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Solvability of a stationary nonlinear Black-Scholes equation under conditions on the potential

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Abstract. In this work, we study a nonlinear problem suggested by the Black–Scholes model for option pricing with stochastic volatility, namely,

$$\begin{cases} \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 f}{\partial \sigma^2} + \rho\sigma^2 V S \frac{\partial^2 f}{\partial S \partial \sigma} - \frac{1}{2}\rho\sigma^2 V \frac{\partial f}{\partial \sigma} + rS\frac{\partial f}{\partial S} = r\gamma(f) & \text{in } \Omega, \\ f(S,\sigma) = h(S,\sigma) & \text{on } \partial\Omega, \end{cases}$$

where the variables S and σ are respectively the asset value and the market volatility ([1], [2]). In [1], an analogous nonlinear problem has been investigated with a nonlinearity γ of the following type $\gamma(f) = g(f)f$. It has been used an iterative procedure under the hypothesis that $g(f)f$ is nondecreasing, applying upper and lower solutions. The function g was assumed to be C^2 and this regularity was used in computations in the proofs.

We consider a Hölder continuous nonlinearity $\gamma(f)$ and, assuming certain conditions on the potential Γ of γ, we prove the existence of a positive solution. The method of the proof, which is based on the construction of upper and lower solutions, obtained as solutions of an auxiliary initial value problem, also yields information on the localization of f.

Keywords: Nonlinear Black-Scholes equation, condition on the potential, positive stationary solutions, upper and lower solutions.
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1. INTRODUCTION

In this work, we consider a nonlinear elliptic problem suggested by an option pricing model that appears in Finance. In fact, the Black–Scholes partial differential equation for the price of $C = C(S,t)$ of an European call on the asset value S_t reads

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} = rC \quad \text{in } (0,+\infty) \times (0,T),$$

where T is the expiry date, r is the risk-free interest rate and σ is the volatility of the asset value S_t.

An important characteristic of this model is the assumption that the volatility of the underlying security is constant. However practice has pointed out some evidences that are not consistent with the constant volatility assumption made in Black-Scholes [4]. This fact led to further studies. In particular, in [3] a stochastic volatility model was developed, assuming a general correlation (negative, zero, positive) between the asset

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price and its volatility. In that case, the corresponding partial differential equation for the price of a derivative security contingent $f = f(S,\sigma,t)$ on the price S of an underlying asset reads

$$\frac{\partial f}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 f}{\partial S^2}+\frac{1}{2}\sigma^2V^2\frac{\partial^2 f}{\partial \sigma^2}+\rho\sigma^2VS\frac{\partial^2 f}{\partial S\partial\sigma}-\frac{1}{2}\rho\sigma^2V\frac{\partial f}{\partial\sigma}+rS\frac{\partial f}{\partial S}=rf. \quad (1.1)$$

The elements V and ρ are parameters arising from the financial setting (properly estimated - either by confidence intervals or point estimation - from the relevant financial market data). $V > 0$ represents the volatility of the volatility and $-1 < \rho < 1$ is the correlation coefficient between S and σ. In this paper, V and ρ are constant parameters (which corresponds to having been estimated by point estimation).

The above equations are linear. In this paper, we consider a generalization of the Black-Scholes equation (1.1) to a nonlinear situation by replacing in the second member rf by $r\gamma(f)$, which introduces the possibility of a nonlinear perturbation on f. We study the stationary case with Dirichlet boundary conditions.

More precisely, let Ω be an open bounded domain, such that $\overline{\Omega} \subset (]0,+\infty[)^2$ and the boundary $\partial\Omega$ is of class $C^{2,\mu}$, $0 < \mu < 1$. We consider the following problem $(P1)$

$$\begin{cases} \frac{1}{2}\sigma^2S^2\frac{\partial^2 f}{\partial S^2}+\frac{1}{2}\sigma^2V^2\frac{\partial^2 f}{\partial \sigma^2}+\rho\sigma^2VS\frac{\partial^2 f}{\partial S\partial\sigma}-\frac{1}{2}\rho\sigma^2V\frac{\partial f}{\partial\sigma}+rS\frac{\partial f}{\partial S}=r\gamma(f) & \text{in } \Omega, \\ f(S,\sigma)=h(S,\sigma) & \text{on } \partial\Omega, \end{cases}$$

where $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^+$ is a Hölder continuous function. As for the Dirichlet condition, $h \in C^{2,\mu}\left(\overline{\Omega}\right)$ and $h_0 := \min\{h(S,\sigma) : (S,\sigma) \in \partial\Omega\} > 0$. We prove the existence of a positive solution f, on the variables S and σ, assuming certain conditions on the potential Γ of γ.

In [1], a problem of this type with $\gamma(f) = g(f)f$ has been studied by an iterative procedure, under the hypothesis that $g(f)f$ is nondecreasing, and also using upper and lower solutions. There, the function g is assumed to be C^2 and this regularity is used in the computations in the proofs.

In the spirit of [1], our result is one more contribution to the study of problem $(P1)$ and to a better understanding of Black-Scholes equation. It is based however on different assumptions and the arguments are used in a different way. For instance, writing $\gamma(f)$ in the form $g(f)f$, we do not require differentiability on g. As an example, the nonlinearity $\gamma(f) = e^f$ can be considered in our result but not in [1].

The method of the proof, which is based on the construction of upper and lower solutions, obtained as solutions of an auxiliary initial value problem, also yields information on the localization of f. A similar argument has been used in [6] to study a parabolic equation with Dirichlet-periodic boundary conditions assuming some restrictions on the asymptotic behaviour on the potential Γ, namely a Hammerstein-type condition (see also [7]).

In this paper, the condition assumed on the potential Γ of γ is related to the value of Γ on $h_0 := \min\{h(S,\sigma) : (S,\sigma) \in \partial\Omega\}$. It guarantees in some sense that the ratio $\frac{\Gamma(h_0)-\Gamma(s)}{(h_0-s)^2}$ will assume some values under a certain bound which, as we shall see, depends on S,σ,ρ

and V. This type of conditions on the primitive is usually used when variational methods are applied. Although it is not the case in this paper, it will play a crucial role in the proof, namely to obtain a lower solution.

This paper is organized in the following way. In Section 2 the main result is stated. Section 3 deals with an initial value problem for an autonomous second order ODE. An auxiliary lemma that consists of an existence and localization statement is presented. This result will be used to obtain a lower solution for our problem. Section 4 is concerned with the proof of our main theorem. In the last section, we apply that result to study the problem $(P1)$ with $\gamma(f) = e^f$. We shall sketch briefly the proofs of the results. For a complete version with detailed proofs we refer to the forthcoming paper [5].

2. MAIN RESULT

As notation, we will write when convenient that

$$A(S,\sigma,\partial)f = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 f}{\partial \sigma^2} + \rho\sigma^2 V S \frac{\partial^2 f}{\partial S \partial \sigma} - \frac{1}{2}\rho\sigma^2 V \frac{\partial f}{\partial \sigma} + rS\frac{\partial f}{\partial S}.$$

Obviously, the coefficients of $A(S,\sigma,\partial)$ belong to $C^{0,\mu}\left(\overline{\Omega}\right)$, $0 < \mu < 1$. Since $|\rho| < 1$ then A is uniformly elliptic. As referred, the function $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^+$ is Hölder continuous and, as for the Dirichlet condition, $h \in C^{2,\mu}\left(\overline{\Omega}\right)$.

In order to state our result we settle some notation. Let

- $]S_0, S_1[\subset \mathbb{R}^+$ be the projection of Ω on the $S-axis.$
- $]\sigma_0, \sigma_1[\subset \mathbb{R}^+$ be the projection of Ω on the $\sigma - axis.$

Then $\Omega \subset]S_0,S_1[\times]\sigma_0,\sigma_1[$ and, of course, $\overline{\Omega} \subset [S_0,S_1] \times [\sigma_0,\sigma_1] \subset (]0,+\infty[)^2$.
Take the infimums of some of the coefficients in the equation of (P1) as follows

$$\begin{aligned} a_1 &= \frac{1}{2}\sigma_0^2 S_0^2 \\ a_2 &= \frac{1}{2}\sigma_0^2 V^2 \\ b_2 &= -\frac{1}{2}\max\{0,\rho\}\,\sigma_1^2 V \end{aligned}$$

and set

$$\theta := \frac{1}{r}\max\left\{\frac{a_1}{(S_1-S_0)^2}, \frac{a_2}{(\sigma_1-\sigma_0)^2} e^{2\frac{b_2}{a_2}(\sigma_1-\sigma_0)}\right\}. \tag{2.1}$$

Observe that θ is strictly positive. Then we state our main result.

Theorem 2.1. *Let* $h_0 := \min\{h(S,\sigma) : (S,\sigma) \in \partial\Omega\} > 0$ *and set* $\Gamma(s) = \int_0^s \gamma(\zeta)\,d\zeta$ *for* $s \in \mathbb{R}$. *Assume that*

$$\min_{s\in[0,h_0[} 2\frac{\Gamma(h_0)-\Gamma(s)}{(h_0-s)^2} < \theta. \tag{2.2}$$

Then problem $(P1)$ *has a positive solution* $f \in C^{2+\mu}\left(\overline{\Omega}\right)$, *such that*

$$0 < f \leq h_\infty := \max_{\partial\Omega} h.$$

Note that in (2.2) the minimum exists, since the function γ is positive and so the quotient tends to $+\infty$ as s tends to h_0.

The main argument to prove the above theorem relies on the method of upper and lower solutions. In fact, if we prove that problem $(P1)$ has a positive lower solution $\alpha \in C^{2,\mu}(\overline{\Omega})$ and an upper solution $\beta \in C^{2,\mu}(\overline{\Omega})$ satisfying $\alpha \leq \beta$, the thesis will follow by a classical result contained in the first chapter of Mawhin [8].

We recall that a *lower solution* α of $(P1)$ is a function $\alpha \in C^{2,\mu}(\overline{\Omega})$ such that

$$\begin{cases} A(S,\sigma,\partial)\alpha \geq r\gamma(\alpha), & \text{a.e. in } \Omega, \\ \alpha(S,\sigma) \leq h(S,\sigma), & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Similarly, an *upper solution* β of $(P1)$ is defined by reversing the inequalities in (2.3). A *solution* of $(P1)$ is a function u which is simultaneously a lower and an upper solution. We shall prove the existence of ordered lower and upper solutions of problem $(P1)$, more precisely $\alpha \leq \beta$. For that we will use the lemma contained in the next section.

Remark 2.2. Some hypotheses assumed above appear quite naturally due to the fact that the problem is related to an option pricing model. Namely, we look for positive solutions since these solutions are option prices. The function γ is also positive, for it is not admissible that a perturbation of the price of an option would make it negative. Also the fact that S, V and σ are positive is related to their financial meaning. On the other hand, they are assumed greater than some positive constant in order to avoid singularities in the equation. However that condition is not restricitve in the financial framework. The correlation coefficient ρ is prevented from taking the values -1 and $+1$; if not, the operator A would not be uniformly elliptic. From the financial point of view, it means that we are excluding only the extreme cases where the random variables are linearly related with probability 1. Nevertheless they can have a high degree of linearity since it is assumed that ρ may take any value in $]-1,+1[$.

3. AUXILIARY LEMMA

In this section we prove an existence and localization result of an initial value problem for an autonomous second order ODE of the form

$$Au'' + Bu' = r\gamma(u) \quad \text{in} \quad [a,b]. \tag{3.1}$$

As will be shown in the proof of Theorem 2.1, we can associate with the equation of problem $(P1)$ an ODE of the above type. Then, the result of this section will enable us to obtain a lower solution $\alpha \in C^{2,\mu}(\overline{\Omega})$ for problem $(P1)$.

By multiplying both members of (3.1) by $e^{\frac{B}{A}x}$ and dividing by A, it is easy to see that the above ODE can be written in the equivalent form

$$(pu')' = q\varphi(u),$$

where $p(x)=e^{\frac{B}{A}x}$, $q(x)=\frac{e^{\frac{B}{A}x}}{A}$ and $\varphi(u)=r\gamma(u)$.

We will now be concerned with initial value problems for ODE's of such type. More precisely, for $d\in\mathbb{R}$, consider the initial value problem

$$\begin{cases} (pu')'=q\varphi(u) \text{ in }]a,b[, \\ u(a)=d, \\ u'(a)=0, \end{cases} \tag{3.2}$$

where $\varphi:\mathbb{R}\to\mathbb{R}^+$ is a continuous positive function, $p,q:[a,b]\longrightarrow\mathbb{R}$ are two non-negative real functions, such that p is of class C^1 with

$$p_0 := \min_{x\in[a,b]} p(x) > 0,$$

and q is continuous.

By a solution of (3.2) we mean a function u of class C^2, defined on some interval $I=[a,x_0]\subset[a,b]$, which satisfies the equation on I and the initial conditions.
Set $\Psi(s)=\int_0^s \varphi(\zeta)\,d\zeta$ for $s\in\mathbb{R}$, and to fix notations put

$$p_\infty := \max_{x\in[a,b]} p(x) \quad \text{and} \quad q_\infty := \max_{x\in[a,b]} q(x).$$

Then, the following result holds.

Lemma 3.1. *Let $K\in\mathbb{R}^+$ and assume that*

$$\min_{s\in[0,K[} 2\frac{\Psi(K)-\Psi(s)}{(K-s)^2} < \left(\frac{p_0^2}{p_\infty q_\infty}\right)\frac{1}{(b-a)^2}. \tag{3.3}$$

Then, there is $d\in\mathbb{R}$, with $0<d<K$, such that problem (3.2) has a solution u defined on $[a,b]$ that satisfies

$$K\ge u(x)\ge d \quad \text{and} \quad u'(x)\ge 0.$$

Proof. We sketch briefly the main ideas of the proof. By (3.3) we can take $d\in\mathbb{R}$ such that $0<d<K$ and

$$2\frac{\Psi(K)-\Psi(d)}{(K-d)^2} < \left(\frac{p_0^2}{p_\infty q_\infty}\right)\frac{1}{(b-a)^2}. \tag{3.4}$$

For such d consider the problem (3.2). Let u be a solution for (3.2). Since $u(a)=d<K$, it is obvious that for some $\varepsilon>0$ we have $u<K$ on $[a,a+\varepsilon]$. So define ω_d such that

$$\omega_d := \sup\{x>a : u \text{ is defined and } u<K \text{ on } [a,x]\}.$$

It is clear that $\omega_d>a$. If we integrate the equation of (3.2) in $[a,x]$ for some $x\in]a,\omega_d[$ we can easily see that

$$u'(x)>0, \text{ for all } x\in]a,\omega_d[,$$

and, since $u(a) = d > 0$,

$$u(x) > d, \text{ for all } x \in [a, \omega_d[.$$

We claim that $\omega_d \geq b$, which will be enough to guarantee that the thesis holds. Assume by contradiction that

$$\omega_d < b.$$

By the definition of ω_d,

$$\lim_{x \to \omega_d^-} u(x) = K =: u(\omega_d),$$

and by the mean value theorem, for some $\xi \in]a, \omega_d[$,

$$(K-d)^2 = [u(\omega_d) - u(a)]^2 = \left[u'(\xi)(\omega_d - a)\right]^2.$$

But, taking $x \in]a, \omega_d[$, multiplying the equation in (3.2) by pu' and integrating between a and x, it is easy to obtain

$$|u'(x)|^2 \leq 2\frac{p_\infty q_\infty}{p_0^2}(\Psi(K) - \Psi(d)).$$

Therefore,

$$(K-d)^2 \leq 2\frac{p_\infty q_\infty}{p_0^2}(\Psi(K) - \Psi(d))(b-a)^2,$$

and so

$$\frac{1}{(b-a)^2}\frac{p_0^2}{p_\infty q_\infty} < \frac{2(\Psi(K) - \Psi(d))}{(K-d)^2},$$

which yields a contradiction to (3.4). So, $\omega_d \geq b$. Hence, there is a parameter $0 < d < K$ for which the problem (3.2) has a solution u such that $K \geq u(x) \geq d$ and $u'(x) \geq 0$ in $[a,b]$. □

4. SKETCH OF THE PROOF OF THEOREM 2.1

As referred in section 2, if we prove that problem $(P1)$ has a positive lower solution $\alpha \in C^{2,\mu}(\overline{\Omega})$ and an upper solution $\beta \in C^{2,\mu}(\overline{\Omega})$ satisfying $\alpha \leq \beta$, the thesis will follow by a result contained in Mawhin [8]. That will be done in three steps:

Step 1: Problem $(P1)$ has a positive lower solution $\alpha \in C^{2,\mu}(\overline{\Omega})$.

Consider the equation of problem $(P1)$

$$\frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} + \frac{1}{2}\sigma^2 V^2\frac{\partial^2 f}{\partial \sigma^2} + \rho\sigma^2 VS\frac{\partial^2 f}{\partial S \partial \sigma} - \frac{1}{2}\rho\sigma^2 V\frac{\partial f}{\partial \sigma} + rS\frac{\partial f}{\partial S} = r\gamma(f). \tag{4.1}$$

Fix an interval $[a,b]$ and constants A and B according to the value of θ defined in (2.1). That is:

- if $\theta = \frac{1}{r}\frac{a_1}{(S_1-S_0)^2}$, put $[a,b] := [S_0,S_1]$ and $A := a_1, B := 0$,
- if $\theta = \frac{1}{r}\frac{a_2}{(\sigma_1-\sigma_0)^2}e^{2\frac{b_2}{a_2}(\sigma_1-\sigma_0)}$, put $[a,b] := [\sigma_0,\sigma_1]$ and $A := a_2, B := b$.

Consider in $[a,b]$ the autonomous ordinary differential equation

$$Au'' + Bu' = r\gamma(u),$$

which, as referred in the previous section, is equivalent to

$$\left(pu'\right)' = q\varphi(u),$$

with $p(x) = e^{\frac{B}{A}x}$, $q(x) = \frac{e^{\frac{B}{A}x}}{A}$ and $\varphi(u) = r\gamma(u)$.

Set $\Psi(u) = \int_0^u r\gamma(s)ds$, that is, $\Psi(u) = r\Gamma(u)$. For both possible values of θ, condition (2.2) implies that Ψ satisfies (3.3). So, we can apply Lemma 3.1 and state that there is d with $0 < d < h_0$ such that the problem

$$\begin{aligned} Au'' + Bu' &= r\gamma(u) \\ u(a) &= d \\ u'(a) &= 0 \end{aligned}$$

has a solution u_h verifying

$$h_0 \geq u_h(x) \geq d \quad \text{and} \quad u_h'(x) \geq 0.$$

Moreover, since $Au_h'' + Bu_h' = r\gamma(u_h)$, $A > 0$ and $B \leq 0$, then

$$u_h'' = \frac{1}{A}\left(-Bu_h' + r\gamma(u_h)\right) \geq 0.$$

According to the framewok established above, either $[a,b] = [S_0,S_1]$ and the solution will be on the variable S, $u_h(S)$, or $[a,b] = [\sigma_0,\sigma_1]$ and the solution will be on the variable σ, $u_h(\sigma)$. So, put respectively

$$\alpha(S,\sigma) = u_h(S) \qquad \text{or} \qquad \alpha(S,\sigma) = u_h(\sigma), \qquad \text{for } (S,\sigma) \in \overline{\Omega}.$$

We have that $\alpha \in C^{2,\mu}(\overline{\Omega})$. By plugging the function u_h into equation (4.1) it can be seen that we have, for each $(S,\sigma) \in \Omega$,

$$\begin{aligned} &\frac{1}{2}\sigma^2 S^2 \frac{\partial^2\alpha}{\partial S^2} + \frac{1}{2}\sigma^2 V^2 \frac{\partial^2\alpha}{\partial\sigma^2} + \rho\sigma^2 VS \frac{\partial^2\alpha}{\partial S\partial\sigma} - \frac{1}{2}\rho\sigma^2 V \frac{\partial\alpha}{\partial\sigma} + rS\frac{\partial a}{\partial S} \\ \geq\ & r\gamma(u_h) \\ =\ & r\gamma(\alpha(S,\sigma)). \end{aligned}$$

So, since $u_h \leq h_0 = \min_{\partial\Omega} h(S,\sigma)$,

$$\begin{aligned} A(S,\sigma,\partial)\alpha &\geq r\gamma(\alpha) \text{ in } \Omega \\ \alpha(S,\sigma) &\leq h(S,\sigma) \text{ on } \partial\Omega, \end{aligned}$$

what shows that α is a lower solution of problem $(P1)$.

Step 2: Problem $(P1)$ has an upper solution $\beta \in C^{2,\mu}(\overline{\Omega})$.

Take $h_\infty = \max_{\partial\Omega} h(S,\sigma)$ and put

$$\beta(S,\sigma) \equiv h_\infty \qquad \text{for } (S,\sigma) \in \overline{\Omega}.$$

Of course $\beta \in C^{2,\mu}(\overline{\Omega})$ and

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \beta}{\partial S^2} + \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 \beta}{\partial \sigma^2} + \rho\sigma^2 VS \frac{\partial^2 \beta}{\partial S \partial \sigma} - \frac{1}{2}\rho\sigma^2 V \frac{\partial \beta}{\partial \sigma} + rS\frac{\partial \beta}{\partial S} = 0 \leq r\gamma(\beta).$$

It is trivial that

$$\begin{aligned} A(S,\sigma,\partial)\beta &\leq r\gamma(\beta) \text{ in } \Omega \\ \beta(S,\sigma) &\geq h(S,\sigma) \text{ on } \partial\Omega. \end{aligned}$$

Step 3: Problem $(P1)$ has a positive solution $f(S,\sigma)$.

It is clear that

$$\alpha(S,\sigma) \leq \beta(S,\sigma).$$

So, by the above considerations problem $(P1)$ has a solution f such that

$$0 < d \leq f(S,\sigma) \leq \max_{\partial\Omega} h(S,\sigma).$$

5. EXAMPLE

Take $\gamma(f) = e^f$. That is, consider the following problem $(P2)$

$$\begin{cases} \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 f}{\partial \sigma^2} + \rho\sigma^2 VS \frac{\partial^2 f}{\partial S \partial \sigma} - \frac{1}{2}\rho\sigma^2 V \frac{\partial f}{\partial \sigma} + rS\frac{\partial f}{\partial S} = r\,e^f & \text{in } \Omega, \\ f(S,\sigma) = h(S,\sigma) & \text{on } \partial\Omega, \end{cases}$$

in the general setting assumed before. It is clear that $\gamma(f)$ can not be written in the form $g(f)f$, with g differentiable. But applying Theorem 2.1 we can state the following result.

Corollary 5.1. Take $h_0 := \min\{h(S,\sigma) : (S,\sigma) \in \partial\Omega\} > 0$. If

$$2\frac{e^{h_0} - 1}{h_0^2} < \theta, \tag{5.1}$$

then problem $(P2)$ has a positive solution $f \in C^{2+\mu}\left(\overline{\Omega}\right)$ such that

$$0 < f \leq h_\infty := \max_{\partial\Omega} h.$$

Proof. Since $\gamma(f) = e^f$, then $\Gamma(s) = \int_0^s \gamma(\zeta)\, d\zeta = e^s - 1$. Consider the function

$$\eta(s) = 2\frac{e^{h_0} - e^s}{(h_0 - s)^2}.$$

Easy computations show that $\eta'(s) \geq 0$ for $s \in [0, h_0]$. So, we have that

$$\inf_{s\in[0,h_0[} \eta(s) = \min_{s\in[0,h_0[} \eta(s) = \eta(0).$$

Therefore,

$$\inf_{s\in[0,h_0[} 2\frac{e^{h_0} - e^s}{(h_0 - s)^2} = \eta(0) = 2\frac{e^{h_0} - 1}{h_0^2},$$

and by (5.1) the condition (2.2) is satisfied for $\Gamma(s) = e^s - 1$. So, by Theorem 2.1, the thesis holds. □

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