European Journal of Operational Research 39 (1989) 213–222
North-Holland

Theory and Methodology

A structural lagrangean relaxation for two-duty period bus driver scheduling problems

José PAIXÃO
Faculdade de Ciências, Universidade de Lisboa, Av. 24 de Julho 134-5°, 1300 Lisboa, Portugal

Margarida PATO
Instituto Superior de Economia, Universidade Técnica de Lisboa, Rua Miguel Lupi 20, 1200 Lisboa, Portugal

Abstract: The two-duty period bus driver scheduling problem is a particular case of the generalized set covering problem, $\min\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \geqslant \boldsymbol{b},\ \boldsymbol{0} \leqslant \boldsymbol{x} \leqslant \boldsymbol{h}$ and integer$\}$ where, each column of the boolean matrix $\boldsymbol{A}$ consists of at most two strings of consecutive ones. Such a denomination for the problem is due to several real life applications, in particular for bus crew scheduling.

In this paper, we present a 'structural' lagrangean relaxation and penalties for improving the bounds on the optimum for the problem. Two other lagrangean relaxation approaches, previously reported in the literature, are considered too.

A computational study relative to these relaxations was carried out with both randomly generated test problems and real life cases from Rodoviária Nacional, a large mass transport operator in Portugal. The results reported in the paper evidence a better performance for the new lagrangean relaxation approach which, combined with greedy heuristics, yield a reasonably good and fast procedure for tackling real life problems.

Keywords: Lagrangean relaxation, heuristics, generalized set covering, network flows, bus driver scheduling

1. Introduction

This paper is mainly devoted to determining lower bounds for the optimal value of a particular case of the generalized set covering problem, which is the following integer program:

$$\text{(GSCP)}\quad \min \sum_{j\in N} c_j x_j,$$

$$\text{s.t.} \quad \sum_{j\in N} a_{ij} x_j \geqslant b_i \quad (i \in M), \tag{1.1}$$

$$0 \leqslant x_j \leqslant h_j \text{ and integer} \quad (j \in N). \tag{1.2}$$

The particular case of GSCP that we consider in the present paper, is related to real life bus driver scheduling applications [8,12] where:

- N is the index set for the feasible driver workdays or work shifts, each one of them being assigned a cost, c_j;
- the set M corresponds to time periods requiring a minimal number of drivers defined through vector $\boldsymbol{b} = (b_i)_{i\in M}$;
- each column $(a_{ij})_{i\in M}$ consists of one or two strings of consecutive ones according to the fact

Received December 1987; revised March 1988

Research for this paper was supported by the Instituto Nacional de Investigação Científica (INIC) and Junta Nacional de Investigação Ceintífica e Tecnológica (JNICT).

of the j-th column corresponding, respectively, to a single-duty or a two-duty work shift;
- h_j is the maximum number of workers that can be assigned to the shift j.

Due to its particular structure, the problem has been referred in the literature as the two-duty period bus driver scheduling problem which we will denote, for short, as SP.

According to the usual nomenclature, ν(SP) stands for the optimal value for SP. For a particular $i \in M$, we will denote by N_i the set of variables having a positive coefficient in the row, that is the set $N_i = \{j \in N: a_{ij} = 1\}$. Similarly, we define, for a specific $j \in N$, the set $M_j = \{i \in M: a_{ij} = 1\}$.

When the values h_j $(j \in N)$ are all equal to 1 the SP becomes a specific multiple covering problem and if, on top of this, the b_i are all equal to 1, then the SP is a particular case of the set covering problem. Hence, a natural approach for the SP consists of generalizing the techniques previously developed for those well known models such as greedy heuristics and lagrangean relaxation [1,4,7,11], for obtaining both upper and lower bounds on ν(SP).

In previous reports we have described a set of primal–dual greedy heuristics, for the SP [9,10]. There, we have studied several heuristic procedures both from a theoretical and experimental point of view. These greedy heuristics, briefly described in Section 5, provide bounds, from above and from below, on the optimum ν(SP), which, from our own experience, proved to be reasonably accurate.

In this paper, we address the question of sharpening the lower bounds and, consequently, closing the gap for the optimum. Hence, in Section 2, we present a special lagrangean relaxation for the SP, initially reported in [9], which consists of adding first a set of 'structural' constraints and, then, relaxing other ones in order to obtain a network flow problem.

For computational testing of what we call the structural lagrangean relaxation, we consider other two similar approaches that have been presented in the literature. One, which is described in Section 3, was first used for a crew scheduling set partitioning problem [11] and has been referred in the literature as splitting relaxation [6] or decomposition relaxation [3]. The second one, consists of relaxing all the covering constraints (1.1) and is briefly addressed in Section 4.

For all the three lagrangean relaxations considered in this paper for the SP, penalties are deduced in order to tighten the variable-bounds h_j.

In Section 5, a procedure, embedding heuristics and lagrangean relaxation in a subgradient search algorithm, is presented.

Computational experience, carried out on a VAX/VMS 750 and relative to the application of the three lagrangean relaxation approaches, is reported in Section 6. The test problems that we tried out correspond both to randomly generated examples and real life bus driver scheduling problems from Rodoviária Nacional, one of the largest mass transport operators in Portugal. As will be shown, the final procedure is capable of optimally solving a reasonable number of test problems with little computational effort. For the nonoptimally solved test problems, the procedure provides a tight final gap between the lower and upper bound on ν(SP). Hence, as pointed out in Section 7, the procedure seems very suitable for embedding in a three-search method for solving to the optimality some crew scheduling problems.

2. Structural lagrangean relaxation ($\mathbf{R}^s$)

In this section, we present a structural lagrangean relaxation for the SP that results from ignoring the break periods for the work shifts. Mathematically, that is done by firstly adding a set of redundant constraints in such way that the SP becomes a minimum cost network flow problem with side constraints. Then, these are relaxed in a typical lagrangean fashion.

Now, let R_j be the index set of rows corresponding to the 'hole' of column j, that is, the break periods for the j-th work shift $R_j = \{i \in M - M_j: \exists i_1, i_2 \in M_j, \text{ with } i_1 < i < i_2\}$. We define $\tilde{M} = \bigcup_{j \in N} R_j$ and, for a particular row index i, $C_i = \{j \in N: i \in R_j\}$. In words, C_i is the set of work shifts for which the row index i corresponds to a break period.

Then, the scheduling problem SP can be reformulated as a network flow problem with additional constraints as follows:

$$\text{(SP)}\quad \min \sum_{j \in N} c_j x_j,$$

$$\text{s.t.}\quad \sum_{j \in N_i} x_j \geqslant b_i \quad (i \in \tilde{M}), \tag{2.1}$$

$$\sum_{j \in N_i \cup C_i} x_j \geqslant b_i \quad (i \in M),$$

$$0 \leqslant x_j \leqslant h_j \text{ and integer} \quad (j \in N).$$

Relaxing the constraints (2.1) in a normal lagrangean fashion one gets, for a specific vector of multipliers, $\boldsymbol{\lambda} \geqslant \mathbf{0}$, the following network flow problem:

$$(\text{SPR}^{\text{s}}_{\lambda})\quad \min \sum_{j \in N} \left(c_j - \sum_{i \in \tilde{M}_j} \lambda_i \right) x_j + \sum_{i \in \tilde{M}} \lambda_i b_i,$$

$$\text{s.t.}\quad \sum_{j \in N_i \cup C_i} x_j \geqslant b_i \quad (i \in M), \tag{2.2}$$

$$0 \leqslant x_j \leqslant h_j \quad (j \in N),$$

where $\tilde{M}_j = M_j \cap \tilde{M}$ and the integrality constraints for the variables need not to be explicitly considered.

Obviously, the optimal value for $\text{SPR}^{\text{s}}_{\lambda}$, denoted by $\nu(\text{SPR}^{\text{s}}_{\lambda})$, is a lower bound on $\nu(\text{SP})$.

If an upper bound on $\nu(\text{SP})$ is available then penalties can be obtained from $\text{SPR}^{\text{s}}_{\lambda}$ for tightening the variable-bounds h_j. This is stated by Result 2.1 but, first, let us define

$$\hat{r}_j = c_j - \sum_{i \in \tilde{M}_j} \lambda_i - \sum_{i \in \hat{M}_j} \hat{u}_i \quad (j \in N), \tag{2.3}$$

where $\hat{M}_j = M_j \cup R_j$ and $(\hat{u}_i)_{i \in M}$ are the optimal linear dual variables corresponding to the constraints (2.2) in $\text{SPR}^{\text{s}}_{\lambda}$.

Result 2.1. Let z_{u} be a known upper bound on $\nu(\text{SP})$, $\boldsymbol{\lambda}$ a nonnegative $|\tilde{M}|$-dimensional real vector, $\hat{r}_j$ the dual reduced costs given by (2.3) and $H^{\text{s}}_j = [z_{\text{u}} - \nu(\text{SPR}^{\text{s}}_{\lambda}) + h_j \min(0, \hat{r}_j)]/\hat{r}_j$ for all j. For a particular $k \in N$:

(a) if $\hat{r}_k > 0$ then

$$x_k \leqslant \lfloor H^{\text{s}}_k \rfloor \quad \text{(the largest integer strictly less than } H^{\text{s}}_k\text{)}$$

in any feasible solution better than z_{u};

(b) if $\hat{r}_k < 0$ then

$$x_k \geqslant \lceil H^{\text{s}}_k \rceil \quad \text{(the least integer strictly greater than } H^{\text{s}}_k\text{)}$$

in any feasible solution better than z_{u}.

Results 2.1(a) and (b), come from considering, respectively, in and out-penalties for the variables. That is, in-penalties correspond to additional costs when a variable is forced into the solution at a specific level; out-penalties correspond to additional costs incurred by forcing a variable to have a value in the solution less than a specific level.

3. Decomposition lagrangean relaxation ($\mathbf{R}^{\mathbf{d}}$)

In this section, we refer to a different lagrangean relaxation presented by Shepardson and Marsten [11] for two-duty period bus driver scheduling set partitioning problems.

First, let us remind that each column of the 0–1 matrix is given by one or two strings of consecutive ones. Then, each variable x_j can be split into two new variables, x^1_j and x^2_j corresponding respectively to the first and the second duty period of the j-th work shift. Now, a linking constraint (3.1) is required and the SP gains the following formulation:

$$\text{(SP)}\quad \min \sum_{j \in N} c^1_j x^1_j + \sum_{j \in N} c^2_j x^2_j,$$

$$\text{s.t.}\quad \sum_{j \in N} a^1_{ij} x^1_j + \sum_{j \in N} a^2_{ij} x^2_j \geqslant b_i \quad (i \in M),$$

$$0 \leqslant x^p_j \leqslant h_j \text{ and integer} \quad (j \in N,\ p = 1, 2),$$

$$x^1_j = x^2_j \quad (j \in N), \tag{3.1}$$

where c^1_j and c^2_j are such that $c_j = c^1_j + c^2_j$, the matrix $A^1 = (a^1_{ij})_{i \in M, j \in N}$ represents the first string of ones and $A^2 = (a^2_{ij})_{i \in M, j \in N}$ represents the second string, when exists.

If the linking constraints (3.1) are embedded into the objective function associated to $|N|$ real multipliers, $\boldsymbol{\lambda}$, one obtains again a minimum cost network flow problem but different from $\text{SPR}^{\text{s}}_{\lambda}$. This type of lagrangean relaxation is usually known as decomposition or splitting relaxation [3,6], which we denote by R^d:

$$(\mathrm{SPR}^d_\lambda) \quad \min \quad \sum_{j\in N} (c^1_j - \lambda_j) x^1_j + \sum_{j\in N} (c^2_j + \lambda_j) x^2_j,$$
$$\text{s.t.} \quad \sum_{j\in N} a^1_{ij} x^1_j + \sum_{j\in N} a^2_{ij} x^2_j \geq b_i \quad (i \in M), \tag{3.2}$$
$$0 \leq x^p_j \leq h_j \quad (j \in N,\ p = 1, 2).$$

It is clear that every optimal value $\nu(\mathrm{SPR}^d_\lambda)$ is a lower bound on ν(SP), and again, penalties from SPR^d_λ can be used for tightening the variable-bounds according to the following result:

Result 3.1. Let z_u be a known upper bound on ν(SP), λ a real $|N|$-dimensional vector and the dual reduced costs,

$$\hat{r}^p_j = c^p_j + (-1)^p \lambda_j - \sum_{i\in M} a^p_{ij} u^*_i \quad (j \in N,\ p = 1, 2),$$
$$r^*_j = c_j - \sum_{i\in M_j} u^*_i \quad (j \in N),$$

where $(u^*_i)_{i\in M}$ are the optimal linear dual variables corresponding to the constraints (3.2) in SPR^d_λ. Considering for each j the value

$$H^d_j = \left[z_u - \nu(\mathrm{SPR}^d_\lambda) + h_j \sum_{p=1,2} \min(0,\ \hat{r}^p_j) \right] / r^*_j$$

then, for any feasible solution having a better value than z_u, one has:

(a) $x_k \leq \lfloor H^d_k \rfloor$ for a variable k such that $r^*_k > 0$;
(b) $x_k \geq \lceil H^d_k \rceil$ for a variable k such that $r^*_k < 0$.

4. Full lagrangean relaxation (R^f)

In this section, we consider a third possibility for applying lagrangean relaxation to the SP. In fact, if all the constraints (1.1) are relaxed one obtains, for a particular $|M|$-dimensional multiplier vector, $\lambda \geq 0$, the following problem:

$$(\mathrm{SPR}^f_\lambda) \quad \min \quad \sum_{j\in N} \left(c_j - \sum_{i\in M_j} \lambda_i \right) x_j + \sum_{i\in M} \lambda_i b_i,$$
$$\text{s.t.} \quad 0 \leq x_j \leq h_j \quad (j \in N).$$

An optimal solution for SPR^f_λ is trivially given by

$$x^{\mathrm{opt}}_j = \begin{cases} h_j & \text{if } r_j < 0, \\ 0 & \text{if } r_j \geq 0, \end{cases} \quad (j \in N)$$

where

$$r_j = c_j - \sum_{i\in M_j} \lambda_i. \tag{4.1}$$

Again, the optimal value for SPR^f_λ, $\nu(\mathrm{SPR}^f_\lambda)$, is less than or equal to ν(SP). Penalties, both in and out, can be easily obtained taking into account the following:

Result 4.1. Let z_u be an upper bound on ν(SP), λ a nonnegative $|M|$-dimensional real vector, r_j defined as in (4.1) and

$$H^f_j = \left[z_u - \nu(\mathrm{SPR}^f_\lambda) + h_j \min(0,\ r_j) \right] / r_j$$

for all j.

Then, for any feasible solution having a better value than z_u, one has:

(a) $x_k \leq \lfloor H^f_k \rfloor$ for a variable k such that $r_k > 0$;
(b) $x_k \geq \lceil H^f_k \rceil$ for a variable k such that $r_k < 0$.

5. Subgradient optimization and heuristics

When considering any of the lagrangean relaxation approaches described in the previous sections, one aims to find the best corresponding multipliers. That is, one looks for the optimal solution for

$$\left(\underset{i=\mathrm{f,d,s}}{\mathrm{DSPR}^i} \right) \quad \max \quad \nu(\mathrm{SPR}^i_{\lambda^i}),$$
$$\text{s.t.} \quad \lambda^i \geq \mathbf{0} \quad \text{in case } i = \mathrm{f, s},$$
$$\lambda^i \gtrless \mathbf{0} \quad \text{in case } i = \mathrm{d}.$$

The numerical search for $\nu(\mathrm{DSPR}^i)$ with i = f, d, s and for the optimal multipliers can be made through the subgradient optimization method [5] and, since all the three lagrangean subproblems $\mathrm{SPR}^f_{\lambda^f}$, $\mathrm{SPR}^d_{\lambda^d}$ and $\mathrm{SPR}^s_{\lambda^s}$ have the integrality property [2], it is well known that

$$\nu(\mathrm{DSPR}^f) = \nu(\mathrm{DSPR}^d) = \nu(\mathrm{DSPR}^s) = \nu(\mathrm{D\overline{SP}}),$$

where $\mathrm{D\overline{SP}}$ denotes the dual problem for the linear relaxation for SP.

Therefore, theoretically the three lagrangean approaches are equivalent but, in practice, the speed of convergence for the subgradient optimization can be much different for them.

Moreover, if one considers a specific feasible solution for $\overline{\text{DSP}}$, $[(u_i)_{i\in M}, (v_j)_{j\in N}]$ non null and such that $v_j = \max(0, \Sigma_{i\in M_j} u_i - c_j)$ for all j, it is easy to show that

$$z_\ell = \sum_{j\in M} u_i b_i - \sum_{j\in N} v_j h_j = \nu(\text{SPR}_{\lambda^{\text{f}}}^{\text{f}}) \leqslant \min(\nu(\text{SPR}_{\lambda^{\text{d}}}^{\text{d}}),\ \nu(\text{SPR}_{\lambda^{\text{s}}}^{\text{s}})), \tag{5.1}$$

where

$$\lambda_i^{\text{f}} = u_i \quad (i\in M),$$

$$\lambda_i^{\text{s}} = u_i \quad (i\in \tilde{M}),$$

$$\lambda_j^{\text{d}} = \left(c_j^1 \sum_{i\in M} a_{ij}^2 u_i - c_j^2 \sum_{i\in M} a_{ij}^1 u_i\right) \Big/ \sum_{i\in M_j} u_i \quad (j\in N).$$

The above results were confirmed by computational experience with a composed procedure, combining the primal–dual greedy heuristics (Phase 1) with each one of the lagrangean relaxation approaches (Phase 2). As shown later in this paper, and although not theoretically proved, empirical results evidence a better behaviour for the structural approach relatively to the decomposition technique.

As pointed out above the combined procedure consists of two main phases. Hence, in Phase 1, an heuristic procedure incorporating the techniques described in [10] (primal and dual greedy heuristics improved by local search), is used in order to produce bounds, both from below and from above.

Concerning to the upper bounds we consider a set of greedy heuristics each one of them with a local search procedure and following the general pattern:

– First, a feasible solution is constructed step by step, selecting a row in M to be covered and, then, a variable to cover the unsatisfied demand for that row. Several row and variable selection criteria are combined for building up feasible solutions. Relatively to the value given to the selected variable different options can be taken too (see [10]).

– After producing a greedy upper bound on the optimal value, improvements are attempted through a local search procedure based on three main steps: decreasing redundant variables and producing a prime solution; replacing one variable in the current solution by another one that covers the same demand at a cheaper cost; replacing a pair of variables in the current solution by a single variable which covers the same at lower cost.

Also in Phase 1, dual greedy heuristics are considered for obtaining lower bounds on ν(SP) and following the scheme:

– In each iteration, and according to a previously specified criterion, a particular element, say index i^*, is selected from a subset row $R \subseteq M$ and the maximum feasible value is assigned to the corresponding dual variable, u_{i^*}. The row index i^* is then removed from R and the process repeats until this becomes empty. The rows in $M - R$ are then considered in a similar way. The dual variables associated to the variable-bound constraints, v_j $(j \in N)$, are set always equal to zero.

The last step of Phase 1 consists in attempting to improve the bounds, both from below and from above, by trying to impose the linear complementary conditions to the corresponding solutions. Two different kind of algorithms are described in [10] for performing the linear complementary improvement tests.

Then, in Phase 2 of the combined procedure, lagrangean relaxation is used considering the initial multipliers set equal to the dual variables relative to the best heuristic lower bound obtained in Phase 1. From that, one knows (condition (5.1)) that the lagrangean relaxation lower bound is, at least, as good as the heuristic one.

Figure 1 summarizes this procedure.

6. Computational results

The procedure briefly described in the previous section was tried out for 30 randomly generated problems (15 with 100 columns, R1–R15; 15 with 865 columns, G1–G15) and 4 real life bus crew scheduling problems from Rodoviária Nacional (RN), a large bus transport company in Portugal, denoted by RN1–RN4 ($|N| = 865$). All the test problems have the same number of constraints corresponding to 36 half an hour working periods from 6 a.m. to midnight. The cost of each work shift is defined according to the rules in use at RN

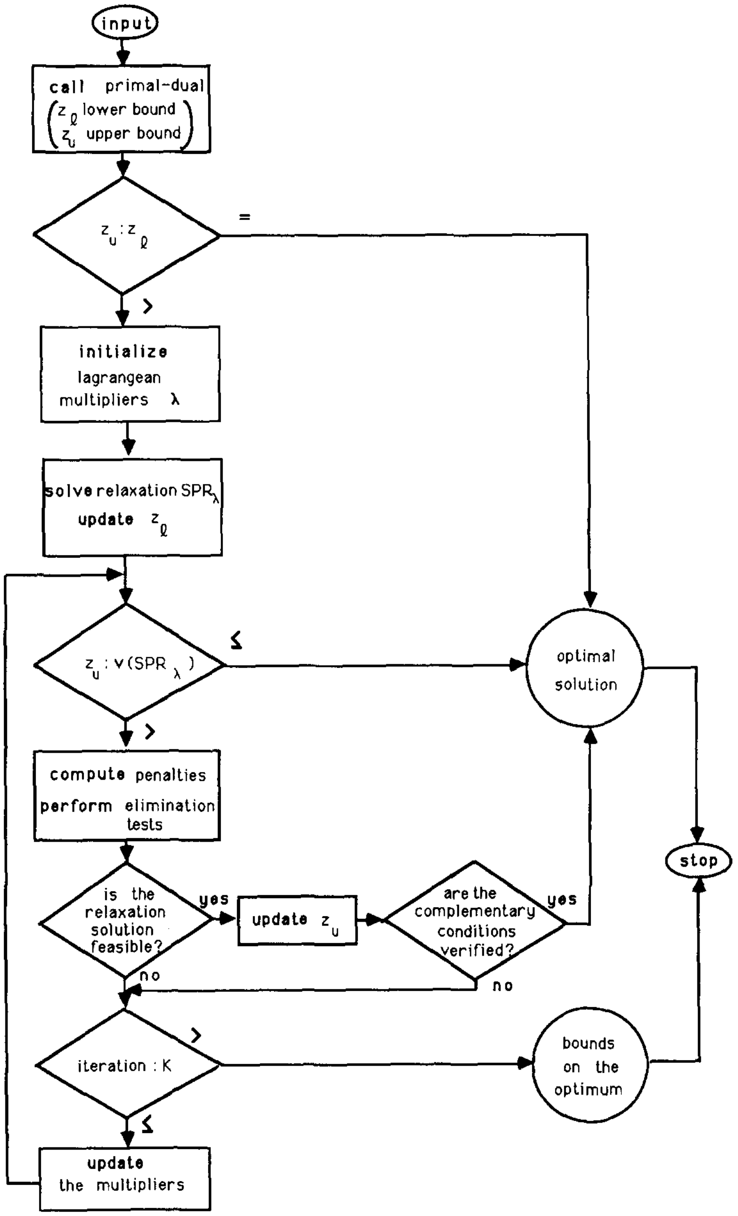

Figure 1. Heuristic-lagrangean procedure

and consists of a fixed cost added by extra costs related either to the period of the day or overtime working periods. The demand patterns of problems R1–R15 and G1–G15 were generated in order to follow 3 different types of distribution: unimodal, bimodal and irregular. For each i, the corresponding problems Ri and Gi have the same right-hand-side vector, differing only on the columns of the matrix $(a_{ij})_{i \in M, j \in N}$. For these set of test problems the variable bounds were set as $h_j = \max_{i \in M_j} b_i$ for $j \in N$.

Table 1 reports on the computational experience mentioned above. In that table, column (1) identifies the two-duty period bus driver scheduling instance whose dimensions — $|M|$, $|N|$ and density (number of ones over $|M| \times |N|$) — and optimal value are shown, respectively, in columns (2) and (3).

Table 1
Computing times and quality of the bounds obtained with the heuristic-lagrangean procedure

Problem (1)	Dimens. density (2)	Optimal value (3)	Primal–dual heuristic			Lagrangean relaxation R^f			Lagrangean relaxation R^d			Lagrangean relaxation R^s		
			lower % (4)	upper % (5)	time (6)	lower % (7)	upper % (8)	time (9)	lower % (10)	upper % (11)	time (12)	lower % (13)	upper % (14)	time (15)
R1	$\|M\| = 36$	51726	90.5	3.2	6.0	95.4	3.2	21.6	91.5	3.2	44.5	99.4	3.2	18.5
R2	$\|N\| = 100$	52268	96.3	0.7	6.3	96.5	0.7	19.3	96.6	0.7	41.5	99.1	0.7	16.4
R3	55%	84184	99.7	2.2	7.1	99.7	2.2	19.8	99.7	2.2	50.3	99.7	2.2	17.5
R4		85080	93.8	4.4	9.7	96.1	4.4	23.1	96.3	4.4	58.2	99.8	1.4	24.6
R5		120930	99.5	0.1	6.8	99.5	0.1	19.4	100.0	0.1	31.6	99.5	0.1	15.1
R6		48990	100.0	0.0	2.3	–	–	–	–	–	–	–	–	–
R7		27062	98.4	1.3	4.8	98.6	1.3	18.3	98.5	0.0	28.2	99.0	0.5	10.7
R8		55236	96.6	2.9	6.4	98.0	2.9	21.6	97.7	2.9	40.7	99.3	2.9	16.2
R9		47470	98.9	1.0	5.8	99.7	1.0	16.0	99.7	0.1	31.2	99.0	1.0	11.0
R10		26612	87.5	1.1	4.2	93.3	1.1	19.1	91.0	1.1	35.3	98.9	0.4	16.6
R11		98930	95.8	1.3	5.5	96.5	1.3	20.0	95.8	1.0	36.4	97.6	1.3	12.2
R12		51248	89.3	0.0	4.3	92.2	0.0	19.6	90.3	0.0	27.3	100.0	0.0	0.6
R13		49096	93.2	0.0	6.1	97.0	0.0	19.9	99.3	0.0	32.9	97.6	0.0	12.5
R14		40862	97.1	4.2	4.8	98.0	4.2	19.3	98.7	0.7	37.6	99.2	4.2	14.7
R15		188598	94.4	3.6	6.6	98.0	3.6	18.7	99.5	3.6	50.3	99.4	3.6	15.9
G1	$\|M\| = 36$	49972	92.2	5.2	31.6	94.4	5.2	181.9	92.2	4.8	225.6	99.9	3.6	137.1
G2	$\|N\| = 865$	49326	96.3	0.8	28.6	97.0	0.8	159.2	96.3	0.8	242.1	100.0	0.4	152.0
G3	56%	80248	99.0	1.0	32.2	99.1	1.0	149.3	99.0	1.0	248.1	100.0	1.0	100.4
G4		79670	92.1	3.0	39.1	93.4	3.0	168.0	92.1	3.0	236.9	99.9	2.3	143.7
G5		115540	100.0	0.4	36.2	100.0	0.0	112.0	100.0	0.4	270.7	100.0	0.0	3.6
G6		48434	100.0	0.0	16.8	–	–	–	–	–	–	–	–	–
G7		24084	91.2	2.6	21.1	94.3	2.6	147.9	96.1	2.6	169.8	99.6	2.6	91.2
G8		51900	96.3	1.2	30.1	97.6	1.2	144.9	96.4	1.2	216.5	99.9	1.2	149.7
G9		45658	99.8	0.5	21.9	99.8	0.5	132.3	99.8	0.0	142.8	99.8	0.5	55.3
G10		25248	91.3	3.0	26.4	93.1	3.0	205.0	91.3	3.0	192.5	99.7	2.7	114.1
G11		93078	99.9	0.1	22.1	99.9	0.1	143.4	99.9	0.1	173.4	99.9	0.1	62.5
G12		51134	89.2	1.0	28.9	91.1	1.0	149.1	89.2	1.0	185.2	100.0	0.0	13.4
G13		47720	89.6	0.0	17.2	93.3	0.0	151.4	92.7	0.0	156.0	100.0	0.0	5.5
G14		36520	100.0	0.0	9.2	–	–	–	–	–	–	–	–	–
G15		167728	99.99	0.1	23.2	99.99	0.1	132.2	99.99	0.0	140.9	100.0	0.0	13.3
RN1	$\|M\| = 36$	70418	96.0	0.1	40.4	96.0	0.1	191.0	96.0	0.1	325.3	99.9	0.1	79.1
RN2	$\|N\| = 865$	45952	98.3	4.4	36.8	98.3	4.4	164.1	98.3	1.9	257.6	99.98	1.7	108.3
RN3	56%	33360	83.4	0.5	31.2	90.4	0.5	145.0	89.9	0.5	280.3	100.0	0.0	18.9
RN4		80316	100.0	0.0	29.1	–	–	–	–	–	–	–	–	–
Average			95.5	1.5		96.5	1.6		96.1	1.3		99.5	1.3	
Worst value			83.4	5.2		90.4	5.2		89.2	4.8		97.6	4.2	
Number of optimal values			5	7		1	4		2	6		8	7	

The remaining columns of Table 1 show the results obtained with the different algorithms coded in FORTRAN and implemented on a VAX/VMS 750 (with FPA). Columns (4)–(6) refer to the heuristic lower and upper bounds, that is the Phase 1 of the procedure. Hence, in column (4), the value $[z_\ell/v(\mathrm{SP})] \times 100$ is shown for each test problem in order to evaluate the greedy heuristic lower bound z_ℓ. For evaluating the heuristic upper bounds, the value $[(z_u - v(\mathrm{SP}))/v(\mathrm{SP})] \times 100$ is given in column (5). Column (6) presents the computing time, in seconds, taken by the primal-dual greedy heuristic procedure.

Also, Table 1 reports on the experiments with the structural and the other two lagrangean relaxations corresponding to Phase 2 of the procedure. There, similar information to the one given for the heuristic procedure is provided by columns (7)–(9), (10)–(12) and (13)–(15) respectively for R^f, R^d and R^s. For each one of them we fixed a value of K for the maximum number of subgradient iterations (R^f: $K = 140$; R^d: $K = 28$ and R^s: $K = 30$), in order to make a fair comparison between them.

Some statistics are provided at the bottom rows of Table 1. From that, one can see that the heuristic lower bound was relatively poor for most of the test problems. In fact, it remains 4.4% in average under the optimum and, in particular for the real life problem RN3, it was 83.4% of the optimal value. However, these results were substantially improved by the lagrangean relaxations as columns (7), (10) and (13), show, respectively for R^f, R^d and R^s.

Additionally, from Table 1, it becomes clear that structural relaxation R^s outperforms the other two. In fact the average quality of the lower bound provided by R^f is 96.5%, by R^d is only

Table 2
Efficiency of the penalties

Problem	Heuristic		R^f		R^d		R^s	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
R1	0	31.9	0	44.9	0	40.6	0	58.6
R2	0	58.7	0	60.3	0	62.7	11	79.3
R3	0	58.3	0	58.5	0	58.5	0	65.0
R4	0	35.6	0	39.5	0	44.7	0	72.9
R5	20	80.9	0	81.0	58	86.8	0	82.2
R6	–	–	–	–	–	–	–	–
R7	5	69.2	0	69.7	17	78.7	8	73.6
R8	0	51.5	0	56.0	0	55.6	0	62.9
R9	20	68.3	1	73.6	39	86.5	0	74.0
R10	0	34.4	0	44.8	0	53.2	4	75.9
R11	0	51.2	0	55.0	0	55.2	0	58.4
R12	0	34.9	0	52.5	0	46.2	–	–
R13	0	47.6	16	66.9	39	83.4	15	73.8
R14	0	32.3	0	34.1	24	65.8	0	45.2
R15	0	23.1	0	30.2	0	31.3	0	41.1
G1	0	32.1	0	37.4	0	32.6	0	69.9
G2	0	78.6	0	80.0	0	79.2	666	96.0
G3	60	87.7	55	88.2	0	88.1	128	91.6
G4	0	48.1	0	49.9	0	48.7	0	71.5
G5	415	94.3	0	94.3	0	94.3	–	–
G6	–	–	–	–	–	–	–	–
G7	0	71.0	3	75.6	80	75.8	199	86.6
G8	0	75.8	0	79.7	0	77.2	234	89.2
G9	666	96.0	0	96.0	162	97.4	23	96.2
G10	0	33.5	0	37.9	0	34.8	5	77.8
G11	670	91.2	0	91.2	0	93.6	0	91.2
G12	0	56.9	0	57.9	0	58.1	–	–
G13	0	77.3	0	83.8	0	82.4	–	–
G14	–	–	–	–	–	–	–	–
G15	677	95.8	0	95.8	150	98.8	–	–
RN1	0	64.7	0	64.9	0	66.8	714	94.0
RN2	0	62.3	0	62.8	46	75.0	129	84.0
RN3	0	27.0	0	43.6	0	46.4	–	–
RN4	–	–	–	–	–	–	–	–
Average		59.0		63.5		66.6		75.5
Worst		23.1		30.2		31.3		41.2

96.1%, whereas the corresponding value for R^s is 99.5%. The worst lower bound values are 90.4%, 89.2% and 97.6% respectively for R^f, R^d and R^s. Also, for 8 test problems, the lower bound reached the optimum for R^s, while such situation occurred only once for R^f and twice for R^d.

Figures referring to the upper bounds (columns (5), (8), (11) and (14)) come also, in favour of the structural version R^s. Actually, the improvements on the already good upper bounds produced by the heuristic procedure were, in average, slightly better for R^s (column (14)) than for R^f or for R^d (columns (8) and (11) respectively).

For this set of instances the computing times (columns (6), (9), (12) and (15)) are much more favourable with the R^s-version. Let us note that the algorithm applied for solving the minimum cost network flow lagrangean subproblems, SPR^d_λ and SPR^s_λ, was a straightforward version of the out-of-kilter method and we have good reasons to believe that the computing time for R^s can be further improved when using a more sophisticated network flow code.

Finally, Table 2, reports on the efficiency of the penalties (see Results 2.1, 3.1 and 4.1) measured in two ways:

(i) the number of variables fixed at 0 level at the end of the procedure (columns (2), (4), (6) and (8));

(ii) the average improvement on the variable-bounds (columns (3), (5), (7) and (9)) defined as

$$\left[1/|N| \sum_{j\in N} \left(h_j - \lfloor H_j^i \rfloor\right)/h_j\right] \times 100,$$

where h_j is the initial upper bound for x_j (constraint (1.2)) and $\lfloor H_j^i \rfloor$ is the best bound on the value for the j-th variable, for $i =$ f, d, s.

Naturally, these figures were not computed for the problems optimally solved by the procedure (e.g., R6, G6, G14 and RN4). It is also worthwhile to refer the fact that the part (b) of Results 2.1, 3.1, 4.1 has never been active for the test problems that we considered.

It is interesting to see that the figure in column (3) means, for instance for problem R9, that the variable bounds were reduced, with the heuristic procedure, in average, by 68.3%. Whereas in column (2), for the same problem, we can see that 20 variables were fixed at zero level, that is, were conceptually removed from the problem.

Although, for most of the problems no variables were removed, one must remark that, in most of the cases, the reduction on the bounds was over 23% for the heuristic and over 30% for the composed procedure with anyone of the lagrangean relaxations.

Also, concerning to the penalties, Table 2 shows the higher quality of the structural lagrangean relaxation, since it manages to tighten the variable bounds more than the other two lagrangean relaxations. In fact, the average improvement on the variable-bounds came from 59% with the heuristics to 75.5% with R^s (63.5% for R^f and 66.6% for R^d). The number of variables removed was slightly better for the large problems ($|N| = 865$) with the R^s-version than to the others. However, for the small size problems ($|N| = 100$) relaxation R^d outperformed R^s in terms of number of variables removed.

At last, we can conclude, from this set of test problems, that it seems worthwhile to use the combined procedure with the structural lagrangean relaxation R^s, after performing the primal–dual greedy heuristic procedure described with more details in [10]. In fact, recalling Table 1, let us point out that for 12 out of 34 test problems the optimal solution was found and the average final gap was less than 1.6%. For the real life two-duty period bus driver scheduling problems, RN1–RN4, the computing results proved even better with 0.5% for the average final gap and 2 out of 4 problems being optimally solved.

7. Final remarks

In this paper we have presented a new lagrangean relaxation for tightening bounds, both from below and from above, on the optimal value for generalized set covering problems related to crew scheduling applications. We named it structural lagrangean relaxation since some new redundant constraints are added to the original problem and, after relaxing another set of constraints in a lagrangean way, the resulting problem has a network flow structure.

Computational experience, carried out for different type of test problems, shows a very reasonable performance for a procedure combining heuristics and the structural lagrangean relaxation.

At present, further research is being carried out on the development of a branch-and-bound scheme where this bounding procedure is embedded.

References

[1] Balas, E., and Ho, A., "Set covering algorithms using cutting planes, heuristics and subgradient optimization: A computational study", *Mathematical Programming* 12 (1980) 37–60.

[2] Geoffrion, A.M., "Lagrangean relaxation for integer programming", *Mathematical Programming Study* 2 (1974) 82–114.

[3] Guinard-Spielberg, M., and Kim, S., "Lagrangean decomposition: A model yielding stronger Lagrangean bounds", Technical Report #73, Wharton School, University of Pennsilvanya, 1986.

[4] Hall, N., and Hochbaum, D., "The multicovering problem: The use of heuristics, cutting planes and subgradient optimization for a class of integer programs", Working paper WPS 85-73, College of Administrative Science, The Ohio State University, 1985.

[5] Held, M., Wolfe, P., and Crowder, H.P., "Validation of subgradient optimization", *Mathematical Programming* 6 (1974) 62–88.

[6] Jörnsten, K., Näsberg, M., and Smeds, P., "Variable splitting—A new Lagrangean relaxation approach to some mathematical programming models", LITH-MAT-R-85-04, Department of Math., Linköping Institute of Technology, Sweden, 1986.

[7] Paixão, J.P., "Algorithms for large scale set covering problems", Ph.D. thesis, Department of Management Science, Imperial College, London, 1983.

[8] Paixão, J.P., Branco, M.I., Captivo, M.E., Pato, M.V., Eusébio, R., and Amado, L., "Bus and crew scheduling on a microcomputer", in: J.D. Coelho and L.V. Tavares (eds.), *OR Models on Microcomputers*, North-Holland, Amsterdam, 1986.

[9] Paixão, J.P., and Pato, M.V., "Heuristics, lagrangean relaxation and penalties for two-duty period scheduling problems", Nota 1/87 CEAUL, Faculdade de Ciências da Universidade de Lisboa, presented at the EURO VIII Congress, Lisboa, 1986.

[10] Paixão, J.P., and Pato, M.V., "Primal and dual greedy heuristics for the generalized set covering problem", Nota 12/87 CEAUL, Faculdade de Ciências da Universidade de Lisboa, 1987.

[11] Shepardson, F., and Marsen, R.E., "A Lagrangean relaxation algorithm for the two duty period scheduling problem", *Management Science* 26 (1980) 274–281.

[12] Wren, A. (ed.), *Computer Scheduling of Public Transportation: Urban Passenger Vehicle and Crew Scheduling*, North-Holland, Amsterdam, 1981.