $\mathscr{L}_0^p = L^p$ and $\mathscr{L}_1^p = W^{1,p}$ using the Gagliardo-Nirenberg interpolation inequality

$$\|v\|_{X^{\gamma-1/2}} \equiv \|v\|_{\mathscr{L}_{2\gamma-1}^p} \le C_0\|v\|_{L^p}^\theta\|v\|_{W^{1,p}}^{1-\theta}, \quad \text{where } 2\gamma - 1 = 0\cdot\theta + 1\cdot(1-\theta),$$

(cf. [46, Section 1.6]) and applying Lemma 3.6.4 we obtain

$$\|f[u_{BS}(\tau,\cdot)]\|_{L^p} \le C\|v(\tau,\cdot)\|_{X^{\gamma-1/2}} \le C_0\tau^{-(2\gamma-1)\left(\frac{1}{2}-\frac{1}{2p}\right)}, \qquad 0<\tau\le T,$$

as claimed.

In order to prove the remaining estimates, let us estimate the norm $\|\partial_\tau v(\tau,\cdot)\|_{X^{\gamma-1/2}}$. As $\partial_\tau d_1 = -\tau^{-3/2}x/(2\sigma)+\tau^{-1/2}(r+\sigma^2/2)/(2\sigma) = -\tau^{-1}d_1/2+\tau^{-1/2}(r+\sigma^2/2)/\sigma$ we have

$$\partial_\tau v = rv + Ke^{r\tau+x}N'(d_1)\partial_\tau d_1 = rv + w(-\tau^{-1/2}\sigma d_1/2 + r + \sigma^2/2).$$

Using the estimate (3.101) with $k=0,1$ we obtain

$$\|\partial_\tau v(\tau,\cdot)\|_{L^p} \le C_0\tau^{-1+\frac{1}{2p}}, \qquad 0<\tau\le T.$$

To estimate the $W^{1,p}$ norm of $\partial_\tau v$ we recall that $\partial_x v = v + w$. Thus

$$\begin{aligned}
\partial_x\partial_\tau v &= \partial_\tau v + \partial_\tau w = \partial_\tau v + rw + Ke^{r\tau+x}\left(\frac{N''(d_1)}{\sigma\sqrt{\tau}}\partial_\tau d_1 - \frac{N'(d_1)}{2\sigma\tau^{3/2}}\right)\\
&= \partial_\tau v + rw + w\left(-d_1\partial_\tau d_1 - \tau^{-1}/2\right)\\
&= \partial_\tau v + rw + w\left(d_1^2\tau^{-1}/2 - \tau^{-1}/2 - \tau^{-1/2}d_1(r+\sigma^2/2)/\sigma\right),
\end{aligned}$$

as $N''(d_1) = -d_1N'(d_1)$. Using the estimate (3.101) with $k=0,1,2$ we obtain

$$\|\partial_\tau v(\tau,\cdot)\|_{W^{1,p}} \le C_0\tau^{-\frac{3}{2}+\frac{1}{2p}}, \qquad 0<\tau\le T.$$

Again, using the Gagliardo-Nirenberg interpolation inequality

$$\|\partial_\tau v\|_{X^{\gamma-1/2}} \equiv \|\partial_\tau v\|_{\mathscr{L}^p_{2\gamma-1}} \le C_0\|\partial_\tau v\|^{\theta}_{L^p}\|\partial_\tau v\|^{1-\theta}_{W^{1,p}}, \quad \text{where } 2\gamma-1 = 0\cdot\theta + 1\cdot(1-\theta)$$

and applying Lemma 3.6.4 we obtain

$$\|\partial_\tau f[u_{BS}(\tau,\cdot)]\|_{L^p} \le C\|\partial_\tau v(\tau,\cdot)\|_{X^{\gamma-1/2}} \le C_0\tau^{-\gamma-\frac{1}{2}+\frac{1}{2p}}, \qquad 0<\tau\le T,$$

as claimed in the second statement of lemma.

Finally,

$$\begin{aligned}
&\|f[u_{BS}(\tau_1,\cdot)] - f[u_{BS}(\tau_2,\cdot)]\|_{L^p} = \|\int_{\tau_1}^{\tau_2}\partial_\tau f[u_{BS}(\tau,\cdot)]d\tau\|_{L^p}\\
&\le \left|\int_{\tau_1}^{\tau_2}\|\partial_\tau f[u_{BS}(\tau,\cdot)]\|_{L^p}d\tau\right| \le C_0|\tau_1-\tau_2|^{-\gamma+\frac{p+1}{2p}}, \qquad 0<\tau_1,\tau_2\le T,
\end{aligned}$$

and the function $f[u_{BS}(\tau,\cdot)]$ is Hölder continuous with the Hölder exponent $-\gamma+\frac{p+1}{2p}>0$. The proof of lemma follows. ◇

Combining the previous Lemmas 3.6.4, 3.6.6, sectoriality of the operator $A=-\partial_x^2$ in $X=L^p(\mathbb{R})$ (see Lemma 3.6.3), and Proposition 3.6.5 we obtain the following existence and uniqueness result for the linear PIDE (3.98), and, consequently, for the linear option pricing model (3.97):

Theorem 3.6.7 *Assume ν is an admissible activity Lévy measure with the shape parameters $\alpha < 3$ and either $\mu > 0, D^{\pm} \in \mathbb{R}$, or $\mu = 0$ and $D^- + 1 < 0 < D^+$. Let $X^\gamma = \mathscr{L}^p_{2\gamma}(\mathbb{R})$ be the space of Bessel potentials where $\frac{1}{2} \le \gamma < 1$ and $\frac{\alpha-1}{2} < \gamma < \frac{p+1}{2p}$.*

Then, for any $T > 0$, the linear PIDE (3.98) has the unique solution u such that the difference $U = u - u_{BS}$ satisfies $U \in C([0,T], X^\gamma)$. Moreover, $U(\tau, \cdot) \in X^1 = \mathscr{L}^p_2(\mathbb{R}) \subseteq W^{2,p}(\mathbb{R})$ and $\partial_\tau U(\tau, \cdot) \in X = L^p(\mathbb{R})$ for any $\tau \in (0,T)$.

Proof. Since the Black-Scholes solution u_{BS} solves the linear PDE (3.99) the difference $U = u - u_{BS}$ of a solution u to (3.98) and u_{BS} satisfies the PIDE:

$$\begin{aligned} \frac{\partial U}{\partial \tau} &= \frac{\sigma^2}{2}\frac{\partial^2 U}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial U}{\partial x} + f[U] + f[u_{BS}], \\ U(0,x) &= 0, \quad x \in \mathbb{R}, \tau \in (0,T). \end{aligned}$$

This PIDE equation can be rewritten in the abstract form:

$$\frac{\partial U}{\partial \tau} + AU = F(U) + h(\tau), \quad U(0) = 0, \tag{3.102}$$

where the linear operators A and f were defined in (3.94) and (3.95). The functions $F = F(U)$ and $h = h(\tau)$, $F : X^\gamma \to X$, $h : (0,T] \to X$ are defined as follows:

$$F(U) = (r - \sigma^2/2)\frac{\partial U}{\partial x} + f[U], \qquad h(\tau) = f[u_{BS}(\tau, \cdot)].$$

With regard to Lemma 3.6.4, F is a bounded linear mapping, and, consequently Lipschitz continuous from the space X^γ into X provided that $\gamma \ge 1/2$ and $\gamma > (\alpha - 1)/2$.

Taking into account Lemma 3.6.6 we obtain

$$\|h(\tau_1) - h(\tau_2)\|_{L^p} = \|f[u_{BS}(\tau_1, \cdot)] - f[u_{BS}(\tau_2, \cdot)]\|_{L^p} \le C_0|\tau_1 - \tau_2|^{-\gamma + \frac{p+1}{2p}},$$

for any $0 < \tau_1, \tau_2 \le T$. Since $\gamma < \frac{p+1}{2p}$ the mapping $h : [0,T] \to X \equiv L^p(\mathbb{R})$ is Hölder continuous. Moreover,

$$\int_0^T \|h(\tau)\|_{L^p} d\tau = \int_0^T \|f[u_{BS}(\tau, \cdot)]\|_{L^p} d\tau \le C_0 \int_0^T \tau^{-(2\gamma-1)\left(\frac{1}{2} - \frac{1}{2p}\right)} d\tau < \infty,$$

because $(2\gamma - 1)\left(\frac{1}{2} - \frac{1}{2p}\right) < 1$. The rest of the proof now follows from Theorem 3.6.5. ◇

The following corollary is a consequence of embedding of the Bessel potential space into the space of Hölder continuous functions.

Corollary 3.6.1 *Suppose that an admissible activity Lévy measure ν fulfills assumptions of Theorem 3.6.7. Then, for any $T > 0$, linear PIDE (3.98) has the unique solution $u \in C([0,T], C^\kappa_{loc}(\mathbb{R}))$, with the Hölder exponent $\kappa > 0$ satisfying $\alpha - 1 - 1/p < \kappa < 1$.*

Proof. Recall continuity of the embedding

$$X^\gamma = \mathscr{L}^p_{2\gamma}(\mathbb{R}) \hookrightarrow C^\kappa_{loc}(\mathbb{R}),$$

where $\kappa = 2\gamma - 1/p$ (cf. [46, Section 1.6]), i.e. $\gamma = \kappa/2 + 1/(2p)$. Now, there exists $1/2 \le \gamma < 1$ such that $\frac{\alpha-1}{2} < \gamma < \frac{p+1}{2p}$ if and only if $\alpha - 1 - 1/p < \kappa < 1$, as claimed. Therefore $U = u - u_{BS}$ belongs to $C([0,T], C^\kappa_{loc}(\mathbb{R}))$.

The solution $u_{BS} = u_{BS}(\tau, x)$ is a real analytic function in the τ and x variables for any $\tau > 0$ and $x \in \mathbb{R}$. As $u_{BS}(0,x)$ represents the transformed call or put payoff diagram we have $u_{BS} = u_{BS}(0,x)$ is locally Lipschitz continuous in the x variable. Hence $u_{BS} \in C([0,T], C^\kappa_{loc}(\mathbb{R}))$. Therefore the solution $u = U + u_{BS}$ to the linear PIDE (3.98) belongs to $C([0,T], C^\kappa_{loc}(\mathbb{R}))$, as claimed. $\diamondsuit$

Remark 7 *The conditions $\frac{1}{2} \le \gamma < 1$ and $\frac{\alpha-1}{2} < \gamma < \frac{p+1}{2p}$ are fulfilled for a power $p \ge 1$ provided that either $\alpha \in [0,2]$ and $p \ge 1$, or $\alpha \in (2,3)$ and $1 \le p < 1/(\alpha - 2)$. It means that if the Lévy measure ν has a strong singularity of the order $\alpha \in (2,3)$ at the origin then we can find a solution in the framework of fractional power spaces of the Banach space $X = L^p(\mathbb{R})$ where p is limited by the order α.*

3.6.2 Existence results for nonlinear PIDE option pricing models

In this section we present an application of the general existence and uniqueness result for the penalized version of the PIDE for solving the linear complementarity problem arising in pricing American style of a put option on an underlying asset following Lévy stochastic process.

In [15] Bensoussan and Lions proved results which allow to characterize price of a put option in terms of a solution of a system of partial-integro differential inequalities (see also [52]). In [86] and [85] Wang *et al.* investigated the penalty method for solving a linear complementarity problem using a power penalty term for the case without jumps in underlying asset dynamics. In [54] Lesman and Wang proposed a power penalty method for solving the free boundary problem for pricing American options under transaction costs. Penalty methods for American option pricing under stochastic volatility models are studied in the paper [90] by Zvan, Forsyth and Vetzal. In [30] d'Halluin, Forsyth, and Labahn investigated a penalty method for American options on jump diffusion underlying processes.

Recall that American style options can be exercised anytime before the maturity time T. In the case of an American put option the state space $\{(t,S),\, t \in [0,T], S > 0\}$ can be divided into the so-called early exercise region $\mathcal{E}$ and continuation region $\mathcal{C}$ where the put option should be exercised and hold, respectively. These regions are separated by the early exercise boundary defined by a function $t \mapsto S_f(t)$, such that $0 < S_f(t) \le K$, and

$$\mathcal{E} = \{(t,S),\, t \in [0,T], 0 < S \le S_f(t)\}, \quad \mathcal{C} = \{(t,S),\, t \in [0,T], S_f(t) < S\}.$$

We refer the reader to [51], [80], [53], [89] for overview of qualitative properties of the early exercise boundary for the case of pricing American style of put options for the Black-Scholes PDE with no integral part.

In the continuation region $\mathcal{C}$ the put option price is strictly greater than the pay-off diagram, i.e. $V(t,S) > \Phi(S) = (K-S)^+$ for $S_f(t) < S$. In the exercise region $\mathcal{E}$ the put option price is given by its pay-off diagram, i.e. $V(t,S) = \Phi(S) = (K-S)^+$. Moreover, the put option price $V(t,S)$ is a decreasing function in the S variable. Hence in the exercise region where $0 < S < S_f(t) \leq K$, for the price $V(t,S) = K - S$ we obtain

$$\begin{aligned}
\frac{\partial V}{\partial t} + L^S[V] &\equiv \frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \\
&\quad + \int_{\mathbb{R}} \left[V(t, Se^y) - V(t,S) - S\left(e^y - 1\right)\frac{\partial V}{\partial S}(t,S)\right]\nu(\,\mathrm{d}y) \\
&= -rK + \int_{-\infty}^{0} \left[V(t,Se^y) - (K-S) - S\left(e^y-1\right)(-1)\right]\nu(\,\mathrm{d}y) \\
&\quad + \int_0^\infty \left[V(t,Se^y) - (K-S) - S\left(e^y-1\right)(-1)\right]\nu(\,\mathrm{d}y) \\
&= -rK + \int_0^\infty \left[V(t,Se^y) - (K-S) + S\left(e^y-1\right)\right]\nu(\,\mathrm{d}y) \\
&\leq -rK + S\int_0^\infty \left(e^y - 1\right)\nu(\,\mathrm{d}y)
\end{aligned}$$

because $S \mapsto V(t,S)$ is a decreasing function, and thus $V(t,Se^y) \leq V(t,S) = K - S$ for $y \geq 0$, and $V(t,Se^y) = K - Se^y$ for $y \leq 0$.

Let us assume that the admissible activity Lévy measure ν satisfies the inequality:

$$\int_0^\infty \left(e^y - 1\right)\nu(\,\mathrm{d}y) \leq r. \tag{3.103}$$

Then the price $V(t,S)$ of an American put option satisfies the inequality $\partial_t V(t,S) + L^S[V](t,S) \leq 0$ for $0 < S \leq S_f(t) \leq K$, i.e. for $(t,S) \in \mathcal{E}$. On the other hand, for $(t,S) \in \mathcal{C}$ the price $V(t,S)$ is obtained from the Black-Scholes PIDE equation $\partial_t V(t,S) + L^S[V](t,S) = 0$.

In summary, we have shown the following result.

Theorem 3.6.8 *Let $V(t,S)$ be the price of an American style put on underlying asset S following a geometric Lévy process with an admissible activity Lévy measure ν satisfying the structural inequality (3.103). Then V is a solution to the linear complementarity problem:*

$$\partial_t V(t,S) + L^S[V](t,S) \leq 0, \qquad V(t,S) \geq \Phi(S), \tag{3.104}$$

$$\left(\partial_t V(t,S) + L^S[V](t,S)\right)\cdot\left(V(t,S) - \Phi(S)\right) = 0, \tag{3.105}$$

for any $t \in [0,T), S > 0$, and $V(T,S) = \Phi(S) = (K-S)^+$.

A standard method for solving the linear complementarity problem (3.104)–(3.105) is based on construction of an approximate solution by means of the penalty method. It consists in construction of a suitable nonnegative penalty function $\mathcal{G}_\varepsilon(t,V)$ penalizing negative values of the difference $V(t,S)-\Phi(S)$. For example, one can consider the penalty function of the form:

$$\mathcal{G}_\varepsilon(t,V)(S) = \varepsilon^{-1}\min(S/K,\,1)(\Phi(S) - V(t,S))^+,$$

where $0 < \varepsilon \ll 1$ is a small parameter. Clearly, $\mathcal{G}_\varepsilon(t,V)(S) = 0$ if and only if $V(t,S) \geq \Phi(S)$. Then the penalized problem for the approximate solution $V = V_\varepsilon$ to (3.104)–(3.105) reads as follows:

$$\begin{aligned} &\partial_t V + L^S[V] + \mathcal{G}_\varepsilon(t,V) = 0, \quad S > 0, t \in [0,T), \\ &V(T,S) = \Phi(S). \end{aligned} \tag{3.106}$$

In terms of the transformed function $u(\tau,x) = e^{r\tau}V(T-\tau, Ke^x)$ and the shifted function $U = u - u_{BS}$ the penalized PIDE problem (3.106) can be rewritten as follows:

$$\frac{\partial U}{\partial \tau} + AU = F(U) + h(\tau) + g_\varepsilon(\tau,U), \quad U(0) = 0. \tag{3.107}$$

The penalty term g_ε can be deduced from $\mathcal{G}_\varepsilon$, i.e.

$$g_\varepsilon(\tau,U) = \varepsilon^{-1}e^{x^-}(w(\tau,x) - U)^+, \quad \text{where } w(\tau,x) = e^{r\tau}\Phi(Ke^x) - u_{BS}(\tau,x).$$

Recall that the linear operators A and f were defined in (3.94) and (3.95) and

$$F(U) = (r - \sigma^2/2)\frac{\partial U}{\partial x} + f[U], \qquad h(\tau) = f[u_{BS}(\tau,\cdot)].$$

Before proving existence and uniqueness of a solution to the penalized PIDE equation (3.107) we need the following auxiliary lemma.

Lemma 3.6.9 *The penalty function $g_\varepsilon : [0,T]\times X \to X$ is Lipschitz continuous in the U variable and Hölder continuous in the τ variable, i.e. there exists a constant $C_0 > 0$ such that*

$$\|g_\varepsilon(\tau,U_1) - g_\varepsilon(\tau,U_2)\|_X \leq \varepsilon^{-1}\|U_1 - U_2\|_X, \quad \|g_\varepsilon(\tau_1,U) - g_\varepsilon(\tau_2,U)\|_X \leq \varepsilon^{-1}C_0|\tau_1-\tau_2|^{\frac{p+1}{2p}}$$

for any $U, U_1, U_2 \in X$ and $\tau, \tau_1, \tau_2 \in [0,T]$.

Proof. Note the inequality $|a^+ - b^+| \leq |a-b|$ for all $a,b \in \mathbb{R}$. As $e^{x^-} \leq 1$, we obtain

$$\begin{aligned} \|g_\varepsilon(\tau,U_1) - g_\varepsilon(\tau,U_2)\|^p_{L^p} &\leq \varepsilon^{-p}\int_{-\infty}^{\infty} \left|(w(\tau,x) - U_1(x))^+ - (w(\tau,x) - U_2(x))^+\right|^p \mathrm{d}x \\ &\leq \varepsilon^{-p}\int_{-\infty}^{\infty} |U_1(x) - U_2(x)|^p \,\mathrm{d}x = \varepsilon^{-p}\|U_1 - U_2\|^p_{L^p}. \end{aligned}$$

Moreover, it is easy to verify that the function $e^{x^-}w(\tau,x)$ belongs to $X = L^p$ and

$$w(\tau,x) = e^{r\tau}\Phi(Ke^x) - KN(-d_2(\tau,x)) + Ke^{r\tau+x}N(-d_1(\tau,x)).$$

Hence $g_\varepsilon(\tau,0) \in X = L^p$ and $g_\varepsilon(\tau,\cdot) : X \to X$ is well defined and Lipschitz continuous mapping for any $\tau \in [0,T]$.

Recall that $d_1 - d_2 = \sigma\sqrt{\tau}$, $d_1 + d_2 = 2(x + r\tau)/\sigma\sqrt{\tau}$, and, consequently, $e^{r\tau+x}N'(-d_1) - N'(-d_2) = 0$. Since $N(-d_1) = 1 - N(d_1)$ we obtain

$$\begin{aligned}
\partial_\tau w &= re^{r\tau}\Phi(Ke^x) + rKe^{r\tau+x}N(-d_1) - KN'(-d_2)\frac{\sigma}{2\sqrt{\tau}} \\
&= re^{r\tau}\Phi(Ke^x) - rv - K\frac{e^{-d_2^2/2}}{\sqrt{2\pi}}\frac{\sigma}{2\sqrt{\tau}}
\end{aligned}$$

where the auxiliary function v was defined as in (3.100). Therefore

$$\begin{aligned}
\|e^{x^-}\partial_\tau w\|_{L^p} &\le re^{r\tau}\|e^{x^-}\Phi(Ke^x)\|_{L^p} + r\|e^{x^-}v\|_{L^p} + \frac{K\sigma}{2\sqrt{\tau}}\left(\int_{-\infty}^{\infty} e^{px^-}\frac{e^{-pd_2^2/2}}{(2\pi)^{p/2}}\,\mathrm{d}x\right)^{1/p} \\
&\le rKe^{r\tau}\|e^{x^-}1_{x\le 0}\|_{L^p} + r\|v\|_{L^p} + \frac{K\sigma}{2\sqrt{\tau}}\left(\int_{-\infty}^{\infty}\frac{e^{-p\xi^2/2}}{(2\pi)^{p/2}}\sigma\sqrt{\tau}\,\mathrm{d}\xi\right)^{1/p} \\
&\le C_0\tau^{\frac{1}{2p}-\frac{1}{2}},
\end{aligned}$$

where $C_0 > 0$ is a constant independent of $\tau \in (0,T]$. Thus

$$\begin{aligned}
\|g_\varepsilon(\tau_1,U) - g_\varepsilon(\tau_2,U)\|_{L^p}^p &= \varepsilon^{-p}\int_{-\infty}^{\infty} e^{px^-}\left|(w(\tau_1,x) - U(x))^+ - (w(\tau_2,x) - U(x))^+\right|^p\,\mathrm{d}x \\
&\le \varepsilon^{-p}\int_{-\infty}^{\infty} e^{px^-}|w(\tau_1,x) - w(\tau_2,x)|^p\,\mathrm{d}x \\
&= \varepsilon^{-p}\|e^{x^-}(w(\tau_1,\cdot) - w(\tau_2,\cdot))\|_{L^p}^p.
\end{aligned}$$

Hence

$$\|g_\varepsilon(\tau_1,U) - g_\varepsilon(\tau_2,U)\|_{L^p} \le \varepsilon^{-1}\int_{\tau_1}^{\tau_2}\|e^{x^-}\partial_\tau w(\tau,\cdot)\|_{L^p}\,\mathrm{d}\tau \le \varepsilon^{-1}C_0|\tau_1 - \tau_2|^{\frac{p+1}{2p}},$$

as claimed. The proof of lemma follows. ◇

Similarly as in the case of a linear PIDE, applying Lemmas 3.6.4, 3.6.6, 3.6.3, and Proposition 3.6.5 we obtain the following existence and uniqueness result for the nonlinear penalized PIDE (3.107).

Theorem 3.6.10 *Assume ν is an admissible activity Lévy measure with the shape parameters $\alpha < 3$ and either $\mu > 0, D^\pm \in \mathbb{R}$, or $\mu = 0$ and $D^- + 1 < 0 < D^+$. Let $X^\gamma = \mathscr{L}^p_{2\gamma}(\mathbb{R})$ be the space of Bessel potentials where $\frac{1}{2} \le \gamma < 1$ and $\frac{\alpha-1}{2} < \gamma < \frac{p+1}{2p}$. Suppose that the structural condition (3.103) is fulfilled for the measure ν.*

Then, for any $\varepsilon > 0$ and $T > 0$, the nonlinear penalized PIDE (3.107) has the unique solution $U_\varepsilon \in C([0,T), X^\gamma)$. Moreover, $U_\varepsilon(\tau,\cdot) \in X^1 = \mathscr{L}^p_2(\mathbb{R}) \hookrightarrow W^{2,p}(\mathbb{R})$ and $\partial_\tau U_\varepsilon(\tau,\cdot) \in L^p(\mathbb{R})$ for any $\tau \in (0,T)$.

In this section we analyzed existence and uniqueness of solutions to a partial integro-differential equation (PIDE) in the Bessel potential space. As a model we considered a model for pricing vanilla call and put options on underlying assets following a geometric Lévy stochastic process. Using the theory of abstract semilinear parabolic equations we proved existence and uniqueness of solutions in the Bessel potential space representing a fractional power space of the space of Lebesgue p-integrable functions with respect to the second order Laplace differential operator. We generalized known existence results for a wider class of Lévy measures including those having strong singular kernel. We also proved existence and uniqueness of solutions to the penalized PIDE representing approximation of the linear complementarity problem arising in pricing American style of options.

Chapter 4

Numerical Methods

The aim of this chapter is to propose numerical schemes for solving PIDEs . The methods of discretization are based on Finite Difference methods and Galerkin methods.

4.1 Finite Difference Methods

4.1.1 Implicit-explicit numerical discretization scheme for the Classical PIDE

Our aim is to solve numerically $L_{PIDE}V^{PIDE}(t,S_t)=0$ i.e

$$\begin{aligned}&\frac{\partial V^{PIDE}}{\partial t}+\frac{1}{2}\frac{\partial^2 V^{PIDE}}{\partial^2 S}S_{t^-}^2\sigma^2+S_{t^-}r\frac{\partial V^{PIDE}}{\partial S}-rV \\ &+\int_{\mathbb{R}}V^{PIDE}(t,S_{t^-}+y)-V^{PIDE}(t^-,S_{t^-})-y\frac{\partial V^{PIDE}}{\partial S}\nu(\,\mathrm{d}y)=0.\end{aligned} \tag{4.1}$$

We make the following transformations: $V^{PIDE}(t,S_t)=e^{-r\tau}u(\tau,x)$, where $\tau=T-t, x=\ln(\frac{S_t}{S_0})$ and get

$$\frac{\partial u}{\partial \tau}=\frac{1}{2}\frac{\partial^2 u}{\partial^2 x}\sigma^2+\left(r-\frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial x}+\int_{\mathbb{R}}u(\tau,x+z)-u(\tau,x)-(e^z-1)\frac{\partial u}{\partial x}\nu(\,\mathrm{d}z).$$

In order to solve this equation numerically, the domain of integration of the integral term needs to be truncated into a bounded interval and because the Variance Gamma process is a jump process of infinite activity, the small jumps of the initial Lévy process need to be approximated by a process of finite activity, namely the Brownian Motion. The Lévy process obtained has a new characteristic triplet given by $(\gamma(\epsilon),\sqrt{\sigma^2(\epsilon)+\sigma^2},\nu 1_{|x|>\epsilon})$, where $\sigma^2(\epsilon)=\int_{-\epsilon}^{\epsilon}y^2\nu(\,\mathrm{d}y)$ and the drift is given by the associated martingale condition. The scheme proposed in [83] is the explicit-implicit finite difference scheme. The idea is to separate Lu into two parts, the differential part Du and the integral part Ju. The operator then becomes in this case:

$$Lu(\tau,x)=Du(\tau,x)+Ju(\tau,x), \tag{4.2}$$

where

$$Du\tau, x) = -\left(\frac{\sigma^2(\epsilon)+\sigma^2}{2} - r + \alpha\right)\frac{\partial u}{\partial x}(\tau, x) + \frac{\sigma^2(\epsilon)+\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}(\tau, x) - \lambda u(\tau, x), \quad (4.3)$$

$$Ju(\tau, x) = \int_{B_l}^{B_r} u(\tau, x+y)\,\nu(\,dy)\,1_{|y|>\epsilon}, \quad (4.4)$$

$$\alpha = \int_{B_l}^{B_r}(e^y - 1)\nu(\,dy)1_{|y|>\epsilon}, \lambda = \int_{B_l}^{B_r}\nu(\,dy)1_{|y|>\epsilon}. \quad (4.5)$$

The localized problem becomes:

$$\frac{\partial f}{\partial \tau} - Lu = 0,\ (\tau, x) \in [0, T] \times (-A, A) \quad (4.6)$$

$$u(0, x) = h(x),\ x \in (-A, A), \quad (4.7)$$

$$u(\tau, x) = g(\tau, x),\ x \notin (-A, A). \quad (4.8)$$

In [83], it is shown that the best choice for $g(\tau, x)$ is $h(x + r\tau)$. Let $\{u_i^n\}$ be the numerical solution of the scheme proposed and define a uniform grid:
$Q_{\Delta t, \Delta x} = \left\{(\tau_n, x_i) : \tau_n = n\Delta t, n = 0, 1, ...M, x_i = -A + i\Delta x, i \in \mathbb{Z}, \Delta t = \frac{T}{M}, \Delta x = \frac{2A}{N}\right\}$
and choose K_l, K_r such that $[B_l, B_r] \subset [(K_l - 1/2)\Delta x, (K_r + 1/2)\Delta x]$.

Then,

$$\alpha \approx \hat{\alpha} = \sum_{j=K_l}^{K_r}(e^{y_j} - 1)\nu_j 1_{|y_j|>\epsilon}, \lambda \approx \hat{\lambda} = \sum_{j=K_l}^{K_r}\nu_j 1_{|y_j|>\epsilon}, \quad (4.9)$$

$$\int_{B_l}^{B_r} u(\tau, x_i+y)\,\nu(\,dy)\,1_{|y|>\epsilon} \approx \sum_{j=K_l}^{K_r}\nu_j u_{i+j} 1_{|y_j|>\epsilon}, \quad (4.10)$$

$$\text{where } \nu_j \approx \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x}\nu(\,dy) \approx 0.5\Delta x(\nu((j-1/2)\Delta x) + \nu((j+1/2)\Delta x)) \quad (4.11)$$

$$(\frac{\partial u}{\partial x})_i \approx \begin{cases} \frac{u_{i+1}-u_i}{\Delta x} & \text{if } (\frac{\sigma^2(\epsilon)}{2} - r + \hat{\alpha}) < 0, \\ \frac{u_i - u_{i-1}}{\Delta x} & \text{if } (\frac{\sigma^2(\epsilon)}{2} - r + \hat{\alpha}) \geq 0 \end{cases} \quad (4.12)$$

In order to approximate $\frac{\partial u}{\partial x}(\tau, x+y)$ we need the finite difference approximations

$$(\frac{\partial u}{\partial x})_{i+j} \approx \begin{cases} \frac{u_{i+j+1}-u_{i+j}}{\Delta x} & \text{if } (\frac{\sigma^2(\epsilon)}{2} - r + \hat{\alpha}) < 0, \\ \frac{u_{i+j} - u_{i+j-1}}{\Delta x} & \text{if } (\frac{\sigma^2(\epsilon)}{2} - r + \hat{\alpha}) \geq 0 \end{cases} \quad (4.13)$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}, \quad (4.14)$$

$$\left(\frac{\partial u}{\partial \tau}\right)_i \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}. \quad (4.15)$$

Then replacing all these quantities in the problem (4.6)-(4.8) the algorithm becomes:

$$\text{Initialization:} \tag{4.16}$$
$$u_i^0 = h(x_i), i \in \{0, 1..., N\}, \tag{4.17}$$
$$u_i^0 = g(0, x_i), i \notin \{0, 1..., N\}. \tag{4.18}$$
$$\text{For n=0,....M-1:} \tag{4.19}$$
$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = (D_\Delta u^{n+1})_i + (J_\Delta u^n)_i, i \in \{0, 1..., N\}, \tag{4.20}$$
$$u_i^{n+1} = g((n+1)\Delta t, x_i), i \notin \{0, 1..., N\}, \tag{4.21}$$

where

$$(D_\Delta u^{n+1}) = -\left(\frac{\sigma^2(\epsilon) + \sigma^2}{2} - r + \hat{\alpha}\right) \frac{u_{i+1}^{n+1} - u_i^{n+1}}{\Delta x} + \frac{\sigma^2(\epsilon) + \sigma^2}{2} \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2}$$
$$-\hat{\lambda} u_i^{n+1}, \tag{4.22}$$
$$(J_\Delta u^n)_i = \sum_{j=K_l}^{K_r} \nu_j u_{i+j}^n 1_{|y_j|>\epsilon}. \tag{4.23}$$

Remember that according to [83], the sum

$$(J_\Delta u^n)_i = \sum_{j=K_l}^{K_r} \nu_j u_{i+j}^n 1_{|y_j|>\epsilon}, i = 0, 1, 2..., N. \tag{4.24}$$

requires a lot of computational effort. More specifically it takes $O(N^2)$ operations, because in fact when we discretize the domain of the truncated integral we use the same step Δx. Let

$$x = (x_1, x_2, x_3, ..., x_n), \tag{4.25}$$

then its discrete Fourier transform is

$$\langle x \rangle_k = \sum_{j=1}^{n} x_j e^{-\frac{2\pi i}{n}(j-1)(k-1)}, \qquad k = 1, 2, 3..., n. \tag{4.26}$$

Also we can define the discrete inverse Fourier Transform

$$x_j = \langle\langle x \rangle\rangle_j^{-1} = \sum_{k=1}^{n} \langle x \rangle_k e^{\frac{2\pi i}{n}(j-1)(k-1)}, \qquad j = 1, 2, 3..., n. \tag{4.27}$$

If $y = (y_1, y_2, y_3, ..., y_n)$ is another vector then the discrete convolution of x and y is given by

$$c_j = \sum_{k=1}^{n} x_k y_{j+1-k}, \qquad j = 1, 2, 3..., n. \tag{4.28}$$

The indices of y are taken modulo n, i.e $y_0 = y_n, y_{-1} = y_{n-1}, ..., y_{2-n} = y_2$. We have the following property which is the discrete analogue of the convolution theorem:

$$\langle c \rangle_k = \langle x \rangle_k \langle y \rangle_k , \qquad k = 1, 2, 3..., n. \tag{4.29}$$

This enable us to have

$$c_j = \langle \langle x \rangle \langle y \rangle \rangle_j^{-1} , \qquad , j = 1, 2, 3..., n, \tag{4.30}$$

We are going to use this property to compute $(J_\Delta u^n)_i$ in a faster way. In order to reduce the number of operations to $O(N \ln(N))$, we can use the Fast Fourier Transform method to c_j. Let us build two vectors μ,v of size $\hat{N} = N + K_r - K_l$

$$\mu = (\nu_{K_r}, ..., \nu_{K_l}, 0, ..., 0) \tag{4.31}$$
$$v = (u_{K_r+1}, ..., u_{K_r+N-1}, u_{K_l}, ..., u_{K_r}) \tag{4.32}$$

Then we can express (4.24) in terms of these two vectors.

4.1.2 Implicit-explicit numerical discretization scheme for the nonlinear PIDE

The aim of this section is to propose a full time-space discretization scheme for solving the nonlinear PIDE (3.81). The method of discretization is based on a finite difference approximation of all derivatives occurring in (3.81) and approximation of the integral term by means of the trapezoidal integration rule on a truncated domain.

In order to solve (3.81) we transform it into a nonlinear parabolic PIDE defined on the whole $\mathbb{R}$. Indeed, using the following standard transformations $V(t, S) = e^{-r\tau} u(\tau, x)$, $\phi(t, S) = \psi(\tau, x)$ where $\tau = T - t, x = \ln(\frac{S}{K})$ we conclude that $V(t, S)$ is a solution to (3.81) if and only if the function $u(\tau, x)$ solves the following nonlinear parabolic equation:

$$\begin{aligned} \frac{\partial u}{\partial \tau} &= \frac{1}{2} \frac{\sigma^2}{(1 - \rho \frac{\partial \psi}{\partial x})^2} \frac{\partial^2 u}{\partial^2 x} + \left(r - \frac{1}{2} \frac{\sigma^2}{(1 - \rho \frac{\partial \psi}{\partial x})^2} \right) \frac{\partial u}{\partial x} \\ &+ \int_{\mathbb{R}} u(\tau, x + \xi(\tau, z, x)) - u(\tau, x) - H(T - \tau, z, Ke^x) \frac{1}{K} e^{-x} \frac{\partial u}{\partial x}(\tau, x) \nu(\,\mathrm{d}z), \end{aligned} \tag{4.33}$$
$$u(0, x) = h(x) \equiv \Phi(Ke^x), \quad (\tau, x) \in [0, T] \times \mathbb{R}, \tag{4.34}$$

and

$$H(t, z, S) = S(e^z - 1) + \rho S[\phi(t, S + H(t, z, S)) - \phi(t, S)], \tag{4.35}$$
$$\xi(\tau, z, x) = \ln(1 + \frac{1}{K} e^{-x} H(T - \tau, z, Ke^x)). \tag{4.36}$$

4.1.3 Numerical scheme for solving nonlinear PIDEs with finite activity Lévy measures

We first consider the case when the Lévy measure ν has finite activity, i.e. $\nu(\mathbb{R}) < \infty$. Let us denote

$$\lambda = \int_{\mathbb{R}} \nu(\,\mathrm{d}z), \quad \text{and} \quad \omega(\tau,x) = \int_{\mathbb{R}} H(T-\tau, z, S_0 e^x)\frac{1}{S_0}e^{-x}\,\nu(\,\mathrm{d}z).$$

We have $\lambda < \infty$. Observe that (4.33) is equivalent to

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\frac{\sigma^2}{(1-\rho\frac{\partial\psi}{\partial x})^2}\frac{\partial^2 u}{\partial^2 x} + \left(r - \frac{1}{2}\frac{\sigma^2}{(1-\rho\frac{\partial\psi}{\partial x})^2} - \omega\right)\frac{\partial u}{\partial x} - \lambda u + \int_{\mathbb{R}} u(\tau, x+\xi(\tau,z,x))\nu(\,\mathrm{d}z). \tag{4.37}$$

We proceed to solve (4.37) by means of the semi-implicit finite difference scheme proposed in [83]. The idea is to separate the right-hand side into two parts: the differential part and the integral part.

Let $u_i^j = u(\tau_j, x_i), \tau_j = j\Delta\tau, x_i = z_i = i\Delta x$ for $i = -N+1, \cdots, N-1$ and $j = 1, \cdots, M$. We approximate the differential part implicitly except of $\psi(\tau,x)$

$$\left(\frac{\partial u}{\partial x}\right)_i^j \approx \begin{cases} \frac{u_{i+1}^{j+1}-u_i^{j+1}}{\Delta x}, & \text{if } \frac{(\sigma_i^j)^2}{2} - r + \omega_i^j < 0, \\ \frac{u_i^{j+1}-u_{i-1}^{j+1}}{\Delta x}, & \text{if } \frac{(\sigma_i^j)^2}{2} - r + \omega_i^j \geq 0, \end{cases} \qquad \sigma_i^j = \frac{\sigma}{1-\rho D\psi_i^j},$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i^j \approx \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{(\Delta x)^2},$$

$$\left(\frac{\partial u}{\partial \tau}\right)_i^j \approx \frac{u_i^{j+1} - u_i^j}{\Delta t}. \qquad \left(\frac{\partial \psi}{\partial x}\right)_i^j \approx \frac{\psi_{i+1}^j - \psi_i^j}{\Delta x} = D\psi_i^j.$$

As for the integral operator, first we have to truncate the integration domain to a bounded interval $[B_l, B_r]$. We approximate this integral by choosing integers K_l and K_r such that $[B_l, B_r] \subset [(K_l - 1/2)\Delta x, (K_l + 1/2)\Delta x]$. Then

$$\int_{B_l}^{B_r} u(\tau_j, x_i + \xi(\tau_j, z_i, x_i))\,\nu(\,\mathrm{d}z) \approx \sum_{k=K_l}^{K_r} u(\tau_j, x_i + \xi(\tau_j, z_k, x_i))\nu_k, \tag{4.38}$$

where $\nu_k = \frac{1}{2}\left(\nu(z_{k+1/2}) + \nu(z_{k-1/2})\right)\Delta x$. Analogously,

$$\omega_i^j \approx \frac{e^{-x_i}}{K}\sum_{k=K_l}^{K_r} H(T-\tau_j, z_k, Ke^{x_i})\nu_k, \quad \text{and} \quad \lambda \approx \sum_{k=K_l}^{K_r}\nu_k,$$

where $\xi(\tau,z,x)$ is given as in (4.36).

Inserting the finite difference approximations of derivatives of u into (4.37) we obtain

$$\begin{aligned} \frac{u_i^{j+1} - u_i^j}{\Delta t} &= \frac{1}{2}(\sigma_i^j)^2\frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{(\Delta x)^2} - \lambda u_i^{j+1} \\ &\quad + (r - \frac{1}{2}(\sigma_i^j)^2 - \omega_i^j)\frac{u_{i+1}^{j+1} - u_i^{j+1}}{\Delta x} + \sum_{k=K_l}^{K_r} u(\tau_j, x_i + \xi(\tau_j, z_k, x_i))\nu_k, \end{aligned} \tag{4.39}$$

provided that $\frac{1}{2}(\sigma_i^j)^2 - r + \omega_i^j < 0$. Similarly, we can derive difference equation for the case $\frac{1}{2}(\sigma_i^j)^2 - r + \omega_i^j \geq 0$. If we define coefficients $\beta_{i\pm}^j$, and β_i^j as follows:

$$\beta_{i\pm}^j = -\frac{\Delta\tau}{2(\Delta x)^2}(\sigma_i^j)^2 - \frac{\Delta\tau}{\Delta x}\left(r - \frac{1}{2}(\sigma_i^j)^2 - \omega_i^j\right)^{\pm}, \tag{4.40}$$

$$\beta_i^j = 1 + \Delta\tau\lambda - (\beta_{i-}^j + \beta_{i+}^j), \tag{4.41}$$

where $(a)^+ = \max(a, 0)$, $(a)^- = \min(a, 0)$, then the tridiagonal system of linear equations for the solution $u^j = (u_{-N+1}^j, \cdots, u_{N-1}^j)^T,\ j = 0, \cdots, M$, reads as follows:

$$\begin{aligned}
u_i^0 &= h(x_i),\ \text{for } i = -N+1, \cdots, N-1,\\
u_i^{j+1} &= g(\tau_{j+1}, x_i),\ \text{for } i = -N+1, \cdots, -N/2-1,\\
\beta_{i+}^j u_{i+1}^{j+1} + \beta_i^j u_i^{j+1} + \beta_{i-}^j u_{i-1}^{j+1} &= u_i^j + \Delta\tau \sum_{k=K_l}^{K_r} u(\tau_j, x_i + \xi(\tau_j, z_k, x_i))\nu_k, \qquad (4.42)\\
&\quad \text{for } i = -N/2+1, \cdots, N/2-1,\\
u_i^{j+1} &= g(\tau_{j+1}, x_i),\ \text{for } i = N/2, \cdots, N-1,
\end{aligned}$$

where

$$\xi(\tau_j, z_k, x_i) = \ln(1 + S_0^{-1} e^{-x_i} H(T - \tau_j, z_k, S_0 e^{x_i})),$$

and g is a function of points x_i lying outside the localization interval. Following Proposition 4.3.1 in [83] , the recommended choice is $g(\tau, x) = h(x + r\tau) = \Phi(S_0 e^{r\tau + x})$. The term $u(\tau_j, x_i + \xi(\tau_j, z_k, x_i))$ entering the sum in the right-hand side of (4.42) is approximated by means of the first order Taylor series expansion:

$$u(\tau_j, x_i + \xi(\tau_j, z_k, x_i)) \approx u_i^j + \frac{u_{i+1}^j - u_i^j}{\Delta x}\xi(\tau_j, z_k, x_i).$$

4.1.4 Numerical scheme for solving nonlinear PIDEs with infinite activity Lévy measures

Next we consider the case when the Lévy measure has infinite activity, e.g. the Variance Gamma process where its Lévy density explodes at zero and $\nu(\mathbb{R}) = \infty$. Equation (4.33) is equivalent to

$$\begin{aligned}
\frac{\partial u}{\partial \tau} &= \frac{1}{2}\frac{\sigma^2}{(1 - \rho\frac{\partial \psi}{\partial x})^2}\frac{\partial^2 u}{\partial^2 x} + \left(r - \frac{1}{2}\frac{\sigma^2}{(1 - \rho\frac{\partial \psi}{\partial x})^2} - \omega\right)\frac{\partial u}{\partial x}\\
&\quad + \int_{\mathbb{R}} u(\tau, x + \xi(\tau, z, x)) - u(\tau, x)\nu(\,dz). \qquad (4.43)
\end{aligned}$$

Equation (4.43) differs from (4.37) as the term $u(\tau, x)$ is contained in the integral part because $\lambda = \int_{\mathbb{R}} \nu(\,dz) = \infty$. Proceeding similarly as for discretization of (4.37) we can solve (4.43) numerically by means of the semi-implicit finite difference scheme. If the coefficients $\beta^j_{i\pm}$ are defined as in (4.40) and $\beta^j_i = 1 - (\beta^j_{i-} + \beta^j_{i+})$, then the solution vector $u^j = (u^j_{-N+1}, \cdots, u^j_{N-1})^T$, $j = 0, \cdots, M$, is a solution to the following tridiagonal system of linear equations:

$$
\begin{aligned}
u^0_i &= h(x_i), \text{ for } i = -N+1, \cdots, N-1, \\
u^{j+1}_i &= g(\tau_{j+1}, x_i), \text{ for } i = -N+1, \cdots, -N/2-1, \qquad (4.44) \\
\beta^j_{i+} u^{j+1}_{i+1} + \beta^j_i u^{j+1}_i + \beta^j_{i-} u^{j+1}_{i-1} &= u^j_i + \Delta\tau \sum_{k=K_l}^{K_r} \left(u(\tau_j, x_i + \xi(\tau_j, z_k, x_i)) - u(\tau_j, x_i)\right) \nu_k, \\
& \quad \text{for } i = -N/2+1, \cdots, N/2-1, \\
u^{j+1}_i &= g(\tau_{j+1}, x_i), \text{ for } i = N/2, \cdots, N-1.
\end{aligned}
$$

The term $u(\tau_j, x_i + \xi(\tau_j, z_k, x_i)) - u(\tau_j, x_i)$ entering the sum in the right-hand side of (4.44) is again approximated by means of the first order Taylor series expansion, i.e.

$$
u(\tau_j, x_i + \xi(\tau_j, z_k, x_i)) - u(\tau_j, x_i) \approx \frac{u^j_{i+1} - u^j_i}{\Delta x} \xi(\tau_j, z_k, x_i).
$$

4.1.5 Numerical results

In this section we present results of numerical experiments using the finite difference scheme described in Section 4.1 for the case of an European put option, i.e. $\Phi(S) = (K - S)^+$. As for the Lévy process, we considered the Variance Gamma process with parameters $\theta = -0.33, \sigma = 0.12, \kappa = 0.16$, and other option pricing model parameters: $r = 0, K = 100, T = 1$. Numerical discretization parameters were chosen as follows: $\Delta x = 0.01, \Delta t = 0.005$. Since the Variance Gamma process has infinite activity, we employ the numerical discretization scheme described in Section 4.1.2. In what follows, we present various option prices computed by means of the finite-difference numerical scheme described in Section 4.1.1 for Black–Scholes ($\rho = 0$) and Frey–Stremme model ($\rho > 0$) and their jump-diffusion PIDE generalizations.

In Fig. 4.1 we show the comparison of European put option prices between the classical PIDE and the linear Black–Scholes model, and comparison between the classical PIDE and the Frey–Stremme PIDE model for the case when the large trader's influence is small, $\rho = 0.001$. In Fig. 4.2 we depict the dependence of the implied volatilities as decreasing functions of the strike price K for the Frey–Stremme model and its PIDE generalizations. We can observe that the implied volatilities for the Frey–Stremme PIDE model are always higher when varying the strike price of the European Put option.

Numerical values of option prices for various models and parameter settings are summarized in Tables 4.1 and 4.2. The numerical results confirm our expectation that assuming risk arising from sudden jumps in the underlying asset process yields a higher option price when comparing to the Frey–Stremme model option price.

In Fig. 4.3 (left) we compare European put option prices $V(0, S)$ computed by means of the Black–Scholes and Frey-Stremme models depending on the parameter ρ measuring influence of a large trader. We can observe that the price of the European put option

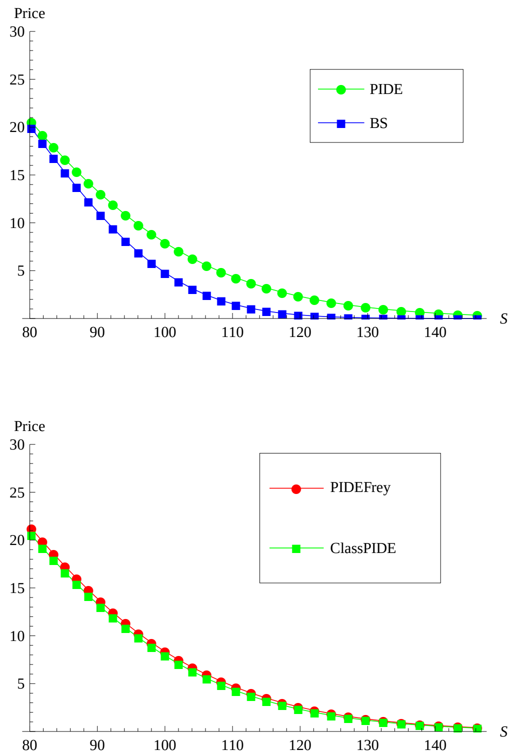

Figure 4.1: Comparison of European put option prices between the classical PIDE and the linear Black–Scholes model (left). Comparison between the classical PIDE and the Frey–Stremme PIDE model (right).

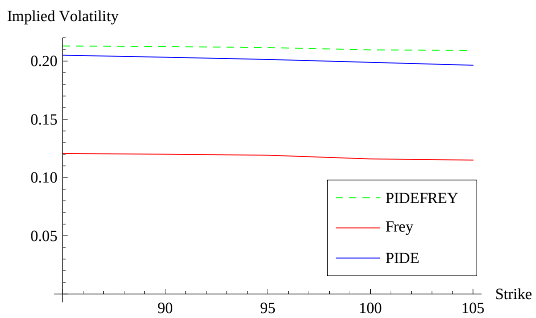

Figure 4.2: Comparison of implied volatilities between the Frey–Stremme model, classical PIDE and Frey–Stremme PIDE generalization.

Table 4.1: European put option prices $V(0,S)$ for the Black-Scholes and Frey–Stremme models with $\rho = 0.2$ and their PIDE generalizations.

S	B-S $\nu = 0, \rho = 0$	F-S $\nu = 0, \rho \neq 0$	B-S PIDE $\nu \neq 0, \rho = 0$	F-S PIDE $\nu \neq 0, \rho \neq 0$
61.8783	38.1217	38.1258	38.2297	38.8234
67.032	32.9691	32.9763	33.4319	34.1889
72.6149	27.3972	27.4207	28.4887	29.4425
78.6628	21.4275	21.5118	23.5224	24.6911
85.2144	15.2547	15.4835	18.6979	20.0701
92.3116	9.42895	9.85754	14.2078	15.7321
100.	4.78444	5.32697	10.243	11.8282
108.329	1.88555	2.34727	6.95353	8.48304
117.351	0.550422	0.814477	4.41257	5.77178
127.125	0.114716	0.216426	2.60009	3.70615
137.713	0.016615	0.043112	1.41444	2.2351

Table 4.2: European put option prices $V(0,S)$ for the Frey–Stremme and Frey-Stremme PIDE models for various values of ρ.

S	F-S $\rho = 0.1$	F-S PIDE $\rho = 0.1$	F-S $\rho = 0.2$	F-S PIDE $\rho = 0.2$	F-S $\rho = 0.3$	F-S PIDE $\rho = 0.3$
61.8783	38.1257	38.4958	38.1258	38.8234	38.1373	39.2259
67.032	32.9759	33.7763	32.9763	34.1889	33.019	34.6865
72.6149	27.4191	28.9293	27.4207	29.4425	27.5623	30.049
78.6628	21.5061	24.0698	21.5118	24.6911	21.8893	25.4118
85.2144	15.4688	19.3477	15.4835	20.0701	16.2645	20.896
92.3116	9.83127	14.9344	9.85754	15.7321	11.0916	16.6367
100.	5.29421	10.9999	5.32697	11.8282	6.8043	12.7672
108.329	2.31882	7.68096	2.34727	8.48304	3.68338	9.4005
117.351	0.797286	5.05246	0.814477	5.77178	1.72932	6.61053
127.125	0.209195	3.11214	0.216426	3.70615	0.693804	4.41995
137.713	0.040995	1.78547	0.043112	2.2351	0.234949	2.79821

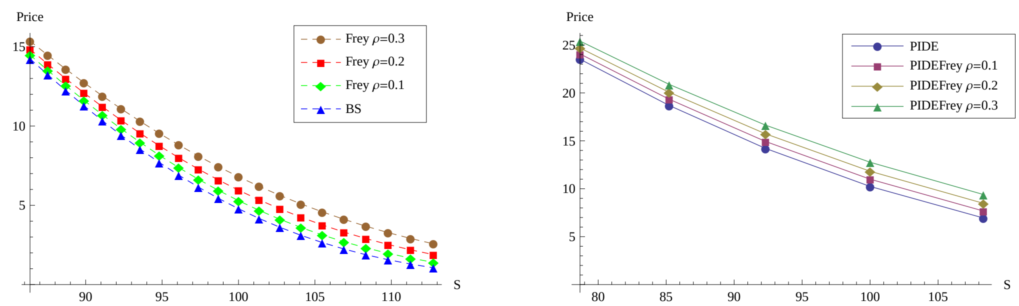

Figure 4.3: Comparison of European put option prices for the Black–Scholes and the Frey–Stremme models (left) and Frey-Stremme PIDE model for various ρ.

increases with respect to ρ, as expected. Furthermore, the price computed from the Frey–Stremme PIDE model is larger than the one obtained from the linear Black–Scholes equation. Moreover, the price computed from Frey–Stremme PIDE model is higher than the one computed by means of the nonlinear Frey–Stremme model. This is due to the fact that the jump part of the underlying asset process enhances risk, and, consequently increases the option value. Fig. 4.3 (right) shows comparison of the option prices for the Black–Scholes and Frey–Stremme PIDE model for various values of ρ.

4.2 Galerkin Methods

4.2.1 Classical PIDE

Galerkin methods consist of representing the solution u using a basis of functions (see for example [24]):

$$u(\tau, x) = \sum_{i \geq 1} a_i(\tau) e_i(x). \tag{4.45}$$

Then we approximate the solution by restricting to a finite number of functions:

$$u_N(\tau, x) = \sum_{i \geq 1}^{N} a_i(\tau) e_i(x). \tag{4.46}$$

If the basis functions $e_i(x)$ have derivatives known in closed form then this representation has the advantage of being able to estimate the Greeks of the options and to compute values of the solution in points that are not necessarily on the uniform grid .

4.2.1.1 Radial Basis Function Interpolation Scheme

Originally proposed by Kansa (1990) in order to approximate partial derivatives using Radial Basis Functions, this numerical scheme became more popular with Fausshauer et al. (2004a,b), Larsson et al. (2008), Pettersson et al. (2008) and Hon and Mao (1999) which have used this meshless technique to solve Black-Scholes equations to price European and American options.

In order to begin the numerical scheme we must first obtain an approximation of the payoff function using the RBF interpolant. So the idea is to choose the interpolant points $x_j, j = 1, 2, ...N$ and approximate the solution $u(\tau, x)$ using the RBF interpolant for any fixed τ:

$$u(\tau, x) \approx \sum_{j=1}^{N} p_j(\tau) \phi(||x - x_j||_2). \tag{4.47}$$

Then it follows that

$$\frac{\partial u(\tau,x)}{\partial \tau} \approx \sum_{j=1}^{N} \frac{\partial p_j(\tau)}{\partial \tau}\phi(|x-x_j|), \tag{4.48}$$

$$\frac{\partial u(\tau,x)}{\partial x} \approx \sum_{j=1}^{N} p_j(\tau)\frac{\partial \phi(|x-x_j|)}{\partial x}, \tag{4.49}$$

$$\frac{\partial^2 u(\tau,x)}{\partial x^2} \approx \sum_{j=1}^{N} p_j(\tau)\frac{\partial^2 \phi(|x-x_j|}{\partial x^2}. \tag{4.50}$$

As in [22] we are going to use the cubic spline, i.e $\phi(|x-x_j|) = |x-x_j|^3$. Then

$$\frac{\partial \phi(|x-x_j|)}{\partial x} = \begin{cases} 3|x-x_j|^2 & \text{if} \quad x-x_j > 0 \\ -3|x-x_j|^2 & \text{if} \quad x-x_j < 0, \end{cases} \tag{4.51}$$

$$\frac{\partial^2 \phi(|x-x_j|)}{\partial x^2} = 6|x-x_j|. \tag{4.52}$$

So we start by choosing equally spaced points

$$x_j = -A + j\Delta x, j = 0,1,2...N-1, \tag{4.53}$$

where $\Delta x = \frac{x_{Max}-x_{Min}}{N}$. Also we construct the following matrices Φ, Φ_x, Φ_{xx}, which are the matrices of the partial derivatives of the cubic spline and finally a matrix valued function $\Phi(y)$ which has generic element $(\phi(|x_i+y-x_j|))_{i,j=0,1,..,N}$.

In order to solve the usual PIDE we make the following transformations $V^{PIDE}(t,S_t) = e^{-r\tau}u(\tau,x)$, where $\tau = T-t, x = \ln(\frac{S_t}{S_0})$ and get

$$\frac{\partial u}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial^2 x} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial x} + \int_{\mathbb{R}} u(\tau,x+z) - u(\tau,x) - (e^z-1)\frac{\partial u}{\partial x}\nu(\,\mathrm{d}z). \tag{4.54}$$

Defining

$$\lambda = \int_{\mathbb{R}} \nu(\,\mathrm{d}z), \quad \alpha = \int_{\mathbb{R}} (e^z-1)\,\nu(\,\mathrm{d}z), \tag{4.55}$$

we arrive at the following equation

$$\frac{\partial u}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial^2 x} + \left(r - \frac{\sigma^2}{2} - \alpha\right)\frac{\partial u}{\partial x} - \lambda u(\tau,x) + \int_{\mathbb{R}} u(\tau,x+z)\nu(\,\mathrm{d}z). \tag{4.56}$$

Since $u(\tau,x_1) = K$ for $x_1 = -A$ and $u(\tau,x_{N+1}) = 0$ for $x_{N+1} = A$, we must have

$$\sum_{j=1}^{N+1} p_j(\tau)|x_1-x_j|^3 = K, \tag{4.57}$$

$$\sum_{j=1}^{N+1} p_j(\tau)|x_{N+1}-x_j|^3 = 0. \tag{4.58}$$

Then if we substitute $\tilde{u}(\tau, x)$ by $u(\tau, x)$ in (4.56) and demand that the PIDE be satisfied in each interpolation point x_j, we arrive at the following system of equations for the vector $p(\tau) = (p_1(\tau), p_2(\tau), .., p_N(\tau), p_{N+1}(\tau))$

$$\left(\Phi\frac{\partial p(\tau)}{\partial\tau}\right)_1 = K, \tag{4.59}$$

$$\left(\Phi\frac{\partial p(\tau)}{\partial\tau}\right)_i = \frac{\sigma^2}{2}\left(\Phi_{xx}p(\tau)\right)_i + \left(r - \frac{\sigma^2}{2} - \alpha\right)\left(\Phi_x p(\tau)\right)_i - \lambda\left(\Phi p(\tau)\right)_i$$
$$+ \int_{\mathbb{R}} \left(\Phi(z)p(\tau)\right)_i \nu(\,\mathrm{d}z), i = 2, 3, ...N, \tag{4.60}$$

$$\left(\Phi\frac{\partial p(\tau)}{\partial\tau}\right)_{N+1} = 0, \tag{4.61}$$

$$\Phi\frac{\partial p(\tau)}{\partial\tau} = \left(\frac{\sigma^2}{2}\Phi_{xx} + \left(r - \frac{\sigma^2}{2} - \alpha\right)\Phi_x - \lambda\Phi + \int_{\mathbb{R}}\Phi(z)\nu(\,\mathrm{d}z)\right)p(\tau),$$
$$\frac{\partial p(\tau)}{\partial\tau} = D\left(\frac{\sigma^2}{2}\Phi u_{xx} + \left(r - \frac{\sigma^2}{2} - \alpha\right)\Phi_x - \lambda\Phi + \int_{\mathbb{R}}\Phi(z)\nu(\,\mathrm{d}z)\right)p(\tau), \tag{4.62}$$

where $D = \Phi^{-1} = P^{-1}U^{-1}P^{-1}$ and P is an $N \times N$ matrix as it was shown in ([18])

$$P = \begin{bmatrix} |x_1 - x_1| & |x_1 - x_2| & |x_1 - x_3| & \dots & |x_1 - x_N| \\ |x_2 - x_1| & |x_2 - x_2| & |x_1 - x_3| & \dots & |x_2 - x_N| \\ \vdots & & & \ddots & \vdots \\ |x_N - x_1| & |x_N - x_2| & |x_N - x_3| & \dots & |x_N - x_N|, \end{bmatrix} \tag{4.63}$$

and U is a near tri-diagonal matrix $N \times N$ matrix.

$$U = \begin{bmatrix} \Delta x - (N-1)\Delta x & \frac{\Delta x}{2} & 0 & \dots & 0 & \frac{(N-1)\Delta x}{2} \\ \frac{\Delta x}{2} & 2\Delta x & \frac{\Delta x}{2} & 0 & \dots & 0 \\ 0 & \frac{\Delta x}{2} & 2\Delta x & \frac{\Delta x}{2} & \dots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & \dots & \frac{\Delta x}{2} & 2\Delta x & \frac{\Delta x}{2} \\ \frac{(N-1)\Delta x}{2} & 0 & \dots & 0 & \frac{\Delta x}{2} & \Delta x - (N-1)\Delta x \end{bmatrix} \tag{4.64}$$

and P^{-1} is also known in explicit form

$$P^{-1} = \begin{bmatrix} \frac{\Delta x - (N-1)\Delta x}{2(N-1)(\Delta x)^2} & \frac{1}{2\Delta x} & 0 & \dots & 0 & \frac{1}{2(N-1)\Delta x} \\ \frac{1}{2\Delta x} & -\frac{1}{\Delta x} & \frac{1}{2\Delta x} & 0 & \dots & 0 \\ 0 & \frac{1}{2\Delta x} & -\frac{1}{\Delta x} & \frac{1}{2\Delta x} & \dots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & \dots & \frac{1}{2\Delta x} & -\frac{1}{\Delta x} & \frac{1}{2\Delta x} \\ \frac{1}{2(N-1)\Delta x} & 0 & \dots & 0 & \frac{1}{2\Delta x} & \frac{\Delta x - (N-1)\Delta x}{2(N-1)(\Delta x)^2} \end{bmatrix}. \tag{4.65}$$

Also

$$\int_{\mathbb{R}} \Phi(z)\nu(\,\mathrm{d}z) \approx \int_{B_l}^{B_r} \Phi(z)\nu(\,\mathrm{d}z) \approx \begin{bmatrix} G(x_1-x_1) & G(x_1-x_2) & \dots & G(x_1-x_N) \\ G(x_2-x_1) & G(x_2-x_2) & \dots & G(x_2-x_N) \\ G(x_3-x_1) & G(x_3-x_2) & \dots & G(x_3-x_N) \\ \vdots & & \ddots & \\ G(x_N-x_1) & G(x_N-x_2) & \dots & G(x_N-x_N) \end{bmatrix},$$

where

$$\int_{B_l}^{B_r} u\left(\tau, x_i - x_j + z\right) \nu\left(\,\mathrm{d}z\right) 1_{|z|>\epsilon} \approx \sum_{k=K_l}^{K_r} \nu_k u_{i-j+k} 1_{|y_k|>\epsilon} = G(\Delta x(i-j)), \quad (4.66)$$

and as usual

$$\nu_j \approx \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \nu\left(\,\mathrm{d}y\right) \approx 0.5\Delta x(\nu((j-1/2)\Delta x) + \nu((j+1/2)\Delta x)). \quad (4.67)$$

Then, discretising also on time we obtain the implicit scheme

$$\sum_{j=1}^{N+1} p_j(\tau)|x_1 - x_j|^3 = K, \quad (4.68)$$

$$\frac{p(\tau_{j+1}) - p(\tau_j)}{\Delta \tau} = D\left(\frac{\sigma^2}{2}\Phi u_{xx} + \left(r - \frac{\sigma^2}{2} - \alpha\right)\Phi_x - \lambda\Phi + \int_{\mathbb{R}} \Phi(z)\nu(\,\mathrm{d}z)\right) p(\tau_{j+1}),$$

$$j = 1, ...M, \quad (4.69)$$

$$\sum_{j=1}^{N+1} p_j(\tau)|x_{N+1} - x_j|^3 = 0. \quad (4.70)$$

Or in matrix notation

$$\Phi_1 p(\tau_{j+1}) = K, \quad (4.71)$$

$$(I - \Delta\tau\Theta)p(\tau_{j+1}) = p(\tau_j), j = 1, ...M, \quad (4.72)$$

$$\Phi_{N+1} p(\tau_{j+1}) = 0. \quad (4.73)$$

where

$$\Theta = D\left(\frac{\sigma^2}{2}\Phi u_{xx} + \left(r - \frac{\sigma^2}{2} - \alpha\right)\Phi_x - \lambda\Phi + \int_{\mathbb{R}} \Phi(z)\nu(\,\mathrm{d}z)\right). \quad (4.74)$$

Or in a more compact way

$$Bp(\tau_{j+1}) = p(\tau_j) + b, j = 1, ...M, \quad (4.75)$$

where

$$B = \begin{pmatrix} \dots & \Phi_1 & \dots \\ \dots & (I - \Delta\tau\Theta)_{N-1,N+1} & \dots \\ \dots & \Phi_{N+1} & \dots \end{pmatrix} \quad (4.76)$$

and

$$b = \begin{pmatrix} K \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4.77}$$

4.2.1.2 Different choice of the Interpolant

Since the matrix A is ill conditioned and the former interpolant is conditionally positive definite of order 2, one alternative is to consider the following radial basis functions interpolants

$$u(\tau, x) \approx \sum_{j=1}^{N} p_j(\tau)|x - x_j|^3 + \sum_{k=1}^{2} \gamma_k(\tau) g_k(x), \tag{4.78}$$

where $g_1(x) = 1, g_2(x) = x$.

Then as before we demand that at every point (τ_i, x_i) the equation (4.56) is satisfied and if we impose the additional set of constraints $\sum_{j=1}^{N} \dot{p}_j(\tau) g_k(x_j) = 0, k = 1, 2$ we arrive at an augmented system of equations.

If we define $p^*(\tau) = (p_1(\tau), p_2(\tau), .., p_N(\tau), \gamma_1(\tau), \gamma_2(\tau))$.

$$\begin{aligned}
&\frac{\partial}{\partial \tau}\left(\sum_{j=1}^{N} p_j(\tau)|x - x_j|^3 + \sum_{k=1}^{2} \gamma_k(\tau) g_k(x)\right) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\left(\sum_{j=1}^{N} p_j(\tau)|x - x_j|^3 + \sum_{k=1}^{2} \gamma_k(\tau) g_k(x)\right) \\
&+\left(r - \frac{\sigma^2}{2} - \alpha\right)\frac{\partial}{\partial x}\left(\sum_{j=1}^{N} p_j(\tau)|x - x_j|^3 + \sum_{k=1}^{2} \gamma_k(\tau) g_k(x)\right) \\
&-\lambda\left(\sum_{j=1}^{N} p_j(\tau)|x - x_j|^3 + \sum_{k=1}^{2} \gamma_k(\tau) g_k(x)\right) \\
&+\int_{\mathbb{R}} \sum_{j=1}^{N} p_j(\tau)|x + z - x_j|^3 + \sum_{k=1}^{2} \gamma_k(\tau) g_k(x + z)\nu(\,\mathrm{d}z), x = x_i, i = 1, 2, .., N,
\end{aligned} \tag{4.79}$$

$$\sum_{j=1}^{N} \dot{p}_j(\tau) g_k(x_j) = 0, k = 1, 2. \tag{4.80}$$

Then

$$\begin{pmatrix} \Phi_{N\times N} & g_1 & g_2 \\ g_1^T & 0 & 0 \\ g_2^T & 0 & 0 \end{pmatrix} \dot{p}^* = \begin{pmatrix} \frac{\sigma^2}{2}\Phi^*_{xx} + \beta\Phi^*_x - \lambda\Phi^* + \int_{\mathbb{R}} \Phi^*(z)\nu(\,\mathrm{d}z) \\ \mathcal{O}_{2\times N+2} \end{pmatrix} p^*, \tag{4.81}$$

where $\beta = r - \frac{\sigma^2}{2} - \alpha, \Phi^* = (\Phi, g(x)), \Phi^*_x = (\Phi_x; \mathcal{O}_{N\times 2}), \Phi^*_{xx} = (\Phi_x; \mathcal{O}_{N\times 1}; 1_{N\times 1}), g(x) = (g_1(x), g_2(x))$ and g_1, g_2 are N-dimensional column vectors.

4.2.2 Application to Illiquid Options Market

As in [22] we are going to use the cubic spline. Then

$$\frac{\partial \phi_i(|x-x_j|)}{\partial x} = \begin{cases} 3|x-x_j|^2 & \text{if} \quad x-x_j > 0 \\ -3|x-x_j|^2 & \text{if} \quad x-x_j < 0, \end{cases} \tag{4.82}$$

$$\frac{\partial^2 \phi_i(|x-x_j|)}{\partial x^2} = 6|x-x_j|. \tag{4.83}$$

So we start by choosing equally spaced points

$$x_j = -A + j\Delta x, j = 0, 1, 2...N-1, \tag{4.84}$$

where $\Delta x = \frac{x_{Max} - x_{Min}}{N}$.

Also we construct the following matrices Φ, Φ_x, Φ_{xx}, which are the matrices of the partial derivatives of the cubic spline and finally a matrix valued function $\Phi(y)$ with generic element $(\phi_i(|x_i + y - x_j|))_{i,j=1,2,..,N}$.

The equation we need to solve is

$$\begin{aligned} \frac{\partial u}{\partial \tau} &= \frac{1}{2}\frac{\partial^2 u}{\partial^2 x}\frac{\sigma^2}{1-\rho\frac{\partial \psi(\tau,x)}{\partial x}} + \left(r - \frac{1}{2}\frac{\sigma^2}{1-\rho\frac{\partial \psi(\tau,x)}{\partial x}}\right)\frac{\partial u}{\partial x} \\ &+ \int_{\mathbb{R}} u(\tau, x+\xi(z,x)) - u(\tau,x) - H(t,z,S_0e^x(e^z-1))\frac{1}{S_0}e^{-x}\frac{\partial u}{\partial x}\nu(\,\mathrm{d}z). \end{aligned} \tag{4.85}$$

Defining

$$\alpha = \int_{B_l}^{B_r} H(t,z,S_0e^x)\frac{1}{S_0}e^{-x}\nu(\,\mathrm{d}z)1_{|z|>\epsilon}, \lambda = \int_{B_l}^{B_r}\nu(\,\mathrm{d}z)1_{|z|>\epsilon}, \sigma^2(\epsilon) = \int_{\mathbb{R}} y^2\nu(\,\mathrm{d}y),$$

we arrive at the alternative format

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\frac{\partial^2 u}{\partial^2 x}\frac{\sigma^2+\sigma^2(\epsilon)}{1-\rho\frac{\partial \psi(\tau,x)}{\partial x}} + \left(r - \frac{1}{2}\frac{\sigma^2+\sigma^2(\epsilon)}{1-\rho\frac{\partial \psi(\tau,x)}{\partial x}} - \alpha\right)\frac{\partial u}{\partial x} \tag{4.86}$$

$$-\lambda u(\tau,x) + \int_{\mathbb{R}} u(\tau, x+\xi(z,x))\nu(\,\mathrm{d}z). \tag{4.87}$$

$$\Sigma = \begin{pmatrix} \frac{1}{2}\frac{\sigma^2+\sigma^2(\epsilon)}{1-\rho\frac{\partial\psi(\tau_1,x_1)}{\partial x}} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2}\frac{\sigma^2+\sigma^2(\epsilon)}{1-\rho\frac{\partial\psi(\tau_2,x_2)}{\partial x}} & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{2}\frac{\sigma^2+\sigma^2(\epsilon)}{1-\rho\frac{\partial\psi(\tau_N,x_N)}{\partial x}} \end{pmatrix}, \tag{4.88}$$

where

$$\frac{\partial \psi(\tau_i, x_i)}{\partial x} \approx \frac{\psi(\tau_i, x_{i-1} + \Delta x) - \psi(\tau_i, x_{i-1})}{\Delta x}. \tag{4.89}$$

Also

$$\int_{B_l}^{B_r} H(t, z, S_0 e^x) \frac{1}{S_0} e^{-x} \nu(\,\mathrm{d}z) 1_{|z|>\epsilon} \approx \sum_{d=1}^{N} H(\tau_j, z_d, S_0 e^{x_i}) \nu(z) 1_{|z_d|>\epsilon} = T_{j,i}, \tag{4.90}$$

with

$$T = \begin{pmatrix} T_{1,1} & T_{1,2} & T_{1,3} & \dots & T_{1,N} \\ T_{2,1} & T_{2,2} & T_{2,3} & \dots & T_{2,N} \\ \vdots & & \ddots & & \vdots \\ T_{N,1} & T_{N,2} & T_{N,3} & \dots & T_{N,N} \end{pmatrix}. \tag{4.91}$$

If we substitute $\tilde{u}(\tau, x)$ by $u(\tau, x)$ in (4.56) and demand that the PIDE be satisfied in each interpolation point x_j, then we arrive at the following system of equations for the vector $p(\tau) = (p_1(\tau), p_2(\tau), .., p_N(\tau))$

$$\begin{aligned} \Phi \frac{\partial p(\tau)}{\partial \tau} &= \left(\Sigma \Phi_{xx} + (rI - \Sigma - T)\, \Phi_x - \lambda \Phi + \int_{\mathbb{R}} \Phi(z) \nu(\,\mathrm{d}z) \right) p(\tau), \\ \frac{\partial p(\tau)}{\partial \tau} &= D \left(\Sigma \Phi_{xx} + (rI - \Sigma - T)\, \Phi_x - \lambda \Phi + \int_{\mathbb{R}} \Phi(z) \nu(\,\mathrm{d}z) \right) p(\tau). \end{aligned} \tag{4.92}$$

Since we need to find the inverse of Φ which may be ill conditioned, we factorize this matrix as $\Phi = PUP$. This way we define $D = \Phi^{-1} = P^{-1} U^{-1} P^{-1}$, where P is an $N \times N$ matrix

$$P = \begin{bmatrix} |x_1 - x_1| & |x_1 - x_2| & |x_1 - x_3| & \dots & |x_1 - x_N| \\ |x_2 - x_1| & |x_2 - x_2| & |x_1 - x_3| & \dots & |x_2 - x_N| \\ \vdots & & \ddots & & \vdots \\ |x_N - x_1| & |x_N - x_2| & |x_N - x_3| & \dots & |x_N - x_N| \end{bmatrix} \tag{4.93}$$

and U is a near tri-diagonal matrix $N \times N$ matrix

$$U = \begin{bmatrix} \Delta x - (N-1)\Delta x & \frac{\Delta x}{2} & 0 & \dots & 0 & \frac{(N-1)\Delta x}{2} \\ \frac{\Delta x}{2} & 2\Delta x & \frac{\Delta x}{2} & 0 & \dots & 0 \\ 0 & \frac{\Delta x}{2} & 2\Delta x & \frac{\Delta x}{2} & \dots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & \dots & \frac{\Delta x}{2} & 2\Delta x & \frac{\Delta x}{2} \\ \frac{(N-1)\Delta x}{2} & 0 & \dots & 0 & \frac{\Delta x}{2} & \Delta x - (N-1)\Delta x \end{bmatrix} \tag{4.94}$$

and P^{-1} is also known in explicit form

$$P^{-1} = \begin{bmatrix} \frac{\Delta x-(N-1)\Delta x}{2(N-1)(\Delta x)^2} & \frac{1}{2\Delta x} & 0 & \dots & 0 & \frac{1}{2(N-1)\Delta x} \\ \frac{1}{2\Delta x} & -\frac{1}{\Delta x} & \frac{1}{2\Delta x} & 0 & \dots & 0 \\ 0 & \frac{1}{2\Delta x} & -\frac{1}{\Delta x} & \frac{1}{2\Delta x} & \dots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & \dots & \frac{1}{2\Delta x} & -\frac{1}{\Delta x} & \frac{1}{2\Delta x} \\ \frac{1}{2(N-1)\Delta x} & 0 & \dots & 0 & \frac{1}{2\Delta x} & \frac{\Delta x-(N-1)\Delta x}{2(N-1)(\Delta x)^2} \end{bmatrix}. \tag{4.95}$$

Also

$$\int_{\mathbb{R}} \Phi(z)\nu(\,\mathrm{d}z) \approx \int_{B_l}^{B_r} \Phi(z)\nu(\,\mathrm{d}z) \approx \begin{bmatrix} G(x_1-x_1) & G(x_1-x_2) & \dots & G(x_1-x_N) \\ G(x_2-x_1) & G(x_2-x_2) & \dots & G(x_2-x_N) \\ G(x_3-x_1) & G(x_3-x_2) & \dots & G(x_3-x_N) \\ \vdots & & \ddots & \\ G(x_N-x_1) & G(x_N-x_2) & \dots & G(x_N-x_N) \end{bmatrix},$$

where

$$\begin{aligned} \int_{B_l}^{B_r} u\left(\tau, x_i - x_j + \xi(z,x)\right)\nu\left(\,\mathrm{d}z\right) 1_{|z|>\epsilon} &\approx \sum_{k=1}^{N} \nu_k u(\Delta\tau, \Delta x(i-j) + b(k\Delta x, i\Delta x)) 1_{|z_k|>\epsilon} \\ &= G(\Delta x(i-j)), \end{aligned} \tag{4.96}$$

where as usual

$$\nu_j \approx \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \nu\left(\,\mathrm{d}y\right) \approx 0.5\Delta x(\nu((j-1/2)\Delta x) + \nu((j+1/2)\Delta x)). \tag{4.97}$$

4.3 Study of the nonlinear PIDE when ρ is small

Now we assume ρ is small enough such that equation (3.81) can be thought of as a small perturbation of the classical PIDE (3.84). We compute the first order correction to the solution of the classical PIDE for a European option under the effects of feedback when ρ is small. When $\rho = 0$ we obtain $V^{PIDE}(t, S_t)$, which we denote as the solution of the classical PIDE. Then, constructing a regular perturbation series, next proposition gives us an approximation of $H(t, y, S_{t^-})$ when the large trader is a small fraction of the economy.

Proposition 4.3.1 *Assume that ρ is small. Then*

$$H(t, y, S_{t^-}) \approx y + \rho S_{t^-}\left(\phi(t, S_{t^-} + y) - \phi(t, S_{t^-})\right) \tag{4.98}$$

Proof. If we write $H(t, y, S_{t^-})$ as a a function of ρ and making a first order Taylor expansion, we get

$$g(\rho) \quad = g(0) + \rho g'(0) + \mathcal{O}(\rho^2), \tag{4.99}$$

where

$$g(\rho) = y + \rho S_{t^-} \left(\phi(t, S + g(\rho)) - \phi(t, S)\right). \tag{4.100}$$

Differentiating with respect to ρ yields

$$g'(\rho) = S_{t^-} \left(\phi(t, S + g(\rho)) - \phi(t, S)\right) + \rho S_{t^-} \frac{\partial \phi}{\partial S}(t, S + g(\rho)) g'(\rho). \tag{4.101}$$

Setting $\rho = 0$, we obtain

$$g'(0) = S_{t^-} \left(\phi(t, S + g(0)) - \phi(t, S)\right) + \rho S_{t^-} \frac{\partial \phi}{\partial S}(t, S + g(0)) g'(0). \tag{4.102}$$

Since $g(0) = y$ and since $1 - \rho S \frac{\partial \phi}{\partial S} > 0$ we get

$$g'(0) = S_{t^-} \left(\phi(t, S + y) - \phi(t, S)\right), \tag{4.103}$$

which after inserting into Taylor's formula entails the claim. ∎

Then, constructing a regular perturbation series

$$V(t, S_t) = V_0(t, S_t) + \rho V_1(t, S_t) + \mathcal{O}(\rho^2) \tag{4.104}$$

and defining

$$L^{PIDE} = \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S_{t^-}^2 \frac{\partial^2 V}{\partial S^2} + r S_{t^-} \frac{\partial V}{\partial S} - rV \tag{4.105}$$

$$+ \int_{\mathbb{R}} V(t, S_{t^-} + y) - V(t^-, S_{t^-}) - y \frac{\partial V}{\partial S} \nu(\,\mathrm{d}y), \tag{4.106}$$

we expand (3.81) for small ρ and obtain

$$\mathcal{O}(\rho^2) = L^{PIDE} V(t, S_t) + \sigma^2 S_{t^-}^3 \frac{\partial \phi}{\partial S}(t, S_{t^-}) \rho \frac{\partial^2 V}{\partial S^2}(t, S_t)$$
$$+ \int_{\mathbb{R}} \rho S_{t^-} \left(\frac{\partial V}{\partial S}(t, S_{t^-} + y) - \frac{\partial V}{\partial S}(t, S_{t^-}) \right) \left(\phi(t, S_{t^-} + y) - \phi(t, S_{t^-})\right) \nu, \tag{4.107}$$

since for

$$j(\rho) = v(t, S_{t^-})^2 = \frac{\sigma^2}{\left(1 - \rho S_{t^-} \frac{\partial \phi}{\partial S}(t, S_{t^-})\right)^2}$$

we have

$$j(\rho) = \sigma^2 + 2\sigma^2 \left(S_{t^-} \frac{\partial \phi}{\partial S}(t, S_{t^-}) \right) \rho + \mathcal{O}(\rho^2).$$

Consequently, using (4.104) and since $L^{PIDE} V_0 = 0$, $V_1(t, S_t)$ satisfies

$$L^{PIDE} V_1(t, S_t) = L^{PIDE} \frac{1}{\rho} \left(V(t, S_t) - V_0(t, S_t) - \mathcal{O}(\rho^2) \right).$$

Then,using (4.107) and simplifying we get

$$L^{PIDE}V_1(t,S_t) = -\sigma^2 S_{t^-}^3 \frac{\partial \phi}{\partial S}(t,S_{t^-})\frac{\partial^2 V}{\partial S^2}(t,S_t) - I,$$

where

$$I = \int_{\mathbb{R}} S_{t^-}\left(\frac{\partial V}{\partial S}(t,S_{t^-}+y) - \frac{\partial V}{\partial S}(t,S_{t^-})\right)(\phi(t,S_{t^-}+y)-\phi(t,S_{t^-}))\,\nu(\,\mathrm{d}y).$$

Notice that we can write this as

$$I = I_0 + I_1, \tag{4.108}$$

where

$$I_0 = \int_{\mathbb{R}} S_{t^-}\left(\frac{\partial V_0}{\partial S}(t,S_{t^-}+y) - \frac{\partial V_0}{\partial S}(t,S_{t^-})\right)(\phi(t,S_{t^-}+y)-\phi(t,S_{t^-}))\,\nu, \tag{4.109}$$

$$I_1 = \int_{\mathbb{R}} \rho S_{t^-}\left(\frac{\partial V_1}{\partial S}(t,S_t+y) - \frac{\partial V_1}{\partial S}(t,S_t)\right)(\phi(t,S_{t^-}+y)-\phi(t,S_{t^-}))\,\nu. \tag{4.110}$$

First we need to solve $L^{PIDE}V_0(t,S_t)=0$ i.e

$$\begin{aligned}&\frac{\partial V_0}{\partial t} + \frac{\sigma^2}{2}S_{t^-}^2\frac{\partial^2 V_0}{\partial S^2} + S_{t^-}r\frac{\partial V_0}{\partial S} - rV_0\\ &+\int_{\mathbb{R}} V_0(t,S_{t^-}+y) - V_0(t^-,S_{t^-}) - y\frac{\partial V_0}{\partial S}\nu(\,\mathrm{d}y) = 0.\end{aligned}$$

Then solve the following equation

$$L^{PIDE}V_1(t,S_t) = -\sigma^2 S_{t^-}^3\frac{\partial \phi}{\partial S}(t,S_{t^-})\frac{\partial^2 V_0}{\partial S^2}(t,S_t) - I_0 \tag{4.111}$$

$$-\rho\sigma^2 S_{t^-}^3\frac{\partial \phi}{\partial S}(t,S_{t^-})\frac{\partial^2 V_1}{\partial S^2}(t,S_t) - I_1 \tag{4.112}$$

Finally we obtain the solution when ρ is considered small

$$V(t,S_t) = V_0 + \rho V_1(t,S_t).$$

Now we want to solve (4.112) where I_0 and I_1 are given by (4.109) and (4.110) respectively.

So making the usual set of transformations i.e $\tau = T-t$, $S=\ln(x)$, $u_j(\tau,x) = e^{r\tau}V_j(t,S)$ for $j=0,1$ and $\phi_(t,S) = \psi_(\tau,x)$, we get the equivalent problem

$$\begin{aligned}&\frac{\partial u_1}{\partial \tau} - Lu_1(\tau,x) = f(u_0,u_1)(x),\\ &u(0,x) = \Phi(S_0e^x), x\in\mathbb{R},\end{aligned} \tag{4.113}$$

where

$$Lu(\tau,x) = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}(\tau,x) + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial x}(\tau,x)$$

$$+\int_{\mathbb{R}} u(x+z) - u(x) - (e^z-1)\frac{\partial u}{\partial x}(x)\nu(\mathrm{d}z), \tag{4.114}$$

$$f(u_0,u_1)(x) = -\sigma^2\frac{\partial \psi}{\partial x}(\tau,x)\left(\frac{\partial^2 u_0}{\partial x^2}(x) - \frac{\partial u_0}{\partial x}(x)\right)$$

$$-\rho\sigma^2\frac{\partial \psi}{\partial x}(\tau,x)\left(\frac{\partial^2 u_1}{\partial x^2}(x) - \frac{\partial u_1}{\partial x}(x)\right) - I_0 - I_1, \tag{4.115}$$

and

$$I_0 = \int_{\mathbb{R}}\left(\frac{\partial u_0}{\partial x}(\tau,x+z) - \frac{\partial u_0}{\partial x}(\tau,x)\right)(\psi(\tau,x+z)-\psi(\tau,x))\,\nu(\mathrm{d}z), \tag{4.116}$$

$$I_1 = \int_{\mathbb{R}}\rho\left(\frac{\partial u_1}{\partial x}(\tau,x+z) - \frac{\partial u_1}{\partial x}(\tau,x)\right)(\psi(\tau,x+z)-\psi(\tau,x))\,\nu(\mathrm{d}z). \tag{4.117}$$

4.4 Convergence Results

This section is dedicated to analyze the convergence of the numerical scheme based on the proposed finite difference method. First, we take into account the fact that we can have an infinite activity process. Then, as usual, we prove consistency, stability and monotony of the scheme. At last we obtain a convergence result for the linear PIDE.

4.4.1 Approximation by a finite activity process

Now consider the case when we have an infinite activity process such as the Variance Gamma process described above. We can see that its Lévy density explodes at zero. For that reason one replaces the small jumps of the process by a Brownian motion chosen so that one gets a finite activity process. The Lévy process obtained has a new characteristic triplet given by $(\gamma(\epsilon), \sqrt{\sigma^2(\epsilon)+\sigma^2}, \nu 1_{|x|>\epsilon})$, where $\sigma^2(\epsilon) = \int_{-\epsilon}^{\epsilon} y^2\nu(\mathrm{d}y)$ and the drift is given by the associated martingale condition.

We can write the dynamics of stock's price logarithm in the following way

$$\mathrm{d}X_t = \left(b(t,S_0e^{X_{t^-}}) - \frac{1}{2}v^2(t,S_0e^{X_{t^-}}) + \int_{|x|<1}\ln(1+\frac{H(t,x,S_0e^{X_{t^-}})}{S_0e^{X_{t^-}}})\nu\,(dx)\right)\mathrm{d}t$$

$$+v(t,S_0e^{X_{t^-}})\,\mathrm{d}W_t + \int_{|x|>1}\ln(1+\frac{H(t,x,S_0e^{X_{t^-}})}{S_0e^{X_{t^-}}})J_X\,(ds,dx)$$

$$+\int_{|x|<1}\ln(1+\frac{H(t,x,S_0e^{X_{t^-}})}{S_0e^{X_{t^-}}})\tilde{J}_X\,(ds,dx)\,. \tag{4.118}$$

When ρ is considered small we have

$$X_t = \gamma t + \sigma W_t + \int_0^t \int_{|x|\geq 1} \ln(e^x + \rho(\phi(t, S_{t^-}e^x) - \phi(t, S_{t^-})))J_X(ds, dx)$$
$$+ \int_0^t \int_{|x|<1} \ln(e^x + \rho(\phi(t, S_{t^-}e^x) - \phi(t, S_{t^-})))\widetilde{J}_X(ds, dx), \tag{4.119}$$

where $\gamma = \left(b(t, S_0 e^{X_{t^-}}) - \frac{1}{2}v^2(t, S_0 e^{X_{t^-}}) + \int_{|x|<1} \ln(e^x + \rho(\phi(t, S_{t^-}e^x) - \phi(t, S_{t^-})))\nu(dx)\right)$.

But defining the following process we get a finite activity process

$$X_t^\epsilon = \gamma(\epsilon)t + \sigma(\epsilon)B_t + \sigma W_t + \int_0^t \int_{|z|\geq 1} \ln(e^z + \rho(\phi(t, S_{t^-}e^z) - \phi(t, S_{t^-})))J_X(ds, dz)$$
$$+ \int_0^t \int_{|z|>\epsilon} \ln(e^z + \rho(\phi(t, S_{t^-}e^z) - \phi(t, S_{t^-})))\widetilde{J}_X(ds, dz), \tag{4.120}$$

where

$$\gamma(\epsilon) = -\frac{\sigma^2 + \sigma^2(\epsilon)}{2} - \int_{|z|>\epsilon} e^{k(s,z)} - 1 - k(s,z)1_{\{|z|<1\}}\nu(\mathrm{d}z), \tag{4.121}$$
$$\sigma(\epsilon) = \int_{|z|<\epsilon} z^2 \nu(dz), \tag{4.122}$$
$$k(s,z) = \ln(e^z + \rho(\phi(t, S_{t^-}e^z) - \phi(t, S_{t^-}))), \tag{4.123}$$

and if $\rho = 0$ then

$$X_t^\epsilon = \gamma(\epsilon)t + \sigma(\epsilon)B_t + \sigma W_t + \int_0^t \int_{|z|\geq 1} zJ_X(ds, dz) + \int_0^t \int_{|z|>\epsilon} z\widetilde{J}_X(ds, dz). \tag{4.124}$$

$\gamma(\epsilon)$ and $\sigma(\epsilon)$ are chosen so that the $e^{X_t^\epsilon}$ still remains a martingale and to keep the total variance. B_t and W_t are independent Brownian motions. The characteristic triplet is given by $\left(\gamma(\epsilon), \sqrt{\sigma^2(\epsilon) + \sigma^2}, \nu 1_{|x|>\epsilon}\right)$. Then we just have to proceed as in (4.42) but where σ^2 is replaced by $\sigma^2(\epsilon) + \sigma^2$ and ν by $\nu 1_{|x|>\epsilon}$. We have seen that when we have a process of infinite activity i.e. $\int_{\mathbb{R}} \nu(\mathrm{d}z) = \infty$ we can approximate the infinite activity process by a finite activity process by replacing ν and σ^2 appropriately. In this section we study the error of this approximation.

Now let X_t^ϵ be given by (4.120) and define

$$f^\epsilon(\tau, x) = \mathbb{E}[h(x + r\tau + X_\tau^\epsilon)], \tag{4.125}$$
$$f(\tau, x) = \mathbb{E}[h(x + r\tau + X_\tau)]. \tag{4.126}$$

As we have seen $f^\epsilon(\tau, x)$ satisfies

$$\frac{\partial f^\epsilon(\tau, x)}{\partial \tau} = Lf^\epsilon(\tau, x), \quad (0, T] \times \mathbb{R}, \tag{4.127}$$
$$f(0, x) = h(x), \quad \forall x \in \mathbb{R} \tag{4.128}$$

where

$$Lf^{\epsilon}(\tau,x) = \frac{\sigma^2+\sigma^2(\epsilon)}{\left(1-\rho\frac{\partial\psi(\tau,x)}{\partial x}\right)^2}\frac{\partial^2 f}{\partial x^2} + \left(r - \frac{1}{2}\frac{\sigma^2+\sigma^2(\epsilon)}{\left(1-\rho\frac{\partial\psi(\tau,x)}{\partial x}\right)^2} - \alpha\right)\frac{\partial f}{\partial x}$$
$$-\lambda(\epsilon) f + \int_{|x|>\epsilon} f\left(x+\xi(z,x)\right)\nu\left(\,\mathrm{d}x\right), \tag{4.129}$$

and

$$\alpha(\epsilon) = \int_{|z|>\epsilon} H(t,z,S_0e^x)\frac{1}{S_0}e^{-x},\ \nu\left(\,\mathrm{d}z\right) \tag{4.130}$$

$$\lambda(\epsilon) = \int_{|z|>\epsilon} \nu\left(\,\mathrm{d}z\right), \tag{4.131}$$

and $\sigma^2(\epsilon)$ is given by (4.122).

Proposition 4.4.1 *Let h be a Lipschitz function and let f and f^{ϵ} be defined by* (4.125) *and* (4.126) *respectively. Then*

$$|f(\tau,x) - f^{\epsilon}(\tau,x)| \leq \left(C + \frac{e^{\epsilon}}{6} + \sigma^2(\epsilon)\right)\epsilon + C\left(\int_{|z|>\epsilon} e^{2z}\nu(\,\mathrm{d}z)\right)^{1/2}(\lambda(\epsilon))^{1/2}$$
$$+C\lambda(\epsilon) + C\sigma(\epsilon)\left(\int_{1>|z|>\epsilon} 1\nu(\,\mathrm{d}z)\right)^{1/2} \tag{4.132}$$

Proof. Define $W_\tau = Y_\tau - (\gamma-\gamma(\epsilon))\tau$. Then

$$\begin{aligned}|f(\tau,x) - f^{\epsilon}(\tau,x)| &= |\mathbb{E}[h(x+Y_\tau)] - \mathbb{E}[h(x+Y^{\epsilon}_\tau)]| \\ &\leq |\mathbb{E}[h(x+Z_\tau)] - \mathbb{E}[h(x+Y^{\epsilon}_\tau)]| \\ &+ |\mathbb{E}[h(x+Z_\tau+(\gamma-\gamma(\epsilon))\tau)] - \mathbb{E}[h(x+Z_\tau]| \\ &\leq Kc\frac{\int_{|z|<\epsilon}|z|^3\nu(\,\mathrm{d}z)}{\sigma^2(\epsilon)} + |\mathbb{E}[h(x+Z_\tau+(\gamma-\gamma(\epsilon))\tau)] - \mathbb{E}[h(x+Z_\tau]| \\ &\leq Kc\frac{\int_{|z|<\epsilon}|z|^3\nu(\,\mathrm{d}z)}{\sigma^2(\epsilon)} + c|\gamma-\gamma(\epsilon)|\tau,\end{aligned}$$

where the first term of the inequality follows from [24] Proposition 6.2 and the second since h is Lipschitz. Recall that

$$\gamma = -\frac{\sigma^2}{2} - \int_{\mathbb{R}} e^z - 1 - z1_{\{|z|<1\}}\nu(\,\mathrm{d}z).$$

Then as in Theorem 5.1 in [25], we have for the first term of this inequality

$$\begin{aligned}|\gamma-\gamma(\epsilon)| &\leq \left|\frac{\sigma^2(\epsilon)}{2} - \int_{|z|<\epsilon} e^z - 1 - z\,\nu(\,\mathrm{d}z)\right| + \left|\int_{|z|>\epsilon} e^{k(s,z)} - e^z - (k(s,z)-z)1_{|z|<1}\nu(\,\mathrm{d}z)\right| \\ &\leq \frac{e^{\epsilon}}{6}\int_{|z|<\epsilon}|z|^3\nu(\,\mathrm{d}z) + \left|\int_{|z|>\epsilon} e^{k(s,z)} - e^z - (k(s,z)-z)1_{|z|<1}\nu(\,\mathrm{d}z)\right|.\end{aligned}$$

As for the second term we have by definition of $k(s,z)$, given in, that (4.123)

$$\begin{aligned}\left|\int_{|z|>\epsilon} e^{k(s,z)} - e^z - (k(s,z)-z)1_{|z|<1}\nu(\,\mathrm{d}z)\right| &\le \int_{|z|>\epsilon} \left|e^{k(s,z)} - e^z\right| \nu(\,\mathrm{d}z) \\ &+ \int_{|z|>\epsilon} |k(s,z)-z|1_{|z|<1}\nu(\,\mathrm{d}z) \\ &\le I_1 + I_2,\end{aligned}$$

where

$$\begin{aligned}I_1 &= \int_{|z|>\epsilon} \rho|\phi(t, S_{t^-}e^z) - \phi(t, S_{t^-})|\nu(\,\mathrm{d}z), \\ I_2 &= \int_{|z|>\epsilon} |\ln(1+\rho e^{-z}(\phi(t, S_{t^-}e^z) - \phi(t, S_{t^-})))|1_{|z|<1}\nu(\,\mathrm{d}z).\end{aligned}$$

As for I_1 we have

$$\begin{aligned}I_1 \le \int_{|z|>\epsilon} |e^z - 1|\nu(\,\mathrm{d}z) &\le \tilde{K}\int_{|z|>\epsilon} e^z + 1\nu(\,\mathrm{d}z) \\ &\le \tilde{K}\left(\int_{|z|>\epsilon} e^{2z}\nu(\,\mathrm{d}z)\right)^{1/2}\left(\int_{|z|>\epsilon} 1\nu(\,\mathrm{d}z)\right)^{1/2} + \left(\int_{|z|>\epsilon} 1\nu(\,\mathrm{d}z)\right),\end{aligned}$$

since $\rho|\phi(t, S_{t^-}e^z) - \phi(t, S_{t^-})| \le \rho|S_{t^-}\frac{\partial\phi}{\partial S}||e^z - 1|$.
As for I_2 we have

$$\begin{aligned}I_2 &\le \int_{1>|z|>\epsilon} \frac{1}{1+\xi}e^{-z}|e^z - 1|\nu(\,\mathrm{d}z) \\ &\le Ce^{-\epsilon}\int_{1>|z|>\epsilon} |z|\nu(\,\mathrm{d}z) \\ &\le Ce^{-\epsilon}\sigma(\epsilon)\left(\int_{1>|z|>\epsilon} 1\nu(\,\mathrm{d}z)\right)^{1/2},\end{aligned}$$

where the last inequality follows from Hölder's inequality. Then, combining these inequalities yields

$$\begin{aligned}|f(\tau,x) - f^\epsilon(\tau,x)| &\le C\frac{\int_{|z|<\epsilon}|z|^3\nu(\,\mathrm{d}z)}{\sigma^2(\epsilon)} + \frac{e^\epsilon}{6}\int_{|z|<\epsilon}|z|^3\nu(\,\mathrm{d}z) \\ &+C\left(\int_{|z|>\epsilon} e^{2z}\nu(\,\mathrm{d}z)\right)^{1/2}(\lambda(\epsilon))^{1/2} + C\lambda(\epsilon) + C\sigma(\epsilon)\left(\int_{1>|z|>\epsilon} 1\nu(\,\mathrm{d}z)\right)^{1/2} \\ &\le C\epsilon + \frac{e^\epsilon}{6}\epsilon\sigma^2(\epsilon) + C\left(\int_{|z|>\epsilon} e^{2z}\nu(\,\mathrm{d}z)\right)^{1/2}(\lambda(\epsilon))^{1/2} \\ &+C\lambda(\epsilon) + C\sigma(\epsilon)\left(\int_{1>|z|>\epsilon} 1\nu(\,\mathrm{d}z)\right)^{1/2}.\end{aligned}$$

■

4.4.2 Consistency of the implict-explicit scheme

Consider the following PIDE operator $\mathcal{H}$

$$\mathcal{H}u = \frac{\partial u}{\partial \tau} - \frac{v^2}{2}\frac{\partial^2 u}{\partial x^2} - \left(r - \frac{1}{2}v^2 - \alpha\right)\frac{\partial u}{\partial x} + \lambda u - \int_{\mathbb{R}} u(\tau, x + \xi(z,x))\nu(\,\mathrm{d}z), \tag{4.133}$$

where as before we have

$$\lambda = \int_{\mathbb{R}} \nu(\,\mathrm{d}z), \alpha = \int_{\mathbb{R}} H(t,z,S_0e^x)\frac{1}{S_0}e^{-x}\,\nu(\,\mathrm{d}z), \tag{4.134}$$

and

$$\xi(z,x) = \ln(e^z + \rho\frac{1}{S_0}e^{-x}H(t,S_0e^x,S_0e^x\,(e^z-1))), \tag{4.135}$$

$$v^2 \equiv v^2(\tau,x) = \frac{\sigma^2}{\left(1-\rho\frac{\partial\psi(\tau,x)}{\partial x}\right)^2}, \tag{4.136}$$

and also define for every $i = 1,2,..,N$ and $j = 1,2,..,M$ the discretized scheme operator

$$\mathcal{P}_{k,h}u = \frac{u_i^{j+1} - u_i^j}{k} - \left(\frac{(v_i^j)^2}{2}\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} + \left(r - \frac{1}{2}(v_i^j)^2 - \alpha\right)\frac{u_{i+1}^j - u_i^j}{h} + \lambda u_i^j\right)$$

$$-\Delta\tau\sum_{k=K_l}^{K_r} u(\tau_j, x_i + \xi(z_k,x_i))\nu_k 1_{|z_k|>\epsilon}, \tag{4.137}$$

where

$$\nu_k = \frac{1}{2}\nu\left((k-\Delta x)\,0.5\right)1_{|k\Delta x|>\epsilon} + \frac{1}{2}\nu\left((k+\Delta x)\,0.5\right)1_{|k\Delta x|>\epsilon}$$

and

$$v_i^j = \frac{\sigma^2 + \sigma^2(\epsilon)}{\left(1-\rho D\psi_i^j\right)^2}.$$

The following proposition tells us that the finite difference scheme constructed above is consistent

Proposition 4.4.2 *Let $\mathcal{P}_{k,h}$ be defined as in* (4.137) *and $\mathcal{H}$ as in* (4.133)*. Then we have*

$$|\mathcal{P}_{k,h}u(\tau_j,x_i) - \mathcal{H}u(\tau,x)| \to 0 \quad as \quad (h,k)\to 0. \tag{4.138}$$

Proof.

As we have seen before we can decompose operator $\mathcal{H}$ into two operators, the differential and integral operator as follows

$$\mathcal{H}u = \mathcal{D}u + \mathcal{J}u, \tag{4.139}$$

where

$$\mathcal{D}u = \frac{\partial u}{\partial \tau} - \left(\frac{v^2}{2}\frac{\partial^2 u}{\partial x^2} + \left(r - \frac{1}{2}v^2 - \alpha \right) \frac{\partial u}{\partial x} - \lambda u \right), \tag{4.140}$$

$$\mathcal{J}u = - \int_{\mathbb{R}} u(\tau, x + \xi(z,x))\nu(\,\mathrm{d}z). \tag{4.141}$$

As before let $u_i^j = u(\tau_j, x_i), \tau_j = jk, x_i = ih, h = \Delta x, k = \Delta \tau$ and consider the discretized scheme operator defined above (4.137), which is the sum of the differential and integral operator i.e $\mathcal{P}_{k,h} = \mathcal{D}_{k,h} + \mathcal{J}_{k,h}$, where

$$\mathcal{D}_{k,h}u = \frac{u_i^{j+1} - u_i^j}{k} - \left(\frac{(v_i^j)^2}{2}\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} + \left(r - \frac{1}{2}(v_i^j)^2 - \alpha \right) \frac{u_{i+1}^j - u_i^j}{h} - \lambda u_i^j \right)$$

and

$$\mathcal{J}_{k,h} = \Delta\tau \sum_{k=K_l}^{K_r} u(\tau_j, x_i + \xi(z_k, x_i))\nu_k 1_{|z_k|>\epsilon}.$$

So, making the following Taylor Expansion around the point (τ_j, x_i) we get

$$\begin{aligned}
\frac{u_i^{j+1} - u_i^j}{k} &= \frac{\partial u}{\partial \tau} + \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2}k + o(k^2), \\
\frac{u_{i+1}^j - u_i^j}{h} &= \frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}h + \frac{1}{6}\frac{\partial^2 u}{\partial x^3}h^2 + o(h^3), \\
\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} &= \frac{\partial^2 u}{\partial x^2} + o(h^2).
\end{aligned}$$

Then with $\beta_i^j = r - \frac{1}{2}(v_i^j)^2 - \alpha$ we obtain

$$\begin{aligned}
\mathcal{D}_{k,h}u &= \frac{u_i^{j+1} - u_i^j}{k} - \left(\frac{(v_i^j)^2}{2}\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} + \left(r - \frac{1}{2}(v_i^j)^2 - \alpha \right) \frac{u_{i+1}^j - u_i^j}{h} - \lambda u_i^j \right) \\
&= \frac{\partial u}{\partial \tau} + \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2}k + o(k^2) - \frac{(v_i^j)^2}{2}\left(\frac{\partial^2 u}{\partial x^2} + o(h^2) \right) - \beta_i^j \left(\frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}h + \frac{1}{6}\frac{\partial^2 u}{\partial x^3}h^2 + o(h^3) \right) \\
&+ \lambda u_i^j \\
&= \frac{\partial u}{\partial \tau} + \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2}k - \frac{(v_i^j)^2}{2}\frac{\partial^2 u}{\partial x^2} - \beta_i^j \left(\frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}h + \frac{1}{6}\frac{\partial^2 u}{\partial x^3}h^2 \right) \\
&+ \lambda u_i^j + o(k^2) - \frac{\sigma^2}{2}o(h^2) - \beta_i^j o(h^3).
\end{aligned}$$

Then

$$\mathcal{D}_{k,h}u - \mathcal{D}u = \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2}k - \beta_i^j\left(\frac{1}{2}\frac{\partial^2 u}{\partial x^2}h + \frac{1}{6}\frac{\partial^2 u}{\partial x^3}h^2\right) + o(k^2) - \frac{(v_i^j)^2}{2}o(h^2) - \beta o(h^3).$$

We see then that as the time and space step approaches zero the difference operator approaches the differential operator as long as $u \in C^4([0,T]\times\mathbb{R})$

$$(k,h) \to 0 \Rightarrow |\mathcal{D}_{k,h}u - \mathcal{D}u| \to 0. \tag{4.142}$$

As for the integral part we get

$$\begin{aligned}
&\left|\sum_{k=K_l}^{K_r} u_{i+k}^j \nu_k - \int_{B_l}^{B_r} u(\tau, x+\xi(z,x))\nu(\,\mathrm{d}z)\right| \\
&= \left|\sum_{k=K_l}^{K_r} u_{i+k}^j \nu_k - \sum_{k=K_l}^{K_r}\int_{(k-\Delta x)0.5}^{(k+\Delta x)0.5} u(\tau, x+\xi(z,x))\nu(\,\mathrm{d}z)\right| \\
&= \left|\sum_{k=K_l}^{K_r}\left(u_{i+k}^j \nu_k - \int_{(k-\Delta x)0.5}^{(k+\Delta x)0.5} u(\tau, x+\xi(z,x))\nu(\,\mathrm{d}z)\right)\right| \\
&= \left|\sum_{k=K_l}^{K_r}\left(\int_{(k-\Delta x)0.5}^{(k+\Delta x)0.5} u_{i+k}^j - u(\tau, x+\xi(z,x))\nu(\,\mathrm{d}z)\right)\right| \\
&\le \sum_{k=K_l}^{K_r}\left(\int_{(k-\Delta x)0.5}^{(k+\Delta x)0.5} \left|u_{i+k}^j - u(\tau, x+\xi(z,x))\right|\nu(\,\mathrm{d}z)\right).
\end{aligned}$$

But since in the numerical scheme we localize the problem in the interval (A_l, A_r) we must define

$$w(\tau,x) = u(\tau,x)1_{x\in(A_l,A_r)} + u(\tau,x)1_{x\notin(A_l,A_r)} = u(\tau,x)1_{x\in(A_l,A_r)} + g(\tau,x)1_{x\notin(A_l,A_r)} \tag{4.143}$$

Then in fact we must have

$$
\begin{aligned}
&|\mathcal{J}_{k,h}u(\tau_j,x_i)-\mathcal{J}u(\tau,x)| \\
&=\left|\sum_{k=K_l}^{K_r}\left(\int_{(k-\Delta x)0.5}^{(k+\Delta x)0.5} u(\tau_j,x_i+\xi(z_k,x_i))-u(\tau,x+\xi(z,x))\nu(\,\mathrm{d}z)\right)\right| \\
&=\left|\sum_{k=K_l}^{K_r}\left(\int_{(k-\Delta x)0.5}^{(k+\Delta x)0.5} \left(u(\tau_j,x_i+\xi(z_k,x_i))-u(\tau,x+\xi(z,x))\right)1_{x\in(A_l,A_r)}\right.\right. \\
&\left.\left.+\left(g(\tau_j,x_i+\xi(z_k,x_i))-g(\tau,x+\xi(z,x))\right)1_{x\notin(A_l,A_r)}\nu(\,\mathrm{d}z)\right)\right| \\
&\leq\sum_{k=K_l}^{K_r}\left|\int_{(k-\Delta x)0.5}^{(k+\Delta x)0.5} w(\tau_j,x_i+\xi(z_k,x_i))-w(\tau,x+\xi(z,x))\nu(\,\mathrm{d}z)\right| \\
&\leq\sum_{k=K_l}^{K_r}\left|\int_{(k-\Delta x)0.5}^{(k+\Delta x)0.5} C_\tau k+C_x h\nu(\,\mathrm{d}z)\right| \\
&\leq\sum_{k=K_l}^{K_r} C_\tau k+C_x h\int_{(k-\Delta x)0.5}^{(k+\Delta x)0.5}\nu(\,\mathrm{d}z) \\
&=\sum_{k=K_l}^{K_r}\left(C_\tau k^*+C_x h\right)\nu_k=\left(C_\tau k^*+C_x h\right)\lambda.
\end{aligned}
$$

When $(k^*,h)\to 0$, by making the following Taylor expansion

$$
\begin{aligned}
w(\tau_j,x_i+\xi(z_k,x_i))-w(\tau,x+\xi(z,x))=&\frac{\partial w}{\partial\tau}(\tau-\tau_j)+\frac{\partial w}{\partial x}(x-x_i) \\
&+\frac{\partial w}{\partial x}\left(\xi(z_k,x_i)-\xi(z,x)\right)+o(\|h,k\|),
\end{aligned}
\tag{4.144}
$$

we see that

$$
\begin{aligned}
|w(\tau_j,x_i+\xi(z_k,x_i))-w(\tau,x+\xi(z,x))|\leq&\left|\frac{\partial w}{\partial\tau}\right||(\tau-\tau_j)|+\left|\frac{\partial w}{\partial x}\right||(x-x_i)| \\
&+\left|\frac{\partial w}{\partial x}\right|\left|\ln\left(\frac{1+\rho(\psi(\tau_j,x_i+z_k)-\psi(\tau_j,x_i))}{1+\rho(\psi(\tau,x+z)-\psi(\tau,x))}\right)\right| \\
&\leq C_\tau k^*+\left(C_x+C^*\right)h,
\end{aligned}
\tag{4.145}
$$

where

$$
k^*=\tau-\tau_j,h=x-x_i,C_\tau=\sup_{s\in(\tau_j,\tau)}\frac{\partial w}{\partial\tau}(\tau,x+\xi(z,x)),
\tag{4.146}
$$

$$
C_x=\sup_{x\in(x_i+\xi(z_k,x_i),x+\xi(z,x))}\left|\frac{\partial w}{\partial x}\right|,
\tag{4.147}
$$

$$
\begin{aligned}
&\ln\left(\frac{1+\rho(\psi(\tau_j,x_i+z_k)-\psi(\tau_j,x_i))}{1+\rho(\psi(\tau,x+z)-\psi(\tau,x))}\right) && (4.148)\\
&\approx \frac{\rho(\psi(\tau_j,x_i+z_k)-\psi(\tau,x)+\psi(\tau,x+z)-\psi(\tau_j,x_i))}{1+\rho(\psi(\tau,x+z)-\psi(\tau,x))} && (4.149)\\
&\le \rho(\psi(\tau_j,x_i+z_k)-\psi(\tau,x)+\psi(\tau,x+z)-\psi(\tau_j,x_i))\\
&\le \rho\,(\tau_j-\tau)\left|\frac{\partial\psi}{\partial\tau}\right|+\rho\left|\frac{\partial\psi}{\partial x}\right|(x_i+z_k-x)+\rho\,(\tau_j-\tau)\left|\frac{\partial\psi}{\partial\tau}\right|+\rho\,(x+z-x_i)\left|\frac{\partial\psi}{\partial x}\right|\\
&\le \rho k^*\left(c_\tau^1+c_\tau^2\right)+\rho\left|h^1\right|c_x^1+\rho\left|h^2\right|c_x^2=C^*, && (4.150)
\end{aligned}
$$

where

$$
k^*=\tau-\tau_j, h^1=x_i+z_k-x, h^2=x+z-x_i, c_\tau^1=\sup_{s\in(\tau_j,\tau)}\frac{\partial\psi}{\partial\tau}(\tau,x), c_\tau^2=\sup_{s\in(\tau_j,\tau)}\frac{\partial\psi}{\partial\tau}(\tau,x_i),
$$

$$
c_x^1=\sup_{x\in(x,x_i+z_k)}\left|\frac{\partial\psi}{\partial x}\right|, c_x^2=\sup_{x\in(x_i,x+z)}\left|\frac{\partial\psi}{\partial x}\right|. \qquad (4.151)
$$

But we can do this as long as $w(\tau,x)$ does not contain discontinuity points in the interval $((k-\Delta x)\,0.5,(k+\Delta x)\,0.5)$ and if A_l or A_r does not belong to this interval. In those cases we can proceed similarly

$$
\begin{aligned}
&\left|\int_{(k-\Delta x)0.5}^{(k+\Delta x)0.5} w(\tau_j,x_i+\xi(z_k,x_i))-w(\tau,x+\xi(z,x))\nu(\,\mathrm{d}z)\right| && (4.152)\\
&\le \int_{(k-\Delta x)0.5}^{(k+\Delta x)0.5}\left|w(\tau_j,x_i+\xi(z_k,x_i))-w(\tau,x+\xi(z,x))\right|\nu(\,\mathrm{d}z)\\
&\le \int_{(k-\Delta x)0.5}^{(k+\Delta x)0.5}\left|w(\tau_j,x_i+\xi(z_k,x_i))\right|+\left|w(\tau,x+\xi(z,x))\right|\nu(\,\mathrm{d}z)\\
&\le \left|w(\tau_j,x_i+\xi(z_k,x_i))\right|\int_{(k-\Delta x)0.5}^{(k+\Delta x)0.5}\nu(\,\mathrm{d}z)\\
&+\int_{(k-\Delta x)0.5}^{(k+\Delta x)0.5}\sup_{z\in((k-\Delta x)0.5,(k+\Delta x)0.5)}\left|w(\tau,x+\xi(z,x))\right|\nu(\,\mathrm{d}z)\\
&=\left(\left|w(\tau_j,x_i+\xi(z_k,x_i))\right|+\sup_{z\in((k-\Delta x)0.5,(k+\Delta x)0.5)}\left|w(\tau,x+\xi(z,x))\right|\right)\nu_k\\
&\le C(\tau,x)\nu_k.
\end{aligned}
$$

So we see that if ν has not a singularity at zero when $\Delta x\to 0$, then

$$
\nu_k=\int_{(k-\Delta x)0.5}^{(k+\Delta x)0.5}\nu(\,\mathrm{d}z)\approx\frac{\Delta x}{2}\left(\nu\left((k-\Delta x)\,0.5\right)+\nu\left((k+\Delta x)\,0.5\right)\right)\to 0. \qquad (4.153)
$$

Then summing up these terms we get as $(h,k)\to 0$

$$
\left|\mathcal{J}_{k,h}u(\tau_j,x_i)-\mathcal{J}u(\tau,x)\right|\le\lambda\left(C_\tau k^*+\left(C_x+C^*\right)h\right)+C(\tau,A_r)\nu_{K_r}+C(\tau,A_l)\nu_{K_l}\to 0.
$$

■

4.4.3 Stability and Monotony

Definition 4.4.1 *The scheme proposed in the previous section is stable if for every h and g bounded it has a unique bounded solution independent of the time and space step uniformly over* $[0,T]\times\mathbb{R}$ *:*

$$\exists C>0, \forall \Delta t, \Delta x>0, i\in\mathbb{Z}, j\in\{0,1,2..,M\}: |u_i^j|\leq C.$$

Definition 4.4.2 *The scheme proposed is monotone if for every* u^j *and* v^j *are two solutions of the scheme with initial conditions h and* h^* *and conditions on the borders g and* g^* *respectively and if*

$$h>h^*, g>g^*$$

we have

$$\forall j\geq 1, u^j\geq v^j.$$

Next proposition shows that the scheme proposed is stable and monotone.

Proposition 4.4.3 *The scheme* (4.42) *is stable and monotone.*

Proof.
If we define

$$\begin{aligned}
\alpha_1 &= -\frac{1}{(\Delta x)^2}\left(\frac{1}{2}\frac{\sigma^2}{(1-\rho D\psi_i^n)^2}\right)-\frac{1}{\Delta x}\left(r-\frac{1}{2}\frac{\sigma^2}{(1-\rho D\psi_i^n)^2}-\alpha\right),\\
\alpha_2 &= 1+2\frac{1}{(\Delta x)^2}\left(\frac{1}{2}\frac{\sigma^2}{(1-\rho D\psi_i^n)^2}\right)+\frac{1}{\Delta x}\left(r-\frac{1}{2}\frac{\sigma^2}{(1-\rho D\psi_i^n)^2}-\alpha\right)+\lambda,\\
\alpha_3 &= -\frac{1}{(\Delta x)^2}\left(\frac{1}{2}\frac{\sigma^2}{(1-\rho D\psi_i^n)^2}\right),
\end{aligned}$$

we can write scheme (4.42) i.e

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=(D_\Delta u^{n+1})_i+(J_\Delta u^n)_i$$

for $i\in\{0,1...,N\}$, in the following way

$$-\alpha_3\Delta t u_{i-1}^{n+1}+(1+\alpha_1\Delta t)u_i^{n+1}-\alpha_2\Delta t u_{i+1}^{n+1}=u_i^n+\Delta t\sum_{k=K_l}^{K_r}u(\tau_j,x_i+\xi(z_k,x_i))\nu_k 1_{|z_k|>\epsilon}.$$

This set of equations defines a tridiagonal system of equations which has diagonal strictly dominant, which ensures the uniqueness of the system. To see this notice that $\alpha_1=\alpha_2+\alpha_3+\lambda$, which implies

$$(1+\alpha_1\Delta t)=1+(\alpha_2+\alpha_3)\Delta t+\lambda\Delta t>(\alpha_2+\alpha_3)\Delta t.$$

We can show by recurrence, that if $C = \max\{||g||, ||h||\} \leq \infty$ then for any $n > 0$ we have $||u^n|| \leq C$.

Let it hold for n. We have some $j \in \{0, 1, 2, ..., N-1\}$ such that $||u^{n+1}||_\infty = |u_j^{n+1}|$ and $\forall i \neq j$ $|u_i^{n+1}| \leq |u_j^{n+1}|$.

Then

$$\begin{aligned}
||u^{n+1}||_\infty = |u_j^{n+1}| &= |u_j^{n+1}| \left(1 + (\alpha_1 - \alpha_2 - \alpha_3 - \lambda)\Delta t\right) \\
&\leq -\alpha_3 \Delta t |u_{j-1}^{n+1}| + (1 + \alpha_1 \Delta t) |u_j^{n+1}| - \alpha_2 \Delta t |u_{j+1}^{n+1}| - \lambda \Delta t |u_j^{n+1}| \\
&\leq | - \alpha_3 \Delta t u_{j-1}^{n+1} + (1 + \alpha_1 \Delta t) u_j^{n+1} - \alpha_2 \Delta t u_{j+1}^{n+1}| - \lambda \Delta t |u_j^{n+1}| \\
&= |u_j^n + \Delta t \sum_{k=K_l}^{K_r} u(\tau_j, x_i + \xi(z_k, x_i)) \nu_k 1_{|z_k| > \epsilon}| - \lambda \Delta t |u_k^{n+1}| \\
&\leq (1 + \lambda \Delta t|)\, ||u^n||_\infty - \lambda \Delta t |u_k^{n+1}| \leq C.
\end{aligned}$$

Now to prove monotony consider two solutions u^n and v^n with respectively border conditions

$$h(x) \geq h^*(x), \forall x \in (A_l, A_r), g(\tau, x) \geq g^*(\tau, x), \forall (\tau, x) \in [0, T] \times (A_l, A_r)^c.$$

For $n = 0$ we have by construction $u^0 \geq v^0$. Let $u^n \geq v^n$ for $n > 0$. Since for any $i \notin \{1, 2, .., N\}$, $g(\tau_{n+1}, x_{n+1}) \geq g^*(\tau_{n+1}, x_{n+1})$ we have for $i \in \{1, 2, .., N\}$ and $y_i^n = u_i^n - v_i^n$

$$\begin{aligned}
\inf_i y_i^{n+1} = y_j^{n+1} &= y_j^{n+1} \left(1 + (\alpha_1 - \alpha_2 - \alpha_3 - \lambda)\Delta t\right) \\
&\geq -\alpha_3 \Delta t y_{j-1}^{n+1} + (1 + \alpha_1 \Delta t) y_j^{n+1} - \alpha_2 \Delta t y_{j+1}^{n+1} - \lambda \Delta t y_j^{n+1} \\
&= y_j^n + \Delta t \sum_{k=K_l}^{K_r} y(\tau_j, x_i + \xi(z_k, x_i)) \nu_k 1_{|z_k| > \epsilon} - \lambda \Delta t y_j^{n+1} \\
&\geq y_j^n + \Delta t y_j^n \sum_{k=K_l}^{K_r} \nu_j 1_{|y_k| > \epsilon} - \lambda \Delta t y_j^{n+1} \\
&= y_j^n + \Delta t y_j^n \lambda - \lambda \Delta t y_j^{n+1} \geq 0.
\end{aligned}$$

Then we conclude that $y_i^n = u_i^n - v_i^n \geq 0$. ■

4.4.4 Convergence of the Classical PIDE

Using the same notation as in [83] we rewrite our scheme in the following way

$$\begin{aligned}
&B(\Delta t, \Delta x, n+1, i, u_i^{n+1}, \tilde{u}) = -\alpha_3 \Delta t u_{i-1}^{n+1} + (1 + \alpha_1 \Delta t) u_i^{n+1} - \alpha_2 \Delta t u_{i+1}^{n+1} \\
&-u_i^n - \Delta t \sum_{0 \leq i+j \leq N-1} \nu_j u_{i+j}^n 1_{|y_j| > \epsilon} - \Delta t \sum_{0 \leq i+j \leq N-1} \nu_j g(\tau_n, x_{i+j}) 1_{|y_j| > \epsilon}
\end{aligned} \tag{4.154}$$

where

$$\tilde{u} = \left(u_{i+1}^{n+1}, u_{i-1}^{n+1}, u^n\right)$$

.

We have the following monotony property assuming

$$u_{i+1}^{n+1} = v_{i+1}^{n+1}, u^n \leq v^n$$

$$B(\Delta t, \Delta x, n+1, i, u_i^{n+1}, \tilde{u}) \geq B(\Delta t, \Delta x, n+1, i, v_i^{n+1}, \tilde{v}) \tag{4.155}$$

We have using the consistency of the scheme the following property: for any $\phi \in C_\infty([0,T]\times(A_l, A_r))$ and $(\tau, x) \in [0,T]\times(A_l, A_r)$ when $(\Delta t, x) \to (0,0), (\tau_{n+1}, x_i) \to (\tau, x)$

$$\frac{1}{\Delta t} B(\Delta t, \Delta x, n+1, i, u_i^{n+1}, \tilde{u}) \to \frac{\partial \phi}{\partial \tau}(\tau, x) - L\phi(\tau, x) \tag{4.156}$$

For $i = 1, .., N-1$ and s a constant function on the grid we have

$$\begin{aligned} &B(\Delta t, \Delta x, n+1, i, u_i^{n+1} + s, \tilde{u} + s) = B(\Delta t, \Delta x, n+1, i, u_i^{n+1}, \tilde{u}) \\ &- \sum_{0 \leq i+j \leq N-1} s\nu_j 1_{|y_j|>\epsilon} \end{aligned} \tag{4.157}$$

We define also a constant intepolation over the grid

$$u^{(\Delta t, \Delta x)}(\tau, x) = u_j^n, \tau \in [\tau_n, \tau_{n+1}), x \in [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}). \tag{4.158}$$

and also

$$\underline{u}(\tau, x) = \lim_{(\Delta t, \Delta x)\to 0} \inf_{(t,y)\to(\tau,x)} u^{(\Delta t, \Delta x)}(t, y) \tag{4.159}$$

$$\overline{u}(\tau, x) = \lim_{(\Delta t, \Delta x)\to 0} \sup_{(t,y)\to(\tau,x)} u^{(\Delta t, \Delta x)}(t, y) \tag{4.160}$$

We will need the following two lemmas

Lemma 4.4.3 *The function $\overline{u}$ is upper semi-continuous i.e*

$$\lim_{(t,y)\to(\tau,x)} \sup \overline{u}(t, y) \leq \overline{u}(\tau, x) \tag{4.161}$$

Proof. We have to show that for $\forall \epsilon > 0$, $\exists$ $(t, y) \in V(\tau, x)$ such that $\forall (t, y) \in V(\tau, x), \overline{u}(t, y) < \overline{u}(t, x) + \epsilon$. We have by definition

$$\overline{u}(t, y) = \lim_{(\Delta t, \Delta x)\to 0} \sup_{(r,w)\to(t,y)} u^{(\Delta t, \Delta x)}(r, w) \tag{4.162}$$

Then over any neighbourhood of (t_k, y_k) we can find $(\Delta t_k, \Delta x_k, r_k, w_k) \to (0, 0, \tau, x)$

$$\overline{u}(t_k, y_k) \leq u^{(\Delta t_k, \Delta x_k)}(r_k, w_k) + \frac{1}{k} \tag{4.163}$$

Then

$$\lim_{k\to\infty} \sup u^{(\Delta t_k, \Delta x_k)}(r_k, w_k) \leq \lim_{(\Delta t, \Delta x)\to 0} \sup_{(r,w)\to(\tau,x)} u^{(\Delta t, \Delta x)}(r, w) \tag{4.164}$$

which implies

$$\overline{u}(t, y) \leq \lim_{(\Delta t, \Delta x)\to 0} \sup_{(r,w)\to(\tau,x)} u^{(\Delta t, \Delta x)}(r, w) + \epsilon = \overline{u}(t, x) + \epsilon \tag{4.165}$$

This way $u \in USC$ and the proof is complete. ■

We also need the following lemma presented in [83]

Lemma 4.4.4 *There are sequences* $(\Delta t_k, \Delta x_k) \to 0$ *and* n_k, i_k *such that when* $k \to \infty$

$$(\tau_{n_k}, x_{i_k}) \to (\tau, x) \tag{4.166}$$
$$u^{(\Delta t_k, \Delta x_k)}(\tau_{n_k}, x_{i_k}) \to \overline{u}(t, x) \tag{4.167}$$

and for every k

$$\max_{(\tau_n, x_j) \in [0,T]\times\Omega} \left\{ u^{(\Delta t_k, \Delta x_k)}(\tau_n, x_j) - \phi(\tau_n, x_j) \right\} = \left(u^{(\Delta t_k, \Delta x_k)} - \phi \right)(\tau_{n_k}, x_{i_k}) \tag{4.168}$$

We have then the following theorem of convergence of the finite difference scheme which makes use of the concept of viscosity solutions.

Theorem 4.4.5 *Let* $h(x), g(\tau, x)$ *be Lipschitz bounded functions such that* $g(0, A_r) = h(A_r), g(0, A_l) = h(A_l)$. *Suppose that the localized problem verifies the comparison principle for semi-continuous viscosity solutions. If* ν *has a singularity we impose a restriction on the choice of test functions* $\phi = g$ *apart from* (A_l, Ar). *Then the solution* $u^{(\Delta t, \Delta x)}(\tau, x)$ *of the scheme converges uniformly over any compact of* $[0,T] \times \mathbb{R}$ *to the unique viscosity solution of the localized problem.*

Proof. We have to show that $\overline{u}$ is a subsolution and $\underline{u}$ is a supersolution since by construction $\underline{u} \leq \overline{u}$ and by comparison principle for viscosity solutions $\overline{u} \leq \underline{u}$.

Note that $\overline{u}$ is uniformly bounded, since by the proposition of stability and monotony for every $\Delta t > 0, \Delta x > 0\, \forall(\tau, x) \in [0,T]\times\mathbb{R}$,we have $|u^{(\Delta t, \Delta x)}(\tau, x)| \leq \max\{||h||_\infty, ||g||_\infty\}$, which implies $\overline{u} \leq C$. Also by lemma 4.4.3 we know that $\overline{u} \in USC$, then $\overline{u} \in USC \cap C_p^+([0,T] \times \mathbb{R})$. So it remains to check the conditions in the definition of a viscosity subsolution. If $x \notin [A_l, A_r]$ then $\overline{u}(\tau, x) = g(\tau, x)$ since then by construction $\exists \epsilon > 0$ such that for $\Delta t < \epsilon, \Delta x < \epsilon, |\,(\tau_n, x_i - (\tau, x))\,| < \epsilon$ we have $u^{(\Delta t, \Delta x)}(\tau_n, x_i) = g(\tau_n, x_i)$ and by the definition of upper limit.

Let us choose a sequence as in lemma 4.4.4 and define $\rho_k = u^{(\Delta t_k, \Delta x_k)}(\tau_{n_k}, x_{i_k}) - \phi(\tau_{n_k}, x_{i_k})$, $\phi_i^n = \phi(\tau_n, x_i), (u^{\Delta_k})_i^n \equiv u^{(\Delta t_k, \Delta x_k)}(\tau_{n_k}, x_{i_k})$. By construction

$$u^{(\Delta t_k, \Delta x_k)}(\tau_{n_k}, x_{i_k}) = \phi_{i_k}^{n_k} + \rho_k, \tag{4.169}$$
$$u^{(\Delta t_k, \Delta x_k)}(\tau_{n_k}, x_{i_k+j}) \leq \phi_{i_k+j}^{n_k} + \rho_k, n = 0, 1, 2, .., M, j = K_l, .., K_r, \tag{4.170}$$
$$\rho_k \to 0. \tag{4.171}$$

Then using (4.155), (4.157), (4.169) and (4.169) we have

$$\begin{aligned}
0 &= \frac{1}{\Delta t_k} B(\Delta t_k, \Delta x_k, n_k, i_k, u^{(\Delta t_k, \Delta x_k)}(\tau_{n_k}, x_{i_k}), \tilde{u}^{\Delta_k}) \\
&= \frac{1}{\Delta t_k} B(\Delta t_k, \Delta x_k, n_k, i_k, \phi_{i_k}^{n_k} + \rho_k, \tilde{u}^{\Delta_k}) \\
&\geq \frac{1}{\Delta t_k} B(\Delta t_k, \Delta x_k, n_k, i_k, \phi_{i_k}^{n_k} + \rho_k, \tilde{\phi}^{\Delta_k} + \rho_k) \\
&= \frac{1}{\Delta t_k} B(\Delta t_k, \Delta x_k, n_k, i_k, \phi_{i_k}^{n_k}, \tilde{\phi}^{\Delta_k}) - \sum_{0 \leq i_k + j \leq N-1} \rho_k \nu_j 1_{|y_j| > \epsilon} \to \left(\frac{\partial \phi}{\partial \tau} - L\phi \right)(\tau, x)
\end{aligned}$$

because the scheme is consistent and the sum term is uniformly bounded since

$$|\sum_{0\leq i_k+j\leq N-1} s\nu_j 1_{|y_j|>\epsilon}| \leq \lambda |u^{(\Delta t_k, \Delta x_k)}(\tau_{n_k}, x_{i_k}) - \phi_{i_k}^{n_k}| \leq \lambda \left(K + ||\phi||_{[0,T]\times[A_l,A_r]}\right)$$

Then for $\overline{u} \leq \phi$ over $[0,T]\times\mathbb{R}\setminus(\tau,x)$ we have proven

$$\left(\frac{\partial\phi}{\partial\tau} - L\phi\right)(\tau,x) \leq 0 \tag{4.172}$$

Now for the case $\tau = 0$ we consider a sequence $(\tau_{n_k}, x_{i_k}) \to (\tau, x)$ such that $(\tau_{n_k}, x_{i_k}) \in [0,T]\times[A_l, A_r]$, in which case we obtain the same conclusion as before, i.e

$$\left(\frac{\partial\phi}{\partial\tau} - L\phi\right)(\tau,x) \leq 0 \tag{4.173}$$

If $\exists k^* : k > k^*, \tau_{n_k} = 0, x_{i_k} \in (A_l, A_r)$, then we have $(u^{\Delta_k})_{i_k}^{n_k} = h(x_{i_k})$ and passing to the limit we obtain $\overline{u}(\tau,x) = h(x)$.

If $\exists k^* : k > k^*, x_{i_k} = A_l \cup x_{i_k} = A_r$, then we have $(u^{\Delta_k})_{i_k}^{n_k} = g(\tau_{n_k}, x_{i_k})$ which implies $\overline{u}(\tau,x) = g(0,x) = h(x)$.

If $\tau \neq 0 \cap x_{i_k} = A_l \cup x_{i_k} = A_r$, then we have $\left(\frac{\partial\phi}{\partial\tau} - L\phi\right)(\tau,x) \leq 0$ or $\overline{u}(\tau,x) = g(\tau,x)$.

This way we have proven

$$\min\left\{\left(\frac{\partial\phi}{\partial\tau} - L\phi\right)(\tau,x), \overline{u}(\tau,x) - h(x)\right\} \leq 0, \tau = 0 \cap x \in [A_l, A_r], \tag{4.174}$$

$$\min\left\{\left(\frac{\partial\phi}{\partial\tau} - L\phi\right)(\tau,x), \overline{u}(\tau,x) - g(\tau,x)\right\} \leq 0, \tau \neq 0 \cap x = A_l \cup x = A_r. \tag{4.175}$$

Then we have shown that $\overline{u}(\tau,x)$ is a viscosity sub-solution and proceeding in the same way we can prove that $\underline{u}(\tau,x)$ is a viscosity super-solution.

It remains to show that the convergence is uniform over every compact interval $[0,T]\times\mathbb{R}$. In order to show it we use Dini's theorem which states that if we have $v_n : X \to \mathbb{R}$,$v_n \in USC$ and for any $x \in X$ $v_n(x) \to 0$ and decreasing in n, then $v_n \to 0$ uniformly over any compact on $\mathbb{R}$.

Therefore if we let $X = [0,T]\times\mathbb{R}$ and if we define

$$\overline{v}_n(\tau,x) = \sup_{\substack{||(\Delta t,\Delta x)||\leq\frac{1}{n}\\ ||(r,w)-(\tau,x)||\leq\frac{1}{n}}} u^{(\Delta t,\Delta x)}(r,w), \tag{4.176}$$

we see that both $u(\tau,x)$ and $\overline{v}_n(\tau,x)$ are decreasing and is upper semi-continuous. So we can use Dini's theorem to conclude that $v_n(\tau,x) = \overline{v}_n(\tau,x) - u(\tau,x) \to 0$ uniformly over every compact of X, which then leads to

$$\lim_{\substack{(\Delta t,\Delta x)\to 0\\ (r,w)\to(\tau,x)}} u^{(\Delta t,\Delta x)}(r,w) = u(\tau,x), \tag{4.177}$$

uniformly over every compact of $[0,T]\times\mathbb{R}$. ■

Corollary 4.4.1 *For every $\tau > 0, x \in \mathbb{R}$ and for the case of an European option and if $h(x)$ is discontinuous except for a finite number of points, then the solution of the scheme converges to the solution of*

$$\left(\frac{\partial u}{\partial \tau} - Lu\right)(\tau, x) = 0.$$

Proof. Let $\overline{h}, \underline{h} \in C^{\infty}(\mathbb{R})$ such that $\underline{h} \leq h \leq \overline{h}$. Also define

$$u_{\underline{h}}(\tau, x) = E^{\mathbb{Q}}[\underline{h}(x + Y_\tau)], u_{\overline{h}}(\tau, x) = E^{\mathbb{Q}}[\overline{h}(x + Y_\tau)] \tag{4.178}$$

and the solutions of the scheme proposed using initial conditions $\underline{h}, \overline{h}$ respectively by $u_{\underline{h}}^{(\Delta t,\Delta x)}(t, y), u_{\overline{h}}^{(\Delta t,\Delta x)}(t, y)$. We have by monotony that $u_{\underline{h}}^{(\Delta t,\Delta x)}(t, y) \leq u^{(\Delta t,\Delta x)}(t, y) \leq u_{\overline{h}}^{(\Delta t,\Delta x)}(t, y)$. The consequence is that $u^{(\Delta t,\Delta x)}(t, y) \to u^*(\tau, x)$ because $u_{\underline{h}}^{(\Delta t,\Delta x)}(t, y) \to u_{\underline{h}}(\tau, x)$ and $u_{\overline{h}}^{(\Delta t,\Delta x)}(t, y) \to u_{\overline{h}}(\tau, x)$.

Let us then prove that $u^*(\tau, x) = u(\tau, x)$. For that consider $a_1, a_2, ..., a_n$ the discontinuity points of h and suppose the jumps are bounded by a constant C. Then for a given $\epsilon > 0$, we have

$$|\underline{h}(\tau, x) - \overline{h}(\tau, x)| \leq \epsilon, \forall x \notin \cup_{i=1}^{n} (a_i - \epsilon, a_i + \epsilon), \tag{4.179}$$
$$|\underline{h}(\tau, x) - \overline{h}(\tau, x)| \leq C, \forall x \in \cup_{i=1}^{n} (a_i - \epsilon, a_i + \epsilon). \tag{4.180}$$

Then

$$\begin{aligned}
&|u_{\underline{h}}(\tau, x) - u_{\overline{h}}(\tau, x)| = |E^{\mathbb{Q}}(\underline{h}(x + Y_\tau)) - E^{\mathbb{Q}}\left(\overline{h}(x + Y_\tau)\right)| \\
&= |E^{\mathbb{Q}}\left(\left(\underline{h}(x + Y_\tau) - \overline{h}(x + Y_\tau)\right)\left(1_{x+Y_\tau \in \cup_{i=1}^{n}(a_i-\epsilon,a_i+\epsilon)} + 1_{x+Y_\tau \notin \cup_{i=1}^{n}(a_i-\epsilon,a_i+\epsilon)}\right)\right)| \\
&\leq \epsilon \mathbb{Q}\left(x + Y_\tau \in \cup_{i=1}^{n}(a_i - \epsilon, a_i + \epsilon)\right) + K\mathbb{Q}\left(x + Y_\tau \notin \cup_{i=1}^{n}(a_i - \epsilon, a_i + \epsilon)\right) \\
&\leq \epsilon + K\mathbb{Q}\left(x + Y_\tau \in \cup_{i=1}^{n}(a_i - \epsilon, a_i + \epsilon)\right).
\end{aligned}$$

Now if we define $A_\epsilon = \{x + Y_\tau \in \cup_{i=1}^{n}(a_i - \epsilon, a_i + \epsilon)\}$, then

$$\lim_{\epsilon \to 0} \mathbb{Q}(A_\epsilon) = \mathbb{Q}(\cap_\epsilon A_\epsilon) = \mathbb{Q}(x + Y_\tau \in \{a_1, a_2, .., a_n\}) = 0, \tag{4.181}$$

because Y_t has an absolutely continuous distribution and $\sigma > 0$.

The consequence is that $|u_{\underline{h}}(\tau, x) - u_{\overline{h}}(\tau, x)| \leq \epsilon$ and since $u_{\underline{h}}(\tau, x) \leq u(\tau, x) \leq u_{\overline{h}}(\tau, x)$ and $u_{\underline{h}}(\tau, x) \leq u^*(\tau, x) \leq u_{\overline{h}}(\tau, x)$, we can conclude that $u^*(\tau, x) = u(\tau, x)$. ■

Chapter 5

Interest Rate Derivatives under the Martingale Approach

Convexity adjustment is used by practitioners to value non standard products by using plain vanilla products. The real world interbank market is not populated completely by riskless banks. However, market operators assumed that the risk in the interbank lending market was negligible when dealing with interest rate sensitive products to build zero-coupon bonds curves. After August 2007, the Libor rate $L(t,T)$ was no longer considered a good approximation to the truly default-free interest rate. Therefore we define a convexity adjustment to value a contract called the Forward Rate Agreement. First, we consider only an affine term structure (ATS) and then we combine an ATS with a shot-noise process.

5.1 Introduction

To value nonstandard products in a fixed income market, practitioners usually use the price of standard (plain vanilla) products corrected by an adjustment called the convexity adjustment. This adjustment is made to plain vanilla products whose price can be computed in closed form or obtained in the market, to correct the deviation introduced in prices due to the complex nature of non-standard products. Another way to see this is to think of convexity adjustment as a consequence of measure change since pricing non-standard products is equivalent to compute prices under the wrong measure.

5.2 Problem Formulation

Let π denote the no arbitrage price, at time t, of a derivative paying Φ at time T. Then

$$\pi_t(\Phi) = \mathbb{E}_t^{\mathbb{Q}}\left[\Phi(T)e^{-\int_t^T r_s\,\mathrm{d}s}\right] = p(t,T)\mathbb{E}_t^T\left[\Phi(T)\right], \tag{5.1}$$

where $p(t,T)$ represents the price of a zero coupon bond, at time t, with maturity T. $\mathbb{E}_t^{\mathbb{Q}}[.]$ and $\mathbb{E}_t^T[.]$ denotes the conditional expectation, given the information available at time t under the risk neutral measure and the T-forward measure respectively. We denote B_t, the price process of a risk-free asset, at time t.

We are interested in computing the expected value of our payoff $\Phi(T)$ under the T-forward measure, i.e $\mathbb{E}_t^T[\Phi(T)]$, where the numeraire is $p(t,T)$. However, we have situations in which our payoff is not a martingale under the T-forward measure. If the payoff Φ_t is a martingale under some $\mathbb{Q}^U$ martingale measure then $\Phi_t = \mathbb{E}_t^{\mathbb{Q}^U}[\Phi_T]$ and the convexity adjustment can be defined as

$$\mathbb{E}_t^T[\Phi_T] = \mathbb{E}_t^{\mathbb{Q}^U}[\Phi_T] + CC^U(t). \tag{5.2}$$

We study only the convexity adjustment for the Forward Rate Agreement contract which is defined in the following way

Definition 5.2.1 *The Forward Rate Agreement contract is a contract entered at time t, between two entities where the buyer of the contract at time T, obtains the amount $\Phi(r(T))$. At time T the buyer pays the amount Fra_t which is determined at time t. The forward price for a T-claim contracted at time t is defined as the value Fra_t which gives the contract the value of zero at time t.*

We consider that the payoff is known and is computed in the following way: the interests due to the difference between the par FRA rate and the Libor rate, $L(S,T)$, accrued over the period $[S,T]$, discounted between times S and T using the Libor rate. Notice that since the FRA is sold at par we have

$$0 = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{B_t}{B_S}\frac{Fra_t(S,T) - L(S,T)}{1+(T-S)L(S,T)}\right]. \tag{5.3}$$

Thus, the price of a forward rate agreement is given by

$$Fra_t(S,T) = \frac{\mathbb{E}_t^{\mathbb{Q}}\left[\frac{B_t}{B_S}\frac{L(S,T)}{1+(T-S)L(S,T)}\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[\frac{B_t}{B_S}\frac{1}{1+(T-S)L(S,T)}\right]}. \tag{5.4}$$

5.2.1 Classical Approach

Before August 2007, market operators thought in terms of one single term structure of riskless interest rates in the sense that credit and liquidity risk were considered negligible. Then

$$p(t,T) = \frac{1}{1+R(t,T)\alpha_{t,T}}, \tag{5.5}$$

where $p(t,T)$ is the zero-coupon bond price, $R(t,T)$ is the deposit interest rate and $\alpha_{t,T}$ is the year fraction between t and T. The real world interbank market is not populated completely by riskless banks. However, market operators assumed that the risk in the interbank lending market was negligible when dealing with interest rate sensitive products to build zero-coupon bonds curves. This was justified by the perceived low level of risk for the large majority of banks and by the fact that most interest rate derivatives products were indexed by Libor rates. [1]

Thus the Libor rate $L(t,T)$ was considered a good approximation to the truly default-free interest rate $R(t,T)$ in the sense that one could treat $L(t,T)$ as the riskless rate. This way, under the single curve assumption, the price of a zero coupon bond at time S is given by

$$p(S,T) = \frac{1}{1+(T-S)L(S,T)}. \tag{5.6}$$

Notice that we are assuming that Libor rate is a good approximation for the interbank deposit rate. Also we assume that the differences of deposit quotes due to credit lines or volume can be neglected.

This assumption was widely used before the crisis but nowadays is no longer a market practice.

Result 5.2.2 *Under the single curve assumption the price of a forward rate agreement contract is given by*

$$Fra_t(S,T) = \mathbb{E}_t^T\left[L(S,T)\right], \tag{5.7}$$

where $L(S,T)$ is given as the solution of (5.6).

Proof.

By the definition of a forward rate agreement price (5.4) and (5.6) we have

$$F_t(S,T) = \frac{\mathbb{E}_t^{\mathbb{Q}}\left[\frac{\frac{B_t}{B_S}L(S,T)}{1+(T-S)L(S,T)}\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[\frac{\frac{B_t}{B_S}}{1+(T-S)L(S,T)}\right]} = \frac{\mathbb{E}_t^{\mathbb{Q}}\left[\frac{B_t}{B_S}\frac{1-p(S,T)}{T-S}\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[\frac{B_t}{B_S}p(S,T)\right]}.$$

Notice that by risk neutral valuation and Bayes theorem we have

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}\left[\frac{B_t}{B_S}p(S,T)\right] &= \mathbb{E}_t^{\mathbb{Q}}\left[\frac{B_t}{B_S}\mathbb{E}_S^{\mathbb{Q}}\left[\frac{B_S}{B_T}\right]\right] \\ &= \mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{E}_S^{\mathbb{Q}}\left[\frac{B_t}{B_S}\frac{B_S}{B_T}\right]\right] \\ &= \mathbb{E}_t^{\mathbb{Q}}\left[\frac{B_t}{B_T}\right] = p(t,T).\end{aligned}$$

Then finally using the following change of measure

$$\frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{Q}} = \frac{p(S,T)}{B_S}\frac{B_t}{p(t,T)},$$

[1] Libor is an average rate of the rates at which banks believe they can obtain unsecured funding.

we have once again using the definition of (5.6)

$$F_t(S,T) = \frac{\mathbb{E}_t^T\left[\frac{B_t}{B_S}\frac{1-p(S,T)}{T-S}\frac{p(t,T)}{B_t}\frac{B_S}{p(S,T)}\right]}{p(t,T)} = \mathbb{E}_t^T\left[L(S,T)\right].$$

■

We see that in the single-curve assumption the price of a forward rate agreement contract is equal to $F_t(S,T)$, i.e the forward rate, which is defined by

$$F_t(S,T) = \frac{1}{T-S}\left(\frac{p(t,S)}{p(t,T)} - 1\right), \tag{5.8}$$

5.2.2 Multiple-Curve Approach

After August 2007, the liquidity crisis increased the difference between deposit rates and overnight interest rates (OIS) for the same maturity. This led to a larger difference between forward rates, implied by two deposits, and the quoted FRA rate or the forward rates implied by OIS quotes. Then, what the market usually does is to segment market rates with respect to their application period, thus constructing different zero-coupon bonds for each possible rate length considered.

Then, in the multiple-curve framework the forward rate agreement price is given in the following result.

Result 5.2.3 *Under the multiple-curve framework the price of a forward rate agreement is given by*

$$Fra_t(S,T) = F_t(S,T)(1+\gamma^{Fra}),$$

where

$$\gamma^{Fra} = \frac{Cov_t^T\left(F_S(S,T), Q_S(S,T)\right)}{F_t(S,T)Q_t(S,T)},$$

and

$$F_t(S,T) = \mathbb{E}_t^T\left[L(S,T)\right]$$
$$Q_t(S,T) = \mathbb{E}_t^T\left[\frac{1}{1+(T-S)L(S,T)}\frac{B_T}{B_S}\right].$$

To justify the current divergence between market rates that have the same rate length, practitioners usually deal with those differences by segmenting market rates, labeling them according to their maturity. Instead of constructing different zero-coupon curves, one for each possible rate length considered, we try to model the counterparty risk.

5.2.3 Modelling counterparty risk

The first consequence of the credit crunch is that we can no longer assume that bank counterparties are riskless. Thus several credit models have been proposed in the literature to take into account the difference between the forward rate and the FRA rate, which are based on assuming that the generic counterparty is subject to default risk. Then the price of a defaultable bond at time t is

$$\bar{p}(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(s)\,\mathrm{d}s} 1_{\tau>T} | \tau > t\right] = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(s)+\lambda(s)\,\mathrm{d}s}\right],$$

where τ is the default time of the bond issuer.

Then since the Libor rate is the reference lending rate, we can define it in terms of the risky bond $\bar{p}(t,T)$ as

$$L(t,T) = \frac{1}{T-t}\left(\frac{1}{\bar{p}(t,T)} - 1\right). \tag{5.9}$$

Lemma 5.2.4 *The price of a forward rate agreement taking into account counterparty risk is given by*

$$Fra_t(S,T) = \mathbb{E}_t^{\bar{T}}\left[L(S,T)\right], \tag{5.10}$$

where the expectation is taken under the $\mathbb{Q}^{\bar{T}}$ measure, whose numeraire is $\bar{p}(t,T)$.

Proof. By the definition of a forward rate agreement price (5.4) and (5.9), we have

$$Fra_t(S,T) = \frac{\mathbb{E}_t^{\mathbb{Q}}\left[\frac{\frac{B_t}{B_S}L(S,T)}{1+(T-S)L(S,T)}\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[\frac{\frac{B_t}{B_S}}{1+(T-S)L(S,T)}\right]} = \frac{\mathbb{E}_t^{\mathbb{Q}}\left[\frac{B_t}{B_S}\frac{1-\bar{p}(S,T)}{T-S}\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[\frac{B_t}{B_S}\bar{p}(S,T)\right]}. \tag{5.11}$$

Notice that by risk neutral valuation and Bayes theorem we have

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}\left[\frac{B_t}{B_S}\bar{p}(S,T)\right] &= \mathbb{E}_t^{\mathbb{Q}}\left[\frac{B_t}{B_S}\mathbb{E}_S^{\mathbb{Q}}\left[\frac{B_S}{B_T}1_{\tau>T}|\tau>t\right]\right] \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{E}_S^{\mathbb{Q}}\left[\frac{B_t}{B_S}\frac{B_S}{B_T}1_{\tau>T}|\tau>t\right]\right] \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{E}_S^{\mathbb{Q}}\left[\frac{B_t}{B_T}1_{\tau>T}|\tau>t\right]\right] \\
&= \bar{p}(t,T).
\end{aligned}$$

Then finally using the following change of measure

$$\frac{\mathrm{d}\mathbb{Q}^{\bar{T}}}{\mathrm{d}\mathbb{Q}} = \frac{\bar{p}(S,T)}{B_S}\frac{B_t}{\bar{p}(t,T)},$$

the price of a forward rate agreement contract is given by

$$\begin{aligned} Fra_t(S,T) &= \frac{\mathbb{E}_t^{\bar{T}}\left[\frac{\bar{p}(t,T)}{B_t}\frac{B_S}{\bar{p}(S,T)}\frac{B_t}{B_S}\frac{1-\bar{p}(S,T)}{T-S}\right]}{\bar{p}(t,T)} = \mathbb{E}_t^{\bar{T}}\left[\frac{1}{T-S}\left(\frac{1}{\bar{p}(S,T)}-1\right)\right] \\ &= \mathbb{E}_t^{\bar{T}}\left[L(S,T)\right], \end{aligned}$$

where we have used once again (5.9). ∎

This motivates us to consider the following definition.

Definition 5.2.5 *The Forward Rate Agreement convexity adjustment is defined as*

$$\mathbb{E}_t^{T}\left[L(S,T)\right] = \mathbb{E}_t^{\bar{T}}\left[L(S,T)\right] + CC^{Fra}(t,S,T). \tag{5.12}$$

Note that if $p(t,T) = \bar{p}(t,T)$ then the convexity adjustment is zero, i.e $CC^{Fra}(t,S,T) = 0$.

5.3 Affine Term Structure Models

5.3.1 Non-Defaultable bonds

We assume that the risk-free interest rate r is linear on given factors described by a $\mathbb{R}^m$ valued process $(Z_t)_{t\geq 0}$.

Assumption 5.3.1 *Assume a $\mathbb{R}^m$ valued process $(Z_t)_{t\geq 0}$ whose dynamics are given by*

$$\mathrm{d}Z_t = \alpha(t,Z_t)\,\mathrm{d}t + \sigma(t,Z_t)\,\mathrm{d}W_t^{\mathbb{Q}}, \tag{5.13}$$

where W_t is a n-dimensional standard Brownian motion, $\alpha : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^{m\times n}$ such that

$$\alpha(t,z) = d(t) + E(t)z, \tag{5.14}$$

$$\sigma(t,z)\sigma^T(t,z) = k_0(t) + \sum_{i=1}^{m} k_i(t)z_i, \tag{5.15}$$

with smooth functions $d : \mathbb{R}_+ \to \mathbb{R}^m, E : \mathbb{R}_+ \to \mathbb{R}^{m\times m}, k_0 : \mathbb{R}_+ \to \mathbb{R}^{n\times n}, k_i : \mathbb{R}_+ \to \mathbb{R}^{n\times n}, i = 1,2...,m$ map $\mathbb{R}_+$ to $\mathbb{R}^{n\times n}$. Also, the risk-free short rate $(r_t)_{t\geq 0}$ is given by

$$r(t,Z_t) = g^T(t)Z_t + f(t), \tag{5.16}$$

where $g : \mathbb{R}_+ \to \mathbb{R}^m$ and $f : \mathbb{R}_+ \to \mathbb{R}$ are smooth functions.

In this setup the risk-free bond price is given in the following result and can be found in [41].

Result 5.3.2 *Let Assumption* 5.3.1 *holds. Then the price of a non-defaultable zero coupon bond is given by*

$$p(t,T) = e^{A(t,T)+B^T(t,T)Z_t}, \tag{5.17}$$

where A and B are deterministic functions of (t,T) that solve the ODE system,

$$\begin{cases} \frac{\partial A}{\partial t} + d^T(t)B + \frac{1}{2}B^T k_0(t)B = f(t), \\ A(T,T) = 0 \end{cases} \tag{5.18}$$

$$\begin{cases} \frac{\partial B}{\partial t} + E^T(t)B + \frac{1}{2}\overline{B}^T K(t)B = g(t) \\ B(T,T) = 0, \end{cases} \tag{5.19}$$

where

$$\overline{B} = \begin{pmatrix} B & 0 & \cdots & 0 \\ 0 & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B \end{pmatrix}, K(t) = \begin{pmatrix} k_1(t) \\ k_2(t) \\ \vdots \\ k_m(t) \end{pmatrix}, \tag{5.20}$$

and E,d,k_0 are the same as in (5.14)-(5.15)*, while f and g are as in* (5.16)*. A and B are evaluated at (t,T).*

Proof. Let $G(t,Z_t) = \mathbb{E}[e^{-\int_t^T g^T(s)Z_s+f(s)\,\mathrm{d}s}] = e^{A(t,T)+B(t,T)Z_t}$.

If we apply Ito's formula to $Y_t = G(t,Z_t)e^{-\int_0^t g^T(s)Z_s+f(s)\,\mathrm{d}s}$, we get

$$\begin{aligned} \mathrm{d}(G(t,Z_t)e^{-\int_0^t g^T(s)Z_s+f(s)\,\mathrm{d}s}) &= e^{-\int_0^t g^T(s)Z_s+f(s)\,\mathrm{d}s}\left(\frac{\partial G}{\partial t} - \left(g^T(t)Z_t + f(t)\right)G(t,Z_t)\right)\mathrm{d}t \\ &+ \frac{\partial G}{\partial z}e^{-\int_0^t g^T(s)Z_s+f(s)\,\mathrm{d}s}\,\mathrm{d}Z + \frac{1}{2}\frac{\partial^2 G}{\partial z^2}e^{-\int_0^t g^T(s)Z_s+f(s)\,\mathrm{d}s}(\,\mathrm{d}Z)^2. \end{aligned}$$

Plugging in the dynamics of Z given by (5.13), we obtain

$$\begin{aligned} \mathrm{d}(Y_t) = e^{-\int_0^t g^T(s)Z_s+f(s)\,\mathrm{d}s}&\left(\frac{\partial G}{\partial t} - \left(g^T(t)Z_t + f(t)\right)G(t,Z_t)\right. \\ &+ \left.\frac{\partial G}{\partial z}\alpha(t,Z_t) + \frac{1}{2}tr\left(\sigma^T(t,Z_t)\frac{\partial^2 G}{\partial z^2}\sigma(t,Z_t)\right)\right)\mathrm{d}t \\ &+ \frac{\partial G}{\partial z}e^{-\int_0^t g^T(s)Z_s+f(s)\,\mathrm{d}s}\sigma(t,Z_t)\,\mathrm{d}W_t. \end{aligned} \tag{5.21}$$

Since $G(t,Z_t)e^{-\int_0^t g^T(s)Z_s+f(s)\,\mathrm{d}s}$ is a martingale by construction, we have

$$\begin{aligned} &\frac{\partial G}{\partial t} - \left(g^T(t)z + f(t)\right)G(t,Z_t) + \frac{\partial G}{\partial z}\alpha(t,Z_t) + \frac{1}{2}tr\left(\sigma^T(t,Z_t)\frac{\partial^2 G}{\partial z^2}\sigma(t,Z_t)\right) = 0 \\ &G(T,Z_T) = 1. \end{aligned} \tag{5.22}$$

Due to Assumption 5.3.1 and assuming $G(t, Z_t) = e^{A(t,T)+B^T(t,T)Z_t}$, we compute

$$\frac{\partial G}{\partial t} = \frac{\partial A}{\partial t} G(t, Z_t) + \frac{\partial B}{\partial t} Z_t G(t, Z_t), \tag{5.23}$$
$$\frac{\partial G}{\partial z} = B^T(t,T) G(t, Z_t), \tag{5.24}$$
$$\frac{\partial^2 G}{\partial z^2} = B(t,T) B^T(t,T) G(t, Z_t), \tag{5.25}$$

to plug in (5.22) and using the definition of α and σ given by (5.14) and (5.15) respectively, we obtain

$$\frac{\partial A}{\partial t} G(t, Z_t) + \frac{\partial B}{\partial t} z G(t, Z_t) - \left(g^T(t) z + f(t)\right) G(t, Z_t) + B^T(t,T) G(t, Z_t) \left(d(t) + E(t) z\right)$$
$$+ \frac{1}{2} tr \left(B(t,T) B^T(t,T) \left(k_0(t) + \sum_{i=1}^{m} k_i(t) z_i \right) \right) G(t, Z_t) = 0. \tag{5.26}$$

After some reshuffling we get

$$\frac{\partial A}{\partial t} + \frac{\partial B}{\partial t} z - \left(g^T(t) Z_t + f(t)\right) + B^T(t,T) \left(d(t) + E(t) z\right) \tag{5.27}$$
$$+ \frac{1}{2} tr \left(B^T(t,T) k_0(t) B(t,T) \right) + \frac{1}{2} B^T K(t) \overline{B} z = 0. \tag{5.28}$$

Since $G(T, Z_T) = 1$ we must have $A(T,T) = 0$ and $B(T,T) = 0$ and because this has to hold for every z, we end up with the desired system of equations as stated.

■

5.3.2 Defaultable bonds

5.3.2.1 Convexity Adjustment without the shot-noise process

Next we assume that the default intensity is also linear on given factors.

Assumption 5.3.3 *Assume Z_t is a $\mathbb{R}^m$ valued process. The intensity process λ_t is given by*

$$\lambda(t) = g_\lambda^T(t) Z_t + f_\lambda(t). \tag{5.29}$$

Note that there is no loss of generality in assuming the same factors Z_t as in the risk-free process.

Next we want to be able to compute

$$\bar{p}(t,T) = \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^T r(s)+\lambda(s)\,\mathrm{d}s} \right].$$

Next proposition gives us the price of a defaultable zero coupon bond.

Proposition 5.3.1 *Let Assumption 5.3.3 hold. Then the price of a defaultable zero coupon bond is given by*

$$\bar{p}(t,T) = e^{\bar{A}(t,T)+\bar{B}^T(t,T)Z_t}, \tag{5.30}$$

where $\bar{A}(t,T)$ and $\bar{B}(t,T)$ are deterministic functions of (t,T) that solve the ODE system,

$$\begin{cases} \frac{\partial \bar{A}}{\partial t} + d^T(t)\bar{B}(t,T) + \frac{1}{2}\bar{B}^T(t,T)k_0(t)\bar{B}(t,T) = f(t) + f_\lambda(t) \\ \bar{A}(T,T) = 0, \end{cases} \tag{5.31}$$

$$\begin{cases} \frac{\partial \bar{B}}{\partial t} + E^T(t)\bar{B}(t,T) + \frac{1}{2}\overline{\bar{B}}^T K(t)\bar{B} = g(t) + g_\lambda(t) \\ \bar{B}(T,T) = 0, \end{cases} \tag{5.32}$$

where E,d,k_0 are the same as in (5.14)-(5.15). $\overline{B}$,K *are as in* (5.20) *while* g_λ,f_λ *are given in* (5.29).

Proposition 5.3.2 *Suppose Assumption* 5.3.3 *holds . Then the Convexity adjustment of Definition 5.2.5 is obtained in closed-form and is given by*

$$CC^{Fra}(t,S,T) = \frac{1}{T-S}\frac{\bar{p}(t,S)}{\bar{p}(t,T)}\left(e^{F(t,S,T)+G(t,S,T)Z_t} - 1\right), \tag{5.33}$$

where F and G solve the deterministic system of ODE

$$\begin{cases} \frac{\partial F}{\partial t} + \left(\bar{B}^T(t,S) - \bar{B}(t,T)^T\right)k_0(t)\left(B(t,T) - \bar{B}(t,T)\right) \\ +G(t,S,T)d(t) + Gk_0(t)B(t,T) + \frac{1}{2}Gk_0(t)G^T + G(t,S,T)k_0(t)(\bar{B}(t,T) - \bar{B}(t,S)) = 0\,, \\ F(T,S,T) = 0 \end{cases}$$

$$\begin{cases} \frac{\partial G}{\partial t} + \left(I \otimes \left(\bar{B}^T(t,S) - \bar{B}^T(t,T)\right)\right)K(t)\left(B(t,T) - \bar{B}(t,T)\right) + GE(t) + \tilde{G}K(t)B(t,T) \\ +\frac{1}{2}\tilde{G}K(t)G^T + \tilde{G}K(t)(\bar{B}(t,T) - \bar{B}(t,S)) = 0\,, \\ G(T,S,T) = 0 \end{cases}$$

where

$$\tilde{G} = \begin{pmatrix} G & 0 & \cdots & 0 \\ 0 & G & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G \end{pmatrix} \tag{5.34}$$

and I denotes the identity matrix and E,d,k_0 are the same as in (5.14)-(5.15), *while $\bar{B}$ is the solution of the ODE system* (5.31)-(5.32), *F and G are evaluated at (t,S,T). Also $K(t)$ is given as in* (5.20).

Proof. By definition, we have

$$\mathbb{E}_t^T[L(S,T)] = \frac{1}{T-S}\mathbb{E}_t^T\left[\frac{1}{\bar{p}(S,T)} - 1\right] \Leftrightarrow 1 + (T-S)\mathbb{E}_t^T\left[L(S,T)\right] = \mathbb{E}_t^T\left[\frac{\bar{p}(S,S)}{\bar{p}(S,T)}\right].$$

If we define $M(t,S,T) = \mathbb{E}_t^T\left[\frac{\bar{p}(S,S)}{\bar{p}(S,T)}\right] = \frac{\bar{p}(t,S)}{\bar{p}(t,T)} e^{F(t,S,T)+G(t,S,T)Z_t}$ we know that $M(t,S,T)$ is a $\mathbb{Q}^T$- martingale.

By Girsanov theorem with kernel given by $v(t,T) = B^T(t,T)\sigma(t,T)$ we have

$$\mathrm{d}W_t^{\mathbb{Q}} = v^T(t,T)\,\mathrm{d}t + \mathrm{d}W_t^{\mathbb{Q}^T}, \tag{5.35}$$

which gives the dynamics of Z_t under $\mathbb{Q}^T$

$$\mathrm{d}Z_t = \left(\alpha(t,Z_t) + \sigma(t,Z_t)v^T(t,T)\right)\,\mathrm{d}t + \sigma(t,Z_t)\,\mathrm{d}W_t^T. \tag{5.36}$$

Now applying Ito's formula to (5.30) we have under $\mathbb{Q}^T$

$$\begin{aligned}\mathrm{d}\bar{p}(t,T) = \Bigg(&\frac{\partial \bar{A}}{\partial t} + \frac{\partial \bar{B}}{\partial t}z + \bar{B}^T\left(d(t) + E(t)z\right) + \bar{B}^T\sigma(t,Z_t)v^T(t,T)\\ &+\frac{1}{2}\bar{B}^T\sigma\sigma^T\bar{B}\Bigg)\bar{p}(t,T)\,\mathrm{d}t + \bar{p}(t,T)\bar{B}^T\sigma\,\mathrm{d}W_t^T,\end{aligned}$$

by plugging in the dynamics for Z_t and taking into account (5.14) and (5.15).

Since A and B solve (5.31)-(5.32) we have

$$\begin{aligned}\mathrm{d}\bar{p}(t,T) = &\left(\left(g^T(t) + g_\lambda^T(t)\right)z + f_\lambda(t) + f(t) + \bar{B}^T\sigma(t,Z_t)v^T(t,T)\right)\bar{p}(t,T)\,\mathrm{d}t\\ &+\bar{p}(t,T)\bar{B}^T\sigma\,\mathrm{d}W_t^T.\end{aligned}$$

Now using the fact that the intensity process is given by (5.29) and short rate is given by (5.16), we end up with the dynamics for the defaultable zero coupon bond under the forward measure

$$\mathrm{d}\bar{p}(t,T) = \left(r_t + \lambda_t + \bar{B}^T(t,T)\sigma(t,Z_t)v^T(t,T)\right)\bar{p}(t,T)\,\mathrm{d}t + \bar{p}(t,T)\bar{B}^T(t,T)\sigma(t,T)\,\mathrm{d}W_t^T.$$

Now similarly for the zero-coupon bond with maturity S, performing Girsanov theorem with the same kernel $v(t,T) = B^T(t,T)\sigma(t,T)$, we get

$$\mathrm{d}\bar{p}(t,S) = \left(r_t + \lambda_t + \bar{B}^T(t,S)\sigma(t,Z_t)v^T(t,T)\right)\bar{p}(t,T)\,\mathrm{d}t + \bar{p}(t,S)\bar{B}^T(t,S)\sigma(t,S)\,\mathrm{d}W_t^T$$

Define $\bar{v}(t,T) = \bar{B}^T(t,T)\sigma(t,Z_t)$. Applying Ito formula to $Y_t = \frac{\bar{p}(t,S)}{\bar{p}(t,T)}$ we get

$$\mathrm{d}Y_t = Y_t\tilde{v}(t,S,T)\,\mathrm{d}t + Y_t\left(\bar{v}(t,S) - \bar{v}(t,T)\right)\,\mathrm{d}W_t^T,$$

where

$$\tilde{v}(t,S,T) = \left(\bar{v}(t,S) - \bar{v}(t,T)\right)\left(v^T(t,T) - \bar{v}^T(t,T)\right).$$

Applying Ito formula to $M(t,S,T) = \frac{\bar{p}(t,S)}{\bar{p}(t,T)} e^{F(t,S,T)+G(t,S,T)Z_t}$ we get

$$\begin{aligned}\mathrm{d}M = &\frac{\partial M}{\partial t}\,\mathrm{d}t + \frac{\partial M}{\partial y}\left(Y_t\tilde{v}(t,S,T)\,\mathrm{d}t + Y_t\left(\bar{v}(t,S) - \bar{v}(t,T)\right)\,\mathrm{d}W_t^T\right)\\ &+\frac{\partial M}{\partial z}\left(\left(\alpha(t,Z_t) + \sigma(t,Z_t)v(t,T)^T\right)\,\mathrm{d}t + \sigma(t,Z_t)\,\mathrm{d}W_t^T\right)\\ &+\frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 M}{\partial z_i \partial z_j}\sigma_i\sigma_j\,\mathrm{d}t + \frac{\partial^2 M}{\partial z \partial y}\left(\sigma(t,Z_t)\right)\left(Y_t\left(\bar{v}(t,S) - \bar{v}(t,T)\right)\right)\,\mathrm{d}t.\end{aligned} \tag{5.37}$$

Since $M(t,S,T)$ is a $\mathbb{Q}^T$-martingale then

$$\frac{\partial M}{\partial t}+\frac{\partial M}{\partial y}Y_t\tilde{v}(t,S,T)+\frac{\partial M}{\partial z}\left(\alpha(t,Z_t)+\sigma(t,Z_t)v(t,T)^T\right)$$
$$+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 M}{\partial z_i\partial z_j}\sigma_i\sigma_j+Y_t\frac{\partial^2 M}{\partial z\partial y}\left(\sigma(t,Z_t)\right)\left(\bar{v}^T(t,S)-\bar{v}^T(t,T)\right)=0.$$

Noticing that

$$\frac{\partial M}{\partial t}=\left(\frac{\partial F}{\partial t}+\frac{\partial G}{\partial t}Z\right)M,\ \frac{\partial M}{\partial y}=e^{F(t,S,T)+G(t,S,T)Z_t},$$
$$\frac{\partial M}{\partial z}=G(t,S,T)M,\ \frac{\partial^2 M}{\partial z_i\partial z_j}=G_iG_jM,$$
$$\frac{\partial^2 M}{\partial z\partial y}=G(t,S,T)e^{F(t,S,T)+G(t,S,T)Z_t},$$

we end up with

$$\left(\frac{\partial F}{\partial t}+\frac{\partial G}{\partial t}z\right)+\tilde{v}(t,S,T)+\frac{1}{2}G\sigma\sigma^TG^T+G(t,S,T)\left(\alpha(t,Z_t)+\sigma(t,Z_t)v(t,T)^T\right)$$
$$+G(t,S,T)\left(\sigma(t,Z_t)\right)\left(\bar{v}^T(t,S)-\bar{v}^T(t,T)\right)=0.$$

Taking into account the affine dynamics of Z, former PDE has the following form

$$\left(\frac{\partial F}{\partial t}+\frac{\partial G}{\partial t}z\right)+\tilde{v}(t,S,T)+G(t,S,T)\left(d(t)+E(t)z+(k_0(t)+\sum_{i=1}^{m}k_i(t)z_i)B(t,T)\right)$$
$$+\frac{1}{2}G\left(k_0(t)+\sum_{i=1}^{m}k_i(t)z_i\right)G^T+G(t,S,T)\left(k_0(t)+\sum_{i=1}^{m}k_i(t)z_i\right)(\bar{B}(t,T)-\bar{B}(t,S))=0,$$

where now $\tilde{v}(t,S,T)$ is given by

$$\tilde{v}(t,S,T)=\left(\bar{B}(t,S)-\bar{B}(t,T)\right)\left(k_0(t)+\sum_{i=1}^{m}k_i(t)z_i\right)\left(B^T(t,T)-\bar{B}^T(t,T)\right).$$

Separating variables yields the following system of equations

$$\begin{cases}\frac{\partial F}{\partial t}+\left(\bar{B}^T(t,S)-\bar{B}^T(t,T)\right)k_0(t)\left(B(t,T)-\bar{B}(t,T)\right)+\frac{1}{2}Gk_0(t)G^T\\ +G(t,S,T)d(t)+G(t,S,T)k_0(t)B(t,T)+G(t,S,T)k_0(t)(\bar{B}(t,T)-\bar{B}(t,S))=0\,,\\ F(T,S,T)=0\end{cases}$$
$$\begin{cases}\frac{\partial G}{\partial t}+\left(\bar{B}^T(t,S)-\bar{B}^T(t,T)\right)\left(\sum_{i=1}^m k_i(t)z_i\right)\left(B(t,T)-\bar{B}(t,T)\right)\\ +G(t,S,T)E(t)+G(t,S,T)\sum_{i=1}^m k_i(t)z_iB(t,T)\\ +\frac{1}{2}G\sum_{i=1}^m k_i(t)z_iG^T+G(t,S,T)\sum_{i=1}^m k_i(t)z_i(\bar{B}(t,T)-\bar{B}(t,S))=0\,,\\ G(T,S,T)=0.\end{cases}$$

If we define $\tilde{G}$ by (5.34) then we get the claimed PDE's. Moreover, since

$$1 + (T-S)\mathbb{E}_t^T\left[L(S,T)\right] = \frac{\bar{p}(t,S)}{\bar{p}(t,T)} e^{F(t,S,T)+G(t,S,T)Z_t},$$

we can solve for $\mathbb{E}_t^T\left[L(S,T)\right]$ we get

$$\mathbb{E}_t^T\left[L(S,T)\right] = \frac{1}{T-S}\left(\frac{\bar{p}(t,S)}{\bar{p}(t,T)} e^{F(t,S,T)+G(t,S,T)Z_t} - 1\right).$$

Since

$$Fra_t(S,T) = \mathbb{E}_t^{\bar{T}}\left[L(S,T)\right] = L(t,S,T) = \frac{1}{T-S}\left(\frac{\bar{p}(t,S)}{\bar{p}(t,T)} - 1\right),$$

we use the definition 5.2.5 and obtain the claimed Convexity adjustment.

■

Notice that when $\bar{p}(t,T) = p(t,T)$ then $\bar{A}(t,T) = A(t,T)$, $\bar{B}(t,T) = B(t,T)$, meaning that $G = 0$ which implies $F = 0$. This way $CC^{Fra}(t,S,T) = 0$.

Now we present an example when the short rate and the intensity default follow the Vasicek model.

Example 5.3.4

$$\mathrm{d}r_t = (a - br_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^r, \tag{5.38}$$

$$\mathrm{d}\lambda_t = (a_\lambda - b_\lambda\lambda_t)\,\mathrm{d}t + \sigma_\lambda\,\mathrm{d}W_t^\lambda. \tag{5.39}$$

In this case m=2 and

$$Z_t = \begin{bmatrix} r_t \\ \lambda t \end{bmatrix}, g(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, g_\lambda(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, d(t) = \begin{bmatrix} a \\ a_\lambda, \end{bmatrix} \tag{5.40}$$

$$E(t) = \begin{bmatrix} -b & 0 \\ 0 & -b_\lambda \end{bmatrix}, k_0 = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma_\lambda^2 \end{bmatrix}, \tag{5.41}$$

with $f(t) = f_\lambda(t) = 0$,$K = 0$. *Then the solution of the system of ODE's* (5.18) – (5.19) *becomes*

$$\bar{B}_1(t,T) = \frac{1}{b}\left(e^{-b(T-t)} - 1\right), \tag{5.42}$$

$$\bar{B}_2(t,T) = \frac{1}{b_\lambda}\left(e^{-b_\lambda(T-t)} - 1\right), \tag{5.43}$$

$$\bar{A}(t,T) = \left(\bar{B}_1(t,T) + T - t\right)\left(\frac{a}{b} - \frac{\sigma^2}{b^2}\right) - \frac{1}{4}\frac{\sigma^2}{b}\bar{B}_1^2(t,T) + \left(\bar{B}_2(t,T) + T - t\right)\left(\frac{a_\lambda}{b_\lambda} - \frac{\sigma_\lambda^2}{b_\lambda^2}\right) - \frac{1}{4}\frac{\sigma_\lambda^2}{b_\lambda}\bar{B}_2^2(t,T). \tag{5.44}$$

Then the convexity adjustment is given by

$$CC^{Fra}(t,S,T) = \frac{1}{T-S}\frac{\bar{p}(t,S)}{\bar{p}(t,T)}\left(e^{F(t,S,T)+G(t,S,T)Z_t} - 1\right), \tag{5.45}$$

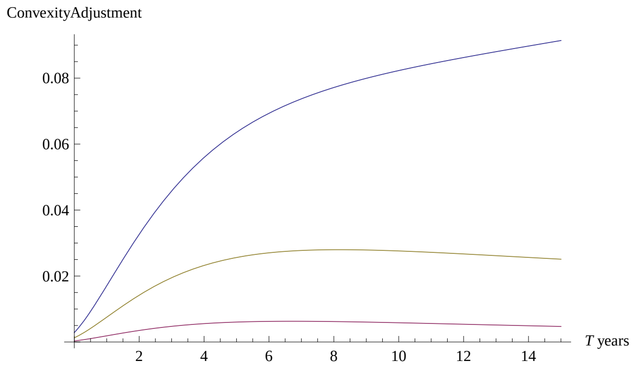

Figure 5.1: The Convexity Adjustment under the Vasicek Model with $a = 0.3, a_\lambda = 0.4, b = 0.05, b_\lambda = 0.1, \sigma_\lambda = 0.05, r = 0.05, \lambda = 0.2$ when $T = S + 0.5$. Blue: σ=0.15,Brown: σ=0.1,Red: σ=0.05.

where F and G solve the deterministic system of ODE

$$\begin{cases} \frac{\partial F}{\partial t} + \left(\bar{B}_1{}^T(t,S) - \bar{B}_1{}^T(t,T)\right)\sigma^2\left(B_1(t,T) - \bar{B}_1(t,T)\right) \\ + \left(\bar{B}_2{}^T(t,S) - \bar{B}_2{}^T(t,T)\right)\sigma_\lambda^2\left(B_2(t,T) - \bar{B}_2(t,T)\right) + G_1 a + G_2 a_\lambda + G_1\sigma^2 B_1(t,T) \\ +G_2\sigma_\lambda^2 B_2(t,T) + \frac{1}{2}G_1\sigma^2 G_1 + \frac{1}{2}G_2\sigma_\lambda^2 G_2 \\ +G_1\sigma^2(\bar{B}_1(t,T) - \bar{B}_1(t,S)) + G_2\sigma_\lambda^2(\bar{B}_2(t,T) - \bar{B}_2(t,S)) = 0, \\ F(T,S,T) = 0 \end{cases}$$

$$\begin{cases} \frac{\partial G_1}{\partial t} - bG_1 = 0, \\ \frac{\partial G_2}{\partial t} - b_\lambda G_2 = 0, \\ G(T,S,T) = 0. \end{cases}$$

But this way $G_1 = G_2 = 0$, which implies that F is given by the solution of the following equation satisfying $F(T,S,T) = 0$

$$\begin{aligned} \frac{\partial F}{\partial t} &+ \left(\bar{B}_1{}^T(t,S) - \bar{B}_1{}^T(t,T)\right)\sigma^2\left(B_1(t,T) - \bar{B}_1(t,T)\right) \\ &+ \left(\bar{B}_2{}^T(t,S) - \bar{B}_2{}^T(t,T)\right)\sigma_\lambda^2\left(B_2(t,T) - \bar{B}_2(t,T)\right) = 0. \end{aligned} \tag{5.46}$$

In figure 5.1 we can observe the Convexity Adjustment when $a = 0.3, a_\lambda = 0.4, b = 0.05, b_\lambda = 0.1, \sigma = 0.15, \sigma_\lambda = 0.05, r = 0.05, \lambda = 0.2$ for several maturities.

Now we present an example when the short rate follows the CIR model and the intensity default follows the Vasicek model.

Example 5.3.5

$$\mathrm{d}r_t = (a - br_t)\,\mathrm{d}t + \sigma\sqrt{r_t}\,\mathrm{d}W_t^r, \tag{5.47}$$

$$\mathrm{d}\lambda_t = (a_\lambda - b_\lambda\lambda_t)\,\mathrm{d}t + \sigma_\lambda\,\mathrm{d}W_t^\lambda. \tag{5.48}$$

In this case m=2 and

$$Z_t = \begin{bmatrix} r_t \\ \lambda t \end{bmatrix}, g(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, g_\lambda(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, d(t) = \begin{bmatrix} a \\ a_\lambda \end{bmatrix}, \tag{5.49}$$

$$E(t) = \begin{bmatrix} -b & 0 \\ 0 & -b_\lambda \end{bmatrix}, k_0 = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_\lambda^2 \end{bmatrix}, k_1 = \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.50}$$

with $f(t) = f_\lambda(t) = 0$. *Then the solution of the system of ODE's* $(5.18) - (5.19)$ *becomes*

$$\bar{B}_1(t,T) = \frac{2\gamma_1\gamma_2}{\sigma^2}\left(\frac{e^{-\gamma_2(T-t)} - e^{-\gamma_1(T-t)}}{\gamma_1 e^{-\gamma_2(T-t)} - \gamma_2 e^{-\gamma_1(T-t)}}\right), \tag{5.51}$$

$$\bar{B}_2(t,T) = \frac{1}{b_\lambda}\left(e^{-b_\lambda(T-t)} - 1\right), \tag{5.52}$$

$$\bar{A}(t,T) = -\frac{a}{\sigma^2}\log\left(\frac{e^{-\gamma_2(T-t)} - e^{-\gamma_1(T-t)}}{\gamma_1 - \gamma_2}\right) + \left(\bar{B}_2(t,T) + T - t\right)\left(\frac{a_\lambda}{b_\lambda} - \frac{\sigma_\lambda^2}{b_\lambda^2}\right)$$
$$-\frac{1}{4}\frac{\sigma_\lambda^2}{b_\lambda}\bar{B}_2^2(t,T), \tag{5.53}$$

where

$$\gamma_1 = \frac{b + \sqrt{b^2 + 2\sigma^2}}{2}, \gamma_2 = \frac{b - \sqrt{b^2 + 2\sigma^2}}{2}. \tag{5.54}$$

Then the convexity adjustment is given by

$$CC^{Fra}(t,S,T) = \frac{1}{T-S}\frac{\bar{p}(t,S)}{\bar{p}(t,T)}\left(e^{F(t,S,T)+G(t,S,T)Z_t} - 1\right), \tag{5.55}$$

where F *and* G *solve the deterministic system of ODE*

$$\begin{cases} \frac{\partial F}{\partial t} + \left(\bar{B_2}^T(t,S) - \bar{B_2}^T(t,T)\right)\sigma_\lambda^2\left(B_2(t,T) - \bar{B}_2(t,T)\right) \\ +G_1 a + G_2 a_\lambda + G_1\sigma_\lambda^2 B_2(t,T) + \frac{1}{2}G_2\sigma_\lambda^2 G_2 + G_2\sigma_\lambda^2(\bar{B}_2(t,T) - \bar{B}_2(t,S)) = 0, \\ F(T,S,T) = 0, \end{cases}$$

$$\begin{cases} \frac{\partial G_1}{\partial t} + \left(\bar{B_1}^T(t,S) - \bar{B_1}^T(t,T)\right)\sigma^2\left(B_1(t,T) - \bar{B}_1(t,T)\right) \\ -bG_1 + G_1\sigma^2\bar{B}_1(t,T) + \frac{1}{2}G_1^2\sigma^2 + G_1\left(\bar{B}_1(t,T) - \bar{B}_1(t,S)\right)\sigma^2 = 0, \\ \frac{\partial G_2}{\partial t} - b_\lambda G_2 = 0, \\ G(T,S,T) = 0. \end{cases}$$

In figure (5.2) *we can observe the Convexity Adjustment when* $a = 0.3, a_\lambda = 0.4, b = 0.05, b_\lambda = 0.1, \sigma = 0.15, \sigma_\lambda = 0.05, r = 0.05, \lambda = 0.2$ *for several maturities under the CIR model.*

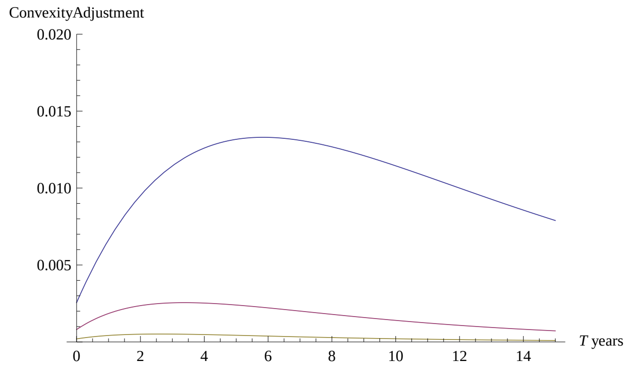

Figure 5.2: The Convexity Adjustment under the CIR model with $a = 0.3, a_\lambda = 0.4, b = 0.05, b_\lambda = 0.1, \sigma_\lambda = 0.05, r = 0.05, \lambda = 0.2$ when $T = S + 0.5$. Blue: σ=0.15,Red: σ=0.1,Brown: σ=0.05.

5.3.2.2 Convexity Adjustment with the shot-noise process

Now we model the default intensity combining an affine term with a shot noise process. We consider a reduced-form type of model that still keeps some tractability of classical ATS setting and is able to model portfolio credit risk. With the shot-noise process we can incorporate realistic features such as clustering of defaults within firms and correlation of defaults across firms.

Assumption 5.3.6 *Assume Z_t is a $\mathbb{R}^m$ valued process. The intensity process μ_t is given by*

$$\mu_t = g_\lambda^T(t) Z_t + f_\lambda(t) + \sum_{\tilde{\tau}_i \le t} Y_i h(t - \tilde{\tau}_i), \tag{5.56}$$

where $\tilde{\tau}_1, \tilde{\tau}_2, ...$ are the jumping times of a Poisson process N with intensity ν. $Y_i, i = 1, 2, ..$ are i.i.d, independent of W and N. $g_\lambda^T(t) : \mathbb{R}_+ \to \mathbb{R}^m$, $f_\lambda(t) : \mathbb{R}_+ \to \mathbb{R}$ and $h : \mathbb{R}_+ \to \mathbb{R}$ are smooth functions. The default time τ is a doubly stochastic random time with intensity $(\mu_t)_{t\ge 0}$.

Proposition 5.3.3 *Let Assumption 5.3.6 hold. Then the price of a defaultable zero coupon bond is given by*

$$\bar{p}(t, T) = e^{\bar{A}(t,T) + \bar{B}^T(t,T) Z_t} \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^T \sum_{\tilde{\tau}_i \le s} Y_i h(s - \tilde{\tau}_i)\,\mathrm{d}s} \right], \tag{5.57}$$

where $\bar{A}(t, T)$ and $\bar{B}(t, T)$ are deterministic functions of (t, T) that solve the ODE system (5.31)-(5.32)

and where

$$\mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^T \sum_{\tilde{\tau}_i \le s} Y_i h(s - \tilde{\tau}_i)\,\mathrm{d}s} \right] = e^{-\sum_{\tilde{\tau}_i \le t} Y_i \int_{t-\tilde{\tau}_i}^{T-\tilde{\tau}_i} h(u)\,\mathrm{d}u} e^{\nu(T-t)(D(T-t)-1)},$$

$$D(T - t) = \int_0^1 \mathbb{E}^{\mathbb{Q}} \left[e^{-Y \int_0^{(T-t)\xi} h(u)\,\mathrm{d}u} \right] \mathrm{d}\xi. \tag{5.58}$$

Proof.

Observe that after change of variables $u = s - \tilde{\tau}_i, t \leq s \leq T \Rightarrow t - \tilde{\tau}_i \leq u \leq T - \tilde{\tau}_i$, we have

$$e^{-\int_t^T \sum_{\tilde{\tau}_i \leq t} Y_i h(s-\tilde{\tau}_i)\,\mathrm{d}s} = e^{-\sum_{\tilde{\tau}_i \leq t} Y_i \int_{t-\tilde{\tau}_i}^{T-\tilde{\tau}_i} h(u)\,\mathrm{d}u},$$

which gives us

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \sum_{\tilde{\tau}_i \leq s} Y_i h(s-\tilde{\tau}_i)\,\mathrm{d}s}\right] &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \sum_{\tilde{\tau}_i \leq t} Y_i h(s-\tilde{\tau}_i) + \sum_{\tilde{\tau}_i \in (t,s)} Y_i h(s-\tilde{\tau}_i)\,\mathrm{d}s}\right] \\
&= e^{-\int_t^T \sum_{\tilde{\tau}_i \leq t} Y_i h(s-\tilde{\tau}_i)\,\mathrm{d}s} \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \sum_{\tilde{\tau}_i \in (t,s)} Y_i h(s-\tilde{\tau}_i)\,\mathrm{d}s}\right] \\
&= e^{-\sum_{\tilde{\tau}_i \leq t} Y_i \int_{t-\tilde{\tau}_i}^{T-\tilde{\tau}_i} h(u)\,\mathrm{d}u} \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \sum_{\tilde{\tau}_i \in (t,s)} Y_i h(s-\tilde{\tau}_i)\,\mathrm{d}s}\right].
\end{aligned}$$

The second term can be computed in the following way

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \sum_{\tilde{\tau}_i \in (t,s)} Y_i h(s-\tilde{\tau}_i)\,\mathrm{d}s}\right] &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \sum_{\tilde{\tau}_i \in (t,T)} 1_{\tilde{\tau}_i \leq s} Y_i h(s-\tilde{\tau}_i)\,\mathrm{d}s}\right] \\
&= \sum_{k=0}^{\infty} e^{-\nu(T-t)} \frac{(\nu(T-t))^k}{k!} \mathbb{E}^{\mathbb{Q}}\left[e^{-\sum_{i=1}^k \int_{\tilde{\tau}_i}^T Y_i h(s-\tilde{\tau}_i)\,\mathrm{d}s}\right] \\
&= \sum_{k=0}^{\infty} e^{-\nu(T-t)} \frac{(\nu(T-t))^k}{k!} \mathbb{E}^{\mathbb{Q}}\left[e^{-\sum_{i=1}^k Y_i \int_0^{T-\tilde{\tau}_i} h(u)\,\mathrm{d}u}\right] \\
&= \sum_{k=0}^{\infty} e^{-\nu(T-t)} \frac{(\nu(T-t))^k}{k!} \mathbb{E}^{\mathbb{Q}}\left[e^{-\sum_{i=1}^k Y_i \int_0^{(T-t)(1-\eta_i)} h(u)\,\mathrm{d}u}\right],
\end{aligned}$$

since $u = s - \tilde{\tau}_i, \tilde{\tau}_i \leq s \leq T \Rightarrow 0 \leq u \leq T - \tilde{\tau}_i$ and because the distribution of jump times conditional on the number of jumps follows the distribution of order statistics of uniform *i.i.d* random variables denoted by $\eta_i, i = 1, 2, ..$ over the interval.

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[e^{-\sum_{i=1}^k Y_i \int_0^{(T-t)(1-\eta_i)} h(u)\,\mathrm{d}s}\right] &= \left(\mathbb{E}^{\mathbb{Q}}\left[e^{-Y \int_0^{(T-t)(1-\eta_1)} h(u)\,\mathrm{d}u}\right]\right)^k \\
&= \left(\int_0^1 \mathbb{E}^{\mathbb{Q}}\left[e^{-Y \int_0^{(T-t)\xi} h(u)\,\mathrm{d}u}\right] \mathrm{d}\xi\right)^k \\
&= D(T-t)^k,
\end{aligned}$$

since $\xi = 1 - \eta_i, 0 \leq u \leq (T-t)(1-\eta_i) \Rightarrow 0 \leq u \leq (T-t)\xi$. Therefore

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \sum_{\tilde{\tau}_i \in (t,s)} Y_i h(s-\tilde{\tau}_i)\,\mathrm{d}s}\right] = e^{\nu(T-t)(D(T-t)-1)}. \tag{5.59}$$

Then

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \sum_{\tilde{\tau}_i \leq s} Y_i h(s-\tilde{\tau}_i)\,\mathrm{d}s}\right] = e^{-\sum_{\tilde{\tau}_i \leq t} Y_i \int_{t-\tilde{\tau}_i}^{T-\tilde{\tau}_i} h(u)\,\mathrm{d}u} e^{\nu(T-t)(D(T-t)-1)}.$$

Finally by risk neutral valuation we have

$$\begin{aligned}
\bar{p}(t,T) &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(s)+\lambda(s)+\sum_{\tilde{\tau}_i \leq s} Y_i h(s-\tilde{\tau}_i)\,\mathrm{d}s}\right] \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(s)+\lambda(s)\,\mathrm{d}s}\right] \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \sum_{\tilde{\tau}_i \leq s} Y_i h(s-\tilde{\tau}_i)\,\mathrm{d}s}\right],
\end{aligned}$$

and since we already computed the first term in Proposition (5.3.2) given by (5.30), we get (5.57). ■

As proven in [43] the shot noise process is Markovian if and only if is of the form $h(t) = ae^{-bt}$. We will consider only the special case when $a = 1$ and $b = 0$.

Proposition 5.3.4 *Suppose Assumption* 5.3.6 *holds for the case when there is a shot noise Process. Then the Convexity adjustment of Definition 5.2.5 is obtained in closed-form and is given by*

$$CC^{Fra}(t,S,T) = \frac{1}{T-S}\frac{\bar{p}(t,S)}{\bar{p}(t,T)}\left(e^{F(t,S,T)+G(t,S,T)Z_t+\int_0^t\int_{\mathbb{R}}\tilde{K}(s,y)N(\,\mathrm{d}s,\mathrm{d}y)}-1\right), \quad (5.60)$$

where F and G solve the deterministic system of ODE

$$\begin{cases} \frac{\partial F}{\partial t} + \left(\bar{B}^T(t,S) - \bar{B}^T(t,T)\right)k_0(t)\left(B(t,T) - \bar{B}(t,T)\right) + G^T d(t) + G^T k_0(t) B(t,T) \\ +\frac{\partial \bar{C}}{\partial t}(t,S) - \frac{\partial \bar{C}}{\partial t}(t,T) + \frac{1}{2}G^T k_0(t) G + G^T k_0(t)(\bar{B}(t,T) - \bar{B}(t,S)) \\ +\int_{\mathbb{R}} e^{\tilde{K}(s,y)} - 1\nu(\,\mathrm{d}y) = 0 \\ F(T,S,T) = 0, \end{cases}$$

$$\begin{cases} \frac{\partial G}{\partial t} + \left(I\otimes\left(\bar{B}^T(t,S) - \bar{B}^T(t,T)\right)\right)K(t)\left(B(t,T) - \bar{B}(t,T)\right) + G^T E(t) + \tilde{G}^T K(t) B \\ +\frac{1}{2}\tilde{G}^T K(t) G + \tilde{G}^T K(\bar{B}(t,T) - \bar{B}(t,S)) = 0, \\ G(t,S,T) = 0, \end{cases}$$

where F and G should be evaluated at (t,S,T), I is the identity matrix and $\tilde{G}$ is defined in the same way as in (5.34)*. Also $K(t)$ is given as in* (5.20)*. Also $\tilde{K}(u,y) = -\int_u^T yh(s-\tilde{\tau}_i)\,\mathrm{d}s$.*

Proof. By definition we have

$$\mathbb{E}_t^T[L(S,T)] = \frac{1}{T-S}\mathbb{E}_t^T\left[\frac{1}{\bar{p}(S,T)} - 1\right] \Leftrightarrow 1 + (S-T)\mathbb{E}_t^T\left[L(S,T)\right] = \mathbb{E}_t^T\left[\frac{\bar{p}(S,S)}{\bar{p}(S,T)}\right].$$

If we define $M(t,S,T) = \mathbb{E}_t^T\left[\frac{\bar{p}(S,S)}{\bar{p}(S,T)}\right] = \frac{\bar{p}(t,S)}{\bar{p}(t,T)}e^{F(t,S,T)+G(t,S,T)Z_t+\int_0^t\int_{\mathbb{R}}\tilde{K}(s,y)N(\,\mathrm{d}s,\mathrm{d}y)}$ we know that $M(t,S,T)$ is a $\mathbb{Q}^T$- martingale.

$$\bar{p}(t,T) = e^{\bar{A}(t,T)+\bar{B}^T(t,T)Z_t-\int_t^T\sum_{i=1}^{N_t}Y_i h(s-\tilde{\tau}_i)\,\mathrm{d}s+\bar{C}(t,T)}, \quad (5.61)$$

or changing variables in the jump term we have

$$\bar{p}(t,T) = e^{\bar{A}(t,T)+\bar{B}^T(t,T)Z_t+\bar{C}(t,T)+\int_0^t\int_{\mathbb{R}}\tilde{K}(s,y)N(\,\mathrm{d}s,\mathrm{d}y)}, \quad (5.62)$$

where $\tilde{K}(u,y) = \int_u^T -Y_i h(s-\tilde{\tau}_i)\,\mathrm{d}s$ and $\bar{C}(t,T) = \nu(T-t)(D(T-t)-1)$.

In order to get the dynamics of $\bar{p}(t,T)$ under the T-measure we apply Ito's lemma to $\bar{p}(t,T)$ and obtain

$$\mathrm{d}\bar{p}(t,T) = a(t,T)\bar{p}(t,T)\,\mathrm{d}t + \bar{p}(t,T)\bar{B}^T\sigma\,\mathrm{d}W_t^{\mathbb{Q}} + \bar{p}(t,T)\int_{\mathbb{R}} e^{\tilde{K}(s,y)} - 1\,\tilde{N}(\,\mathrm{d}s,\,\mathrm{d}y),$$

where

$$a(t,T) = \frac{\partial \bar{A}}{\partial t} + \frac{\partial \bar{B}}{\partial t} Z_t + \frac{\partial \bar{C}}{\partial t} + \bar{B}^T \alpha(t, Z_t) \\ + \frac{1}{2} \bar{B}^T \sigma(t, Z_t) \sigma^T(t, Z_t) \bar{B}(t,T) + \int_{\mathbb{R}} e^{\tilde{K}(s,y)} - 1\, \nu(\,\mathrm{d}y).$$

Similar to the proof of Proposition (5.3.2), by Girsanov theorem and remembering that $\bar{A}$ and $\bar{B}$ solve the ODE system (5.31)-(5.32) as well as the affine setup for the short rate and default intensity, we get

$$\mathrm{d}\bar{p}(t,T) = \left(r_t + \lambda_t + \bar{v}(t,T) v^T(t,T) + \frac{\partial \bar{C}}{\partial t}(t,T) + \int_{\mathbb{R}} e^{\tilde{K}(s,y)} - 1\, \nu(\,\mathrm{d}y) \right) \bar{p}(t,T)\, \mathrm{d}t \\ + \bar{p}(t,T) \bar{B}^T \sigma\, \mathrm{d}W_t^T + \bar{p}(t,T) \int_{\mathbb{R}} e^{\tilde{K}(s,y)} - 1\, \tilde{N}(\,\mathrm{d}s,\,\mathrm{d}y).$$

We can also deduce the dynamics of a zero-coupon bond with maturity S

$$\mathrm{d}\bar{p}(t,S) = \left(r_t + \lambda_t + \bar{v}(t,S) v^T(t,T) + \frac{\partial \bar{C}}{\partial t}(t,S) + \int_{\mathbb{R}} e^{\tilde{K}(s,y)} - 1 \nu(\,\mathrm{d}y) \right) \bar{p}(t,S)\, \mathrm{d}t \\ + \bar{p}(t,S) \bar{B}^T(t,S) \sigma\, \mathrm{d}W_t^T + \bar{p}(t,S) \int_{\mathbb{R}} e^{\tilde{K}(s,y)} - 1 \tilde{N}(\,\mathrm{d}s,\,\mathrm{d}y).$$

Applying Ito's formula to $Y_t = \frac{\bar{p}(t,S)}{\bar{p}(t,T)}$ yields after some calculations

$$\mathrm{d}Y_t = \bar{p}(t,S)\, \mathrm{d}\frac{1}{\bar{p}(t,T)} + \frac{1}{\bar{p}(t,T)}\, \mathrm{d}\bar{p}(t,S) + \,\mathrm{d}\left[\bar{p}(t,S), \frac{1}{\bar{p}(t,T)} \right] \\ = Y_t \left(\left(\bar{v}(t,S) - \bar{v}(t,T) \right) \left(v^T(t,T) - \bar{v}^T(t,T) \right) + \frac{\partial \bar{C}}{\partial t}(t,S) - \frac{\partial \bar{C}}{\partial t}(t,T) \right) \mathrm{d}t \\ + Y_t \left(\bar{B}^T(t,S)\sigma - \bar{B}^T(t,T)\sigma \right) \mathrm{d}W_t^T,$$

where $\bar{v}(t,T) = \bar{B}^T(t,T)\sigma(t,Z_t)$.

The dynamics of the following process $X_t = e^{F(t,S,T) + G(t,S,T)^T Z_t + \int_0^t \int_{\mathbb{R}} \tilde{K}(s,y) N(\,\mathrm{d}s,\,\mathrm{d}y)}$ under $\mathbb{Q}^T$ is given by

$$\mathrm{d}X_t = a_X(t,T) X_t\, \mathrm{d}t + G(t,S,T)^T \sigma(t,T) X_t\, \mathrm{d}W_t^T + X_t \int_{\mathbb{R}} e^{\tilde{K}(s,y)} - 1 \tilde{N}(\,\mathrm{d}s,\,\mathrm{d}y),$$

where $a_X(t,T)$ is given by

$$a_X(t,T) = \frac{\partial F}{\partial t} + \frac{\partial G}{\partial t} z + G(t,S,T)^T \left(d(t) + E(t) z \right) + G(t,S,T)^T \sigma(t,T) v^T(t,T) \\ + \frac{1}{2} G^T \sigma \sigma^T G + \int_{\mathbb{R}} e^{\tilde{K}(s,y)} - 1 \nu(\,\mathrm{d}y).$$

Applying Ito's formula to $M_t = Y_t X_t$

$$\begin{aligned}
&\mathrm{d}M_t = Y_t\,\mathrm{d}X_t + X_t\,\mathrm{d}Y_t + \mathrm{d}\left[Y_t, X_t\right] \\
&= a_M(t,T)M_t\,\mathrm{d}t + 2M_t\int_{\mathbb{R}} e^{\tilde{K}(s,y)} - 1\tilde{N}(\mathrm{d}s,\,\mathrm{d}y) \\
&+ \left(M_t G(t,S,T)^T\sigma(t,T) + M_t\left(\bar{v}(t,S)-\bar{v}(t,T)\right) + M_t G(t,S,T)^T\sigma(t,T)X_t\right)\mathrm{d}W_t^T,
\end{aligned}$$

where $a_M(t,T)$ is given by

$$\begin{aligned}
a_M(t,T) = &\frac{\partial F}{\partial t} + \frac{\partial G}{\partial t}z + G^T\left(d(t)+E(t)z\right) + G^T\sigma(t,T)v^T(t,T) \\
&+ \frac{1}{2}G^T\sigma\sigma^T G + \int_{\mathbb{R}} e^{\tilde{K}(s,y)} - 1\nu(\mathrm{d}y) + \left(\bar{v}(t,S)-\bar{v}(t,T)\right)\left(v^T(t,T)-\bar{v}^T(t,T)\right) \\
&+ \frac{\partial\bar{C}}{\partial t}(t,S) - \frac{\partial\bar{C}}{\partial t}(t,T) + G^T\sigma(t,T)\left(\bar{v}(t,S)-\bar{v}(t,T)\right).
\end{aligned}$$

Since M_t is a $T-$ martingale we must have $a_M(t,T)=0$.

By separation of variables, we get

$$\begin{cases}
\frac{\partial F}{\partial t} + \left(\bar{B}^T(t,S)-\bar{B}^T(t,T)\right)k_0(t)\left(B(t,T)-\bar{B}(t,T)\right) + G^T d(t) + G^T k_0(t)B(t,T) \\
+\frac{\partial\bar{C}}{\partial t}(t,S) - \frac{\partial\bar{C}}{\partial t}(t,T) + \frac{1}{2}G^T k_0(t)G + G^T k_0(t)(\bar{B}(t,T)-\bar{B}(t,S)) \\
+\int_{\mathbb{R}} e^{\tilde{K}(s,y)} - 1\nu(\mathrm{d}y) = 0, \\
F(T,S,T) = 0,
\end{cases}$$

$$\begin{cases}
\frac{\partial G}{\partial t} + \left(\bar{B}^T(t,S)-\bar{B}^T(t,T)\right)\sum_{i=1}^m k_i(t)z_i\left(B(t,T)-\bar{B}(t,T)\right) + G^T E(t) \\
+G^T\sum_{i=1}^m k_i(t)z_i B(t,T) + \frac{1}{2}G^T\sum_{i=1}^m k_i(t)z_i G \\
+G^T\sum_{i=1}^m k_i(t)z_i(\bar{B}(t,T)-\bar{B}(t,S)) = 0, \\
G(t,S,T) = 0,
\end{cases}$$

Defining $\tilde{G}$ as in (5.34) we get (5.61).

Similarly, as seen in the case whithout the shot-noise process, since

$$1 + (T-S)\mathbb{E}_t^T\left[L(S,T)\right] = M(t,S,T),$$

we can solve for $\mathbb{E}_t^T\left[L(S,T)\right]$. Also, since

$$Fra_t(S,T) = \mathbb{E}_t^{\bar{T}}\left[L(S,T)\right] = L(t,S,T) = \frac{1}{T-S}\left(\frac{\bar{p}(t,S)}{\bar{p}(t,T)} - 1\right),$$

we can use the definition 5.2.5 and obtain the claimed Convexity adjustment. ■

In this framework the price of a forward rate agreement taking into account counterparty credit risk is now given by

$$Fra_t(S,T) = L(t,S,T) + CC^{Fra}(t,S,T), \tag{5.63}$$

where $L(t,S,T) = \frac{1}{T-S}\left(\frac{p(t,S)}{p(t,T)} - 1\right)$ is the Libor rate and $CC^{Fra}(t,S,T)$ is given by (5.60).

Chapter 6

Conclusion

In this project thesis we try to model feedback effects using Lévy Processes, therefore relaxing Black-Scholes's model assumptions of market liquidity and completeness. The contribution would be to study option pricing in illiquid markets with jumps and the associated hedging strategy. The basic idea of this thesis is to extend the models already used in the literature and extend them using Lévy Processes. We arrive at a partial integro-differential equation which is nonlinear and where the solution, if it exists, should be the function representing the derivative's security price. The objective is to study the existence and uniqueness of solution of that partial integro-differential equation and then develop numerical schemes to solve it and study its consistency, stability and convergence. Also we would like to study the equation when the influence of the large trader is small, in order to compare it to the already well established classical PIDE.

In this dissertation we showed that if the payoff function and the Lévy process satisfy some conditions, then we can obtain the option price as a solution of a certain partial integro-differential equation. Also, if a solution of a certain PIDE is smooth enough and if the Lévy process satisfies a exponential moment condition, then we can apply the Feynman-Kač formula for option pricing in a Lévy market. In Chapter 3 we present this formula for the case of a pure jump process. Two of the possible methods that can be used to compute the option price numerically are the Fast Fourier technique and the finite difference method. In this dissertation we present the latter in the form proposed by Cont and Voltchkova [25].

We could see that the price function of a binary option was not smooth when we used the Generalized Hyperbolic process. So we can not apply the results of Chapter 3. In this dissertation we present also a proof for the continuity of an up-and-out and down-and-out options when the Lévy process is of type A, besides the cases of type B and C presented in [83] and [26].

We saw different approaches to solve numerically a PIDE. One approach is through the well known Finite difference implicit-explicit scheme. The other one uses Galerkin methods through a variational formulation of the PIDE in study, for the finite and infinite activity cases.

In this thesis we studied an extension of the Black-Scholes model in which the perfect liquid market assumption is relaxed. Also we take into account the jumps that might occur in the stock's price. So we provide an option pricing formula in an illiquid market context in which the stock price follows a jump-diffusion model. Also we provide the associated investor's strategy. Existence result is proven for a certain integro-differential equation. A Numerical method based on a finite difference scheme is presented and some numerical results are given as well as consistency and monotony results for the scheme.

We analyzed existence and uniqueness of solutions to a partial integro-differential equation (PIDE) in the Bessel potential space. As a model we considered a model for pricing vanilla call and put options on underlying assets following Lévy stochastic process. Using the theory of abstract semilinear parabolic equations we proved existence and uniqueness of solutions in the Bessel potential space representing a fraction power space of the space of Lebesgue p-integrable functions with respect to the second order Laplace differential operator. We generalized known existence results for a wide class of Lévy measures including those having strong singular kernel. We also proved existence and uniqueness of solutions to the penalized PIDE representing approximation of the linear complementarity problem arising in pricing American style of options.

In this thesis pricing of Interest rate derivatives is given through Convexity adjustments that are used by practitioners to value non standard products using information on plain vanilla products.
More concretely, we computed the interbank convexity adjustment of FRAs (Forward Rate Agreements), combining the classical affine term structure (ATS) framework with shot-noise process thus being able to capture the counter-party risk of interbank contracts.

As future research it would be interesting to study alternative numerical methods for PIDEs such as the Analytic method of lines, finite element methods because they allow to compute the price of American options, unlike the finite difference methods. One of the reasons to use numerical methods for partial integro-differential equations is that they are computationally efficient in the case of single-asset options. However, in the case of three or more assets these methods become inefficient and the most used method to price American or barrier options is the Monte Carlo method. So it seems that additional study to overcome these difficulties is needed when we consider three or more assets. Moreover, extend the results of existence and uniqueness in the framework of Bessel potential spaces. It seems useful to extend the results presented here to consider the case of transaction costs. The potential theory could also be an interesting topic for future research because it explores the deep connection between partial integro-differential operators and Markov processes with jumps. Another issue that could be interesting to study in the future is the hedging in incomplete markets. An interesting topic is to measure the effects of the small perturbation in terms of Black-Scholes implied volatility, i.e the adjusted volatility parameter that should be used in the Black-Scholes formula to best approximate, for example in the least squares sense, the price of the option which has to be adjusted due to the presence of feedback effects. This way we could see if the results support the observed increased volatility phenomenon.

Appendix A

A.1 Proof of Proposition 3.3.1.

Proof. First, we need to prove the continuity with respect to x.

$$\begin{aligned}|f(\tau,x+\Delta x)-f(\tau,x)| &= |\mathbb{E}[H(S_0e^{x+\Delta x+r\tau+X_\tau})-H(S_0e^{x+r\tau+X_\tau})]| \\ &\leq \mathbb{E}[c|S_0e^{x+\Delta x+r\tau+X_\tau})-S_0e^{x+r\tau+X_\tau}|]=c\mathbb{E}[S_0e^{x+r\tau+X_\tau}|e^{\Delta x}-1|] \\ &= cS_0e^{x+r\tau}\mathbb{E}[e^{X_\tau}]|e^{\Delta x}-1|=cS_0e^{x+r\tau}|e^{\Delta x}-1|,\end{aligned}$$

because $\mathbb{E}[e^{X_\tau}]=1$ since e^{X_τ} is a martingale.

Then,

$$\lim_{\Delta x\to 0} f(\tau,x+\Delta x)-f(\tau,x)\leq \lim_{\Delta x\to 0} cS_0e^{x+r\tau}|e^{\Delta x}-1|=0,$$

which means that $f(\tau,x)$ is continuous in x.

Second, we need to prove the continuity in τ.

$$\begin{aligned}|f(\tau+\Delta\tau,x)-f(\tau,x)| &= |\mathbb{E}[H(S_0e^{x+r(\tau+\Delta)\tau+X_{\tau+\Delta\tau}})-H(S_0e^{x+r\tau+X_\tau})]| \\ &\leq \mathbb{E}[c|S_0e^{x+r(\tau+\Delta)+X_{\tau+\Delta}})-S_0e^{x+r\tau+X_\tau}|] \\ &= cS_0e^{x+r\tau}\mathbb{E}[e^{X_\tau}]\mathbb{E}|e^{r\Delta\tau+X_{\Delta\tau}}-1|.\end{aligned}$$

But

$$\begin{aligned}\mathbb{E}|e^{r\Delta\tau+X_{\Delta\tau}}-1| &= \begin{cases}\mathbb{E}[e^{r\Delta\tau+X_{\Delta\tau}}-1] & \text{if } e^{r\Delta\tau+X_{\Delta\tau}}-1>0,\\ \mathbb{E}[1-e^{r\Delta\tau+X_{\Delta\tau}}] & \text{if } e^{r\Delta\tau+X_{\Delta\tau}}-1<0\end{cases} \\ &= \begin{cases}e^{r\Delta\tau}-1 & \text{if } e^{r\Delta\tau+X_{\Delta\tau}}-1>0,\\ 1-e^{r\Delta\tau} & \text{if } e^{r\Delta\tau+X_{\Delta\tau}}-1<0\end{cases} \\ &= e^{r\Delta\tau}-1+2\begin{cases}\mathbb{E}[1-e^{r\Delta\tau+X_{\Delta\tau}}] & \text{if } 1-e^{r\Delta\tau+X_{\Delta\tau}}>0,\\ 0 & \text{if } 1-e^{r\Delta\tau+X_{\Delta\tau}}<0\end{cases} \\ &= e^{r\Delta\tau}-1+2\mathbb{E}[(1-e^{r\Delta\tau+X_{\Delta\tau}})^+].\end{aligned}$$

Then, because $e^{r\Delta\tau}-1\to 0$ when $\Delta\tau\to 0$, we only have to prove that:

$$\mathbb{E}[(1-e^{r\Delta\tau+X_{\Delta\tau}})^+]\to 0, \text{ when } \Delta\tau\to 0.$$

Let $C_0(\mathbb{R}) = \{h : h \text{ is continuous and vanishing at infinity}\}$. The Feller property tells us that, for any $h \in C_0(\mathbb{R})$ we have:

$$P_\tau h(0) = \mathbb{E}[h(r\tau + X_\tau)] \to h(0) \text{ as } \tau \to 0.$$

But, in this case $h(x) = (1-e^x)^+$ does not belong to $C_0(\mathbb{R})$. Then we try to approximate h with the function $g(x)$ such that:

$$g(x) = h(x), \text{ if } x \geq -1, g(x) = 0, \text{ if } x \leq -2, 0 \leq g(x) \leq h(x),$$

and $g(x)$ is continuously interpolated between -2 and -1, that is $g(x) = h(-1)x + 2h(-1)$ for $-2 \leq x \leq -1$. This way $g \in C_0(\mathbb{R})$.

$$\begin{aligned}\mathbb{E}[(1-e^{r\Delta\tau + X_{\Delta\tau}})^+] &= |P_\tau h(0)| = |P_\tau h(0) - P_\tau g(0) + P_\tau g(0)| \leq |P_\tau h(0) - P_\tau g(0)| + |P_\tau g(0)| \\ &= |\mathbb{E}[(h(r\Delta\tau + X_{\Delta\tau}) - g(r\Delta\tau + X_{\Delta\tau}))1_{r\Delta\tau + X_{\Delta\tau} < -1}]| + |P_\tau g(0)| \\ &\leq \mathbb{E}[1_{r\Delta\tau + X_{\Delta\tau} < -1}] + |P_\tau g(0)| = \mathbb{Q}[r\Delta\tau + X_{\Delta\tau} < -1] + |P_\tau g(0)| \\ &\leq \mathbb{Q}[X_{\Delta\tau} \leq -1] + |P_\tau g(0)|,\end{aligned}$$

because $g = h$ when $r\Delta\tau + X_{\Delta\tau} \geq -1$ and $h(x) \leq 1$, $g(x) \geq 0$ by construction.

Since $|P_\tau g(0)| \to g(0) = 0$ as $\Delta\tau \to 0$, we only have to prove that: $\mathbb{Q}[X_{\Delta\tau} \leq -1] \to 0$ as $\Delta\tau \to 0$.

Defining $M_\tau^- = \sup_{0 \leq s \leq \tau}(-X_s)$ we have $\mathbb{Q}[X_{\Delta\tau} \leq -1] = \mathbb{Q}[(-X_{\Delta\tau}) \geq 1] \leq \mathbb{Q}[M_\tau^- \geq 1]$.

Consider $\tau_n \downarrow 0$ and define $\Omega_n = \{\omega \in \Omega : M_{\tau_n}^-(\omega) \geq 1\}$. This way, the sequence Ω_n is decreasing. Therefore,

$$\begin{aligned}\lim_{n\to\infty} \mathbb{Q}[\Omega_n] &= \lim_{n\to\infty} \mathbb{Q}[M_{\tau_n}^- \geq 1] = \mathbb{Q}[\bigcap_{n=1}^{\infty} \{\omega \in \Omega : M_{\tau_n}^-(\omega) \geq 1\}] \\ &= \mathbb{Q}[M_0^-(\omega) \geq 1] = 0,\end{aligned}$$

since $M_0^- = -X_0$ and $X_0 = 0$ a.s . Then $\mathbb{Q}[M_\tau^- \geq 1] \to 0$ since τ_n is arbitrary. Therefore $\mathbb{Q}[X_{\Delta\tau} \leq -1] \to 0$.

In order to show continuity for any $(\tau, x) \in [0,T] \times \mathbb{R}$, we use the triangular inequality:

$$\begin{aligned}|f(\tau + \Delta\tau, x + \Delta x) - f(\tau, x)| &\leq |f(\tau + \Delta\tau, x + \Delta x) - f(\tau + \Delta\tau, x)| + |f(\tau + \Delta\tau, x) - f(\tau, x)| \\ &\leq cS_0 e^{x + r(\tau + \Delta\tau)}|e^{\Delta x} - 1| + cS_0 e^{x + r\tau}\mathbb{E}[|e^{r\Delta\tau + X_{\Delta\tau}} - 1|] \to 0.\end{aligned}$$

Then $f(\tau, x)$ is continuous on $[0,T] \times \mathbb{R}$. ■

A.2 Proof of Proposition 3.3.2.

First step: We prove that the density function of $r\tau + X_\tau$, $p_\tau(x) \in C^\infty$.

The condition

$$\liminf_{\epsilon \to 0} \epsilon^{-\beta} \int_{-\epsilon}^{\epsilon} |x|^2 \nu(\,dx) > 0$$

implies that

$$\exists_{c_1>0} \int_{-\epsilon}^{\epsilon} |x|^2 \nu(\,\mathrm{d}x) \geq c_1 \epsilon^{\beta},$$

for small ϵ. Following the notation on [83], let $p_\tau(x)$, be the density function of the Lévy process $r\tau + X_\tau$ with characteristic function:

$$\psi_{r\tau+X_\tau}(z) = e^{\tau \phi_{r+X_1}(z)},$$

with

$$\phi_{r+X_1}(z) = -\frac{\sigma^2 z^2}{2} + i\gamma z + \int_{-\infty}^{+\infty} \left(e^{izx} - 1 - izx1_{|x|\leq 1}\right) \nu(dx).$$

Then,

$$\begin{aligned}|\psi_{r\tau+X_\tau}(z)| &= \left| e^{\tau\left(-\frac{\sigma^2 z^2}{2} + i\gamma z + \int_{-\infty}^{+\infty}(e^{izx} - 1 - izx1_{|x|\leq 1})\nu(dx)\right)} \right| \\ &= \left| e^{\tau(i(\gamma z + \int_{-\infty}^{+\infty}(\sin(zx) - zx1_{|x|\leq 1})\nu(dx))} e^{-\frac{\sigma^2 z^2}{2} + \int_{-\infty}^{+\infty}(\cos(zx)-1)\nu(dx))} \right| \leq e^{\int_{-\infty}^{+\infty}(\cos(zx)-1)\nu(dx))}.\end{aligned}$$

Notice that $1 - \cos(u) = 1 - \cos(\frac{u}{2} + \frac{u}{2}) = 1 - (\cos^2(\frac{u}{2}) - \sin^2(\frac{u}{2})) = 2\sin^2(\frac{u}{2}) \geq 2(\frac{u}{\pi})^2$ for $\frac{|u|}{\pi} \leq 1$. Then,

$$\begin{aligned}|\psi_{r\tau+X_\tau}(z)| &\leq e^{\int_{-\infty}^{+\infty}(\cos(zx)-1)\nu(dx))} \leq e^{\int_{|x|\leq\frac{\pi}{|z|}} -2(\frac{zx}{\pi})^2 \nu(dx))} \\ &= e^{-\frac{2}{\pi^2} z^2 \int_{|x|\leq\frac{\pi}{|z|}} x^2 \nu(dx))} = e^{-Kz^2 \int_{|x|\leq\frac{\pi}{|z|}} x^2 \nu(dx)}.\end{aligned}$$

But $\int_{-\epsilon}^{\epsilon} |x|^2 \nu(\,\mathrm{d}x) \geq c_1 \epsilon^\beta$, so by choosing $\epsilon = \frac{\pi}{|z|}$, we get $-\int_{|x|\leq\frac{\pi}{|z|}} x^2\nu(dx) \leq -C(\frac{\pi}{|z|})^\beta$. Then,

$$|\psi_{r\tau+X_\tau}(z)| \leq e^{-Kz^2 C(\frac{\pi}{|z|})^\beta} = e^{-c|z|^{2-\beta}} = e^{-c|z|^\alpha} \text{ with } c = KC\pi^\beta \text{ and } \alpha = 2-\beta.$$

Also,

$$\int_{\mathbb{R}} |\psi_{r\tau+X_\tau}(z)| z^n \,\mathrm{d}z \leq \int_{\mathbb{R}} e^{-c|z|^\alpha} z^n \,\mathrm{d}z < \infty, \tag{A.1}$$

and by inversion formula of the Fourier transform,

$$p_\tau(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixz} \psi_{r\tau+X_\tau}(z) \,\mathrm{d}z.$$

Then, the right hand-side is n times differentiable with respect to x and differentiation is possible under the integral sign because of (A.1). In fact,

$$\begin{aligned}\frac{\partial^n p_\tau(x)}{\partial x^n} &= \frac{1}{2\pi} \int_{\mathbb{R}} (-iz)^n e^{-izx} \psi_{r\tau+X_\tau}(z) \,\mathrm{d}z = \frac{1}{2\pi} \int_{\mathbb{R}} |z|^n e^{(\frac{3}{2}n\pi - zx)i} \psi_{r\tau+X_\tau}(z) \,\mathrm{d}z \\ &\leq \frac{1}{2\pi} \int_{\mathbb{R}} |z|^n \psi_{r\tau+X_\tau}(z) \,\mathrm{d}z < \infty.\end{aligned}$$

Then, by proposition 28.1 of [72], the process $r\tau + X_\tau$ has density function $p_\tau(x)$ of class C_∞.

Second step: Let us prove that $f(\tau, x) = \mathbb{E}[h(x + r\tau + X_\tau)] \in C^\infty((0,T)\times\mathbb{R})$.

Defining,$\widetilde{p}_\tau(x) = p_\tau(-x)$, we have

$$\begin{aligned} f(\tau, x) = \mathbb{E}[h(x + r\tau + X_\tau)] = h(x) * \widetilde{p}_\tau(x) = \int_{\mathbb{R}} h(x-z)\widetilde{p}_\tau(z)\,\mathrm{d}z \\ = \int_{\mathbb{R}} h(x-z)p_\tau(-z)\,\mathrm{d}z = \int_{\mathbb{R}} h(x+w)p_\tau(w)\,\mathrm{d}w, \end{aligned}$$

by making the substitution $w = -z$.

So we have to show that $h(x) * \widetilde{p}_\tau(x)$ belongs to C^∞ and for that to happen, $\frac{\partial^n \widetilde{p}_\tau(x)}{\partial x^n}$ has to decrease sufficiently fast at the infinity so that

$$\frac{\partial^n f(\tau, x)}{\partial x^n} = h(x) * \frac{\partial^n \widetilde{p}_\tau(x)}{\partial x^n} = \int_{\mathbb{R}} h(x-y)\frac{\partial^n \widetilde{p}_\tau(y)}{\partial x^n}\,\mathrm{d}y$$

makes sense.

We have

$$\begin{aligned} \phi'_{r+X_1}(z) &= -\sigma^2 z + i\gamma + \int_{\mathbb{R}} iy(e^{iyz} - 1_{|y|\le 1})\nu(\,\mathrm{d}y), \\ \phi''_{r+X_1}(z) &= -\sigma^2 + \int_{\mathbb{R}} (iy)^2 e^{iyz}\nu(\,\mathrm{d}y), \\ \phi^{(k)}_{r+X_1}(z) &= \int_{\mathbb{R}} (iy)^k e^{iyz}\nu(\,\mathrm{d}y), \forall k \ge 3. \end{aligned}$$

Therefore,

$$\begin{aligned} |\phi'_{r+X_1}(z)| &= \left|-\sigma^2 z + i\gamma + \int_{\mathbb{R}} iy(e^{iyz} - 1_{|y|\le 1})\nu(\,\mathrm{d}y)\right| \le \sigma^2|z| + |\gamma| + \left|\int_{\mathbb{R}} iy(e^{iyz} - 1_{|y|\le 1})\nu(\,\mathrm{d}y)\right| \\ &\le \sigma^2|z| + |\gamma| + \int_{\mathbb{R}} \left|iy(e^{iyz} - 1_{|y|\le 1})\right|\nu(\,\mathrm{d}y) \le \sigma^2|z| + |\gamma| + \int_{\mathbb{R}} |y||e^{iyz}|\nu(\,\mathrm{d}y) \\ &= \sigma^2|z| + |\gamma| + \int_{\mathbb{R}} |y|\nu(\,\mathrm{d}y) = \sigma^2|z| + |\gamma| + \int_{|y|\le 1} |y|\nu(\,\mathrm{d}y) + \int_{|y|>1} |y|\nu(\,\mathrm{d}y) < \infty, \end{aligned}$$

because of (3.23).

$$\begin{aligned} |\phi''_{r+X_1}(z)| &= \left|-\sigma^2 + \int_{\mathbb{R}} (iy)^2 e^{iyz}\nu(\,\mathrm{d}y)\right| \le \sigma^2 + \left|\int_{\mathbb{R}} (iy)^2 e^{iyz}\nu(\,\mathrm{d}y)\right| \\ &\le \sigma^2 + \int_{\mathbb{R}} |(iy)^2||e^{iyz}|\nu(\,\mathrm{d}y) = \sigma^2 + \int_{\mathbb{R}} |y|^2\nu(\,\mathrm{d}y) < \infty, \end{aligned}$$

also by (3.23) and (2.21).

$$\begin{aligned} |\phi^{(k)}_{r+X_1}(z)| &= \left|\int_{\mathbb{R}} (iy)^k e^{iyz}\nu(\,\mathrm{d}y)\right| \le \int_{\mathbb{R}} |(iy)^k e^{iyz}|\nu(\,\mathrm{d}y) = \int_{\mathbb{R}} |y|^k|e^{iyz}|\nu(\,\mathrm{d}y) \\ &= \int_{\mathbb{R}} |y|^k\nu(\,\mathrm{d}y) = \int_{|y|\le 1} |y|^k\nu(\,\mathrm{d}y) + \int_{|y|>1} |y|^k\nu(\,\mathrm{d}y) < \infty, \forall k \ge 3. \end{aligned}$$

Then $\phi_{r+X_1}(z) \in C^\infty$ which implies that $\psi_{r\tau+X_\tau}(z) = e^{\tau\phi_{r+X_1}(z)} \in C^\infty$.

Next, we conclude that

$$|\phi'_{r+X_1}(z)| \leq |\sigma^2|z| + |\gamma| + \int_{\mathbb{R}} |y|\nu(\,\mathrm{d}y)| \leq A_1(1+|z|),$$
$$|\phi''_{r+X_1}(z)| \leq |\sigma^2 + \int_{\mathbb{R}} |y|^2\nu(\,\mathrm{d}y)| \leq A_2,$$
$$|\phi^{(k)}_{r+X_1}(z)| \leq |\int_{\mathbb{R}} |y|^k\nu(\,\mathrm{d}y)| \leq A_k, \forall k \geq 3,$$

and also that

$$|\frac{\partial\psi_{r\tau+X_\tau}(z)}{\partial z}| = \tau|\phi'_{r+X_1}(z)|e^{\tau\phi_{r+X_1}(z)} \leq K(1+|z|)\psi_{r\tau+X_\tau}(z) \leq K(1+|z|)e^{-c|z|^\alpha},$$
$$\begin{aligned}|\frac{\partial^2\psi_{r\tau+X_\tau}(z)}{\partial z^2}| &= \left|\tau\phi''_{r+X_1}(z)e^{\tau\phi_{r+X_1}(z)} + \tau^2(\phi'_{r+X_1}(z))^2e^{\tau\phi_{r+X_1}(z)}\right| \\ &\leq \tau A_2e^{\tau\phi_{r+X_1}(z)} + \tau^2A_1(1+|z|)^2e^{\tau\phi_{r+X_1}(z)} \\ &\leq \tau A_2e^{-c|z|^\alpha} + \tau A_1(1+|z|)^2e^{-c|z|^\alpha} \\ &\leq K(1+|z|^2)e^{-c|z|^\alpha}.\end{aligned}$$

So, by recurrence, we get

$$|\frac{\partial^k\psi_{r\tau+X_\tau}(z)}{\partial z^k}| \leq K(1+|z|^k)\psi_{r\tau+X_\tau}(z) \leq K(1+|z|^k)e^{-c|z|^\alpha}, \forall k \geq 0.$$

Also,

- $$\begin{aligned}\left|\frac{d^k}{dz^k}\int_{\mathbb{R}} e^{izx}\frac{\partial}{\partial x}\widetilde{p}_\tau(x)\,\mathrm{d}x\right| &= \left|\frac{d^k}{dz^k}([e^{izx}\widetilde{p}_\tau(x)]^\infty_{-\infty} - \int_{\mathbb{R}}(iz)e^{izx}\widetilde{p}_\tau(x)\,\mathrm{d}x)\right| \\ &= \left|\frac{d^k}{dz^k}(-iz)\int_{\mathbb{R}} e^{izx}\widetilde{p}_\tau(x)\,\mathrm{d}x\right| \leq K|z|^{1+k}e^{-c|z|^\alpha}.\end{aligned}$$

- $$\begin{aligned}\left|\frac{d^k}{dz^k}\int_{\mathbb{R}} e^{izx}\frac{\partial^2}{\partial x^2}\widetilde{p}_\tau(x)\,\mathrm{d}x\right| &= \left|\frac{d^k}{dz^k}([e^{izx}\frac{\partial}{\partial x}\widetilde{p}_\tau(x)]^\infty_{-\infty} - \int_{\mathbb{R}}(iz)e^{izx}\frac{\partial}{\partial x}\widetilde{p}_\tau(x)\,\mathrm{d}x)\right| \\ &= |\frac{d^k}{dz^k}([e^{izx}\frac{\partial}{\partial x}\widetilde{p}_\tau(x)]^\infty_{-\infty} + [(-iz)e^{izx}\widetilde{p}_\tau(x)]^\infty_{-\infty} \\ &\quad - \int_{\mathbb{R}}(iz)^2e^{izx}\widetilde{p}_\tau(x)\,\mathrm{d}x)| = \left|\frac{d^k}{dz^k}(-iz)^2\int_{\mathbb{R}} e^{izx}\widetilde{p}_\tau(x)\,\mathrm{d}x\right| \\ &\leq K|z|^{2+k}e^{-c|z|^\alpha}, \forall k \geq 0,\end{aligned}$$

  because by proposition 28.1 of [72] the partial derivatives of $\widetilde{p}_\tau$ of orders 0,..,n tend to zero as $|x| \to \infty$. Once again by recurrence,

- $$\left|\frac{d^k}{dz^k}\int_{\mathbb{R}} e^{izx}\frac{\partial^n}{\partial x^n}\widetilde{p}_\tau(x)\,\mathrm{d}x\right| = \left|\frac{d^k}{dz^k}(-iz)^n\int_{\mathbb{R}} e^{izx}\widetilde{p}_\tau(x)\,\mathrm{d}x\right| \leq K|z|^{n+k}e^{-c|z|^\alpha},$$

  for all $n, k \geq 0$.

Then,

$$\forall k, n \geq 0, \int_{\mathbb{R}}\left(\frac{d^k}{dz^k}\int_{\mathbb{R}} e^{izx}\frac{\partial^n}{\partial x^n}\widetilde{p}_\tau(x)\,\mathrm{d}x\right)^2 \mathrm{d}z \leq \int_{\mathbb{R}}\left(K|z|^{n+k}e^{-c|z|^\alpha}\right)^2 \mathrm{d}z < \infty,$$

which means that $\frac{d^k}{dz^k}\int_{\mathbb{R}} e^{izx}\frac{\partial^n}{\partial x^n}\widetilde{p}_\tau(x)\,\mathrm{d}x \in L^2(\mathbb{R})$. But this implies that

$$\int_{\mathbb{R}}\left(|x|^k\frac{\partial^n}{\partial x^n}\widetilde{p}_\tau(x)\right)^2 \mathrm{d}x < \infty$$

or that $|x|^k\frac{\partial^n}{\partial x^n}\widetilde{p}_\tau(x) \in L^2(\mathbb{R})$, and this in turn implies that:

$$\begin{aligned}
\int_{\mathbb{R}} |\frac{\partial^n\widetilde{p}_\tau(x)}{\partial x^n}(1+|x|^k)\,\mathrm{d}x| &\leq C\int_{\mathbb{R}}\frac{1}{1+|x|}(1+|x|^{k+1})\frac{\partial^n\widetilde{p}_\tau(x)}{\partial x^n}\,\mathrm{d}x \\
&\leq C\left(\int_{\mathbb{R}}\left(\frac{1}{1+|x|}\right)^2 \mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}}\left((1+|x|^{k+1})\frac{\partial^n\widetilde{p}_\tau(x)}{\partial x^n}\right)^2 \mathrm{d}x\right)^{1/2} \\
&= C\left\|\frac{1}{1+|x|}\right\|_{L^2}\left\|(1+|x|^{k+1})\frac{\partial^n\widetilde{p}_\tau(x)}{\partial x^n}\right\|_{L^2} < \infty.
\end{aligned}$$

Then,

$$\begin{aligned}
\left|\frac{\partial^n f}{\partial x^n}(\tau, x)\right| &= \left|h(x) * \frac{\partial^n\widetilde{p}_\tau(x)}{\partial x^n}\right| = \left|\int_{\mathbb{R}} h(x-z)\frac{\partial^n\widetilde{p}_\tau(z)}{\partial x^n}\,\mathrm{d}z\right| \\
&\leq C\int_{\mathbb{R}}(1+|x-z|^p)\left|\frac{\partial^n\widetilde{p}_\tau(z)}{\partial x^n}\right|\mathrm{d}z \leq C(1+|x|^p)\int_{\mathbb{R}}(1+|z|^p)\left|\frac{\partial^n\widetilde{p}_\tau(z)}{\partial x^n}\right|\mathrm{d}z \\
&\leq C(1+|x|^p)K\left\|\frac{1}{1+|x|}\right\|_{L^2}\left\|(1+|x|^{k+1})\frac{\partial^n\widetilde{p}_\tau(x)}{\partial x^n}\right\|_{L^2} = D(1+|x|^p).
\end{aligned}$$

Then $\frac{\partial^n f}{\partial x^n}(\tau, x)$ is continuous and finite, which means that f is regular with respect to x. To prove the regularity in time we notice that:

$$|\phi_{r+X_1}(z)| = \left|-\frac{\sigma^2 z^2}{2} + i\gamma z + \int_{-\infty}^{+\infty}(e^{izx} - 1 - izx1_{|x|\leq 1})\nu(dx)\right| \leq C(1+|z|^2)$$

and verify by recurrence that

$$\begin{aligned}
\left|\frac{d^k}{dz^k}\int_{\mathbb{R}} e^{izx}\frac{\partial^m}{\partial\tau^m}\widetilde{p}_\tau(x)\,\mathrm{d}x\right| &= \left|\frac{d^k}{dz^k}\frac{\partial^m}{\partial\tau^m}\int_{\mathbb{R}} e^{izx}\widetilde{p}_\tau(x)\,\mathrm{d}x\right| \\
&= \left|\frac{d^k}{dz^k}[\phi_{r+X_1}(z)]^m e^{\tau\phi_{r+X_1}(z)}\right| \leq C|z|^{2m+k}e^{-c|z|^\alpha}.
\end{aligned}$$

Then, $\frac{d^k}{dz^k}\int_{\mathbb{R}} e^{izx}\frac{\partial^m}{\partial\tau^m}\widetilde{p}_\tau(x)\,\mathrm{d}x \in L^2(\mathbb{R})$, which implies that $\frac{\partial^m\widetilde{p}_\tau(x)}{\partial\tau^m}(1+|x|^k)\in L^1(\mathbb{R})$. Therefore,

$$\begin{aligned}\left|\frac{\partial^m f}{\partial\tau^m}(\tau,x)\right| = \left|h(x)*\frac{\partial^m\widetilde{p}_\tau(x)}{\partial\tau^m}\right| = \left|\int_{\mathbb{R}} h(x-z)\frac{\partial^m\widetilde{p}_\tau(z)}{\partial\tau^m}\,\mathrm{d}z\right| &\le C\int_{\mathbb{R}}(1+|x-z|^p)\left|\frac{\partial^m\widetilde{p}_\tau(z)}{\partial\tau^m}\right|\mathrm{d}z\\ &\le C(1+|x|^p)\int_{\mathbb{R}}(1+|z|^p)\left|\frac{\partial^m\widetilde{p}_\tau(z)}{\partial\tau^m}\right|\mathrm{d}z\\ &\le C(1+|x|^p)K\left\|\frac{1}{1+|x|}\right\|_{L^2}\left\|(1+|x|^{k+1})\frac{\partial^m\widetilde{p}_\tau(x)}{\partial\tau^m}\right\|_{L^2}\\ &= D(1+|x|^p),\end{aligned}$$

which means that $\frac{\partial^n f}{\partial\tau^m}(\tau,x)$ is continuous and finite.

In the same way we conclude that:

$$\begin{aligned}\left|\frac{d^k}{dz^k}\int_{\mathbb{R}} e^{izx}\frac{\partial^{n+m}}{\partial x^n\partial\tau^m}\widetilde{p}_\tau(x)\,\mathrm{d}x\right| &= \left|\frac{d^k}{dz^k}(-iz)^n[\phi_{r+X_1}(z)]^m e^{\tau\phi_{r+X_1}(z)}\int_{\mathbb{R}} e^{izx}\widetilde{p}_\tau(x)\,\mathrm{d}x\right|\\ &\le C|z|^{2m+n+k}e^{-c|z|^\alpha}.\\ \left|\frac{\partial^{n+m} f}{\partial x^n\partial\tau^m}(\tau,x)\right| = \left|h(x)*\frac{\partial^{n+m}\widetilde{p}_\tau(x)}{\partial x^n\partial\tau^m}\right| &= \left|\int_{\mathbb{R}} h(x-z)\frac{\partial^{n+m}\widetilde{p}_\tau(z)}{\partial x^n\partial\tau^m}\,\mathrm{d}z\right|\\ &\le C\int_{\mathbb{R}}(1+|x-z|^p)\left|\frac{\partial^{n+m}\widetilde{p}_\tau(z)}{\partial x^n\partial\tau^m}\right|\mathrm{d}z\\ &\le C(1+|x|^p)\int_{\mathbb{R}}(1+|z|^p)\left|\frac{\partial^{n+m}\widetilde{p}_\tau(z)}{\partial x^n\partial\tau^m}\right|\mathrm{d}z\\ &\le C(1+|x|^p)K\left\|\frac{1}{1+|x|}\right\|_{L^2}\left\|(1+|x|^{k+1})\frac{\partial^{n+m}\widetilde{p}_\tau(x)}{\partial x^n\partial\tau^m}\right\|_{L^2}\\ &= D(1+|x|^p).\end{aligned}$$

Then $f(\tau,x)\in C^\infty((0,T],\mathbb{R})$.

A.3 Proof of Proposition 3.3.5.

Proof. Define $M=\sup_{S\in(0,U)} H(S)$. We can do this because H is bounded due to the fact that it is Lipschitz. We will prove first the continuity in x and τ and finally prove the continuity using the triangular inequality.

First step: Prove continuity in x for all $\tau>0$ and $x<u$. Choosing $\delta\in(0,u-x)$ we

get:

$$\begin{aligned}|f_U(\tau,x+\delta)-f_U(\tau,x)| &= |\mathbb{E}[H(S_0e^{x+\delta+Y_\tau})1_{\tau<R_{u-x-\delta}}-H(S_0e^{x+Y_\tau})1_{\tau<R_{u-x}}]|\\ &\leq \mathbb{E}[|H(S_0e^{x+\delta+Y_\tau})1_{\tau<R_{u-x-\delta}}-H(S_0e^{x+Y_\tau})1_{\tau<R_{u-x}}|]\\ &= \mathbb{E}[|(H(S_0e^{x+\delta+Y_\tau})-H(S_0e^{x+Y_\tau}))1_{\tau<R_{u-x-\delta}}\\ &\quad + H(S_0e^{x+Y_\tau})(1_{\tau<R_{u-x-\delta}}-1_{\tau<R_{u-x}})|]\\ &\leq \mathbb{E}[k|(S_0e^{x+\delta+Y_\tau}-S_0e^{x+Y_\tau})|1_{\tau<R_{u-x-\delta}}]+M\mathbb{E}[1_{R_{u-x-\delta}<\tau<R_{u-x}}]\\ &\leq ke^{x+r\tau}\mathbb{E}[S_0e^{X_\tau}]|e^\delta-1|+M\mathbb{Q}[R_{u-x-\delta}<\tau<R_{u-x}]\\ &\leq kS_0e^{x+r\tau}|e^\delta-1|+M\mathbb{Q}[R_{u-x-\delta}<\tau<R_{u-x}],\end{aligned}$$

because by the martingale condition $\mathbb{E}[e^{X_\tau}]=\mathbb{E}[e^{X_0}]=1$.

Then,

$$\lim_{\delta\to0}|f_U(\tau,x+\delta)-f_U((\tau,x))|\leq\lim_{\delta\to0}kS_0e^{x+r\tau}|e^\delta-1|+M\mathbb{Q}[R_{u-x-\delta}<\tau<R_{u-x}]=0,$$

because $|e^\delta-1|\to0$ and $\mathbb{Q}[R_{u-x-\delta}<\tau<R_{u-x}]\to0$ when $\delta\to0$ by Lemma 3.3.5.

Similarly we prove for $x<u$:

$$\lim_{\delta\to0}|f_U(\tau,x-\delta)-f_U((\tau,x))|\leq\lim_{\delta\to0}kS_0e^{x+r\tau}|e^{-\delta}-1|+M\mathbb{Q}[R_{u-x}<\tau<R_{u-x+\delta}]=0,$$

also by Lemma 3.3.5 and by the martingale condition.

As for $x=u$ the right continuity of $f_U(\tau,x)$ is proven easily so:

$$|f_U(\tau,u-\delta)-f_U((\tau,u))|=|\mathbb{E}[H(S_0e^{u-\delta+Y_\tau})1_{\tau<R_\delta}]|\leq M\mathbb{Q}[\tau<R_\delta].$$

Considering $\delta_n\to0$ we have:

$$\mathbb{Q}[\tau<R_\delta]\to\mathbb{Q}[\cap_{n=1}^\infty\{\omega\in\Omega|R_{\delta_n}>\tau\}]=\mathbb{Q}[\tau\leq R_0]=0,$$

because $R_0=0$ $a.s.$ Therefore, we proved the continuity of $f_U(\tau,x)$ for all $x\in\mathbb{R}$.

Second step: Let us prove continuity in time. For $x<u$ and $0\leq s\leq t$:

$$\begin{aligned}|f_U(t,x)-f_U(s,x)| &= |\mathbb{E}[H(S_0e^{x+Y_t})1_{t<R_{u-x}}-H(S_0e^{x+Y_s})1_{s<R_{u-x}}]|\\ &\leq \mathbb{E}[|H(S_0e^{x+Y_t})-H(S_0e^{x+Y_s})|1_{t<R_{u-x}}+|H(S_0e^{x+Y_s})|1_{s\leq R_{u-x}}<t]\\ &\leq kS_0e^{x+rs}\mathbb{E}[|e^{Y_{t-s}}-1|]+M\mathbb{Q}[s\leq R_{u-x}<t].\end{aligned}$$

$$\lim_{t\to s}|f_U(t,x)-f_U(s,x)|\leq\lim_{t\to s}kS_0e^{x+rs}\mathbb{E}[|e^{Y_{t-s}}-1|]+M\mathbb{Q}[s\leq R_{u-x}<t]=0,$$

because we know that, by the proof of the Proposition 3.3.1, $\mathbb{E}[|e^{Y_{t-s}}-1|]\to0$ when $t\to s$ and considering a decreasing set $\Omega_n=\{\omega\in\Omega|s\leq R_{u-x}(\omega)<t_n\}$, $t_n\to s$:

$$\lim_{n\to\infty}\mathbb{Q}[s\leq R_{u-x}(\omega)<t_n]=\mathbb{Q}[\cap_{n=1}^\infty\Omega_n]=\mathbb{Q}[\emptyset]=0.$$

Third step: Use the triangular inequality. Let $(\tau,x)\in[0,T]\times(-\infty,u)$ and $(\Delta\tau,\Delta x)\in\mathbb{R}^2$

$$\begin{aligned}|f_U(\tau+\Delta\tau,x+\Delta x)-f_U(\tau,x)| &\leq |f_U(\tau+\Delta\tau,x+\Delta x)-f_U(\tau,x+\Delta x)|\\ &\quad+|f_U(\tau,x+\Delta x)-f_U(\tau,x)|.\end{aligned}$$

- First term.

  Defining $y = x + \Delta x$ and $t = \tau + \Delta\tau$ with $\Delta\tau > 0$, we obtain:

$$\begin{aligned}
|f_U(t,y) - f_U(\tau,y)| &= |\mathbb{E}[H(S_0 e^{y+Y_t})1_{t<R_{u-y}} - H(S_0 e^{y+Y_\tau})1_{\tau<R_{u-y}}]| \\
&= |\mathbb{E}[(H(S_0 e^{y+Y_t}) - H(S_0 e^{y+Y_\tau}))1_{t<R_{u-y}} \\
&\quad + H(S_0 e^{y+Y_\tau})(1_{t<R_{u-y}} - 1_{\tau<R_{u-y}})]| \\
&\le \mathbb{E}[|H(S_0 e^{y+Y_t}) - H(S_0 e^{y+Y_\tau})|1_{t<R_{u-y}} + |H(S_0 e^{y+Y_\tau})|1_{\tau<R_{u-y}<t}] \\
&\le k\mathbb{E}[|S_0 e^{y+Y_t} - S_0 e^{y+Y_\tau}|1_{t<R_{u-y}}] + M\mathbb{Q}[\tau < R_{u-y} < t] \\
&\le kS_0 e^y \mathbb{E}[|e^{Y_t} - e^{Y_\tau}|] + M\mathbb{Q}[\tau < R_{u-y} < t] \text{ but } Y_t - Y_\tau \stackrel{d}{=} Y_{\Delta\tau} \\
&\le kS_0 e^y \mathbb{E}[|e^{Y_\tau}|e^{Y_{\Delta\tau}} - 1|] + M\mathbb{Q}[\tau < R_{u-y} < t] \\
&= kS_0 e^y \mathbb{E}[e^{Y_\tau}]\mathbb{E}[|e^{Y_{\Delta\tau}} - 1|] + M\mathbb{Q}[\tau < R_{u-y} < t] \\
&= kS_0 e^{y+r\tau}\mathbb{E}[|e^{Y_{\Delta\tau}} - 1|] + M\mathbb{Q}[\tau < R_{u-y} < t].
\end{aligned}$$

  Similarly for the case $\Delta\tau < 0$, we get:

$$\begin{aligned}
|f_U(t,y) - f_U(\tau,y)| &= |f_U(\tau,y) - f_U(t,y)| \\
&= |\mathbb{E}[H(S_0 e^{y+Y_\tau})1_{\tau<R_{u-y}} - H(S_0 e^{y+Y_t})1_{t<R_{u-y}}]| \\
&= |\mathbb{E}[(H(S_0 e^{y+Y_\tau}) - H(S_0 e^{y+Y_t}))1_{\tau<R_{u-y}} \\
&\quad + H(S_0 e^{y+Y_t})(1_{\tau<R_{u-y}} - 1_{t<R_{u-y}})]| \\
&\le \mathbb{E}[|H(S_0 e^{y+Y_\tau}) - H(S_0 e^{y+Y_t})|1_{\tau<R_{u-y}} + |H(S_0 e^{y+Y_t})|1_{t<R_{u-y}<\tau}] \\
&\le k\mathbb{E}[|S_0 e^{y+Y_\tau} - S_0 e^{y+Y_t}|1_{\tau<R_{u-y}}] + M\mathbb{Q}[t < R_{u-y} < \tau] \\
&\le kS_0 e^y \mathbb{E}[|e^{Y_\tau} - e^{Y_t}|] + M\mathbb{Q}[t < R_{u-y} < \tau] \text{ but }, Y_\tau \stackrel{d}{=} Y_{\tau+\Delta\tau} - Y_{\Delta\tau} \\
&\le kS_0 e^y \mathbb{E}[|e^{Y_t}|e^{-Y_{\Delta\tau}} - 1|] + M\mathbb{Q}[t < R_{u-y} < \tau] \\
&= kS_0 e^y \mathbb{E}[e^{Y_\tau}]\mathbb{E}[|e^{Y_{-\Delta\tau}} - 1|] + M\mathbb{Q}[t < R_{u-y} < \tau] \\
&= kS_0 e^{y+r\tau+\Delta\tau}\mathbb{E}[|e^{Y_{-\Delta\tau}} - 1|] + M\mathbb{Q}[t < R_{u-y} < \tau].
\end{aligned}$$

  So for $\Delta x \in \mathbb{R}$,

$$\begin{aligned}
&|f_U(t,y) - f_U(\tau,y)| \\
&\quad \le kS_0 e^y (e^{r\tau}1_{\Delta\tau\ge0} + e^{rt}1_{\Delta\tau<0})\mathbb{E}[|e^{Y_{|\Delta\tau|}} - 1|] + M(\mathbb{Q}[\tau < R_{u-y} \le t]1_{\Delta\tau\ge0} \\
&\qquad + \mathbb{Q}[t < R_{u-y} \le \tau]1_{\Delta\tau<0}) \\
&\quad = kS_0 e^{y+r\tau}(1_{\Delta\tau\ge0} + e^{r\Delta\tau}1_{\Delta\tau<0})\mathbb{E}[|e^{Y_{|\Delta\tau|}} - 1|] + M(\mathbb{Q}[t < R_{u-y} \le \tau]1_{\Delta\tau<0} \\
&\qquad + \mathbb{Q}[\tau < R_{u-y} \le t]1_{\Delta\tau\ge0}) \\
&\quad \le kS_0 e^{y+r\tau}\mathbb{E}[|e^{Y_{|\Delta\tau|}} - 1|] + M(\mathbb{Q}[t < R_{u-y} \le \tau]1_{\Delta\tau<0} \\
&\qquad + \mathbb{Q}[\tau < R_{u-y} \le t]1_{\Delta\tau\ge0}) \\
&\quad = kS_0 e^{y+r\tau}\mathbb{E}[|e^{Y_{|\Delta\tau|}} - 1|] + M(\mathbb{Q}[\Delta\tau < R_{u-y} - \tau \le 0]1_{\Delta\tau<0} \\
&\qquad + \mathbb{Q}[0 < R_{u-y} - \tau \le \Delta\tau]1_{\Delta\tau\ge0})] \\
&\quad = kS_0 e^{y+r\tau}\mathbb{E}[|e^{Y_{|\Delta\tau|}} - 1|] + M(\mathbb{Q}[-\Delta\tau < R_{u-y} - \tau \le 0] \\
&\qquad + \mathbb{Q}[0 < R_{u-y} - \tau \le \Delta\tau])1_{\Delta\tau\ge0} \\
&\quad \le kS_0 e^{y+r\tau}\mathbb{E}[|e^{Y_{|\Delta\tau|}} - 1|] + M\mathbb{Q}[|R_{u-y} - \tau| \le \Delta\tau].
\end{aligned}$$

We would like to apply Lemma 3.3.5, but we can't, because we still have a bound that depends on $\Delta\tau$ and Δx. However, note that $\forall_{\epsilon>0}, \forall_{\Delta x} - \epsilon \leq \Delta x \leq \epsilon, R_{u-x-\epsilon} \leq R_{u-x-\Delta x} \leq R_{u-x+\epsilon}$.

Then,

$$\begin{aligned}\lim_{\Delta\tau,\Delta x)\to 0} \mathbb{Q}[|R_{u-y} - \tau| \leq \Delta\tau] &\leq \lim_{(\Delta\tau,\Delta x)\to 0} (\mathbb{Q}[|R_{u-x-\epsilon} - \tau| \leq \Delta\tau] + \mathbb{Q}[|R_{u-x+\epsilon} - \tau| \leq \Delta\tau] \\ &\quad +\mathbb{Q}[R_{u-x-\epsilon} \leq \tau \leq R_{u-x+\epsilon}]) \\ &= \mathbb{Q}[R_{u-x-\epsilon} = \tau] + \mathbb{Q}[R_{u-x+\epsilon} = \tau] \\ &\quad + \mathbb{Q}[R_{u-x-\epsilon} \leq \tau \leq R_{u-x+\epsilon}] \\ &= \mathbb{Q}[R_{u-x-\epsilon} \leq \tau \leq R_{u-x+\epsilon}]\end{aligned}$$

after Lemma 3.3.4.

- Second term

$$\begin{aligned}|f_U(\tau, y) - f_U(\tau, x)| &= |\mathbb{E}[H(S_0 e^{y+Y_\tau}) 1_{\tau<R_{u-y}} - H(S_0 e^{x+Y_\tau}) 1_{\tau<R_{u-x}}]| \\ &\leq k S_0 e^{x+r\tau} |e^{\Delta x} - 1| + M(\mathbb{Q}[R_{u-y} \leq \tau < R_{u-x}] 1_{\Delta x \geq 0} \\ &\quad + \mathbb{Q}[R_{u-x} \leq \tau < R_{u-y}] 1_{\Delta x<0})\end{aligned}$$

As already demonstrated, this expression tends to zero when $\Delta x \to 0$.

Then,

$$\begin{aligned}\lim_{(\Delta\tau,\Delta x)\to 0} |f_U(\tau + \Delta\tau, x + \Delta x) - f_U(\tau, x)| &\leq \lim_{(\Delta\tau,\Delta x)\to 0} (M\mathbb{Q}[R_{u-x-\epsilon} \leq \tau \leq R_{u-x+\epsilon}] \\ &\quad + k S_0 e^{x+r\tau} |e^{\Delta x} - 1| \\ &\quad + M(\mathbb{Q}[R_{u-y} \leq \tau < R_{u-x}] 1_{\Delta x \geq 0} \\ &\quad + \mathbb{Q}[R_{u-x} \leq \tau < R_{u-y}] 1_{\Delta x<0})) \\ &= M\mathbb{Q}[R_{u-y-\epsilon} \leq \tau \leq R_{u-y+\epsilon}].\end{aligned}$$

So it remains to prove that when $\epsilon \to 0$, which implies $\Delta x \to 0$, that

$$\mathbb{Q}[R_{u-x-\epsilon} \leq \tau \leq R_{u-x+\epsilon}] \to 0.$$

But once again taking $\epsilon_n \to 0$ and if

$$A_n = \{\omega \in \Omega | R_{u-x-\epsilon_n} \leq \tau \leq R_{u-x+\epsilon_n}\},$$

then

$$\lim_{n=\infty} \mathbb{Q}[R_{u-x-\epsilon_n} \leq \tau \leq R_{u-x+\epsilon_n}] = \mathbb{Q}[\cap_{n=1}^{\infty} A_n] = \mathbb{Q}[R_{u-x} = \tau] = 0,$$

after Lemma 3.3.4.

Fourth Step: Let us show the continuity in $x = u$.

$$\begin{aligned}|f_U(\tau+\Delta\tau, u+\Delta x) - f_U(\tau, u)| &= |f_U(\tau+\Delta\tau, u+\Delta x)1_{\Delta x<0}| \\ &= |\mathbb{E}[H(S_0 e^{u+\Delta x+Y_{\tau+\Delta\tau}})1_{\tau+\Delta\tau<R_{-\Delta x}}1_{\Delta x<0}| \\ &\leq M\mathbb{Q}[\tau+\Delta\tau < R_{-\Delta x}]1_{\Delta x<0} = M\mathbb{Q}[\tau+\Delta\tau < R_{|\Delta x|}].\end{aligned}$$

But, for all $\xi > 0$ such that $|\Delta\tau| \leq \xi$, implies:

$$\left\{\omega \in \Omega | \tau + \Delta\tau < R_{|\Delta x|}\right\} \subset \left\{\omega \in \Omega | \tau - \xi < R_{|\Delta x|}\right\},$$

which in turn implies:

$$\mathbb{Q}[\tau + \Delta\tau < R_{|\Delta x|}] \leq \mathbb{Q}[\tau - \xi < R_{|\Delta x|}] \to 0,$$

when $\Delta x \to 0$, because it only depends on Δx. ■

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