Insurance: Mathematics and Economics 46 (2010) 450–460

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

The optimal reinsurance strategy — the individual claim case[☆]

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ARTICLE INFO

Article history:
Received July 2008
Received in revised form
January 2010
Accepted 18 January 2010

Keywords:
Optimal reinsurance
Adjustment coefficient
Expected utility
Exponential utility function
Convex premium principles
Risk
Katz family

ABSTRACT

This paper is concerned with the optimal form of reinsurance when the cedent seeks to maximize the adjustment coefficient of the retained risk (related to the probability of ultimate ruin) – which we prove to be equivalent to maximizing the expected utility of wealth, with respect to an exponential utility with a certain coefficient of risk aversion – and restricts the reinsurance strategies to functions of the individual claims, which is the case for most nonproportional treaties placed in the market.

Assuming that the premium calculation principle is a convex functional we prove the existence and uniqueness of solutions and provide a necessary optimality condition (via needle-like perturbations, widely known in optimal control). These results are used to find the optimal reinsurance policy when the reinsurance loading is increasing with the variance. The optimal contract is described by a nonlinear function, of a similar form than in the aggregate case.

1. Introduction

There are many theoretical results in favour of this or that type of reinsurance, depending on the optimality criterion and the premium principle chosen. Borch (1960) proved that stop loss minimizes the variance of the retained risk if the reinsurer charges a fixed premium dependent only on the expected reinsurance claims. Taking the maximization of the expected utility as the optimality criterion, Arrow (1963) proved a similar result in favour of the stop loss contract. There are some generalizations of Arrow's result, a few of them quite recent (e.g. Kaluszka, 2004). Hesselager (1990) achieved an equivalent result using the adjustment coefficient as optimality criterion.

All these articles in favour of the stop loss contract are based on the assumption that the ceded claims have a fixed expected value, although Borch himself has made, in Borch (1969), a number of negative remarks to his result. In fact the reinsurance premium with the same loading coefficient (=loading divided by the expected claim amount), for all the reinsurance schemes, does not have any practical adherence.

All these articles consider that reinsurance is placed on the aggregate. When we consider that reinsurance is placed on individual terms excess of loss takes the place of the stop loss contract. See, for instance, Bowers et al. (1987), Gerber (1979), Gajek and Zagrodny (2000) and Kaluszka (2001). The comments made about the reinsurance premium can also apply to individual reinsurance. Froot (2001), using over 4000 catastrophe reinsurance layers transacted during the period 1970–1998, showed that the loading coefficient increases for higher layers, as would be expected. This justifies the need of results that deviate from the assumption that the premium is calculated according to the expected value principle.

There are some results on optimal reinsurance that consider that the premium loading is an increasing function with the variance of the ceded risk. Gajek and Zagrodny (2000) and Kaluszka (2001) minimized the variance of the retained risk, when the loading of the reinsurance premium used is based on the expected value and/or on the variance of the reinsured risk and the premium is fixed. Kaluszka (2005) generalized that article to other convex premium calculation principles and other optimality criteria.

Guerra and Centeno (2008) chose as optimality criterion the adjustment coefficient of the retained risk and assumed that the reinsurance premium is a convex functional. Note, however that the amount to pay for the reinsurance arrangements is not fixed, as is the case in all the papers cited above.

In this article we generalize the results obtained in Guerra and Centeno (2008), by considering that the reinsurance strategies are confined to be per claim reinsurance.

[☆] This research has been supported by Fundação para a Ciência e a Tecnologia (FCT) – project PTDC/ECO/66693/2006 – through PIDDAC, partially funded by the Portuguese State Budget. We are also grateful to the IME reviewers for their comments and suggestions.

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doi:10.1016/j.insmatheco.2010.01.002

The adjustment coefficient is related to the ultimate probability of ruin by the Cramér–Lundberg asymptotic formula and by Lundberg's inequality. The minimization of the ultimate probability of ruin would be our first goal if it was not such a difficult function to minimize. On the other hand the numerical comparisons between the ultimate probability of ruin and the adjustment coefficient, for traditional reinsurance forms, have shown that the Lundberg bound is very close to the ruin probability at the optimal retention levels, see Dickson and Waters (1996) and Centeno (1997), even for relatively small values of the initial reserve and for different claim amount distributions. A similar numerical study for the reinsurance arrangement that we propose in Guerra and Centeno (2008) and in the present paper, would be interesting, but out of the scope of this paper, given its length.

The probability of ultimate ruin, and the adjustment coefficient as a proxy, is in our opinion a better measure for reinsurance purposes, then other measures that only concern one period of time, like the probability of survival in one period of time used for instance by Gajek and Zagrodny (2004) or the VaR used by Cai et al. (2008).

Part of the difficulty in studying the problem of maximizing the adjustment coefficient lies on the fact that it is defined in an implicit form and its domain has not a structure appropriate to use arguments based on classical implicit function theorems. In Guerra and Centeno (2008) we overcame that difficulty by showing that to maximize the adjustment coefficient is equivalent to solve a two-step problem. The first step in this new problem consists in maximizing the expected utility of wealth of the retained risk for an exponential utility function, for all positive values of the coefficient of risk aversion. The second step consists in solving a single-variable equation. The optimal adjustment coefficient equals the coefficient of risk aversion for which the maximal expected value of the utility function is −1. The reinsurance policy that maximizes the adjustment coefficient is the treaty that maximizes the expected utility of wealth for that particular value of the risk-aversion coefficient. It turns out that the maximization step in the two-step problem is easier to deal from the mathematical point of view than the original one. Thus, both problems are solved. It was proved that one optimal reinsurance policy always exists and it was given a necessary condition for a policy to be optimal. Stop loss is indeed the optimal form of reinsurance if the reinsurer rates the contracts by the expected value principle, but when the reinsurance loading is an increasing function of the variance (for example, in the variance or standard deviation premium principles), then the optimal form is of a nonlinear type (not an already known typical form), but very easily constructed (see Guerra and Centeno, in press).

The paper is organized as follows. Section 2 contains the formulation of the problem, the basic notation and the blanket assumptions that will be used. Section 3 contains some essentially technical elements that will be used to obtain the main results. In Section 4 we analyse the relationship between the maximization of the adjustment coefficient of the retained risk and the maximization of the expected value of the utility of the insurer's wealth. In Section 5 we prove existence and uniqueness of optimal policies for the expected utility criterion. This result is used in Section 6 to prove existence and uniqueness of a policy which maximizes the adjustment coefficient. A necessary condition for optimality is obtained in Section 7. In Section 8 we assume that the loading on the reinsurance premium is an increasing function of the variance and provide the optimal necessary conditions. We show that the optimal treaty is broadly of the same type as in the aggregate case but some additional issues concerning the structure of optimal treaties arise in the individual claim case. The relationship between this structure and the distribution of claim numbers is discussed in Section 9.

2. Problem setting and assumptions

We consider the classical discrete-time surplus model (see e.g. Kaas et al., 2008, pp. 98).

Time is divided in periods of equal length (say, years). In each period the insurer receives a constant income $c > 0$ from premiums and faces a random number of claims of random sizes. The insurer may acquire reinsurance on a "per claim" basis. This means that a reinsurance treaty is a function $Z : [0, +\infty) \mapsto [0, +\infty)$ mapping each possible claim size to the value refunded under the reinsurance contract. To acquire a reinsurance Z, the insurer must pay a premium $P(Z)$ per time period. We assume that the pricing rule $Z \mapsto P(Z)$ is fixed by the reinsurance company and is known to the direct insurer who seeks to choose the "best possible" treaty taking into account that pricing rule. Let N_i, $i \in \mathbb{N}$, denote the number of claims received by the direct insurer in the ith period and $Y_{i,j}$, $i \in \mathbb{N}$, $0 \le j \le N_i$, denote the amount of the jth claim received in the period i ($Y_{i,0} \equiv 0$ for all $i \in \mathbb{N}$). Assuming that the reinsurance treaty Z is in force, the net result in the period i is the random variable

$$L_{Z,i} = c - P(Z) - \sum_{j=0}^{N_i} (Y_{i,j} - Z(Y_{i,j})).$$

Assuming that the insurer detains a certain amount of initial reserves $u > 0$ and the same policy remains in force forever, the probability of ultimate (also called eventual) ruin is

$$\psi_Z(u) = \Pr\left\{ u + \sum_{i=1}^{n} L_{Z,i} < 0, \text{ for some } n = 1, 2, \ldots \right\}.$$

Thus, one interesting criterion for selection of a reinsurance treaty would be to find the function Z that minimizes $\psi_Z(u)$ among all functions in some suitable class. Although very interesting, it would be a very hard problem.

Assuming that all the $L_{Z,i}$, $i \in \mathbb{N}$, are i.i.d. random variables, it is well known (see for example Kaas et al., 2008) that the probability of ruin satisfies the Lundberg inequality:

$$\psi_Z(u) \le \exp(-uR_Z), \tag{1}$$

where R_Z is the adjustment coefficient of L_Z, a generic r.v. with the same distribution as $L_{Z,i}$. Thus, a large value of the adjustment coefficient provides a comparatively small probability of ruin in an infinite-time horizon.

Dickson and Waters (1996) have computed the optimal retention level that minimizes the ultimate probability of ruin for the quota-share and excess of loss contracts, under the classical model. In all their examples the optimal retention is not far from the value of the retention that maximizes the adjustment coefficient, at least for not very small values of u, and conclude that maximizing the adjustment coefficient is a good alternative to minimizing the ultimate probability of ruin. This is the strategy that we follow in this study.

To give a rigorous formulation of the problem, we need to present the underlying assumptions and basic definitions.[1]

Assumption 1. The number of claims, N_i, $i \in \mathbb{N}$, are i.i.d. $\mathbb{N}_0$-valued random variables.

The claim sizes, $Y_{i,j}$, $i \in \mathbb{N}, j = 1, 2, \ldots$, are i.i.d. non-negative random variables, independent of N_k, $k \in \mathbb{N}$

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[1] One supplementary assumption will be introduced in Section 7. The assumptions stated in this section guarantee existence of solutions to the problem (see Sections 5 and 6). The supplementary assumption is used to obtain necessary optimality conditions and, due to its technical nature it is better dealt when explaining our method to obtain optimality conditions.

The expected aggregate amount of claims per time period is less than the insurer's income:

$$\mathbb{E}\sum_{j=1}^{N_i} Y_{i,j} < c, \quad \forall i \in \mathbb{N}. \quad \square$$

This assumption guarantees that all the $L_{Z,i}$, $i \in \mathbb{N}$, are i.i.d. random variables and thus the Lundberg upper bound (1) holds. Clearly, the same holds if we allow $(N_i, Y_{i,1}, Y_{i,2}, \ldots)$ to be dependent, provided they are i.i.d. to $(N_k, Y_{k,1}, Y_{k,2}, \ldots)$ for all $i \neq k$. Our results in Section 4 to 6 still hold in this more general framework. However, though necessary optimality conditions analogous to the results in Section 7 can be obtained, they are very difficult to compute. This limits practical application to the simpler case stated in Assumption 1, pending further research on numerical solutions for the more general case.

It is clear that if the insurer's income is less than the expected claim amount, then ruin is certain except in the unrealistic situation where the insurer can cede part of his risk for a premium with negative loading.

Taking Assumption 1 into account, we denote by N (resp., Y) a generic random variable with the same distribution as N_i (resp., $Y_{i,j}$). Concerning the distribution of claim sizes, we assume that

Assumption 2. Y is a continuous random variable with density f. $\square$

We provide this assumption in order to simplify the technical content of the text. It plays no role in the arguments proving existence and uniqueness of solution, but it simplifies considerably the arguments that lead to the necessary optimality conditions, retaining the essential features of the approach we propose. If this assumption is omitted, then perturbations at Lebesgue points and perturbations at singular points of the distribution must be treated separately (see Section 7).

We wish to consider very general reinsurance treaties. Thus, the set of all possible reinsurance policies is:

$$\mathcal{Z} = \{Z : [0, +\infty) \mapsto \mathbb{R} | Z \text{ is measurable and } 0 \le Z(y) \le y, \ \forall y \ge 0\}.$$

This reflects the practice of the reinsurance industry: in a typical treaty claims are not refunded above their value and there no situations in which the direct insurer falls under the obligation of paying the reinsurer for a claim.

We do not distinguish between functions which differ only on a set of zero probability with respect to the density f. i.e., two measurable functions, ϕ and ϕ' are considered to be the same whenever $\Pr\left\{\phi(Y) = \phi'(Y)\right\} = 1$. Similarly, a measurable function Z is an element of $\mathcal{Z}$ whenever

$$\Pr\{0 \le Z(Y) \le Y\} = 1.$$

For convenience of notation we identify any measurable function $\phi : [0, +\infty) \mapsto \mathbb{R}$ with the random variable $\phi(Y)$. Hence we write $\mathbb{E}\phi$, $\Pr\{\phi \le x\}$, etc. to denote $\mathbb{E}\phi(Y)$, $\Pr\{\phi(Y) \le x\}$, etc.

For each period of time, the premium charged for a reinsurance policy is computed by a real functional $P : \mathcal{Z} \mapsto \mathbb{R}$. We consider that this functional satisfies the following assumption.

Assumption 3. The reinsurance premium is a convex non-negative functional over $\mathcal{Z}$ such that $P(0) = 0$, $P(Y) < +\infty$ and

$$\Pr\{L_Z < 0\} > 0, \quad \forall Z \in \mathcal{Z} \tag{2}$$

(i.e., no reinsurance policy guarantees non-negative profits).

P is continuous in the mean-squared sense, i.e., $\lim_{k\to\infty} P(Z_k) = P(Z)$ holds for every sequence $\{Z_k \in \mathcal{Z}\}_{k=1,2,\ldots}$ such that

$$\lim_{k\to\infty} \mathbb{E}\left((Z_k - Z)^2\right) = 0. \quad \square$$

The concept of convex premium calculation principles was introduced in the actuarial literature by Deprez and Gerber (1985). Convexity is an important simplifying assumption in optimization, since it plays a major role in existence theory.

The assumptions that premiums are non-negative, $P(0) = 0$ and $P(Y) < +\infty$ are not really necessary from the mathematical point of view but are natural: no reinsurer will pay to accept risk and the direct insurer pays no premium if he buys no reinsurance. Also, if the risk is insured, then it should be reinsurable, even if only for a very high (but still finite) premium.

From a practical point of view, condition (2) just states the common fact that no company can be sure that it will never incur any loss. From a theoretical point of view, it guarantees that the optimal adjustment coefficient must be finite (see proof of Theorem 2, Section 6 below). If N is an unbounded random variable (i.e., $\Pr\{N > n\} > 0$ for all $n \in \mathbb{N}$), then condition (2) is equivalent to $P(Y) > c$, which is a very realistic assumption. If N is bounded, then condition (2) may be difficult to check analytically from the problem data. However, our results still allow for the computation of treaties with arbitrarily large adjustment coefficient in the case where condition (2) fails (see Remark 2 in Section 8 below).

Some kind of continuity of the premium is expected in real life: it would be surprising if "similar" treaties would fetch very different premiums. The choice of continuity in the mean-squared sense is a technical option: By providing the space of admissible treaties with the topology of mean-squared convergence, and taking Assumption 4 (below), $\mathcal{Z}$ becomes a convex closed bounded subset of an Hilbert space. In this setting, continuity and convexity of P make existence results very easy to prove (see Sections 5 and 6 below).

Assumption 4. The claim size has finite second moment:

$$\mathbb{E}(Y^2) < +\infty. \quad \square$$

This assumption requires the claim size distribution to be not too heavy tailed but still holds for many distributions widely used in the trade which do not have finite moment generating function in any neighbourhood of zero.

Consider the map $G : \mathbb{R} \times \mathcal{Z} \mapsto [0, +\infty]$, defined by

$$G(R, Z) = \mathbb{E}\mathrm{e}^{-RL_Z}, \quad R \in \mathbb{R}, \ Z \in \mathcal{Z}.$$

Let R_Z denote the adjustment coefficient of the retained risk for a particular reinsurance policy, $Z \in \mathcal{Z}$. R_Z is usually defined as the strictly positive value of R which solves the equation

$$G(R, Z) = 1, \tag{3}$$

for that particular Z, when such a root exists. Corollary 1 below shows that (3) cannot have more than one positive solution. This means that the map $Z \mapsto R_Z$ is a well defined functional in the set

$$\mathcal{Z}^+ = \{Z \in \mathcal{Z} : (3) \text{ admits a positive solution}\}.$$

We consider a slightly extended definition of the adjustment coefficient. Namely, we set $R_Z = 0$ (resp., $R_Z = +\infty$) for any $Z \in \mathcal{Z}$ such that $G(R, Z) > 1$ (resp., $G(R, Z) \le 1$) for every $R \in (0, +\infty)$.

Our last assumption guarantees that $\mathcal{Z}^+ \neq \emptyset$ (see Corollary 2, below).

Throughout this text p denotes the probability law of N, i.e.,

$$\Pr\{N = n\} = p(n), \quad n = 0, 1, 2, 3, \ldots.$$

Assumption 5. The radius of convergence of the probability generating function

$$\pi(t) = \mathbb{E}t^N = \sum_{n=0}^{+\infty} t^n p(n)$$

is $\rho \in (1, +\infty]$. $\square$

Due to Assumption 1, the functional G can be represented as

$$G(R,Z)=\pi\left(\mathbb{E}\mathrm{e}^{R(Y-Z)}\right)\mathrm{e}^{R(P(Z)-c)},\quad R\in\mathbb{R},Z\in\mathcal{Z}. \tag{4}$$

Hence, if $\rho = 1$ (i.e., the claim number has a heavy-tailed distribution), then the profit must have a heavy negative tail, except if the direct insurer cedes all the risk. This means that no reinsurance placed on individual claims affords effective protection, justifying the assumption.

Taking into account the definitions and assumptions above, our problem can be formalized as:

Problem 1. Find $(R^*,Z^*)\in(0,+\infty)\times\mathcal{Z}^+$ such that

$$R^*=R_{Z^*}=\max\left\{R_Z:Z\in\mathcal{Z}^+\right\}.\quad\square$$

A policy $Z^*\in\mathcal{Z}$ is said to be **optimal for the adjustment coefficient criterion** if $Z^*\in\mathcal{Z}^+$ and (R_{Z^*},Z^*) solves this problem.

3. Preliminaries

In this section we study some properties of the map $(R,Z)\mapsto G(R,Z)$ which will be useful to prove existence of solutions for Problem 1.

First, note that the map $R\mapsto G(R,Z)$ is just the moment generating function of the random variable $-L_Z$. The following Lemma contains the main properties of moment generating functions. The proof can be found, e.g., in Bhattacharya and Waymire (2007).

Lemma 1. *Consider a random variable X and let*

$$\eta=\sup\{R\in\mathbb{R}:\mathbb{E}\mathrm{e}^{RX}<\infty\}.$$

If $\eta>0$, then the map $R\mapsto\mathbb{E}\mathrm{e}^{RX}$ is continuous in $[0,\eta)$, admits derivatives of all orders in $(0,\eta)$, and

$$\frac{\partial^k\mathbb{E}\mathrm{e}^{RX}}{\partial R^k}=\mathbb{E}\left(X^k\mathrm{e}^{RX}\right),\quad\forall R\in(0,\eta),k\geq0. \tag{5}$$

If $\eta<+\infty$, then

$$\lim_{R\to\eta^-}\mathbb{E}\mathrm{e}^{RX}=\mathbb{E}\mathrm{e}^{\eta X}, \tag{6}$$

and $\mathbb{E}\mathrm{e}^{RX}=+\infty$ for all $R>\eta$. $\square$

For each $Z\in\mathcal{Z}$, let $\eta_Z\in[0,+\infty]$ denote the constant

$$\eta_Z=\sup\{R:G(R,Z)<+\infty\}.$$

Lemma 1 has the following immediate Corollary:

Corollary 1. *For every $Z\in\mathcal{Z}$, the map $R\mapsto G(R,Z)$ is strictly convex in $[0,\eta_Z)$ ($[0,\eta_Z]$ when $\eta_Z<+\infty$). Hence, Eq. (3) admits at most one positive solution.*

If $\mathbb{E}L_Z\leq0$, then $R_Z=0$. $\square$

The first part of Corollary 1 guarantees that $\lim_{R\to+\infty}G(R,Z)$ is well defined (possibly infinite) for any $Z\in\mathcal{Z}$. Thus, we use $G(\infty,Z)$ to denote this limit. Under this trivial extension, the map $R\mapsto G(R,Z)$ is strictly convex in $[0,\eta_Z]$ also in the case $\eta_Z=+\infty$.

Corollary 1 shows that the map $Z\mapsto R_Z$ is well defined in $\mathcal{Z}^+$. However, in general $\mathcal{Z}^+$ is a strict subset of $\mathcal{Z}$ and hard to characterize. More precisely, the set of reinsurance policies such that $\eta_Z\in(0,+\infty)$ and $G(\eta_Z,Z)<1$ is in general not empty. Further, if the claims size has an heavy-tailed distribution,[2] then the map $(R,Z)\mapsto G(R,Z)$ is discontinuous at every pair $(R,Z)\in(0,+\infty)\times\mathcal{Z}$ such that $G(R,Z)<+\infty$. These facts indicate that optimization methods based on the implicit function theorem are not suitable to solve Problem 1.[3] The remaining results presented in this section will be used in later sections to overcome this difficulty.

The following Proposition plays an important role in our proof of existence of a solution for Problem 1.

Proposition 1. *For each $Z\in\mathcal{Z}$ and $\eta\in(0,+\infty)$ such that $G(\eta,Z)<1$, there exists $\tilde{Z}\in\mathcal{Z}^+$ such that $R_{\tilde{Z}}>\eta$.* $\square$

Proof. Fix $Z\in\mathcal{Z}$ and $\eta\in(0,+\infty)$ such that $G(\eta,Z)<1$. For each $M\in[0,+\infty]$, let

$$Z_M(y)=\max\{Z(y),\ y-M\}.$$

The dominated convergence theorem guarantees that for every $M_0\in[0,+\infty]$,

$$\lim_{M\to M_0}\int_0^{+\infty}\left(Z_M(y)-Z_{M_0}(y)\right)^2f(y)\,\mathrm{d}y=0.$$

Therefore, Assumption 3 guarantees that the map $M\mapsto P(Z_M)$ is continuous in $[0,+\infty]$ with $P(Z_\infty)=P(Z)$.

For any $M\in(0,+\infty)$ (fixed), the map $R\mapsto\mathbb{E}\mathrm{e}^{R(Y-Z_M)}$ is continuous and finite in $[0,+\infty)$. Also, the map $M\mapsto\mathbb{E}\mathrm{e}^{\eta(Y-Z_M)}$ is continuous in $[0,+\infty[$, with $\mathbb{E}\mathrm{e}^{\eta_Z(Y-Z_0)}=1$, $\lim_{M\to+\infty}\pi\left(\mathbb{E}\mathrm{e}^{\eta(Y-Z_M)}\right)=\pi\left(\mathbb{E}\mathrm{e}^{\eta(Y-Z)}\right)<+\infty$ and $\mathbb{E}\mathrm{e}^{\eta(Y-Z_M)}<\mathbb{E}\mathrm{e}^{\eta(Y-Z)}$ whenever $\Pr\{Z\neq Z_M\}>0$. Hence, we can pick $M_0\in(0,+\infty)$ such that $G\left(\eta_Z,Z_{M_0}\right)<1$ and $\mathbb{E}\mathrm{e}^{\eta\left(Y-Z_{M_0}\right)}<\rho$. For that M_0 we can pick $R>\eta$ such that $G\left(R,Z_{M_0}\right)\leq1$.

Let $W_\lambda=\lambda Y+(1-\lambda)Z_{M_0}$, $\lambda\in[0,1]$. Again, due to the dominated convergence theorem and Assumption 3, the map $\lambda\mapsto P(W_\lambda)$ is continuous and the map $\lambda\mapsto\mathbb{E}\mathrm{e}^{R(Y-W_\lambda)}$ is continuous and bounded by $\mathbb{E}\mathrm{e}^{R(Y-Z_{M_0})}$. Therefore $\lambda\mapsto G(R,W_\lambda)$ is continuous in the interval $[0,1]$, $G(R,W_0)=G(R,Z_{M_0})\leq1$ and $G(R,W_1)=G(R,Y)>1$. Thus, Bolzano's Theorem guarantees existence of $\lambda\in[0,1)$ such that $G(R,W_\lambda)=1$. ■

Proposition 1 has the following Corollary

Corollary 2. *The set $\mathcal{Z}^+$ is nonempty.* $\square$

Proof. Taking Proposition 1 into account, we only need to show that there exists $(R,Z)\in(0,+\infty)\times\mathcal{Z}$ such that $G(R,Z)<1$.

For each $M\in(0,+\infty)$ consider the excess-of-loss treaty $Z_M(y)=\max\{0,y-M\}$. Due to the dominated convergence theorem and Assumption 3, the maps $M\mapsto\mathbb{E}Z_M$, $M\mapsto P(Z_M)$ are continuous. Therefore, we can pick a large but finite $M\in(0,+\infty)$ such that $\mathbb{E}L_{Z_M}>0$. Since $\mathbb{E}\mathrm{e}^{R(Y-Z_M)}\leq\mathrm{e}^{RM}$, we see that $\eta_{Z_M}>0$ and Lemma 1 shows that $G(R,Z_M)<1$ holds for every sufficiently small $R>0$. ■

The following Proposition studies the convexity of the map $Z\mapsto G(R,Z)$.

Proposition 2. *For any $R>0$ the map $Z\mapsto G(R,Z)$ is convex in $\mathcal{Z}$. It is strictly convex in its effective domain,[4] unless the random variable N is concentrated at a unique integer $k\geq1$. In that case the map $\left(\frac{P(Z)}{k}-Z\right)\mapsto G(R,Z)$ is strictly convex in its effective domain.* $\square$

Proof. Equality (4) can be represented as

$$G(R,Z)=\mathrm{e}^{R(P(Z)-c)}\sum_{n=0}^{+\infty}\left(\mathbb{E}\mathrm{e}^{R(Y-Z)}\right)^np(n). \tag{7}$$

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[2] i.e., it does not have finite moment generating function on the right side of zero.

[3] For the implicit function theorem in infinite-dimensional spaces see e.g. Lang (1993).

[4] The set where it takes real finite values.

Consider two reinsurance treaties $Z_1, Z_2 \in \mathcal{Z}$, with $Z_1 \neq Z_2$. Fix $\lambda \in (0, 1)$. Then,

$$G\left(R, \lambda Z_1 + (1-\lambda) Z_2\right) = \mathrm{e}^{R(P(\lambda Z_1+(1-\lambda)Z_2)-c)} p(0) + \sum_{n=1}^{\infty} \left(\mathbb{E}\mathrm{e}^{R\left(\frac{P(\lambda Z_1+(1-\lambda)Z_2)-c}{n}+Y-\lambda Z_1-(1-\lambda)Z_2\right)} \right)^n p\left(n\right).$$

Convexity of the premium implies that

$$G\left(R, \lambda Z_1 + (1-\lambda) Z_2\right) \leq \mathrm{e}^{R(\lambda(P(Z_1)-c)+(1-\lambda)(P(Z_2)-c))} p\left(0\right) + \sum_{n=1}^{+\infty} \left(\mathbb{E}\mathrm{e}^{R\left(\lambda\left(\frac{P(Z_1)-c}{n}+Y-Z_1\right)+(1-\lambda)\left(\frac{P(Z_2)-c}{n}+Y-Z_2\right)\right)} \right)^n p\left(n\right).$$

Therefore, strict convexity of the exponential and of powers x^n $(n \geq 1, x > 0)$ imply

$$\begin{aligned} G\left(R, \lambda Z_1 + (1-\lambda) Z_2\right) &\leq (\lambda \mathrm{e}^{R(P(Z_1)-c)} + (1-\lambda)\mathrm{e}^{R(P(Z_2)-c)}) p(0) \\ &\quad + \sum_{n=1}^{+\infty} \left(\lambda \left(\mathbb{E}\mathrm{e}^{R\left(\frac{P(Z_1)-c}{n}+Y-Z_1\right)} \right)^n \right. \\ &\quad + \left. (1-\lambda) \left(\mathbb{E}\mathrm{e}^{R\left(\frac{P(Z_2)-c}{n}+Y-Z_2\right)} \right)^n \right) p\left(n\right) \\ &= \lambda G\left(R, Z_1\right) + (1-\lambda) G\left(R, Z_2\right), \end{aligned}$$

with strict inequality holding unless

$$P\left(Z_1\right) - nZ_1 = P\left(Z_2\right) - nZ_2$$

holds for every $n \geq 0$ such that $p\left(n\right) \neq 0$.

It is clear that this last condition can be satisfied only when N is a degenerate random variable taking one single value k with probability 1. In that case, (7) reduces to

$$G(R, Z) = \mathrm{e}^{-Rc} \left(\mathbb{E}\mathrm{e}^{R(Y+\frac{P(Z)}{k}-Z)} \right)^k.$$

Thus, $Z \mapsto G(R, Z)$ depends on Z only through the expression $\frac{P(Z)}{k} - Z$. Strict convexity of the map $\frac{P(Z)}{k} - Z \mapsto G(R, Z)$ follows from strict convexity of the exponential. ■

4. Maximization of the expected utility of wealth

It was pointed in the previous Section that the map $Z \mapsto R_Z$ is a functional defined in implicit form, and since the map $Z \mapsto G(R, Z)$ is in general discontinuous, methods based on implicit function theorems are not suitable to deal with Problem 1. The issue is further complicated by the fact that the domain of $Z \mapsto R_Z$ is hard to characterize. It turns out that these theoretical obstacles can be avoided by exploiting the close relationship between maximizing of the adjustment coefficient of the retained risk (Problem 1) and maximizing of the expected utility of wealth with an arbitrary coefficient of risk aversion (Problem 2, below). In this Section we discuss the relationship between the two problems.

Consider the exponential utility function with coefficient of risk aversion $R > 0$:

$$U_R\left(w\right) = -\mathrm{e}^{-Rw}.$$

For any given coefficient of risk aversion, $R > 0$, the expected utility of wealth obtained by the insurance company in a given unit of time is

$$\mathbb{E}\left(U_R\left(L_Z\right)\right) = -G\left(R, Z\right). \tag{8}$$

We consider the maximization problem:

Problem 2. Find $Z^* \in \mathcal{Z}$, such that

$$\mathbb{E}\left(U_R\left(L_{Z^*}\right)\right) = \max\left\{\mathbb{E}\left(U_R\left(L_Z\right)\right) : Z \in \mathcal{Z}\right\}.$$

Here $R > 0$ is a given constant (fixed). □

A policy $Z \in \mathcal{Z}$ is said to be **optimal for the expected utility criterion** with coefficient of risk aversion R if it solves Problem 2 for that particular R. When it is clear from the context which is the coefficient of risk aversion being considered, we will just say that the policy is optimal for the expected utility criterion.

It follows immediately from (8) that a policy is optimal for the expected utility criterion if and only if it is a minimizer of the functional $Z \mapsto G\left(R, Z\right)$, with the same (fixed) value of R being considered. The following relationship between Problem 1 and Problem 2 is the key to our approach.

Proposition 3. *A pair $(R^*, Z^*) \in (0, +\infty) \times \mathcal{Z}$ solves Problem 1 (i.e., Z^* is optimal for the adjustment coefficient criterion) if and only if it satisfies the following conditions:*

1. *Z^* is optimal for the expected utility criterion (i.e., it solves Problem 2) with coefficient of risk aversion $R = R^*$;*
2. *$G(R^*, Z^*) = 1$.* □

Proof. Let $(R^*, Z^*) \in (0, +\infty) \times \mathcal{Z}$ be a solution of Problem 1, and suppose there exists $Z \in \mathcal{Z}$ such that $G(R^*, Z) < G(R^*, Z^*) = 1$. Then Proposition 1 guarantees existence of $\tilde{Z} \in \mathcal{Z}^+$ such that $R_{\tilde{Z}} > R^*$, a contradiction to the optimality of (R^*, Z^*). Hence Z^* must be optimal for the expected utility criterion.

Now, let $Z^* \in \mathcal{Z}$ be a solution of Problem 2 with coefficient of risk aversion $R = R^*$, and $G(R^*, Z^*) = 1$. If there exists $Z \in \mathcal{Z}^+$ such that $R^* < R_Z < +\infty$, then Corollary 1 shows that $G(R^*, Z) < 1$, a contradiction to the optimality of Z^*. Therefore (R^*, Z^*) must be a solution of Problem 1. ■

Proposition 3 suggests that Problem 1 can be solved in two steps:

1. For each $R \in]0, +\infty[$ find Z_R, the respective optimal policy for the expected utility criterion. Equivalently, find
   $$Z_R = \arg\min\left\{G\left(R, Z\right) : Z \in \mathcal{Z}\right\};$$
2. Solve the equation with one single real variable
   $$G\left(R, Z_R\right) = 1. \tag{9}$$

Below we prove that for any $R \in (0, +\infty)$, Problem 2 admits at least one solution and hence that map $R \mapsto G(R, Z_R) = \min\{G(R, Z) : Z \in \mathcal{Z}\}$ is well defined. In the following, Z_R denotes one (arbitrary, if there is more than one) solution of Problem 2 for the particular coefficient of risk aversion R.

5. Existence and uniqueness of optimal policies for the expected utility criterion

Theorem 1. *For each $R \in (0, +\infty)$ there exists an optimal policy for the expected utility criterion.*

If $\Pr\{N = k\} = 1$ for some $k \geq 1$, then given an optimal policy $Z^* \in \mathcal{Z}$, any other policy $\tilde{Z} \in \mathcal{Z}$ is optimal if and only if

$$\Pr\left\{\tilde{Z} - Z^* = \frac{P(\tilde{Z}) - P(Z^*)}{k}\right\} = 1,$$

otherwise the optimal policy is unique. □

Proof. In our proof we consider the equivalent problem of minimizing the functional $Z \mapsto G\left(R, Z\right)$, for the particular value of R being considered.

Notice that Assumptions 1 and 3 guarantee that $\Pr\{N = 0\} < 1$. Fix $R \in (0, +\infty)$. Proposition 2 states that the functional $Z \mapsto G\left(R, Z\right)$ is convex. If $\Pr\left\{N = k\right\} = 1$ for some integer $k \geq 1$ then $G(R, Z)$ depends on Z only through $\frac{P(Z)}{k} - Z$. Thus, the map $\left(\frac{P(Z)}{k} - Z\right) \mapsto G\left(R, Z\right)$ is well defined and strictly convex in its effective domain. Otherwise the map $Z \mapsto G(R, Z)$ is strictly convex in its effective domain. Since $G(R, Y) < +\infty$, we see that the effective domain of $Z \mapsto G(R, Z)$ is nonempty. Since $\mathcal{Z}$ is convex, this proves the part of the Theorem concerning uniqueness of solutions.

Existence of a minimizer is a consequence of the classical Banach–Alaoglu Theorem from functional analysis.[5]

We start by showing that $Z \mapsto G(R, Z)$ is lower semicontinuous.[6] Fix $Z \in \mathcal{Z}$ and pick a sequence $\{Z_k \in \mathcal{Z}\}_{k\in\mathbb{N}}$ such that $\lim \mathbb{E}\left((Z_k - Z)^2\right) = 0$. We wish to prove that

$$G(R, Z) \leq \liminf G(R, Z_k). \tag{10}$$

Without loss of generality, we may assume that $\lim G(R, Z_k)$ exists and $\Pr\{\lim Z_k(Y) = Z(Y)\} = 1$ (take subsequences if necessary). Fatou's Lemma guarantees that $\mathbb{E}e^{R(Y-Z)} \leq \lim \mathbb{E}e^{R(Y-Z_k)}$. Therefore (10) follows from Assumption 3.

The set $\mathcal{Z}$ is convex. By Assumption 4, it is bounded in the mean-squared sense, and it is closed with respect to mean-squared convergence. Therefore, the Banach–Alaoglu Theorem states that it is compact with respect to the topology of weak convergence.[7]

Since Proposition 2 shows that $Z \mapsto G(R, Z)$ is convex and any convex lower semicontinuous functional is lower semicontinuous with respect to weak convergence, the result follows immediately by Weierstrass theorem. ■

6. Existence and uniqueness of optimal policy for the adjustment coefficient criterion

In this Section we use the results above to prove existence and uniqueness of solutions of Problem 1.

Theorem 2. *There exists an optimal policy for the adjustment coefficient criterion.*

If $\Pr\{N = k\} = 1$ *for some* $k \geq 1$, *then given an optimal policy* $Z^* \in \mathcal{Z}^+$, *any other policy* $\tilde{Z} \in \mathcal{Z}$ *is optimal if and only if*

$$\Pr\left\{\tilde{Z} - Z^* = \frac{P(\tilde{Z}) - P(Z^*)}{k}\right\} = 1. \tag{11}$$

If $\Pr\{N = k\} < 1$ *for every* $k \geq 1$, *then the optimal policy is unique.* □

Proof. Uniqueness is a straightforward consequence of Proposition 3 and Theorem 1. To see this, suppose the adjustment coefficient admits two different global maximizers, Z^* and $\tilde{Z} \in \mathcal{Z}$. Proposition 3 states that Z^* and $\tilde{Z}$ are both optimal policies for the expected utility criterion with the particular coefficient of risk aversion $R = R_{Z^*} = R_{\tilde{Z}}$. Then, Theorem 1 states that either the random variable N is such that $\Pr\{N = k\} = 1$ for some $k \geq 1$, in which case Z^* and $\tilde{Z}$ satisfy (11), or $\tilde{Z} = Z^*$.

In order to prove existence, let $R^* = \sup\left\{R_Z : Z \in \mathcal{Z}^+\right\}$. Pick a sequence $\{R_k \in (0, R^*)\}_{k\in\mathbb{N}}$ such that $\lim R_k = R^*$. By Theorem 1, there exists an associated sequence $\{Z_{R_k}\}$. In the proof of Theorem 1, we have seen that $\mathcal{Z}$ is compact with respect to the weak topology. Hence, we can assume that $\{Z_{R_k}\}$ converges in the weak sense to some $Z^* \in \mathcal{Z}$ (take a subsequence, if necessary). Notice that $G(R_k, Z_{R_k}) \leq 1, \forall k \in \mathbb{N}$ and, due to Corollary 1 this implies

$$G(R, Z_{R_k}) \leq 1, \quad \forall k \in \mathbb{N}, R \in (0, R_k).$$

Therefore lower semicontinuity of $Z \mapsto G(R, Z)$ implies that

$$G(R, Z^*) \leq 1, \quad \forall R \in (0, R^*). \tag{12}$$

Assuming that $R^* = +\infty$, this shows that $R_{Z^*} = +\infty$ and hence $\Pr\{L_{Z^*} < 0\} = 0$, which contradicts Assumption 3. Thus, R^* must be finite. Taking into account Lemma 1, the inequality (12) implies $G(R^*, Z^*) \leq 1$, while Proposition 1 implies $G(R^*, Z^*) \geq 1$. Hence $R^* = R_{Z^*}$ and (R^*, Z^*) is a solution of Problem 1. ■

7. Necessary condition for optimality

Taking Proposition 3 into account, a natural way of obtaining optimality conditions for Problem 1 is to search for optimality conditions for Problem 2 and add the supplementary condition $G(R, Z) = 1$.

Proceeding along this line, fix $R \in (0, +\infty)$. We consider needle-like perturbations (see e.g. Gamkrelidze, 1978), i.e., we consider reinsurance policies of type

$$Z_{\upsilon,\alpha,\varepsilon}(y) = \begin{cases} Z_R(y), & \text{if } y \notin [\upsilon, \upsilon + \varepsilon]; \\ \alpha y, & \text{if } y \in [\upsilon, \upsilon + \varepsilon], \end{cases}$$

with $\varepsilon > 0, \alpha \in [0, 1]$.

In what follows, we assume that the expression

$$\Delta P_{Z_R}(\upsilon) = \lim_{\alpha \to \frac{Z(\upsilon)}{\upsilon}} \lim_{\varepsilon \to 0^+} \frac{P\left(Z_{\upsilon,\alpha,\varepsilon}\right) - P(Z_R)}{\varepsilon\left(\alpha\upsilon - Z_R(\upsilon)\right)} \tag{13}$$

is a well defined (real, finite) function in a measurable domain $A \subset [0, +\infty)$ such that $\Pr\{Y \in A\} = 1$.

Indeed, in order to obtain some of the following results we also need to consider compositions of needle-like perturbations. I.e., we will consider treaties of the type

$$\begin{aligned} &Z_{\upsilon_1,\alpha_1,\varepsilon_1|\upsilon_2,\alpha_2,\varepsilon_2}(y) \\ &\quad = \begin{cases} Z_R(y), & \text{if } y \notin [\upsilon_1, \upsilon_1 + \varepsilon_1] \cup [\upsilon_2, \upsilon_2 + \varepsilon_2]; \\ \alpha_1 y, & \text{if } y \in [\upsilon_1, \upsilon_1 + \varepsilon_1]; \\ \alpha_2 y & \text{if } y \in [\upsilon_2, \upsilon_2 + \varepsilon_2], \end{cases} \end{aligned}$$

for arbitrary $\upsilon_1 \neq \upsilon_2$ and sufficiently small $\varepsilon_1, \varepsilon_2 > 0$. We assume that the function ΔP_{Z_R} suitably approximates the effect of double needle-like perturbations on the reinsurance premium, in the sense that the estimate

$$\begin{aligned} P\left(Z_{\upsilon_1,\alpha_1,\varepsilon_1|\upsilon_2,\alpha_2,\varepsilon_2}\right) - P(Z_R) &= \varepsilon_1\left(\alpha_1\upsilon_1 - Z_R(\upsilon_1)\right)\Delta P_{Z_R}(\upsilon_1) \\ &\quad + \varepsilon_1 o\left(\alpha_1\upsilon_1 - Z_R(\upsilon_1)\right) + o(\varepsilon_1) \\ &\quad + \varepsilon_2\left(\alpha_2\upsilon_2 - Z_R(\upsilon_2)\right)\Delta P_{Z_R}(\upsilon_2) \\ &\quad + \varepsilon_2 o\left(\alpha_2\upsilon_2 - Z_R(\upsilon_2)\right) + o(\varepsilon_2) \end{aligned} \tag{14}$$

holds for every $\upsilon_1 \neq \upsilon_2$ chosen in some measurable domain $A \subset [0, +\infty)$ such that $\Pr\{Y \in A\} = 1$. It can be shown that important premium calculation principles like the expected value principle and the variance-related principles studied in the next section satisfy these conditions.

Theorem 3. *Let* $\rho \in (1, +\infty]$ *denote the radius of convergence of the series* $\pi(t) = \sum_{n=0}^{+\infty} t^n p(n)$. *Fix* $R > 0$ *and let* $Z_R \in \mathcal{Z}$ *be optimal for the expected utility criterion.*

If Z_R *satisfies* (14), *then there exists a constant* $C \in (0, +\infty)$ *such that*

$$\begin{cases} \Delta P_{Z_R}(y) \leq Cf(y), & \text{if } Z_R(y) = y; \\ \Delta P_{Z_R}(y) = Ce^{R(y - Z_R(y))}f(y), & \text{if } 0 < Z_R(y) < y; \\ \Delta P_{Z_R}(y) \geq Ce^{Ry}f(y), & \text{if } Z_R(y) = 0, \end{cases} \tag{15}$$

with probability equal to one with respect to the density f.

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[5] For this theorem and details of the functional analytic arguments used below, see e.g. Rudin (1991).

[6] A function f is lower semicontinuous at x_0 if for every $\varepsilon > 0$ there exists a neighbourhood U of x_0 such that $f(x) \geq f(x_0) - \varepsilon, \forall x \in U$.

[7] Recall that a sequence of square-integrable functions $\{\phi_k\}_{k\in\mathbb{N}}$ converges weakly to the square-integrable function ϕ if and only if $\lim \mathbb{E}(\phi_k\psi) = \mathbb{E}(\phi\psi)$ holds for every square-integrable ψ.

Further,

$$C=\frac{\pi'\left(\mathbb{E}\mathrm{e}^{R(Y-Z_R)}\right)}{\pi\left(E\left[\mathrm{e}^{R(Y-Z_R)}\right]\right)}$$

holds if $\mathbb{E}\mathrm{e}^{R(Y-Z_R)}<\rho$, *while*

$$C\geq\frac{\pi'(\rho^-)}{\pi(\rho)} \tag{16}$$

holds if $\mathbb{E}\mathrm{e}^{R(Y-Z_R)}=\rho$. □

Proof. Fix $\upsilon\in(0,+\infty)$, a Lebesgue point[8] of the functions f, $\mathrm{e}^{-RZ_R}f$, with $f(\upsilon)>0$.

Suppose that $Z_R(\upsilon)<\upsilon$ and consider a perturbation $Z_{\upsilon,\alpha,\varepsilon}$ with

$$\alpha\upsilon>Z_R(\upsilon). \tag{17}$$

Optimality of Z_R implies that

$$0\leq G\left(R,Z_{\upsilon,\alpha,\varepsilon}\right)-G(R,Z_R), \tag{18}$$

and (17) implies that $\mathbb{E}\mathrm{e}^{R(Y-Z_{\upsilon,\alpha,\varepsilon})}<\mathbb{E}\mathrm{e}^{R(Y-Z_R)}\leq\rho$. Therefore the function $t\mapsto\pi(t)=\sum_{n=0}^{\infty}t^np(n)$ is continuous in the interval $\left[0,\mathbb{E}\mathrm{e}^{R(Y-Z_R)}\right]$, differentiable in $\left(0,\mathbb{E}\mathrm{e}^{R(Y-Z_R)}\right)$. It follows by the mean-value theorem that for each α satisfying (17) and each sufficiently small $\varepsilon>0$ there exists $\theta\in(0,1)$ such that

$$\begin{aligned}
&G\left(R,Z_{\upsilon,\alpha,\varepsilon}\right)-G(R,Z_R)\\
&\quad=\mathrm{e}^{R\left((1-\theta)P(Z_R)+\theta P\left(Z_{\upsilon,\alpha,\varepsilon}\right)-c\right)}\pi\left((1-\theta)\,\mathbb{E}\mathrm{e}^{R(Y-Z_R)}+\theta\mathbb{E}\mathrm{e}^{R(Y-Z_{\upsilon,\alpha,\varepsilon})}\right)\\
&\qquad\times R\left(P\left(Z_{\upsilon,\alpha,\varepsilon}\right)-P(Z_R)\right)\\
&\qquad+\mathrm{e}^{R\left((1-\theta)P(Z_R)+\theta P\left(Z_{\upsilon,\alpha,\varepsilon}\right)-c\right)}\\
&\qquad\times\pi'\left((1-\theta)\,\mathbb{E}\mathrm{e}^{R(Y-Z_R)}+\theta\mathbb{E}\mathrm{e}^{R(Y-Z_{\upsilon,\alpha,\varepsilon})}\right)\\
&\qquad\times\left(\mathbb{E}\mathrm{e}^{R(Y-Z_{\upsilon,\alpha,\varepsilon})}-\mathbb{E}\mathrm{e}^{R(Y-Z_R)}\right).
\end{aligned}$$

Since υ is a Lebesgue point of the functions f, $\mathrm{e}^{-RZ_R}f$, we have

$$\begin{aligned}
&\mathbb{E}\mathrm{e}^{R(Y-Z_{\upsilon,\alpha,\varepsilon})}-\mathbb{E}\mathrm{e}^{R(Y-Z_R)}=\int_{\upsilon}^{\upsilon+\varepsilon}\left(\mathrm{e}^{R(y-\alpha y)}-\mathrm{e}^{R(y-Z_R(y))}\right)f(y)\,\mathrm{d}y\\
&\quad=\varepsilon\left(\mathrm{e}^{R(\upsilon-\alpha\upsilon)}-\mathrm{e}^{R(\upsilon-Z_R(\upsilon))}\right)f(\upsilon)+o(\varepsilon).
\end{aligned}$$

Therefore, inequality (18) reduces to

$$\begin{aligned}
&\pi'(x_\varepsilon)\left(\left(\mathrm{e}^{R(\upsilon-Z_R(\upsilon))}-\mathrm{e}^{R(\upsilon-\alpha\upsilon)}\right)f(\upsilon)+\frac{o(\varepsilon)}{\varepsilon}\right)\\
&\quad\leq\pi(x_\varepsilon)\,R\frac{P(Z_{\upsilon,\alpha,\varepsilon})-P(Z_R)}{\varepsilon}
\end{aligned}$$

where $x_\varepsilon=(1-\theta)\mathbb{E}\mathrm{e}^{R(Y-Z_R)}+\theta\mathbb{E}\mathrm{e}^{R(Y-Z_{\upsilon,\alpha,\varepsilon})}$. Taking limits when $\varepsilon\to0^+$, we obtain

$$\begin{aligned}
&\pi'\left(\mathbb{E}\mathrm{e}^{R(Y-Z_R)}\right)\frac{\mathrm{e}^{R(\upsilon-Z_R(\upsilon))}-\mathrm{e}^{R(\upsilon-\alpha\upsilon)}}{\alpha\upsilon-Z_R(\upsilon)}f(\upsilon)\\
&\quad\leq\pi\left(\mathbb{E}\mathrm{e}^{R(Y-Z_R)}\right)R\lim_{\varepsilon\to0^+}\frac{P(Z_{\upsilon,\alpha,\varepsilon})-P(Z_R)}{\varepsilon(\alpha\upsilon-Z_R(\upsilon))}.
\end{aligned}\tag{19}$$

Taking limits when $\alpha\to\left(\frac{Z_R(\upsilon)}{\upsilon}\right)^+$, one obtains

$$\pi'\left(\mathbb{E}\mathrm{e}^{R(Y-Z_R)}\right)R\mathrm{e}^{R(\upsilon-Z_R(\upsilon))}f(\upsilon)\leq\pi\left(\mathbb{E}\mathrm{e}^{R(Y-Z_R)}\right)R\Delta P(Z_R)(\upsilon),$$

i.e.,

$$\frac{\pi'\left(\mathbb{E}\mathrm{e}^{R(Y-Z_R)}\right)}{\pi\left(\mathbb{E}\mathrm{e}^{R(Y-Z_R)}\right)}\mathrm{e}^{R(\upsilon-Z_R(\upsilon))}f(\upsilon)\leq\Delta P_{Z_R}(\upsilon) \tag{20}$$

holds for almost every υ such that $Z_R(\upsilon)<\upsilon$.

If $\mathbb{E}\mathrm{e}^{R(Y-Z_R)}<\rho$, then $\mathbb{E}\mathrm{e}^{R\left(Y-Z_{\upsilon,\alpha\varepsilon}\right)}<\rho$ also holds for any $\alpha\in[0,1]$, provided $\varepsilon>0$ is sufficiently small. Therefore in this case, inequality (19) holds for almost every υ such that $Z_R(\upsilon)>0$. Therefore, dividing both sides of (19) by $\alpha\upsilon-Z_R(\upsilon)$ and taking limits when $\alpha\to\left(\frac{Z_R(\upsilon)}{\upsilon}\right)^-$, we obtain

$$\frac{\pi'\left(\mathbb{E}\mathrm{e}^{R(Y-Z_R)}\right)}{\pi\left(\mathbb{E}\mathrm{e}^{R(Y-Z_R)}\right)}\mathrm{e}^{R(\upsilon-Z_R(\upsilon))}f(\upsilon)\geq\Delta P_{Z_R}(\upsilon). \tag{21}$$

This proves the Proposition in the case when $\mathbb{E}\mathrm{e}^{R(Y-Z_R)}<\rho$.

Now, consider the case when $\mathbb{E}\mathrm{e}^{R(Y-Z_R)}=\rho$. In this case we have

$$\mathbb{E}\mathrm{e}^{R(Y-Z_{\upsilon,\alpha,\varepsilon})}>\rho$$

and therefore $G\left(R,Z_{\upsilon,\alpha,\varepsilon}\right)=+\infty$ for any needle-like perturbation with $\alpha\upsilon<Z_R(\upsilon)$ and sufficiently small support. Thus we cannot use the argument above to prove an inequality analogous to (21). Instead we use double needle-like perturbations such that $\mathbb{E}\mathrm{e}^{R(Y-Z_{\upsilon_1,\alpha_1,\varepsilon_1|\upsilon_2,\alpha_2,\varepsilon_2})}=\rho$. Like before, $\upsilon_1,\upsilon_2\in(0,+\infty)$ are Lebesgue points of the functions f, $\mathrm{e}^{-RZ_R}f$, such that $\upsilon_1\neq\upsilon_2$ and $f(\upsilon_i)>0$, $i=1,2$. $\mathbb{E}\mathrm{e}^{R(Y-Z_R)}=\rho$ implies $\Pr\{Z_R<Y\}>0$. Hence we can choose υ_1 such that $Z_R(\upsilon_1)<\upsilon_1$ and fix α_1 such that $\alpha_1\upsilon_1>Z_R(\upsilon_1)$. For any $\upsilon_2\neq\upsilon_1$ such that $Z_R(\upsilon_2)>0$, fix α_2 such that $\alpha_2\upsilon_2<Z_R(\upsilon_2)$. Then, we have

$$\begin{aligned}
&\mathbb{E}\mathrm{e}^{R\left(Y-Z_{\upsilon_1,\alpha_1,\varepsilon_1|\upsilon_2,\alpha_2,\varepsilon_2}\right)}=\mathbb{E}\mathrm{e}^{R(Y-Z_R)}\\
&\quad+\int_{\upsilon_1}^{\upsilon_1+\varepsilon_1}\left(\mathrm{e}^{R(y-\alpha_1y)}-\mathrm{e}^{R(y-Z_R(y))}\right)f(y)\,\mathrm{d}y\\
&\quad+\int_{\upsilon_2}^{\upsilon_2+\varepsilon_2}\left(\mathrm{e}^{R(y-\alpha_2y)}-\mathrm{e}^{R(y-Z_R(y))}\right)f(y)\,\mathrm{d}y.
\end{aligned}$$

Since υ_1, υ_2 are Lebesgue points of the functions f, $\mathrm{e}^{-RZ_R}f$, it follows that

$$\begin{aligned}
&\mathbb{E}\mathrm{e}^{R\left(Y-Z_{\upsilon_1,\alpha_1,\varepsilon_1|\upsilon_2,\alpha_2,\varepsilon_2}\right)}\\
&\quad=\rho+\varepsilon_1\left(\mathrm{e}^{R(\upsilon_1-\alpha_1\upsilon_1)}-\mathrm{e}^{R(\upsilon_1-Z_R(\upsilon_1))}\right)f(\upsilon_1)+o(\varepsilon_1)\\
&\qquad+\varepsilon_2\left(\mathrm{e}^{R(\upsilon_2-\alpha_2\upsilon_2)}-\mathrm{e}^{R(\upsilon_2-Z_R(\upsilon_2))}\right)f(\upsilon_2)+o(\varepsilon_2)\\
&\quad=\rho-\varepsilon_1(\alpha_1\upsilon_1-Z_R(\upsilon_1))\,R\mathrm{e}^{R(\upsilon_1-Z_R(\upsilon_1))}f(\upsilon_1)\\
&\qquad+\varepsilon_1o(\alpha_1\upsilon_1-Z_R(\upsilon_1))+o(\varepsilon_1)\\
&\qquad-\varepsilon_2(\alpha_2\upsilon_2-Z_R(\upsilon_2))\,R\mathrm{e}^{R(\upsilon_2-Z_R(\upsilon_2))}f(\upsilon_2)\\
&\qquad+\varepsilon_2o(\alpha_2\upsilon_2-Z_R(\upsilon_2))+o(\varepsilon_2).
\end{aligned}$$

Therefore, an implicit function-type argument shows that for each sufficiently small $\varepsilon_2>0$, $(\alpha_1\upsilon_1-Z_R(\upsilon_1))$ and $(\alpha_2\upsilon_2-Z_R(\upsilon_2))$ there exists a unique $\varepsilon_1>0$ such that $\mathbb{E}\mathrm{e}^{R\left(Y-Z_{\upsilon_1,\alpha_1,\varepsilon|\upsilon_2,\alpha_2,\varepsilon}\right)}=\rho$. Further, such ε_1 satisfies

$$\begin{aligned}
\varepsilon_1={}&-\frac{\mathrm{e}^{R(\upsilon_2-Z_R(\upsilon_2))}(\alpha_2\upsilon_2-Z_R(\upsilon_2))f(\upsilon_2)}{\mathrm{e}^{R(\upsilon_1-Z_R(\upsilon_1))}(\alpha_1\upsilon_1-Z_R(\upsilon_1))f(\upsilon_1)}\varepsilon_2\\
&+\left(o(\alpha_1\upsilon_1-Z_R(\upsilon_1))+o(\alpha_2\upsilon_2-Z_R(\upsilon_2))\right)\varepsilon_2\\
&+o(\varepsilon_2).
\end{aligned}\tag{22}$$

Since $\mathbb{E}\mathrm{e}^{R(Y-Z_R)}=\mathbb{E}\mathrm{e}^{R\left(Y-Z_{\upsilon_1,\alpha_1,\varepsilon_1|\upsilon_2,\alpha_2,\varepsilon_2}\right)}=\rho$, we have

$$G(R,Z_R)=\mathrm{e}^{R(P(Z_R)-c)}\pi(\rho);$$

$$G\left(R,Z_{\upsilon_1,\alpha_1,\varepsilon_1|\upsilon_2,\alpha_2,\varepsilon_2}\right)=\mathrm{e}^{R\left(P\left(Z_{\upsilon_1,\alpha_1,\varepsilon_1|\upsilon_2,\alpha_2,\varepsilon_2}\right)-c\right)}\pi(\rho).$$

---

[8] Recall that υ is a Lebesgue point of a function $g:\mathbb{R}\mapsto[-\infty,+\infty]$ if:

$$\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon}\int_{\upsilon}^{\upsilon+\varepsilon}g(t)\mathrm{d}t=\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon}\int_{\upsilon-\varepsilon}^{\upsilon}g(t)\mathrm{d}t=g(\upsilon).$$

If g is locally integrable, then almost every point is a Lebesgue point. For details see, e.g., Rudin (1987).

Therefore, optimality of Z_R implies that

$$P\left(Z_{\upsilon_1,\alpha_1,\varepsilon_1|\upsilon_2,\alpha_2,\varepsilon_2}\right) - P\left(Z_R\right) \geq 0.$$

Substituting (22) in (14), this inequality becomes

$$\begin{aligned}&\left(\alpha_2\upsilon_2 - Z_R\left(\upsilon_2\right)\right)\left(\Delta P_{Z_R}\left(\upsilon_2\right) - \frac{e^{R(\upsilon_2-Z_R(\upsilon_2))}f\left(\upsilon_2\right)}{e^{R(\upsilon_1-Z_R(\upsilon_1))}f\left(\upsilon_1\right)}\Delta P_{Z_R}\left(\upsilon_1\right)\right)\\&\quad + o\left(\alpha_1\upsilon_1 - Z_R\left(\upsilon_1\right)\right) + o\left(\alpha_2\upsilon_2 - Z_R\left(\upsilon_2\right)\right) + \frac{o\left(\varepsilon_2\right)}{\varepsilon_2} \geq 0.\end{aligned}$$

Setting $\left(\alpha_1\upsilon_1 - Z_R\left(\upsilon_1\right)\right) = -\left(\alpha_2\upsilon_2 - Z_R\left(\upsilon_2\right)\right)$ and making $\varepsilon_2 \to 0^+$, we obtain

$$\begin{aligned}&\left(\alpha_2\upsilon_2 - Z_R\left(\upsilon_2\right)\right)\left(\Delta P_{Z_R}\left(\upsilon_2\right) - \frac{e^{R(\upsilon_2-Z(\upsilon_2))}f\left(\upsilon_2\right)}{e^{R(\upsilon_1-Z_R(\upsilon_1))}f\left(\upsilon_1\right)}\Delta P_{Z_R}\left(\upsilon_1\right)\right)\\&\quad + o\left(\alpha_2\upsilon_2 - Z_R\left(\upsilon_2\right)\right) \geq 0.\end{aligned}$$

Making $\left(\alpha_2\upsilon_2 - Z_R\left(\upsilon_2\right)\right) \to 0^-$, we see that this implies

$$\frac{\Delta P_{Z_R}\left(\upsilon_2\right)}{e^{R(\upsilon_2-Z_R(\upsilon_2))}f\left(\upsilon_2\right)} \leq \frac{\Delta P_{Z_R}\left(\upsilon_1\right)}{e^{R(\upsilon_1-Z_R(\upsilon_1))}f\left(\upsilon_1\right)}. \tag{23}$$

This shows that (23) holds for any pair of Lebesgue points of the functions $y \mapsto f\left(y\right)$, $y \mapsto e^{-RZ_R(y)}f\left(y\right)$ such that $f\left(\upsilon_i\right) > 0$, $0 \leq Z_R\left(\upsilon_1\right) < \upsilon_1$, $0 < Z_R\left(\upsilon_2\right) \leq \upsilon_2$. Let C denote the infimum value of $\frac{\Delta P_{Z_R}(\upsilon)}{e^{R(\upsilon-Z_R(\upsilon))}f(\upsilon)}$ over all the Lebesgue points of f, $e^{-RZ_R}f$, such that $Z_R\left(\upsilon\right) < \upsilon$. Inequality (20) shows that $C \geq \frac{\pi'(\rho)}{\pi(\rho)}$ and $\Delta P_{Z_R}\left(\upsilon\right) \geq Ce^{R(\upsilon-Z_R(\upsilon))}f\left(\upsilon\right)$ holds whenever $Z_R\left(\upsilon\right) < \upsilon$. Since (23) guarantees that $\Delta P_{Z_R}\left(\upsilon\right) \leq Ce^{R(\upsilon-Z_R(\upsilon))}f\left(\upsilon\right)$ holds whenever $Z_R\left(\upsilon\right) > 0$, the proof is complete. ■

8. The optimal solution for variance-related premium calculation principles

In this section we apply the results obtained in the previous sections to the case where the premium principle $P : \mathcal{Z} \mapsto [0, +\infty)$ is a convex variance-related functional, i.e.

$$P\left(Z\right) = \mathbb{E}\widehat{Z} + g\left(\mathrm{Var}(\widehat{Z})\right), \tag{24}$$

where $\widehat{Z} = \sum_{n=1}^{N} Z_n$ is the aggregate value refunded under the treaty Z and $g : [0, +\infty) \mapsto [0, +\infty)$ is a continuous function, smooth in $(0, +\infty)$, such that

$$g(0) = 0, \quad g'\left(x\right) > 0 \tag{25}$$

and

$$\frac{g''(x)}{g'(x)} \geq -\frac{1}{2x}, \quad \forall x \in (0, B), \tag{26}$$

with $B = \sup\{\mathrm{Var}(\widehat{Z}) : Z \in \mathcal{Z}\}$. Following Guerra and Centeno (in press) we can say that if g is twice differentiable in $(0, B)$ and satisfies (25), then the principle (24) is convex if and only if (26) is fulfilled. It can be checked that such premium principles are continuous in mean-squared sense, as required by Assumption 3. The most important examples of convex variance-related premium principles are the standard deviation and the variance principles.

The following proposition provides an expression for ΔP_Z.

Proposition 4. *If the reinsurance premium is computed by a functional of the form* (24), *then the following equality holds on a set of probability one*

$$\Delta P_Z\left(\upsilon\right) = \begin{cases}\mathbb{E}Nf\left(\upsilon\right) + 2g'\left(\mathrm{Var}(\widehat{Z})\right)\left(\mathbb{E}N\left(Z\left(\upsilon\right) - \mathbb{E}Z\right)\right.\\ \quad \left. + \ \mathrm{Var}(N)\mathbb{E}Z\right)f\left(\upsilon\right), & \text{if } \Pr\left\{Z \neq 0\right\} > 0;\\ \mathbb{E}Nf\left(\upsilon\right), & \text{if } Z \equiv 0 \text{ and } g'\left(0^+\right) < +\infty;\\ +\infty, & \text{if } Z \equiv 0 \text{ and } g'\left(0^+\right) = +\infty.\end{cases} \quad \square$$

Proof. Fix $\upsilon > 0$, a Lebesgue point of the maps $\upsilon \to f\left(\upsilon\right)$, $\upsilon \to Z(\upsilon)f(\upsilon)$ and $\upsilon \to Z(\upsilon)^2 f(\upsilon)$, such that $f\left(\upsilon\right) > 0$.

By (13), we have

$$\begin{aligned}\Delta P_Z\left(\upsilon\right) &= \lim_{\alpha \to \frac{Z(\upsilon)}{\upsilon}} \lim_{\varepsilon \to 0^+} \frac{P\left(Z_{\upsilon,\alpha,\varepsilon}\right) - P\left(Z\right)}{\left(\alpha\upsilon - Z\left(\upsilon\right)\right)\varepsilon}\\ &= \lim_{\alpha \to \frac{Z(\upsilon)}{\upsilon}} \lim_{\varepsilon \to 0^+} \frac{\mathbb{E}\widehat{Z}_{\upsilon,\alpha,\varepsilon} - \mathbb{E}\widehat{Z} + g\left(\mathrm{Var}(\widehat{Z}_{\upsilon,\alpha,\varepsilon})\right) - g\left(\mathrm{Var}(\widehat{Z})\right)}{\left(\alpha\upsilon - Z\left(\upsilon\right)\right)\varepsilon}.\end{aligned} \tag{27}$$

The mean-value theorem states that, for each $\upsilon > 0$, $\alpha \in [0, 1]$, $\varepsilon > 0$, there exists $\theta \in]0, 1[$ such that

$$\begin{aligned}&g\left(\mathrm{Var}(\widehat{Z}_{\upsilon,\alpha,\varepsilon})\right) - g\left(\mathrm{Var}(\widehat{Z})\right)\\&\quad = g'\left(\left(1-\theta\right)\mathrm{Var}(\widehat{Z}) + \theta\mathrm{Var}(\widehat{Z}_{\upsilon,\alpha,\varepsilon})\right) \times \left(\mathrm{Var}(\widehat{Z}_{\upsilon,\alpha,\varepsilon}) - \mathrm{Var}(\widehat{Z})\right).\end{aligned}$$

Recall that the first two moments of $\widehat{Z}$ can be calculated as

$$\mathbb{E}\widehat{Z} = \mathbb{E}N\mathbb{E}Z, \quad \mathrm{Var}(\widehat{Z}) = \mathbb{E}N\mathrm{Var}(Z) + \mathrm{Var}(N)(\mathbb{E}Z)^2.$$

Since υ is a Lebesgue point, we have

$$\begin{aligned}E\left[Z_{\upsilon,\alpha,\varepsilon}\right] - E\left[Z\right] &= \int_{\upsilon}^{\upsilon+\varepsilon} \left(\alpha y - Z\left(y\right)\right)f\left(y\right)\mathrm{d}y\\ &= \varepsilon\left(\alpha\upsilon - Z\left(\upsilon\right)\right)f\left(\upsilon\right) + o\left(\varepsilon\right),\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Var}(Z_{\upsilon,\alpha,\varepsilon}) - \mathrm{Var}(Z) = \left(\mathbb{E}\left(Z^2_{\upsilon,\alpha,\varepsilon}\right) - \mathbb{E}\left(Z^2\right)\right)\\&\quad - \left(\left(\mathbb{E}Z_{\upsilon,\alpha,\varepsilon}\right)^2 - \left(\mathbb{E}Z\right)^2\right)\\&\quad = \int_{\upsilon}^{\upsilon+\varepsilon} \left(\left(\alpha y\right)^2 - Z\left(y\right)^2\right)f\left(y\right)\mathrm{d}y\\&\quad - \left(\mathbb{E}Z_{\upsilon,\alpha,\varepsilon} - \mathbb{E}Z\right)\left(\mathbb{E}Z_{\upsilon,\alpha,\varepsilon} + \mathbb{E}Z\right)\\&\quad = \varepsilon\left(\alpha^2\upsilon^2 - Z\left(\upsilon\right)^2\right)f\left(\upsilon\right) - \varepsilon\left(\alpha\upsilon - Z\left(\upsilon\right)\right)f\left(\upsilon\right)2\mathbb{E}Z + o\left(\varepsilon\right)\\&\quad = \varepsilon\left(\alpha\upsilon - Z\left(\upsilon\right)\right)\left(\alpha\upsilon + Z\left(\upsilon\right) - 2\mathbb{E}Z\right)f\left(\upsilon\right) + o\left(\varepsilon\right).\end{aligned}$$

It follows that

$$\mathbb{E}\widehat{Z}_{\upsilon,\alpha,\varepsilon} - \mathbb{E}\widehat{Z} = \varepsilon\left(\alpha\upsilon - Z\left(\upsilon\right)\right)\mathbb{E}Nf\left(\upsilon\right) + o\left(\varepsilon\right); \tag{28}$$

$$\begin{aligned}&\mathrm{Var}(\widehat{Z}_{\upsilon,\alpha,\varepsilon}) - \mathrm{Var}(\widehat{Z})\\&\quad = \mathbb{E}N\left(\mathrm{Var}(Z_{\upsilon,\alpha,\varepsilon}) - \mathrm{Var}(Z)\right) + \mathrm{Var}(N)\left(\left(\mathbb{E}Z_{\upsilon,\alpha,\varepsilon}\right)^2 - \left(\mathbb{E}Z\right)^2\right)\\&\quad = \varepsilon\left(\alpha\upsilon - Z\left(\upsilon\right)\right)\left(\left(\alpha\upsilon + Z\left(\upsilon\right)\mathbb{E}N - 2\mathbb{E}Z\right)\right.\\&\quad\quad \left. + \ 2\mathrm{Var}(N)\mathbb{E}Z\right)f\left(\upsilon\right) + o\left(\varepsilon\right).\end{aligned} \tag{29}$$

In the case when $\Pr\left\{Z \neq 0\right\} > 0$, substitution of (28)–(29) in (27) yields immediately the desired equality.

In the case when $\Pr\left\{Z \neq 0\right\} = 0$, equalities (28)–(29) show that

$$\begin{aligned}\Delta P_0(\upsilon) &= \lim_{\alpha \to 0^+} \left(\mathbb{E}N + g'\left(0^+\right)\alpha\upsilon\mathbb{E}N\right)f\left(\upsilon\right)\\ &= \begin{cases}\left(\mathbb{E}N + g'\left(0^+\right)\mathbb{E}N\alpha\upsilon\right)f\left(\upsilon\right), & \text{if } g'\left(0^+\right) < +\infty;\\ +\infty, & \text{if } g'\left(0^+\right) = +\infty.\end{cases}\end{aligned}$$

This completes the proof. ■

Using Proposition 4 we can state the following Corollary to Theorem 3:

Corollary 3. *Fix $R > 0$, and let $Z_R \in \mathcal{Z}$ be optimal for the expected utility criterion. If g' is bounded in a neighbourhood of zero, then the following set of conditions holds with probability equal to one with respect to the density f:*

$$y \leq \alpha_1 + \alpha_2, \quad \textit{if } Z\left(y\right) = y; \tag{30}$$

$$y = Z\left(y\right) + \frac{1}{R}\ln\frac{Z\left(y\right) - \alpha_2}{\alpha_1}, \quad \textit{if } 0 < Z\left(y\right) < y; \tag{31}$$

$$y \leq \frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}, \quad \textit{if } Z\left(y\right) = 0. \tag{32}$$

α_1, α_2 are constants satisfying

$$\alpha_1 = \frac{\pi'\left(\mathbb{E}e^{R(Y-Z)}\right)}{\mathbb{E}N\pi\left(\mathbb{E}e^{R(Y-Z)}\right)2g'\left(\mathrm{Var}(\widehat{Z})\right)}, \quad \text{if } \mathbb{E}e^{R(Y-Z)} < \rho; \tag{33}$$

$$\alpha_1 \geq \frac{\pi'\left(\mathbb{E}e^{R(Y-Z)}\right)}{\mathbb{E}N\pi\left(\mathbb{E}e^{R(Y-Z)}\right)2g'\left(\mathrm{Var}(\widehat{Z})\right)}, \quad \text{if } \mathbb{E}e^{R(Y-Z)} = \rho; \tag{34}$$

$$\alpha_2 = \frac{\mathbb{E}N - \mathrm{Var}(N)}{\mathbb{E}N}\mathbb{E}Z - \frac{1}{2g'\left(\mathrm{Var}(\widehat{Z})\right)}, \tag{35}$$

where ρ is the radius of convergence of $\pi(\cdot)$, the probability generating function of N.

If g' is unbounded in any neighbourhood of zero, then the optimal treaty must be either a function of the type described above or $Z \equiv 0$ (no reinsurance at all). □

Remark 1. Notice that in the Corollary above α_1 is always strictly positive while α_2 may be either positive or negative. So, the structure of a nonzero optimal treaty must be one of the following:

(a) If $\alpha_1 + \alpha_2 > 0$, then the right-hand side of the inequality in (32) is not a positive real number. This is to be understood as meaning that $\Pr\{Z = 0\} = 0$ (because the inequality in (32) is the logarithmic version of (37), in the proof below). Hence Z is defined by (30)–(31) alone.

(b) If $\alpha_1 + \alpha_2 < 0$, then (30) shows that $\Pr\{Z = Y\} = 0$ and hence Z is defined by (31)–(32).

(c) If $\alpha_1 + \alpha_2 = 0$, then the right-hand sides of the inequalities in (30) and (32) are both zero. Hence $\Pr\{Z = Y\} = \Pr\{Z = 0\} = 0$ and Z is defined by (31) alone.

Remark 2. Though this may not appear to be obvious, Corollary 3 is sufficient to actually compute the optimal treaty.

First, notice that for any $(R, \alpha_1, \alpha_2) \in (0, +\infty) \times (0, +\infty) \times \mathbb{R}$ (fixed), equality (31) defines one unique function $\hat{Z}_{R,\alpha_1,\alpha_2} : \mathbb{R} \mapsto \mathbb{R}$ and thus a treaty $Z_{R,\alpha_1,\alpha_2} \in \mathcal{Z}$ defined as

$$Z_{R,\alpha_1,\alpha_2}(y) = \min\left\{y, \max\left\{0, \hat{Z}_{R,\alpha_1,\alpha_2}(y)\right\}\right\}, \quad \forall y \in [0, +\infty).$$

Further, it can be shown that for every $y \geq 0$ (fixed) the Newton algorithm applied to the equation

$$z + \frac{1}{R}\ln\frac{z - \alpha_2}{\alpha_1} = y \tag{36}$$

exhibits global quadratic convergence.[9] Thus, computation of $\hat{Z}_{R,\alpha_1,\alpha_2}(y)$ is not significantly harder than computation of the left-hand side of (36) for a given $z > \alpha_2$.

This shows that Corollary 3 reduces the space of possible solutions for Problem 2 to a two-dimensional (nonlinear) space parameterized by (α_1, α_2). So the problem reduces to finding the optimal values of (α_1, α_2). This can be done either by classical mathematical programming techniques (minimizing $G(R, Z_{R,\alpha_1,\alpha_2})$ as a function of (α_1, α_2)), or by solving the equalities (33)–(35). A discussion of methods for numerical solutions of these equations is out of the scope of this paper. We just point to the fact that all integrals required to evaluate the terms of (33)–(35) can be made explicit by the change of variable $y = \zeta + \frac{1}{R}\ln\frac{\zeta - \alpha_2}{\alpha_1}$ (see Guerra and Centeno, in press).

In order to solve Problem 1, we need to find (R, α_1, α_2) simultaneously. Due to Proposition 3, this can be done by a simple search algorithm:

1. Pick an initial $R \in (0, +\infty)$.
2. Solve Problem 2 finding the optimal (α_1, α_2).
3. If $G(R, Z_{R,\alpha_1,\alpha_2}) < 1$: increase R and repeat from step 2.
   If $G(R, Z_{R,\alpha_1,\alpha_2}) > 1$: decrease R and repeat from step 2.
   Stop when $G(R, Z_{R,\alpha_1,\alpha_2}) \approx 1$ within preset error tolerance.

Notice that this scheme also converges when the optimal adjustment coefficient is $+\infty$ (i.e., when condition (2) of Assumption 3 fails) in the sense that the final R will approach infinity when the error tolerance goes to zero.

Proof of Corollary 3. First, suppose that $\Pr\{Z_R > 0\} > 0$ holds, and let $C \in (0, +\infty)$ be as stated in Theorem 3. Due to Proposition 4 the optimality conditions become

$$2g'\left(\mathrm{Var}(\widehat{Z})\right)\left(\mathbb{E}N\left(y - \mathbb{E}Z\right) + \mathrm{Var}(N)\mathbb{E}Z\right) + \mathbb{E}N \leq C, \quad \text{if } Z(y) = y;$$

$$2g'\left(\mathrm{Var}(\widehat{Z})\right)\left(\mathbb{E}N\left(Z(y) - \mathbb{E}Z\right) + \mathrm{Var}(N)\mathbb{E}Z\right) + \mathbb{E}N = Ce^{R(y-Z(y))}$$
$$\text{if } 0 < Z(y) < y;$$

$$2g'\left(\mathrm{Var}(\widehat{Z})\right)\left(-\mathbb{E}N\mathbb{E}Z + \mathrm{Var}(N)\mathbb{E}Z\right) + \mathbb{E}N \geq Ce^{Ry} \quad \text{if } Z(y) = 0.$$

This is

$$y \leq \frac{C}{2g'\left(\mathrm{Var}(\widehat{Z})\right)\mathbb{E}N} + \frac{\mathbb{E}N - \mathrm{Var}(N)}{\mathbb{E}N}\mathbb{E}Z - \frac{1}{2g'\left(\mathrm{Var}(\widehat{Z})\right)},$$
$$\text{if } Z(y) = y;$$

$$e^{R(y-Z(y))} = \frac{Z(y) - \left(\frac{\mathbb{E}N - \mathrm{Var}(N)}{\mathbb{E}N}\mathbb{E}Z - \frac{1}{2g'\left(\mathrm{Var}(\widehat{Z})\right)}\right)}{C/\left(2g'\left(\mathrm{Var}(\widehat{Z})\right)\mathbb{E}N\right)},$$
$$\text{if } 0 < Z(y) < y;$$

$$e^{Ry} \leq -\frac{\frac{\mathbb{E}N - \mathrm{Var}(N)}{\mathbb{E}N}\mathbb{E}Z - \frac{1}{2g'\left(\mathrm{Var}(\widehat{Z})\right)}}{C/\left(2g'\left(\mathrm{Var}(\widehat{Z})\right)\mathbb{E}N\right)}, \quad \text{if } Z(y) = 0. \tag{37}$$

Hence the result follows immediately from Theorem 3 by making $\alpha_1 = \frac{C}{2g'\left(\mathrm{Var}(\widehat{Z})\right)\mathbb{E}N}$, $\alpha_2 = \frac{\mathbb{E}N - \mathrm{Var}(N)}{\mathbb{E}N}\mathbb{E}Z - \frac{1}{2g'\left(\mathrm{Var}(\widehat{Z})\right)}$.

Now, consider the case when $g'(x)$ is unbounded in any neighbourhood of $x = 0$. Then, $\Pr\{|\Delta P_Z| < \infty\} = 1$ if $Z \neq 0$, and $\Delta P_Z \equiv +\infty$ if $Z \equiv 0$. Therefore, Theorem 3 does not exclude the possibility that $Z \equiv 0$ be optimal. ■

9. Structure of the optimal treaty: Dependence on the distribution of claim numbers

According to Corollary 3, if the optimal treaty for a variance-related principle is not identically zero, then it is a member of the family of functions with two parameters $(\alpha_1, \alpha_2) \in (0, +\infty) \times \mathbb{R}$, satisfying (30)–(32). It can be shown that in the aggregate claim case an optimal treaty satisfying $\alpha_1 + \alpha_2 = 0$ always exists (the limit case $\alpha_1 = \alpha_2 = 0$ corresponding to the case when zero reinsurance is optimal) — see Guerra and Centeno (2008).

Below we show that a similar result holds for the individual claim case when the number of claims N follows a distribution belonging to the so-called Katz family, also known in actuarial literature as the Panjer or $(a, b, 0)$ family of distributions. This family is important for practical applications, including some of the most widely used claim number models (the Poisson and the Negative Binomial), see for instances Duvall (1999), Pitrebois et al. (2005) and Viswanthan and Lemaire (2005). However, we also show that both the cases $\alpha_1 + \alpha_2 > 0$ and $\alpha_1 + \alpha_2 < 0$ do occur in individual claim reinsurance, provided the number of claims follows appropriate distributions.

We start with an interesting relationship between the structure of $\pi(\cdot)$ — the probability generating function of N — and the sign of $\alpha_1 + \alpha_2$.

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[9] See e.g., Quarteroni et al. (2000).

Theorem 4. *Suppose that the optimal treaty is not identically zero and satisfies* $\mathbb{E}\mathrm{e}^{R(Y-Z)} < \rho$, *the radius of convergence of the series* $\pi(t)$.

(a) *If the function* $q(t) = \frac{\pi(t)}{\pi'(t)}$ *is convex, then* $\alpha_1 + \alpha_2 \leq 0$ *holds;*
(b) *If* q *is strictly convex, then* $\alpha_1 + \alpha_2 < 0$ *holds;*
(c) *If* q *is concave, then* $\alpha_1 + \alpha_2 \geq 0$ *holds;*
(d) *If* q *is strictly concave, then* $\alpha_1 + \alpha_2 > 0$ *holds.* □

Proof. It easy to check that

$$\mathbb{E}N = \pi'(1), \quad \mathrm{Var}(N) = \pi''(1) + \pi'(1) - \pi'(1)^2.$$

Therefore, Corollary 3 states that

$$\alpha_1 = \frac{q(1)}{q(x)} \frac{1}{2g'\left(\mathrm{Var}(\hat{Z})\right)}; \tag{38}$$

$$\alpha_2 = \frac{q'(1)}{q(1)} \mathbb{E}Z - \frac{1}{2g'\left(\mathrm{Var}(\hat{Z})\right)}, \tag{39}$$

where $x = \mathbb{E}\mathrm{e}^{R(Y-Z)}$. Eq. (39) is equivalent to

$$\frac{1}{2g'\left(\mathrm{Var}(\hat{Z})\right)} = \frac{q'(1)}{q(1)} \mathbb{E}Z - \alpha_2.$$

Substituting in (38), we obtain

$$q(x)\alpha_1 = q'(1)\mathbb{E}Z - q(1)\alpha_2. \tag{40}$$

Now, suppose that q is convex and $\alpha_1 + \alpha_2 \geq 0$ holds. Corollary 3 implies that

$$\begin{aligned} x = \mathbb{E}\mathrm{e}^{R(Y-Z)} &= \int_0^{\alpha_1+\alpha_2} \mathrm{d}F(y) + \int_{\alpha_1+\alpha_2}^{+\infty} \frac{Z(y) - \alpha_2}{\alpha_1} \mathrm{d}F(y) \\ &= \frac{1}{\alpha_1}\left(\mathbb{E}Z - \alpha_2 + \int_0^{\alpha_1+\alpha_2} (\alpha_1 + \alpha_2 - y)\mathrm{d}F(y)\right). \end{aligned}$$

This is

$$\mathbb{E}Z = x\alpha_1 + \alpha_2 - \int_0^{\alpha_1+\alpha_2} (\alpha_1 + \alpha_2 - y)\mathrm{d}F(y).$$

Substituting in (40) and rearranging we obtain

$$\begin{aligned} &\big(q(x) - q(1) - q'(1)(x-1)\big)\,\alpha_1 + \big(q(1) - q'(1)\big)\,(\alpha_1 + \alpha_2) \\ &\quad = -q'(1)\int_0^{\alpha_1+\alpha_2} (\alpha_1 + \alpha_2 - y)\mathrm{d}F(y). \end{aligned} \tag{41}$$

Notice that $\int_0^{\alpha_1+\alpha_2} (\alpha_1 + \alpha_2 - y)\mathrm{d}F(y) \leq \alpha_1 + \alpha_2$, with strict inequality holding unless $\alpha_1 + \alpha_2 = 0$. If $q'(1) \leq 0$ then the second term on the left-hand side of (41) is non-negative and not smaller than the right-hand side term. By convexity, the first term on the left-hand side is non-negative (strictly positive if q is strictly convex). Hence (41) can hold with $q'(1) \leq 0$ only if $\alpha_1 + \alpha_2 = 0$ and q is linear in $[0, x]$.

Now, note that

$$q(1) - q'(1) = \frac{\mathrm{Var}(N)}{(\mathbb{E}N)^2} > 0. \tag{42}$$

Therefore, if $q'(1) > 0$ then the second term in the left-hand side of (41) is non-negative while the term on the right-hand side is nonpositive and (41) can hold only if $\alpha_1 + \alpha_2 = 0$ and q is linear in $[0, x]$. It cannot hold if q is strictly convex.

To prove (c) and (d), suppose that q is concave and $\alpha_1 + \alpha_2 \leq 0$ holds. In this case, Corollary 3 implies that

$$\begin{aligned} x &= \int_0^{\frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}} \mathrm{e}^{Ry}\mathrm{d}F(y) + \int_{\frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}}^{+\infty} \frac{Z(y) - \alpha_2}{\alpha_1}\mathrm{d}F(y) \\ &= \frac{1}{\alpha_1}\left(\mathbb{E}Z - \alpha_2 - \alpha_1 \int_0^{\frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}} \left(\frac{-\alpha_2}{\alpha_1} - \mathrm{e}^{Ry}\right)\mathrm{d}F(y)\right). \end{aligned}$$

This is

$$\mathbb{E}Z = \left(x + \int_0^{\frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}} \left(\frac{-\alpha_2}{\alpha_1} - \mathrm{e}^{Ry}\right)\mathrm{d}F(y)\right)\alpha_1 + \alpha_2.$$

Substituting in (40) and rearranging we obtain

$$\begin{aligned} &q(x) - q(1) - q'(1)(x-1) \\ &\quad = q'(1)\int_0^{\frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}} \left(\frac{-\alpha_2}{\alpha_1} - \mathrm{e}^{Ry}\right)\mathrm{d}F(y) \\ &\qquad + (q(1) - q'(1))\left(\frac{-\alpha_2}{\alpha_1} - 1\right). \end{aligned} \tag{43}$$

By convexity the left-hand side is nonpositive (strictly negative if q is strictly concave). If $q'(1) \geq 0$, then the right-hand side of (43) is non-negative, being zero only if $\alpha_1 + \alpha_2 = 0$. If $q'(1) < 0$ then rearrange the right-hand side of (43) to obtain

$$\begin{aligned} &q(x) - q(1) - q'(1)(x-1) = q(1)\left(\frac{-\alpha_2}{\alpha_1} - 1\right) \\ &\quad - q'(1)\left(\frac{-\alpha_2}{\alpha_1} - 1 - \int_0^{\frac{1}{R}\ln\frac{-\alpha_2}{\alpha_1}} \left(\frac{-\alpha_2}{\alpha_1} - \mathrm{e}^{Ry}\right)\mathrm{d}F(y)\right), \end{aligned}$$

where the right-hand side is obviously non-negative, being zero only if $\alpha_1 + \alpha_2 = 0$. Hence we see that (43) holds only if $\alpha_1 + \alpha_2 = 0$ and q is linear in $[0, x]$. ■

Remark 3. When the distribution of claim values has a heavy tail, some part of the risk must be ceded under the optimal arrangement, irrespective of the distribution of claim numbers (otherwise the adjustment coefficient would not exist). Therefore the possibility of zero reinsurance being optimal indicated at the beginning of the Theorem can be immediately excluded in such cases.

Theorem 4 shows that the case in which the function q is linear has the special property $\alpha_1 + \alpha_2 = 0$. One such case is aggregate claim reinsurance, analysed in Guerra and Centeno (2008). Indeed, it is clear that the random variable $N \equiv 1$ has probability generating function $\pi(t) = t$ and hence $q(t) = t$ is simultaneously concave and convex. It turns out that there exists also an important family of nondegenerate distributions with this property. This is the so-called Katz family (see (Johnson et al., 1993, pg. 38)). It consists of the binomial, Poisson and negative binomial distributions.

Corollary 4. *Assume that the distribution of the number of claims* N *belongs to the Katz family.*

If g' *is bounded in a neighbourhood of zero, then the optimal policy satisfies*

$$\begin{aligned} y &= Z(y) + \frac{1}{R}\ln\frac{Z(y) + \alpha}{\alpha}; \\ \alpha &= \frac{\pi'\left(\mathbb{E}\mathrm{e}^{R(Y-Z)}\right)}{\mathbb{E}N\pi\left(\mathbb{E}\mathrm{e}^{R(Y-Z)}\right)2g'\left(\mathrm{Var}(\widehat{Z})\right)} \end{aligned}$$

If g' *is unbounded near zero, then either* $Z \equiv 0$ *is optimal or the optimal policy satisfies the conditions above.* □

Proof. Recall that Katz-type distributions are characterized by having a probability generating function of the type

$$\pi(t)=\left(\frac{1-at}{1-a}\right)^{-(a+b)/a}, \tag{44}$$

with $a<1, a+b>0$ and $b=-(k+1)a$ for some $k\in\mathbb{N}$ when $a<0$ (in the case $a=0$, it is $\pi(t)=\lim_{a\to 0}\left(\frac{1-at}{1-a}\right)^{-(a+b)/a}=\mathrm{e}^{b(t-1)}$). It follows that q is the linear function

$$q(t)=\frac{1-at}{a+b},$$

hence it is simultaneously concave and convex. The radius of convergence of π is $\rho=\frac{1}{a}$ if $a\in(0,1)$ or $\rho=+\infty$ if $a\leq 0$. In the case when $a\in(0,1)$, we have $\lim_{x\to\rho^-}\pi(x)=+\infty$. It follows that the optimal reinsurance policy must satisfy $\mathbb{E}\mathrm{e}^{R(Y-Z)}<\rho$ and the result follows immediately from Corollary 3 and Theorem 4. ■

In order to see that there are cases where $\alpha_1+\alpha_2<0$ and cases where $\alpha_1+\alpha_2>0$ holds, recall that the Katz family of distributions can be embedded in the larger Sundt and Jewell family (Sundt and Jewell, 1981). These are combinations of a Katz random variable with a random variable concentrated at $N=0$.

The probability generating function of a Sundt and Jewell distribution is

$$\pi(t)=c+(1-c)\left(\frac{1-at}{1-a}\right)^{-(a+b)/a},$$

with a, b like in (44) and $\frac{(1-a)^{(a+b)/a}}{(1-a)^{(a+b)/a}-1}\leq c<1$. Since this is a combination of a Katz random variable with the null random variable, the proof of Corollary 4 shows that the optimal treaty must satisfy $\mathbb{E}\mathrm{e}^{R(Y-Z)}<\rho$. Therefore, due to Theorem 4, we only need to check the convexity of $q=\frac{\pi}{\pi'}$.

A simple computation shows that a Sundt and Jewell distribution satisfies

$$q''(t)=\frac{c(2a+b)}{(1-at)(\pi(t)-c)}.$$

Since $a+b>0$ and $b=-(k+1)a$ must hold when $a<0$, we see that $2a+b>0$ holds except in the case when $a<0, b=-2a$. In this case N is a Bernoulli random variable and hence any mixture with the null random variable is still Bernoulli.

Therefore we see that, except in the Bernoulli case, q is strictly convex (i.e., $\alpha_1+\alpha_2<0$) if $c>0$ and it is strictly concave (i.e., $\alpha_1+\alpha_2>0$) if $\frac{(1-a)^{(a+b)/a}}{(1-a)^{(a+b)/a}-1}\leq c<0$. It is linear (i.e., $\alpha_1+\alpha_2=0$) if $c=0$.

The discussion above shows that the case $\alpha_1+\alpha_2=0$ is a very particular one, being destroyed by small perturbations of the distribution of claim numbers. To see this consider that N follows a Katz distribution and it is not Bernoulli. This is a Sundt and Jewell distribution with $c=0$. By changing the value of c by any small amount (i.e., by changing $\Pr\{N=0\}$, adjusting proportionally the remaining probabilities in such a way to obtain a probability function) we can either obtain the case $\alpha_1+\alpha_2<0$ or $\alpha_1+\alpha_2>0$, according to the sign of c.

Notice that, according to the proof of Theorem 4, the sign of $\alpha_1+\alpha_2$ depends on the sign of the remainder of the Taylor expansion of the function $q(\cdot)$ (convexity being just one condition that ensures that this remainder has the appropriate sign). Therefore, one can expect to find distributions of the number of claims such that the sign of $\alpha_1+\alpha_2$ changes when the reinsurance loading increases.

To see this, consider a family of variance-related principles

$$P_\beta(Z)=\mathbb{E}Z+\beta g\left(\mathrm{Var}(\hat{Z})\right),\quad \beta\in(0,+\infty),$$

where $g:[0,+\infty)\mapsto[0,+\infty)$ is continuous, smooth in $(0,+\infty)$ and satisfies (25)–(26) (say, $g(t)=\sqrt{t}$, in which case $\{P_\beta\}_{\beta>0}$ is the family of standard deviation principles). Assuming that all the remaining data of the problem remains unchanged, one can expect that for small values of the parameter β a large proportion of the risk is ceded under the optimal treaty and hence the value of $x=\mathbb{E}\mathrm{e}^{R(Y-Z)}$ is close to one. Conversely, when β is large, a small part of the total risk is ceded under the optimal treaty, making $x=\mathbb{E}\mathrm{e}^{R(Y-Z)}$ large.

Now suppose that the distribution of the number of claims is such that the function q is (say) strictly concave in the interval $[1,x_0]$ and strictly convex in $[x_0,\rho)$. Clearly, the Taylor remainder $q(x)-q(1)-q'(1)(x-1)$ is negative whenever $x\in(1,x_1)$ for some $x_1>x_0$. However, we cannot exclude that $x_1<+\infty$ and the remainder becomes positive for $x\in(x_1,+\infty)$. In such a case optimal treaties would satisfy $\alpha_1+\alpha_2>0$ for small loadings (i.e., small β and hence a large amount of risk is ceded under the optimal treaty) and $\alpha_1+\alpha_2<0$ for higher loadings (large β, and a small amount of risk is ceded).

References

Arrow, K.J., 1963. Uncertainty and the Welfare of Medical Care. The American Economic Review LIII, 941–973.

Bhattacharya, R., Waymire, E.C., 2007. A Basic Course in Probability Theory. Springer-Verlag, New York.

Borch, K., 1960. An attempt to determine the optimum amount of stop loss reinsurance. In: Transactions of the 16th International Congress of Actuaries, pp. 597–610.

Borch, K., 1969. The optimum reinsurance treaty. ASTIN Bulletin 5, 293–297.

Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A., Nesbitt, C.J., 1987. Actuarial Mathematics. Society of Actuaries, Chicago.

Cai, J., Tan, K.S., Wenga, C., Zhangc, Y., 2008. Optimal reinsurance under VaR and CTE risk measures. Insurance: Mathematics and Economics 43, 185–196.

Centeno, M.L., 1997. Excess of loss reinsurance and the probability of ruin in finite horizon. ASTIN Bulletin 27 (1), 59–70.

Deprez, O., Gerber, H.U., 1985. On convex principles of premium calculation. Insurance: Mathematics and Economics 4, 179–189.

Dickson, D., Waters, H., 1996. Reinsurance and ruin. Insurance: Mathematics and Economics 19, 61–80.

Duvall, R., 1999. A Bayesian Approach to Negative Binomial Parameter Estimation., Casualty Actuarial Society Forum, pp. 377–386.

Froot, K.A., 2001. The Market for catastrophe risk: A clinical examination. Journal of Financial Economics 60, 529–571.

Gajek, L., Zagrodny, D., 2000. Insurer's optimal reinsurance strategies. Insurance: Mathematics and Economics 27, 105–112.

Gajek, L., Zagrodny, D., 2004. Reinsurance arrangements maximizing insurer's survival probability. Journal of Risk and Insurance 71 (3), 421–435.

Gamkrelidze, R.V., 1978. Principles of Optimal Control Theory. Plenum Press.

Gerber, H.U., 1979. An Introduction to Mathematical Risk Theory. In: Huebner Foundation Monographs, vol. 8. Richard D. Irwin Inc. (dist.), Homewood Illinois.

Guerra, M., Centeno, M.L., 2010. Optimal reinsurance for variance related premium calculation principles. ASTIN Bulletin (in press).

Guerra, M., Centeno, M.L., 2008. Optimal reinsurance policy: The adjustment coefficient and the expected utility criteria. Insurance: Mathematics and Economics 42/2, 529–539. http://dx.doi.org/10.1016/j.insmatheco.2007.02.008.

Hesselager, O., 1990. Some results on optimal reinsurance in terms of the adjustment coefficient. Scandinavian Actuarial Journal 1, 80–95.

Johnson, N., Kotz, S., Kemp, A., 1993. Univariate Discrete Distributions, 2nd Ed. John Wiley & Sons.

Kaas, R., Goovaerts, M., Dhaene, J., Denuit, M., 2008. Modern Actuarial Risk Theory: Using R, 2nd Ed. Springer.

Kaluszka, M., 2001. Optimal reinsurance under mean-variance premium principles. Insurance: Mathematics and Economics 28, 61–67.

Kaluszka, M., 2004. An extension of Arrow's result on optimality of a stop loss contract. Insurance: Mathematics and Economics 35, 524–536.

Kaluszka, M., 2005. Optimal reinsurance under convex principles of premium calculation. Insurance: Mathematics and Economics 36, 375–398.

Lang, S., 1993. Real and Functional Analysis, 3rd Ed. Springer-Verlag, New York.

Pitrebois, S., Walkin, J., Denuit, M., 2005. Bonus Malus Systems with Varying Deductibles. ASTIN Bulletin 35, 261–274.

Quarteroni, A., Sacco, R., Saleri, F., 2000. Numerical Mathematics. Springer–Verlag, New York.

Rudin, W., 1987. Real and Complex Analysis. McGraw-Hill International, Singapore.

Rudin, W., 1991. Functional Analysis. McGraw-Hill International, Singapore.

Sundt, B., Jewell, W., 1981. Further results on recursive evaluation of compound distributions. ASTIN Bulletin 12, 27–39.

Viswanthan, K., Lemaire, J., 2005. Bonus-Malus systems in a deregulated environment: Forecasting market shares using diffusion models. ASTIN Bulletin 35, 299–319.