LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

MASTER

ACTUARIAL SCIENCE

MASTER´S FINAL WORK

THESIS

REVIEW OF QUANTITATIVE MODELS OF CREDIT RISK FOR DEBT INSTRUMENTS, INCLUDING CATASTROPHE BONDS

NIKUNJ SHARMA

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SUPERVISION:

WALTHER NEUHAUS

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Abstract:

Prices of financial contracts vary due to two major factors namely market risk and credit risk. Market risk is the risk that the value of a financial contract will decrease due to changes in market factors, these factors can be reduced by diversification into alternative assets classes. Credit risk is the risk that a person or an organisation will fail to make a payment that they have promised.

It is a consensus that effective credit risk analysis is essential for investors seeking to determine whether a firm has the financial ability to meet its financial obligations. This study is primarily focused on the credit risk component.

This study provides a detailed overview of past research in the area of credit risk modelling for defaultable debt instruments. The study mentions strengths, shortcomings and the recent advancement in the field of credit risk modelling. It also provides a review of commonly used model namely: the Merton model (1974), the first-passage-time model (1976), the two-state model and Jarrow-Lando-Turnbull model (1997).

In the second part, we introduce catastrophe bonds by explaining their definitions and structure. The event of a pay-out from a catastrophe bond after a catastrophe event is treated by investors like a credit default. We will therefore extend the use of quantitative credit risk model to catastrophe bonds.

Keywords: market risk, credit risk, defaultable debt instruments, Merton model, first-passage-time model, two-state model, Jarrow-Lando-Turnbull model, catastrophe bonds.

Abstrato:

Os preços de contratos financeiros variam devido a dois principais fatores, nomeadamente o risco de mercado e o risco de crédito. O risco de mercado é o risco do valor de um contrato financeiro diminuir devido a alterações nos fatores do mercado, fatores estes que podem ser reduzidos pela diversificação em classes de ativos alternativas. O risco de crédito é o risco de uma pessoa ou uma organização deixar de efetuar um pagamento que havia prometido.

É de consenso geral que uma análise efetiva do risco de crédito é essencial para investidores que procuram determinar se uma empresa tem capacidade financeira para cumprir as suas obrigações financeiras. Este estudo tem como principal foco o risco de crédito.

Este estudo fornece uma revisão detalhada de pesquisas anteriores na área de modelagem de risco de crédito para instrumentos de dívida inadimplentes. O estudo menciona pontos fortes, deficiências e os recentes avanços no campo da modelagem de risco de crédito. Para além disso, é também feita uma revisão dos modelos cumumente usado: O modelo de Merton (1974), o modelo da primeira passagem (1976), o modelo de dois estados e o modelo de Jarrow-Lando-Turnbull (1997).

Na segunda parte, é introduzido o conceito de catástrofe, explicando a suas definições e estrutura das partes interessadas. O evento de pagamento de um título de catástrofe após um evento de catástrofe é tratado pelos investidores como uma inadimplência de crédito. Assim, é portanto introduzido o uso do modelo quantitativo simples de risco de crédito para catástrofe de títulos.

Palavras-chave: risco de mercado, risco de crédito, instrumentos de dívida inadimplentes, modelo Merton, modelo da primeira passagem, modelo de dois estados, modelo Jarrow-Lando-Turnbull, títulos de catástrofe.

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Introduction:

We will assume that both parties entering into a financial contract (that is, both the seller or writer and the buyer of bonds, derivatives etc.), may not honor their commitments at time of exercise. Therefore, in reality this may be the case: the seller of a corporate bond, or the writer of an option, can default: miss a coupon payment, or fail to deliver the promised claim. Credit risk models take this default risk explicitly into account when pricing a contract. The quantitative methodologies of credit risk are aimed to help professionals manage risks, better modelling and hedging of this kind of credit risk.

The study firstly covers simple quantitative methodologies/models of credit risk for defaultable debt instruments. Approaches to quantitative credit risk modelling fall into two major categories: the structural approach (also called as value-of-the-firm approach) and the reduced-form approach (also called as the intensity-based approach). There is a divergence of opinion in the literature as to the relative merits of the structural and reduced-form approaches. Both approaches have strength, but there are also shortcomings.

The study covers the review of commonly used structural and the reduced-form models approaches. Some of the important credit risk models include are: Merton`s model (1974), the first-passage-time model (1976), the two-state model and Jarrow-Lando-Turnbull model (1997).

The distinction between structural models and reduced-form models can be defined as follows:

> *"Structural models are based on the information set available to the firm's management, which includes continuous time observations of both asset values and liabilities. Reduced form models are based on the information set available to the market, a reduced information set containing discrete time (and perhaps noisy) observations of both asset values and liabilities"*
>
> Guo, Jarrow and Zeng (2005, p.2) [1]

In the second part, we will introduce catastrophe bonds by explaining their definitions, structure and triggers. We will then extend the use of simple quantitative credit risk model to zero-coupon catastrophe bond, as they also have credit ratings to reflect their probability of default. Investors have taken up catastrophe bonds because they give a higher yield but are uncorrelated with the ordinary bond market. It means investors can add expected investment return to their portfolio without increasing volatility to the same degree.

The growing importance of credit risk has made the development of credit risk models an important issue in finance and therefore in section 3 we briefly conclude by suggesting a further extension to the credit risk modelling.

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1 – STRUCTURE: CREDIT RISK MODELS

1.1 INTRODUCTION TO CREDIT RISK

Prices of financial contracts vary due to two major factors namely market risk (such as, for the instance, interest rate risk in the market) and credit risk (the risk that a person or an organisation will fail to make a payment that they have promised). Effective credit risk analysis is essential for investors seeking to determine whether a firm has the financial ability to meet its financial obligations.

1.1.1 Definitions of credit risk

“Credit risk is the possibility of a loss resulting from a borrower's failure to repay a loan or meet contractual obligations. Traditionally, it refers to the risk that a lender may not receive the owed principal and interest, which results in an interruption of cash flows and increased costs for collection” [2].

“Credit risk refers to the probability of loss due to a borrower’s failure to make payments on any type of debt” [3].

1.1.2 Introduction of bonds (defaultable and default free) relating to credit risk

A bond is a securitized form of loan. Companies raise a considerable amount of business debt in the form of issuance of bonds and companies inversely invest in bonds.

The buyer of a bond is the bondholder who lends the money to the bond issuer for a certain period, in the expectation of a predetermined sequence of payments (interest or coupons) and the eventual return of the initially lent money, also known as the principal. The sequence of payments may be fixed (a fixed interest bonds) or the payments may be linked to some variable index such as inflation. Bonds that are linked to an index are called index-linked bonds.

If bonds are default free, this means that their interest payment and capital payment will definitely be made in full and on time. This is a reasonable assumption when considering most government bonds. For most corporate bonds where the bond issuing company is a corporate entity, it is possible that the issuer will be unable to make its promised payment in full and/or on time. Credit risk analysis of such debt obligations and other defaultable financial instruments is therefore an important risk management activity. [4, pp. 197-198 Ch.11]

1.1.3 Key terminology relating to credit risk: Credit event and recovery rates

A credit event is any event that, if it occurs, will have an impact on the ability of the counterparty to fulfil the financial obligation. Credit events could be bankruptcy with a loss of principal, the issuer not making a scheduled interest payment, the issuer not having the required accounting ratios, and/or having its bond's credit ratings downgraded by credit rating agencies.

In such a default event, there could be situations where interest repayments (and/or principal repayments) are reduced, rescheduled, or not made at all. In such a situation, the recoverable fraction of amount is known as "recovery rate". If a bond defaults during its lifetime, then recovery payment is made at either default time or maturity of the bond. [5, pp. 1-5 Ch.18]

1.1.4 Quantitative methodologies of credit risk

The existing literature provides a vast number of quantitative methodologies for modelling, valuing and hedging of the credit risk on debt obligations and other defaultable financial instruments. These quantitative methodologies majorly follow two types of approaches: the structural approach and the reduced-form approach:

1.1.4.1 Structural approach

The structural approach consists of modelling and pricing credit risk specific to a company. This approach primarily aims to utilise the debt & equity values of a company to determine the likelihood of default and is therefore referred as value-of-the-firm approach. It also studies the nature of default if it occurs and the interactions between debtholders & equityholders. Merton's model (1974) is the first structural model. A second approach, within the structural framework was introduced by Black and Cox (1976). [6, pp. 26-28 Ch.1]

1.1.4.2 Reduced-Form approach

The reduced-form approach (also called as the intensity-based approach) consists of analysing market data rather than the company data itself. This approach looks at the credit ratings of bond issuing companies and their movements. The two-state model and Jarrow-Lando-Turnbull (1997) model (the JLT model) are two important models. The JLT model is more realistic and general than the two-state model.

In the next sections, we have detailed both credit risk approaches: Firstly, Sect 1.2 presents the structural approach and Sect 1.3 details the reduced-form approach.

1.2 STRUCTURAL APPROACH

1.2.1 Introduction

The structural approach is a quantitative methodology for pricing and modelling of the credit risk of a specific firm. The classical structural approach is primarily aimed at firms that issue both debt and equity.

1.2.2 Credit event, default time and recovery rates

Under the structural approach, movements of the firm value cause a credit event. The default time is the first time when the firm's value breaks the defaultable barrier. The recovery rate is the relative amount recovered by creditors.

1.2.3 Models of development in the structural approach

The structural approach for credit risk modelling was initiated by Merton in 1974, who used the Black and Scholes (1973) option-pricing model (OPM) to value firm liabilities. Merton's model is the simplest structural model for assessing the credit risk of a specific firm. In Merton's model, the firm's default may arrive only at the maturity of its debt obligations.

Furthermore, variants of the models were produced by Black and Cox (1976) [7], Galai and Masulis (1976) [8] and Geske (1977) [9].

The first-passage-time approach (also known as FPM) was developed by Black and Cox (1976) and in the paper by Galai and Masulis (1976), a combined pricing model was introduced. The combined pricing model was primarily focused on the risk in issuing corporate stock. Essentially, it was a more complete model combining the option-pricing model (OPM) with capital asset pricing model (CAPM). Geske's (1977) paper presents a theory for pricing options on options or compound options. Formulas were derived for valuing corporate bonds and subordinated bonds as compound options.

Later, these approaches were developed in various directions by: [6, pp. 26-28 Ch.1, pp. 31-32 Ch.2], [10, pp. 1–2]

- Brennan and Schwartz (1977, 1980)
- Pitts and Selby (1983)
- Cooper and Mello (1991)
- Rendleman (1992)
- Kim et al. (1993a)
- Nielsen et al. (1993)
- Leland (1994)
- Longstaff and Schwartz (1995)
- Anderson and Sundaresan (1996)

- Leland and Toft (1996)
- Mella-Barral and Tychon (1996)
- Mella-Barral and Perraudin (1997)
- Briys and de Varenne (1997)
- Crouhy (1998)
- Ericsson and Reneby (1998)
- Pirotte (1999)
- Anderson and Sundaresan (2000)
- Ericsson (2000)
- H. Lee, R. Chen and C. Lee in (2009), and others

Merton (1974) [11] described a model giving a formula for the risk premium on risky zero-coupon debt and Lee (1981) [12] re-examined Merton´s model and gave somewhat different graphs for the risk premium as a function of time. Pitts and Selby in (1983) [13] explained a result which settles both the point, in other words the purpose of this paper is to prove two mathematical properties of risk premium which settle the disparities.

Brennan and Schwartz (1977) [14] and (1980) [15] paper extended the work of Black and Scholes and Merton to the pricing of convertible bonds. Convertible securities, especially convertible bonds, are considerably more complex than other securities as it is a hybrid security that, while retaining most of the characteristic of a normal bond, offers, in addition, the upside potential associated with the underlying stock.

Swaps in their simplest form are an agreement between two entities (called counterparties) to exchange two streams of future cash flows. The paper Cooper and Mello (1991) [16] discusses two types of risk in swap transactions: rate risk and default risk. It also discusses the different alternative treatments in default.

Rendleman (1992) [17] develops a multidimensional binomial pricing model. The purpose of this model is to value debt, equity and swap liabilities of both counterparts to an interest rate swap when there is the potential for default risk on both sides of the transaction.

Interest rate swaps are among the most popular financial derivatives. In the market, IRS´s are quoted irrespective of credit ratings of counterparties. In another words, they are considered as default-free. The valuation of default-free IRS´s under a term structure of interest rates is a classical exercise. However, [some?] swap contracts are traded over-the-counter (OTC) and are not backed by the guarantee of a clearing corporation or an exchange. Consequently, each party is exposed to the credit risk. Alan White (1992) [18] in his paper, presents an analytical model for valuing contingent claims subject to default by both parties.

In the first-passage-time model (also known as FPM), the default occurs the first time when the firm value falls below some default barrier. The default barrier is some specified barrier and crossing the barrier is known as triggering event.

In the FPM, the default barrier may be an exogenously pre-specified constant value to protect bondholders (in the papers by Black and Cox (1976) and Longstaff and Schwartz (1995) [19]). Alternatively, stockholders may fix the constant default barrier not only as an attempt to protect themselves but also to maximise the value of the firm (cf. Leland (1994) [20] and Leland and Toft (1996) [21]) [10, p. 2]

Leland and Toft (1996) are concerned with finite maturity corporate debt, as opposed to corporate perpetuity studied by Leland (1994). They also assume that the short-term interest rate is constant. [6, p. 86 Ch.3]

"Other important factors (such as more realistic modelling of bankruptcy, bankruptcy costs and tax benefits) were highlighted in papers by Leland (1994), Leland and Toft (1996), Anderson and Sundaresan (1996) [22] and Mella-Barral and Perraudin (1997) [23]. " [6, p. 84 Ch.3]

Mella-Barral and Tychon (1996) [24] developed an analytical solution for the impact of default risk on the valuation of a wide range of contracts.

In the field of both early default and interest rate risk modelling, the paper of Briys and Varenne (1997) [25] develops a corporate bond valuation model. The main objective of their paper is to correct the defects of the recent models. This paper addresses two major defects. The first occurs where pricing equations do not assume that the payment to bondholders is no greater than firm value upon default. The second situation occurs where the corporation can find itself in a solvent situation (according to the agreed threshold) when the corporate bond reaches maturity, but with insufficient assets to match the face value of the bond. For example, the situation when at the maturity of the corporate bond, the assets are less than the face value of the bond but the asset value is still above the threshold limit.

Ericsson and Reneby (1998) [26] suggested a versatile methodology for the valuation of corporate securities with the use of Black and Scholes (1973) model.

Crouhy (1998) [27] and Anderson and Sundaresan (2000) [28] produced a comparative analysis of current credit risk models.

Jan Ericsson (2000) [29] suggested a continuous time model for firm´s debt and equity valuation and focused for the firm´s capital structure decision, where advantage and maturity decision are chosen optimally by the firm´s management. In this paper, factors that influence the structure of firm´s balance sheet are analysed.

As the name suggest, the paper “Liquidity and Credit Risk” by Ericsson and Renault (2006) [30], describes the framework of valuing risky debt and develop a structural valuation model to simultaneously capture liquidity and credit risk.

Ericsson, Reneby and Wang (2006) [31] and (2007) [32] used a set of structural model for a sample of US corporations to evaluate the price of default protection. These papers particularly use three structural models: Leland 1994, Leland and Toft 1996 and Fan and Sundaresan 2000.

Pirotte (1999) [33] presents a structural model and models the term structure of the credit spreads with the recovery rate being considered stochastic. It provides an argument for the preference of the structural approach to the reduced-form approach and presents the simple model of Merton.

H. Lee, R. Chen and C. Lee in (2009) [34] review empirical evidence and estimation methods of structural credit risk models. Then they present a detailed overview of the current estimation methods and their drawbacks. For the empirical investigation, they adopted the maximum likelihood estimation . The empirical results conclusively show that the simple Merton model outperforms the Brockman and Turtle model in default prediction.

Let us emphasize the characteristic and criticism or shortcoming of important models developed under the framework of structural approach. Sect. 1.2.4 is entirely devoted to the Merton (1974) model, and in Sect. 1.2.5, we present an overview of the first-passage-time (FPM) model.

1.2.4 The Merton model (1974)

In 1974, the economist Robert C. Merton proposed a model for assessing the credit risk of a firm by modelling the firm’s equity as a call option on its assets. The model makes use of Black and Scholes (1973) option pricing model to value firm liabilities.

Merton’s model is a simple example of a structure model, which assumes that the firm has issued two classes of securities: equity and debt. The equity receives no dividends and the debt has the form of a zero-coupon bond (ZCB) with finite maturity. A zero-coupon bond is a bond that pays no coupon. A further important assumption is that all options are European and are exercised only at the time of expiration. In this model, default occurs only if the firm’s value is less than its outstanding ZCB at the maturity date.

1.2.4.1 Introduction

Let us assume that the capital structure of the firm is comprised by equity and debt. The debt consists of only a zero-coupon bond.

Current Time t: The firm's asset value is simply the sum of equity and debt values. The equity and zero-coupon bond value at current time t are denoted by E_t and Z (t, T) respectively, for $0 \leq t \leq T$. The total firm value is denoted by F(t) at current time t, as shown in the figure 1:

$$F(t) = Equity + Debt = E_t + Z(t, T), \text{ at time } t$$

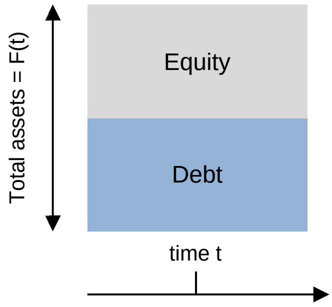

Figure 1: Total firm value F(t) at time t

Future Time T: The firm value has varied up until a future time T. The value of zero-coupon bond at maturity future time is denoted by L. Repayments to both equity and debtholder can result in two possible situations:

Either at maturity T the firm's asset value F(T) is enough to pay back the face value of the debt L, the default does not occur since F(T) > L. The debtholders will therefore receive their due repayments of L in full at maturity T and equityholders will receive F(T)-L, after making debt payments. Equityholders rank behind the debtholders and hence they are repaid only after first repaying the debtholders under the winding up process:

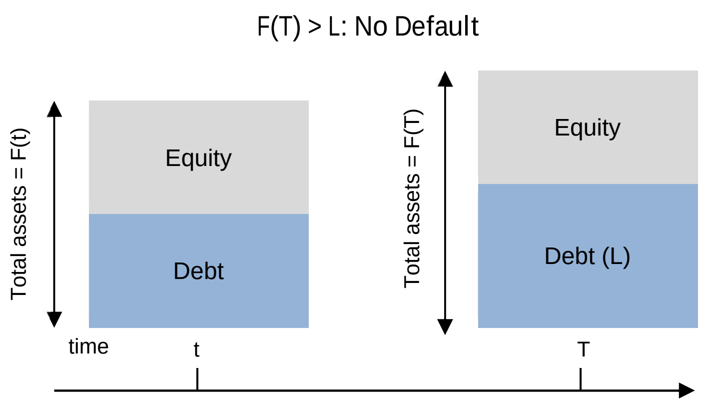

Figure 2: Total firm value F(T) at time T when firm value is more than due debt L

The firm alternatively defaults on due debt at maturity T if the firm's asset value F(T) is not enough to pay back the face value of the debt L as F(T) < L, the debtholder will take control of the firm and equityholder receive nothing:

F(T) < L: Default

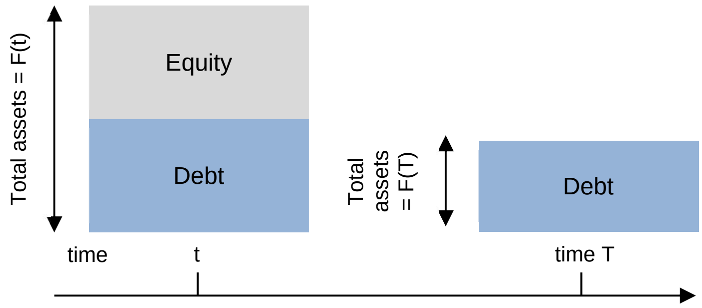

Figure 3: Total firm value F(T) at time T when firm value is less than due debt L

The firm does default in figure 3, and the repayments in this situation to be as follows:

Debtholders : F(T) i.e. repayment not in full

Equityholders : 0 (Nil) i.e. repayment nil

The Merton model assumes that the total firm value F(t) follows a geometric Brownian motion under the risk-neutral probability measures:

$$d\,F(t) = r\,F(t)\,dt + \sigma\,F(t)\,d\,\widetilde{W}_{t}, \qquad F(0) > 0$$

Where, r is a risk-free instantaneous interest rate, σ is the asset volatility and $\widetilde{W}_t$ is a Brownian motion (Brownian motion under the risk-neutral probability measure).[1]

The equityholder and debtholder's funds at current and future time can be written as:

1.2.4.1.1 Equityholders Payoff/Funds:

Future Time T **:** Max [F(T)-L, 0]

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[1] Since we are working under the risk-neutral probability measures, the drift term (r) in the asset value process is the risk-free instantaneous interest rate. If otherwise, under the real-world measures; r and $\widetilde{W}_t$ would be replaced by μ (mean rate of return on the assets) and W_t (Brownian motion under the real-world probability measure without a tilde on a hat) respectively. The assets process otherwise would be:

$$d\,F(t) = \mu\,F(t)\,dt + \sigma\,F(t)\,d\,Wt$$

Under the limited liability, the equityholders have no legal obligation to inject further money into the firm when there is a default on the outstanding debt.

Equityholders of the firm can be seen as having a European call option [2] on the net asset value of the firm. The payoff for equityholder is Max [F(T)-L, 0] which corresponds to a call option with maturity T and a strike price equals to L.

The assumption that the firm can only default at time T allows one to use the Black and Scholes (1973) option pricing formula to calculate the value of the option at the current time.

Current Time t : The value of the call option is equals to c_t

$$\mathbf{c_t} = F(t) \times \phi(d_1) - L \times e^{-r(T-t)} \times \phi(d_2)$$

where:

$$\mathbf{d_1} = \frac{\log(F(t)/L) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}$$

and

$$\mathbf{d_2} = d_1 - \sigma\sqrt{(T-t)} = \frac{\log(F(t)/L) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}} - \sigma\sqrt{(T-t)}$$

and (ϕ) is a phi represents the cumulative standard normal distribution

1.2.4.1.2 Debtholders Payoff/Funds:

Future Time T : Min [F(T), L] can also be written as L – max [L-F(T), 0]

Debtholders of the firm can be treated as having sold/given a European put option on the assets of the company to the equityholders. The term Max [L-F(T), 0] is the value of a put option on F(T) with a strike price equal to L.

Current Time t : The value of the put option is equals to p_t

$$\mathbf{p_t} = L \times e^{-r(T-t)} \times \phi(-d_2) - F(t) \times \phi(-d_1)$$

Therefore, bondholders' funds under the risk-neutral world is the discounted value of L at the risk-free force of interest rate less the value of p_t:

[2] The Black-Scholes formulae for European options is taken from "Formulae and Tables for Examinations of The Faculty of Actuaries and The Institute of Actuaries". For example, call option formula used above is taken from page 47 as $c_t = S(t) \times e^{-q(T-t)} \times \phi(d_1) - L \times e^{-r(T-t)} \times \phi(d_2)$. The formula used above does not include q i.e. dividends on equity as nil dividends payments have been assumed and uses F(t) instead of S(t).

$$L \times e^{-r(T-t)} - p_t$$

In other words, bondholders' funds under the risk-neutral world is the total asset value at time t less the value of c_t:

$$F(t) - c_t$$

1.2.4.1.3 Probability of default and credit spread risk:

Probability of Default = P [F(T) < L | F(t)]

This can also be worked out as probability of not being default as ϕ (d_2) used in first formula above, therefore Probability of Default will be:

$$1 - \phi(d_2) = P [F(T) < L \mid F(t)]$$

Credit spread risk: Credit spread risk is the difference in the yield to maturity between a normal defaultable bond and default free bond in the market. Merton's model can also be used to calculate the spread risk. By definition, the default-free yield-to-maturity Y (t, T) and the defaultable yield-to-maturity Y^d (t, T) satisfy:

Credit spread risk = Y^d (t, T) - Y (t, T)

where:

$$Y^d(t, T) = -\frac{\ln D(t,T)}{T-t}$$

and

$$Y(t, T) = -\frac{\ln B(t,T)}{T-t}$$

Where; D (t, T) and B (t, T) two respective prices of defaultable and default free zero-coupon bonds; formula Y^d (t, T) is valid only prior to default.

For example: considering two respective prices of default free and defaultable zero-coupon bonds with maturity 15 years of 0.7 and 0.5. The yield to maturity can be found by solving the equation: Default free Bond 0.7 = $e^{-15\, Y(t, T)}$; Y (t, T) = 2.37% and defaultable bond 0.5 = $e^{-15\, Y^d(t, T)}$; Y^d (t, T) = 4.62%.

Spread risk in this example is 2.25% i.e. Y^d (t, T) - Y (t, T) (4.62% - 2.37%). Although the yield to maturity on defaultable bond is higher but we have assumed it to be a default free as the return of 4.62% is not expected. [4, pp. 199-201 Ch.11], [5, pp. 6-7 Ch.18], [35, pp. 2–3]

1.2.4.2 Example

Consider a firm initially, at time t total asset value worth €20 million with no debt:

Debt	€0	
Equity	€20m	
Total asset value	€20m	(Net asset value = €20m)

Suppose that, at time t, the firm borrows €10m by issuing bonds. The value of the company's assets will increase to €30m. Since it will now be sitting on €10m in cash at hand, but its net asset value will still be €20m (because it now has a liability of €10m for the bonds).

Debt	€10m	
Equity	€20m	
Total asset value	€30m	(Net asset value = €20m)

In future time at T (i.e. the redemption date of bonds), company's total asset value F(T) likely to be either less or higher than €10m. This will determine the total payoff for the two categories of investor as:

Equityholders	max [F(T)-10,0]
Debtholders	min [F(T),10]

• **F(T) > L: No Default:** The company's total asset value F(T) is more than €10m. The equityholders payoff after paying the bondholders first.

So, when for example F(T) is €11m, the payoff will be:

Equityholders	max [11-10,0]	= 1
Debtholders	min [11,10]	= 10

• **F(T) < L: Default:** The company's total asset value F(T) is less than €10m (i.e. the company's equity is negative), then the equityholders will receive nothing because the bondholders must be paid first. Equityholders are also not liable to inject further money due to the limited liability.

So, when for example F(T) is €7m, the payoff will be:

Equityholders	max [7-10,0]	= 0
Debtholders	min [7,10]	= 7

1.2.4.3 Example: Merton model for pricing defaultable (zero-coupon) bonds

Consider a firm and the parameters:

Table 1: Parameters at time t

Parameters:		
Firm's total asset value	F(t)	**€30m**

Nominal value of debt	L	**€10m**
Term of bonds	T-t	**10 years**
Asset volatility	σ	**30%**
Risk-free rate	r	**1%**

$$\mathbf{d_1} \quad = \frac{\log(F(t)/L)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}$$

$$= \frac{\log(30/10)+(0.01+\frac{1}{2}0.3^2)(10)}{0.3\sqrt{(10)}} = \mathbf{1.7378}$$

and

$$\mathbf{d_2} \quad = d_1 - \sigma\sqrt{(T-t)} = \frac{\log(F(t)/L)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}} - \sigma\sqrt{(T-t)}$$

$$= 1.7378 - 0.3\sqrt{(10)} = \frac{\log(30/10)+(0.01+\frac{1}{2}0.3^2)(10)}{0.3\sqrt{(10)}} - 0.3\sqrt{(10)} = \mathbf{0.7891}$$

Table 2: Black-Scholes parameters

Intermediate Calculations:		
Black-Scholes parameter	d1	1.7378
Black-Scholes parameter	d2	0.7891

Table 3: Parameters for firm at time T (Expected growth rate μ -10%)

Parameters at time T [F(T)>L]:		
Firm's total asset value	F(T)	**€11m**
Nominal value of bonds	L	**€10m**

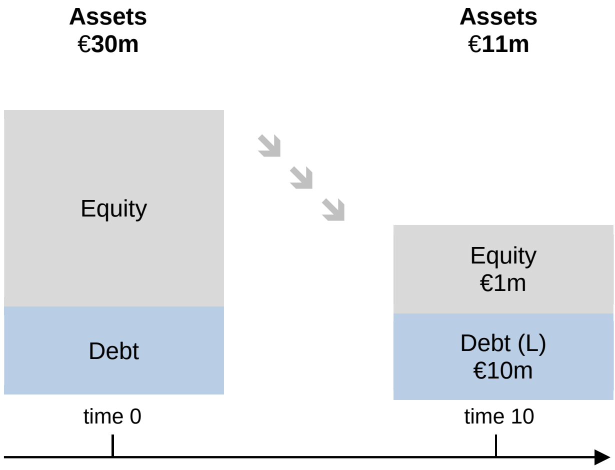

Figure 4: Total asset value movements when F(T) > L

Table 4: Parameters for firm at time T (Expected growth rate μ -15%)

Parameters at time T [F(T)<L]:		
Firm's total asset value	F(T)	**€7m**
Nominal value of bonds	L	**€10m**

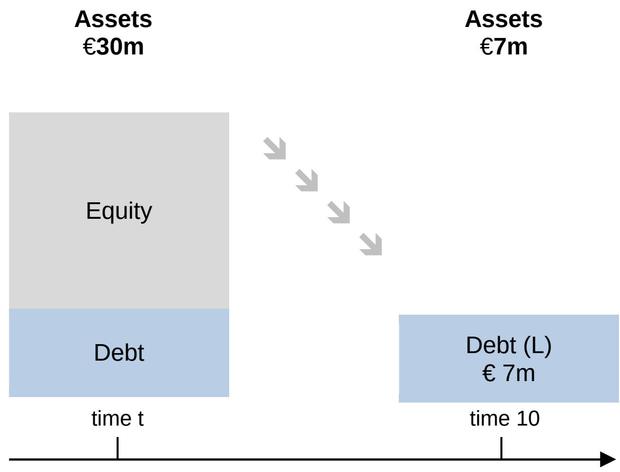

Figure 5: Total asset value movements when F(T) < L

Shareholders receive max[F(T)-L,0] at time T.
So they have a call option on the company's assets.
We can value this option using the Black-Scholes model.
We can then find the bond price and the credit spread.

Value of equity $= \mathbf{c}_{t} = F(t) \times \phi(d_1) - L \times e^{-r(T-t)} \times \phi(d_2)$

$\mathbf{c}_{t} = 30 \times \phi(1.7378) - L \times e^{-0.01(10)} \times \phi(0.7891) =$ **€21.664m**

Probability of Default $= P[F(T) < L \mid F(t)]$ **=** $1 - \phi(d_2)$

$1 - \phi(0.7891)$ **= 22%**

Value of the put option $= \mathbf{p}_{t} = L \times e^{-r(T-t)} \times \phi(-d_2) - F(t) \times \phi(-d_1)$

$= \mathbf{p}_{t} = 10 \times e^{-0.01(10)} \times \phi(-0.7891) - 30 \times \phi(-1.7378) = 0.71191$

Value of the debt = 10 x exp^-0.01(10) **–** 0.71191 = **€8.336m** Or

= F(t) – Value of equity = **€30 - €21.664m** = **€8.336m**

Credit spread risk $= Y^d(t, T) - Y(t, T)$ where $Y^d(t, T) = -\frac{\ln D(t,T)}{T-t}$

Corporate debt price (per €100) = $D(t, T)$ = Value of debt/Nominal value of debt *100

= $D(t, T)$ = **8.336m/10m*100 = €83.36**

Defaultable yield-to-maturity = Y^d (t, T) = - $\frac{\ln 83.36}{10}$ = **1.82%**

Credit spread risk = 1.82% - 1% = **0.82%**

Table 5: Merton model results for pricing defaultable (zero-coupon) bonds

Results:	
Value of equity	€21.664m
Probability of default	22%
Value of debt	€8.336m
Corporate debt price (per €100)	€83.36
Defaultable yield to maturity	1.82%
Credit spread risk	0.82%

1.2.4.4 Drawbacks

There is a trade-off between realistic assumptions and ease of implementation, Merton's model opts for the latter one. The Merton model is very simple and less realistic; therefore, it cannot directly be used to price credit risk. In this section, further critics and extensions of the model are discussed:

1.2.4.4.1 Early default: The assumption that default can occur only at the maturity of the debt is not appropriate, as it rules out the possibility of an early default. It is however an advantage that the model is simple which allows us directly to apply the option theory of Black and Scholes (1973). Nevertheless, at the same time, this is a less realistic assumption as default may occur at any time on or before the maturity date of the debt. Merton's model however gives an insight into the nature of default and interaction between debtholders and shareholders.

1.2.4.4.2 Corporate debt: The classical Merton model assumes that the firm only issues a single zero-coupon bond. This is however not realistic as the firm's capital structure is usually more complicated and is a mixture of different financial instruments. Later Merton himself has proposed extensions to the model where some of the assumptions were relaxed.

Several authors have modified the original approach to cover the following real-life features of corporate debt including:

- Corporate coupon-bonds, Geske (1977, 1978)
- Debt structure (short-term and long-term debt), Vasicek (1984)
- Bond covenants (priority rules, payment schedules, sinking fund, etc.) Ho & Singer (1982, 1984)

- Floating-rate debt, Cox et al. (1980)
- Duration of defaultable zero-coupon bond, Chance (1990) and others [6, p. 58 Ch.2]

Geske's (1977) paper focused on valuing compound options to the risky coupon bond, issued by most corporations. This was later revised in the paper by Geske (1978) [36], where the method is generalised to value many corporate liabilities.

A credit valuation model was presented in paper by Vasicek (1984) [37]. It defines a firm as an entity consisting of its assets (its ongoing business). Assets reflecting not only current liabilities and short-term debt but also long-term debt and equity.

Ho & Singer (1982) [38] in their article examine the effect of alternative bond indenture, and in the Ho & Singer (1984) [39] article, they examined specifically the effects of sinking-fund provisions in corporate bond indentures. Sinking fund provisions are terms, which provide for periodic retirement of a proportion of the corporate debt prior to maturity.

As the name suggests the paper by Chance (1990) [40] examines the duration of a zero coupon bond subject to default risk, applying a contingent claims approach.

1.2.5 The first-passage-time model by Black and Cox (1976)

The original Merton model's assumption that the default may only occur at the maturity of the debt is restrictive and unrealistic.

1.2.5.1 Introduction

"First-passage-time models (FPM)" model the first time when the firm's asset value F(t) hits a lower barrier, and hence allow the default time to occur during the entire lifetime or on the maturity date of the reference security (rather only at the maturity date). First-passage-time models also allow us to model securities with infinite maturities.

The FPM approach was originally developed by Black and Cox in 1976. The first-passage-time models broadly fall into two categories: those that assumes deterministic short-term interest rates and those that assumes stochastic interest rates. [6, pp. 65 & 71 Ch.3]

The simplest first-passage-time model considers, as in the previous section, that the total value F(t) follows a geometric Brownian motion under the risk-neutral probability measures:

$$d\,F(t) = r\,F(t)\,dt + \sigma\,F(t)\,d\,\widetilde{W}_t, \qquad F(0) > 0$$

In addition, the firm is defaulted if the firm's asset value reaches a prespecified lower level of threshold.

Let us assume K denotes the lower barrier. The lower barrier K can be exogenous, constant, time dependent, stochastic or endogenously derived. Under the Black and Cox model, the assumption is that K is a time dependent threshold denoted by K(t).

Let us do a brief work with the constant threshold K i.e. $K > 0$. If at time t, for $0 \leq t \leq T$, the default has not been triggered i.e. $F(t) > K$, then the time of default might be some time denoted as s, for $s \geq t$ given by [10, pp. 7–10]:

$$T = \inf \{ s \geq t \mid F(s) \leq K \}$$

1.2.5.2 Drawbacks

These models are more complex which makes it difficult to obtain the set of required results. The difficulty is even increased when a stochastic interest rate is considered. For an extensive review of FPM see Bielecki and Rutkowski (2002, Chapter 3) and reference therein [6, Ch. 3], [10, pp. 11–14]

1.2.6 Further developments

There have been further developments in the area of "first-passage-time models". The most crucial study features of these are outlined below:

1.2.6.1 Liquidation Process Model (LPM)

In the first-passage-time model, the default event is defined to have occurred when for the first time the firm's asset value goes lower than some specified threshold. This assumes that the firm is liquidated immediately after the default event. In contrast, a firm is not usually liquidated immediately after the default event; rather it is usually a long-lasting liquidation process.

The first-passage-time model was extended to a model labelled as liquidation process model (LPM) to account for the possibility of a lengthy liquidation process.

The LPM model allows for a long-lasting liquidation process. Under the LPM model, the default event is considered as the beginning of a liquidation process rather assuming the firm is liquidated immediately after the default event. After the liquidation process is completed, it might or might not cause the liquidation. [10, pp. 17–21]

1.2.6.2 Default correlation

The structural approaches explained in previous sections were primarily focused for pricing and modelling of credit risk for a specific single firm. The more extended structure approach can also be used between firms where there is a default dependence between the firms. [10, p. 23]

1.2.6.3 Other developments

In this section, further areas of developments of the structural approaches are discussed:

1.2.6.3.1 State variables: It is not uncommon in the first-passage-time modelling to define the default time not in terms of the firm's value, but rather in terms of some other state variables, which reflect the economic fundamentals (such as: the firm's operating earnings, the price of the firm's product etc.). Mella-Barral and Tychon (1996) consider a generic first-passage-time model with a state variable that follows a geometric Brownian motion. [6, p. 88 Ch.3]

1.2.6.3.2 Strategic debt service: It is widely recognized in financial literature that the presence of bankruptcy/liquidation costs may induce creditors to accept deviations from contractual payments, rather than to force the firm's bankruptcy. Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1997) were the first to account for this feature by incorporating the possibility of the debt's renegotiation in case of distress; in other terms they explicitly deal with strategic debt service (recent papers in this vein include Leland (1998) [41], Mella-Barral (1999) [42], and Ericsson (2000)). [6, p. 88 Ch.3].

1.2.7 Structural approach: Criticisms

1.2.7.1 Firm's value

A criticism against the structural approach concerns the assumption that the structural value of the firm can be directly observed at any time. A 1997 study by Duffie and Lando states that the firm value cannot be directly seen by investors as they only get periodic reports. Duffie and Lando commented that in such a situation, modelling of the value of the firm is necessary and can be done through intensity-based approaches. [6, p. 31 Ch.2]

1.2.7.2 Traded assets:

Another perceived flaw of structure approach that it assumes that the debt and equity of the firm are frictionless tradable assets. In the paper by Ericsson and Reneby (1998), is argued that, in theory, it is enough to postulate that at least one of the firm's securities (e.g. common stock) is traded, and that the market is complete; in the sense that the firm's value can be replicated by dynamic trading in some securities. [6, p. 64 Ch.2]

1.3 REDUCED-FORM APPROACH

The reduced-form approach is another quantitative methodology for pricing and modelling of credit risk. The reduced-form approach aims to study the market data rather than the company data itself.

1.3.1 Introduction

In this approach the value of a firm and its capital structure are not modelled at all. This approach studies market data and specifically, it examines the credit ratings of bond issuing companies. These credit ratings are issued by credit rating agencies.

These models are called "reduced-form" as they only use the credit ratings and hence ignore any other information relating to the company issuing a bond. The reduced-form approach is based on a continuous-time model where the movements between different states are modelled using jump-rates called transitions intensities. We will look at the two important models namely: The two-state model and the Jarrow-Lando-Turnbull (JLT) model (1997).

1.3.2 Credit rating classes (or credit grades)

A firm´s credit rating is a measure of its performance and propensity to default. A credit rating is basically an assigned credit class (or credit grade) given to the major corporations based on their financial performance by credit rating agencies such as Moody`s Investors Service or Standard and Poor`s Corporations For example: On Standard & Poor's scale, companies are given one of these grades: AAA, AA, A, BBB, BB, B, CCC, CC, C and D (except D, other ratings can be fine-tuned further with plus (+) or minus (–) sign to show relative standing within the rating categories) [3]. These agencies have also periodic schedules to make changes to the previously assigned ratings based on recent performance. These periodic updates may result in credit ratings being downgraded, upgraded or held.

1.3.3 Credit event and recovery rates

Under the reduced-form models, "credit events" are specified in terms of a jump process. A credit event occurs when the credit rating of the bond issuing company is downgraded below a certain threshold. In such an event, the likelihood increases of interest re-payments and/or principal re-payments being reduced, rescheduled, or not made at all. The recoverable fraction of the bond is known as "recovery rate".

1.3.4 Past model of development on reduced-form approach

[3] Please note these are the "I. General-Purpose Credit Ratings" obtained from republished article on Sept 18, 2019 by Standard & Poor`s Corporations https://www.standardandpoors.com/en_US/web/guest/article/-/view/sourceId/504352

This section is devoted to the reduced-form approach for credit risk, as it looks at the movements on the credit ratings on the bonds over time.

The reduced-form approach was initiated by:

- Pye (1974) and
- Litterman and Iben (1991)

Defaultable bonds' premium yield can be divided into a default premium and a risk premium. The major purpose of the article by Pye (1974) [43] is to provide a means for deriving the default premium. In the paper by Pye, a simple formula is derived for calculating the default premium in terms of parameters describing future anticipation about bond default experience. In addition, this article presents some historical data on default experience of bonds with different ratings that may be helpful in predicting future default experience.

Litterman and Iben (1991) [44] are "presenting a model that, for the first time, recognizes the term structure of credit risk and uses it to value callable corporate bonds". The value of a corporate bond depends on the main three components: the term structure of interest rates, embedded option value of callable bonds, and credit risk. The term structure of credit risk is an important component of a corporate bond valuation. This article explained how to measure the term structure of the credit risk component and how to use it.

Moreover, this approach was then formalized independently by:

- Lando (1994)
- Jarrow and Turnbull (1995) and
- Madan and Unal (1998).

David Lando in 1994 [45] presented an intensity-based approach to modelling default risk and the term structure of credit risky bonds. The starting point is the modelling of prices of bonds issued by a party who may default. The modelling framework is similar to that of Litterman and Iben (1991), Jarrow Lando and Turnbull (1993), Jarrow and Turnbull (1992), and Madan and Unal (1993). In that framework the event of default and recovery rate is not described explicitly.

Jarrow and Turnbull's 1995 [46] paper provides a new methodology for pricing and hedging derivative securities involving credit risk. Two sources of credit risk are identified and analysed. The first is where the assets underlying the derivative security may default, paying off less than promised. The second is the credit risk introduced by the writer of the derivative security, who may also default.

Madan and Unal 1998 [47] models characterize the default risk as composed of arrival and magnitude risks. Arrival is associated with the timing of the event, which is an uncertain future time. Once the event is triggered, the magnitude depends on the value of assets relative to the value of creditor claims. In this paper, a model is introduced where the two default components are explicitly priced as if they were traded in the futures market and the spot price of risky debt is derived consequently.

There has been further development in papers by:

- Hull and White (1995)
- Das and Tufano (1996)
- Duffie et al. (1996)
- Schonbucher (1996)
- Lando (1997, 1998)
- Monkkonen (1997)
- Lotz (1998, 1999)
- Collin-Dufresne and Solnik (2001)
- Bakshi, Madan and Zhang (2001), and others

Das and Tufano in their 1996 paper [48] develop a model for the pricing of credit-sensitive debt contracts. Such debt contracts, including credit-sensitive notes (CSNs), spread adjusted notes (SPANs), and floating rate notes (FRNs), adjust investor's exposures to three risks: interest rate risk, changes in credit risk caused by changes in credit ratings of the issuer of the debt, and changes in credit risk caused by changes in spreads on the debt, even when rating has not changed. The model contains all the three risks with special emphasis on credit risks and incorporates a decomposition of credit spreads into two stochastic elements: the default process and the recovery process in the event of default.

The article by Duffie et al. (1996) [49] presents a model where the fractional recovery upon default and the default hazard rate of defaultable securities may depend on the market value of security itself. It may possibly also depend on the market value of the other securities issued by the firm.

Lando (1997) [50] provided a Markov model for the term structure of credit-risk spreads which was based on the Jarrow and Turnbull (1995) model. This model was following a discrete state space Markov chain in credit ratings.

Collin-Dufresne and Solnik (2001) [51] proposed a model of the default risk imbedded in the swap term structure, that is able to explain the LIBOR – swap spread. It is also argued that the swap contracts are default risk-free, whereas corporate bonds carry default risk.

The paper by Bakshi, Madan and Zhang in (2001) [52] model recovery rate and quantify its impact on defaultable debt. They investigated the model implications using a cross-section of coupon bonds rated BBB by S&Ps. The effect of recovery rate on the price of defaultable debt is analysed. The study was based on the important assumptions, that when aggregate defaults tend to rise, actual recovery rates often decline; other words recovery rates are negatively associated with default probability. It also assumed that interest rate process is governed by single-factor diffusion.

In the next section we have separated the reduced-form models into two sections namely: Intensity-Based Models and Credit Migration Models: [6, pp. 26-28 Ch.1 & VIII Preface]

1.3.5 Credit risk modelling: Intensity-Based Models vs Credit-Migration Models

There has been a separation of the reduced-form approach in two sections: Intensity-Based Models and Credit-Migration Models:

1.3.5.1 Intensity-Based Models

An intensity-based model is a particular type of continuous-time reduced-form model, which are only concerned with default time. It typically models the "jumps" between different states using transitions intensities.

This approach was initiated by Pye in 1974 with the main emphasis in modelling the random time of default time and it was further developed by:

- Ramaswamy and Sundaresan (1986)
- Litterman and Iben (1991)
- Jarrow and Turnbull (1995)
- Duffie et al. (1996)
- Schonbucher (1996, 1998a, 1999b)
- Lando (1997, 1998)
- Monkokonen (1997) and
- Madan and Unal (1998), and others

The value of floating-rate instruments is more stable than that of fixed-rate investments. Ramaswamy and Sundaresan (1986) [53] developed a framework for valuing floating-rate notes.

Schonbucher (1996, 1998a, 1999b) analyses a model of the development of the term structure of defaultable interest rates that is based on a multiple-defaults model.

1.3.5.2 Credit-Migration Models

The reduced-form model has been extended in many ways. Credit-Migration Models are reduced-form models, which model the credit risk using credit ratings, the worst credit rating state being default. The likelihood of jumps between the states is governed by transition rates or intensities.

These models can further be categorised in two types, i.e. Single Credit Rating Models and Multiple Credit Ratings Models:

1.3.5.2.1 Single Credit Rating Models: These models are classified as traditional models where movements between different credit rating classes prior to default are not allowed i.e. only one other single class, namely "default class" exists; and

1.3.5.2.2 Multiple Credit Ratings Models: These models are more recent models, which allow movements between credit rating classes i.e. there are more classes apart from the single "default class". Reference for such models include:

- Das and Tufano (1996)
- Jarrow et al. (1997)
- Arvanitis et al. (1998)
- Kijima (1998)
- Kijima and Komoribayashi (1998)
- Thomas et al. (1998)
- Duffie and Singleton (1999)
- Huge and Lando (1999)
- Bielecki and Rutkowski (2000)
- Lando (2000a) and
- Schonbucher (2000)

There is an extensive analysis available in the literature to use Markov chain models that incorporate a firm´s credit rating as an indicator of the likelihood of default. Such models can be used not only for describing the dynamics but also for valuing risky debt. Kijma's (1998) article uses Markov chain models for credit rating movements. For example, having a prior knowledge of the rating movements leads to the prediction of the future rating movements.

Duffie and Singleton (1999) [54] presented new reduced-form models for the valuation of contingent claims subject to default risk. The existing available reduced-form model directly assumed the discounting of the defaultable bond at an adjusted short rate. The main important result of this article is to provide a particular reduced-form model that justifies this assumption.

Credit derivatives are financial instruments whose payoffs are linked to the credit characteristic of a reference asset´s value. Jarrow, Lando and Turnbull propose using a Markov chain model for valuing risky debt. The drawback of the model is that numerical problems may arise due to highly rated bonds having very low default probabilities within a period, say one year. In their paper, Kijima and Komoribayashi (1998) [55] propose a new risk premium adjustment to overcome the drawback of the Jarrow-Lando-Turnbull model.

In the next sections, we have emphasized the characteristic and shortcoming of two important models developed under the framework of reduced-form approach; Firstly, Sect 1.3.6 is entirely devoted to "The two-state model" from Intensity-Based Models. Sect. 1.3.7 presents "Jarrow-Lando-Turnbull (JLT) model" from *Multiple Credit Ratings Models,* which was published by Jarrow, Lando, and Turnbull in 1997. [6, pp. 26-28 Chapter1 & VIII Preface]

1.3.6 Intensity-based model: The two-state model

The two-state model is a simple example of an intensity-based model and as the name suggests has only two states: default and no default (usually the default class is an absorbing state). This model helps in pricing of corporate zero-coupon bonds (ZCB).

1.3.6.1 Introduction

This is explained below in three steps. In the first step, the probability that the bond does not default is explained, as this will be used for the pricing of ZCB in the final step.

- **Step 1: Estimation of the probability that the bond does not default in time n i.e. ${}_{n}P_0^{NN}$:**

To begin further with, such as a life can be in alive or move to dead state, we can denote the states that a corporate ZCB can possibly be into two states such as; Not defaulted (N) and Defaulted (D).

The λ (lambda) in the diagram below is the transition intensity under the real-world measure P between these two states, from N to D and is assumed to be the constant[4]. Hence, we note that this is a "two-state continuous-time Markov model". [56, Ch. 4–6]

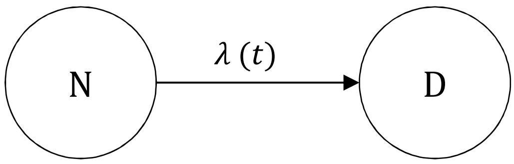

Figure 6: Two states model diagram

We start the derivation by defining the following notations:

i. ${}_{t+h}P_0^{NN}$ = Probability that the bond does not default in time period from 0 to t+h where h is a very small amount of period. Timelines are as follows i.e. $0 < t < n$

ii. ${}_{t}P_0^{NN}$ = Probability that the bond does not default in time period from 0 to t.

iii. ${}_{h}P_t^{NN}$ = Probability that the bond does not default in time period from t to t+h.

$$ {}_{t+h}P_0^{NN} = {}_{t}P_0^{NN} \times {}_{h}P_t^{NN} $$

iv. $h\lambda(t)$ = Probability that the bond does default in time period h. (h is very small period)

[4] The assumption can also be that λ (lambda) is a deterministic function of time t. In this model, transition intensity λ(t) is not allowed to vary over time and is constant. Therefore, this is a "time-homogeneous Markov model". [56, Ch. 5 & 6]

v. ${}_{h}P_{t}^{NN}$ = [1 – h λ (t)] i.e. [1–Probability that the bond does default in time period h]
vi. o(h) = It is a polynomial with h^2 and higher power of h as it makes the difference of approximation and actual probability

$$ {}_{t+h}P_0^{NN} = {}_{t}P_0^{NN} \text{ x } [1 - h\lambda(t)] + o(h) $$

$$ {}_{t+h}P_0^{NN} = {}_{t}P_0^{NN} - h\lambda(t) \times {}_{t}P_0^{NN} + o(h) $$

vii. Dividing equation by h

$$ ({}_{t+h}P_0^{NN} - {}_{t}P_0^{NN})/h = -\lambda(t) \times {}_{t}P_0^{NN} + o(h)/h $$

viii. When h tends to 0

$$ \lim_{h\to 0} o(h)/h = 0 $$

$$ d/dt\ {}_{t}P_0^{NN} = -\lambda(t) \times {}_{t}P_0^{NN} $$

ix. Dividing by ${}_{t}P_0^{NN}$

$$ d/dt\ {}_{t}P_0^{NN} / {}_{t}P_0^{NN} = -\lambda(t) $$

x. Taking the integral on both sides for till time n (n > t)

$$ \int_0^n d/dt\ {}_{t}P_0^{NN} / {}_{t}P_0^{NN}\, dt = -\int_0^n \lambda(t)dt $$

$$ \int_0^n d/dt\ \log {}_{t}P_0^{NN}\, dt = -\int_0^n \lambda(t)dt $$

$$ \log {}_{n}P_0^{NN} - \log {}_{0}P_0^{NN} = -\int_0^n \lambda(t)dt $$

xi. ${}_{0}P_0^{NN}$ is the probability of no default at time 0 i.e. 1 and hence log ${}_{0}P_0^{NN}$ = log 1 = 0

$$ \log {}_{n}P_0^{NN} = -\int_0^n \lambda(t)dt $$

xii. The probability that the bond does not default in n time i.e. ${}_{n}P_0^{NN}$

$$ {}_{n}P_0^{NN} = \exp\left(-\int_0^n \lambda(t)dt\right) $$

- **Step 2: Estimation of a corporate ZCB price at time t:**

Let us say B (t, T) is the price at time t of a corporate ZCB that is due to be paid an amount of 1 at time T. Firstly, we define the assumptions that will be used for a ZCB pricing formula:

i. B (t, T) = ZCB price at time t.
ii. The pricing needs to be done at risk-neutral probability measure under Q [5] so that we will do the discounting using a risk-free force of interest rate which is assumed to be deterministic i.e. r (t) = r is constant for all t.

[5] We have followed the "The Actuarial Education CT8, IFOA Study Material" to define "P as under the real-world probability measure" and "Q as under the risk-neutral probability measures" [5, Ch. 18]

iii. r (t) = r = Risk-free force of interest rate.

iv. Due to assumption (ii), the transition intensity also needs to be modelled with a risk-neutral world measure under Q. If λ(t) used above is the transition intensity under real-world probability measure P, we can use the idea of equivalent probability measures [6] to say that is the transition intensity in the risk-neutral world $\widetilde{\lambda}$(t) , is equal to λ (t).

v. α = the recovery rate. α is the recoverable amount in the occurrence of default. Let us say α is 80%, this example means in the event of defaults only 80% principal payment will only be recovered and in other words they will be reduced by 20% i.e. 1 – α.

- **Step 3 (a): ZCB Pricing formula denoted as B (t, T):** (when λ (t) is deterministic function of t i.e. λ (t) = λ)

i. Let N(t) be number of default at time t.

$$N(T) = 1 = Defaulted$$

$$N(T) = 0 = Not\ Defaulted$$

$$P[N(T) = 1|N(t) = 0] = 1 - \exp\left(-\int_e^T \lambda(t)dt\right)$$

$$P[N(T) = 0|N(t) = 0] = \exp\left(-\int_0^T \lambda(t)dt\right)$$

B (t, T) $= \mathbf{exp}\ (-\mathbf{r}(T-t))\ \mathbf{E_Q}\ [\text{payoff at time T} \mid F_t]$

$= \exp(-r(T-t))\ E_Q[\,1 \times \exp(-\int_t^T \widetilde{\lambda}(s)\,ds) + \alpha \times (1 - \exp(-\int_t^T \widetilde{\lambda}(s)\,ds)\,]$

$= \exp(-r(T-t))\ E_Q[\,(1-\alpha)\exp(-\int_t^T \widetilde{\lambda}(s)\,ds) + \alpha)]$

$= \exp(-r(T-t))\ E_Q[\,1 - (1-\alpha)(1 - \exp(-\int_t^T \widetilde{\lambda}(s)\,ds)\,)\,]$

ii. Since α is a constant value and $\widetilde{\lambda}$(s) is a deterministic function of s. The pricing formula is as:

B (t, T) $= \mathbf{exp}\ (-\mathbf{r}(T-t))\ \{\mathbf{1 - (1-\alpha) \times (1 - exp}\ (-\int_t^T \widetilde{\lambda}(s)\,ds))\}$

This is explained here as follows:

i.	$\exp(-\int_t^T \widetilde{\lambda}(s)\,ds)$	= Probability under measure Q that bond does not default for time t to T. This was derived in Step 1 above but now we are instead using a lambda with tilde $\widetilde{\lambda}$(s) i.e. transition intensity under measure Q. This is the risk-neutral world probability measure, as this would provide a risk-neutral probability instead.
ii.	$(1-\exp(-\int_t^T \widetilde{\lambda}(s)\,ds))$	= A risk-neutral probability that bond does default for time t to T.
iii.	α	= In occurrence of default the recoverable amount.
iv.	(1 – α)	= In occurrence of default the unrecoverable amount i.e. a lost amount
v.	$(1-\alpha) \times (1-\exp(-\int_t^T \widetilde{\lambda}(s)\,ds))$	= Expected default loss in risk-neutral world. This expression therefore, now provides an expected loss amount x probability of default i.e. Expected default loss in risk-neutral world.
vi.	$1-(1-\alpha) \times (1-\exp(-\int_t^T \widetilde{\lambda}(s)\,ds))$	= 1 – Expected default loss = Expected overall payoff

[6] For the probability measures P and Q to be equivalent, the set of times at which transitions are possible must be the same under each measure.

vii. **exp** $(-\mathbf{r}(\boldsymbol{T}-\boldsymbol{t}))$ **{1 – (1 – α) x (1 – exp** $(-\int_t^T \tilde{\boldsymbol{\lambda}}(\mathbf{s})\,\mathbf{ds}))$**}**

= This expression B (t, T), therefore, finally provide the discounted value Expected overall payoff on ZCB i.e. equal to a price of a ZCB at time t.

- **Step 3 (b): ZCB Pricing formula denoted as B (t, T):** (when λ (t) is stochastic function of t)

Please note, we assumed earlier in Step 3 (a) that λ (t) was a deterministic function of t. One can also assume that λ (t) is a stochastic function of t i.e. λ (t) will vary over time (i.e. time-*in*homogeneous Markov model). Then there will be the need to amend the formula by putting the equation in the black box [ ] inside the expectation under risk-neutral world such as: $\mathbf{E_Q}$ **[B (t, T) |** $\mathbf{F_t}$**]** where $_Q$ denotes under risk-neutral probability measure. [5, pp. 8-12 Ch. 18]:

$$B(t,T) = \exp(-r(T-t))\, E_Q[\text{payoff at time T} \mid F_t]$$

$$\mathbf{B(t,T) = \exp(-r(T-t))\, E_Q[1 - (1-\alpha) \times (1 - \exp -(\int_t^T \tilde{\lambda}(s)\,ds)) \mid F_t]}$$

1.3.7 Credit-Migration Model: Jarrow-Lando-Turnbull (JLT) model (1997)

Jarrow-Lando-Turnbull (JLT) model is an example of a multiple credit rating credit risk model. It is a multi-state model and mainly in use to predict the defaulting of bonds where states are usually credit ratings. Jarrow, Lando and Turnbull published the JLT Model in 1997.

The JLT model for credit ratings is an extension of the two-state model as it allows for more than two states rather than the simplistic default/no default model and hence it is more realistic. In this model, there are n-1 credit ratings plus a default credit rating state. [5, pp. 13-17 Ch. 18]

1.3.7.1 Introduction

Similar to the two-state model, JLT model helps in pricing of a corporate zero-coupon bond (ZCB), this is explained below in two steps but in first step, the probability that the bond does not default is explained, as this will be used for the pricing of ZCB in the second step.

- **Step 1: Estimation of the probability that the bond does not default in time:**

We begin with a simple example of JLT model with three states. We have defined the states as credit ratings states that a ZCB can possibly take such as; A, BBB and BB.

The λ (lambda), σ (sigma), μ (mu) and v (nu) in the diagram below are the transition intensities between these credit rating states and assumed to be constant. The assumption that transition intensities are constant is only for the simplicity purpose of the example. They can be deterministic or stochastic functions of time t:

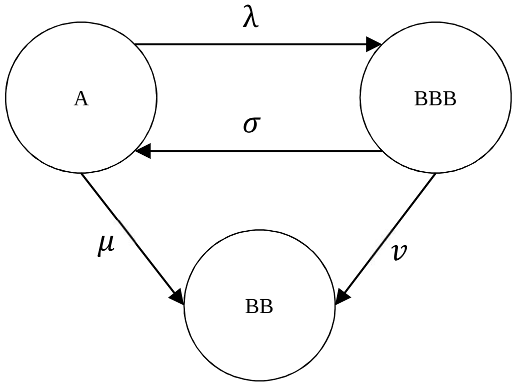

Figure 7: Three states diagram in JLT model

We approach the derivation by defining the followings notations:

i.	A, BBB	= They are called as the investment grade ratings.
ii.	BB	= This is called as junk grade credit rating and is an absorbing state.
iii.	λ, σ, μ, v	= Transition intensities for moving from one state to another and assumed to be under real-world measures P.
iv.	$\tilde{\lambda}$, $\tilde{\sigma}$, $\tilde{\mu}$, $\tilde{v}$	= Transition intensities for moving one state to another and assumed to be under the risk-neutral measures Q.

Let $\tilde{\Lambda}$ be the matrix of constant transition intensities (the "generator" matrix) under the risk-neutral world measures:

$$\tilde{\Lambda} = \begin{matrix} \\ A \\ BBB \\ BB \end{matrix} \begin{matrix} \begin{matrix} A & \quad BBB & \quad BB \end{matrix} \\ \begin{bmatrix} -(\tilde{\lambda}+\tilde{\mu}) & \tilde{\lambda} & \tilde{\mu} \\ \tilde{\sigma} & -(\tilde{\sigma}+\tilde{v}) & \tilde{v} \\ 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

In the two-state model of Sect 1.3.6 the probability formula was of the form as:

$$\mathbf{{}_nP_0^{NN} = exp}\ (-\int_0^n \boldsymbol{\lambda}\,(\mathbf{t})\mathbf{dt}\,)$$

In this case, we have the similar result. Let us define the matrix $\tilde{\Pi}\ (t,T)$ of transition probabilities. The matrix of probabilities under risk-neutral world similarly is:

$$\tilde{\Pi}(t,T) = \exp\ (\int_t^T \tilde{\Lambda}\ (\mathrm{s})\mathrm{ds}\ )$$

Since the assumption that transition intensities are constant:

$$\tilde{\Pi}(t,T) = \exp\ \{\tilde{\Lambda}\ (\mathrm{T}-\mathrm{t})\}$$

To use the Taylor series expansion of the exponential:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \qquad -\infty < x < \infty$$

Therefore $\widetilde{\Pi}\ (t, T)$:

$$\widetilde{\Pi}(t,T) = \mathrm{I} + \frac{\widetilde{\Lambda}\,(\mathrm{T}-\mathrm{t})}{1!} + \frac{\widetilde{\Lambda}\,(\mathrm{T}-\mathrm{t})^2}{2!} + \frac{\widetilde{\Lambda}\,(\mathrm{T}-\mathrm{t})^3}{3!} + \cdots$$

Where I represents an identity matrix.

The denoted probability as $\tilde{p}ij$ of moving between states i to j in time from t and T under the risk-neutral world measures.

$$\widetilde{\Pi}(t,T) = \begin{matrix} & A & BBB & BB \\ A \\ BBB \\ BB \end{matrix} \begin{bmatrix} \tilde{p}11 & \tilde{p}12 & \tilde{p}13 \\ \tilde{p}21 & \tilde{p}22 & \tilde{p}23 \\ \tilde{p}31 & \tilde{p}32 & \tilde{p}33 \end{bmatrix}$$

- **Step 2: Thus, ZCB Pricing formula: X(s) is a current state:**

B (t, T, X(s)) = exp $(-\mathbf{r}(\boldsymbol{T}-\boldsymbol{t}))$ **{1 – (1 – α) x $\tilde{\mathbf{p}}$ij}**

For example, for the probability of default (i.e. rating BB) from current state A will be $\tilde{p}13$ and hence:

B (t, T, A) = exp $(-\mathbf{r}(\boldsymbol{T}-\boldsymbol{t}))$ **{1 – (1 – α) x $\tilde{\mathbf{p}}$13}**

This is explained here as follow:

i.	$\tilde{p}13$	= A risk-neutral probability that bond does default from time credit rating A.
ii.	α	= In default the recoverable amount.
iii.	(1 – α)	= In default the unrecoverable amount i.e. a lost amount.
iv.	(1 – α) x $\tilde{p}13$	= Expected default loss in risk-neutral world. This expression therefore, now provides an expected loss amount x probability of default i.e. Expected default loss in risk-neutral world.
v.	1 – (1 – α) x $\tilde{p}13$	= 1 – Expected default loss = Expected overall payoff
vi.	**exp** $(-\mathbf{r}(\boldsymbol{T}-\boldsymbol{t}))$**{1 – (1 – α) x $\tilde{\mathbf{p}}$13}**	= This expression therefore finally provides the discounted expected value of Expected overall payoff on ZCB i.e. equal to a price of a ZCB at time t.

1.3.8 Other developments

1.3.8.1 Hybrid methodologies

The hybrid approach is a combination of the basic idea from both structural and reduced-form approaches. This is done by directly linking the hazard rate of default to the current value of the firm's assets (or firm's equity). Reduced-form models with this specific feature are referred to as hybrid models. [6, pp. 259–263, Ch.8]

1.3.8.2 Beyond hazard rate

Brody, Hughston, and Macrina (2007) [57] present a new approach to credit risk modelling. Default events are associated directly with failure of obligor to make agreed contractual payments. In other words, a model is presented to consider an objective measure of default, for example missing a coupon payment or principal payment.

1.3.8.3 Random recovery rate

The paper by Millossovich (2002) [58] extends the model of Jarrow, Lando and Turnbull, allowing for a stochastic recovery rate.

1.3.9 Reduced-form approach: Criticism

The main concern is the assumption that the credit quality of corporate debt can be quantified by a finite number of disjoint credit rating classes (grades). The assumption that the fall in credit rating always predicts a default event is not realistic. Another perceived drawback of the reduced-form approach is that it is not well suited to model the rise and fall of credit spreads.

2 – STRUCTURE: CATASTROPHE BOND

2.1 INTRODUCTION TO CATASTROPHE BOND

Global warming is happening, and there is an increased risk for natural disasters to happen. Natural disasters such as hurricane, earthquake, flood, tsunami. Catastrophe bonds were first issued in the mid 1990's and are risk-linked securities, which act as a protection for sponsors (often insurance or reinsurance companies) against such natural disasters. Catastrophe bonds transfer a specified risk from the sponsors to its investors.

2.1.1 Definition

"Catastrophe bonds, also called cat bonds, are an example of insurance securitization, creating risk-linked securities which transfer a specific set of risks (typically catastrophe and natural disaster risks) from an issuer or sponsor (ceding company) to capital market investors.

In this way, the investors take on the risks of a catastrophe loss or named peril event occurring in return for attractive rates of investment return. Should a qualifying catastrophe or named peril event occur, the investors will lose some or all of the principal they invested and the issuer (usually an insurance or reinsurance company, but sometimes a corporate or sovereign entity) will receive that money to cover their losses." [59]

2.1.2 Bond vs catastrophe bond

Insurers typically use catastrophe bonds as an alternative to traditional catastrophe reinsurance. They are structured as other ordinary types of bond but attached with a trigger clause that is linked to specified natural catastrophes. If a catastrophe has occurred, the investors receive reduced coupon and/or principal payments, for the sponsors to be able to pay their losses incurred. In other words, catastrophe bonds are a bet against natural disasters.

These unexpected natural disasters in nature have a low frequency but a very high severity. Due to such risky nature, neither insurance companies are willing to take up the risk nor do the normal re-insurance arrangements offer adequate protection. Therefore, the insurance and reinsurance industry began to look for alternative methods to hedge their risks, and catastrophe bonds were born.

Catastrophe bonds are uncorrelated with the ordinary bond market and they give a higher yield. If the specified disasters do not occur the investors will receive their scheduled principal and coupon payments. It means investors can add expected investment return to their portfolio without increasing volatility to the same degree. Catastrophe bonds are often rated by rating agencies such as Standard & Poor's, Moody's, or Fitch Ratings.

Some of the major catastrophe events that struck in the United States, such as Hurricane Katrina, made a property damage of estimated at $125 billion (USD) in 2005 and about 1,833 fatalities (directly and indirectly), this being as the costliest Atlantic tropical cyclone on record and it was roughly four times the damage estimated by Hurricane Andrew in 1992. Other severe convective storms, damaging winds, tornadoes and earthquakes (such more disasters) cause billions of dollars of damage each year and heavy losses to insurers and reinsurers. [60], [61]

2.1.3 Catastrophe bond structure and triggers

The figure below shows a typical catastrophe bond structure, including triggers and where the capital flows from one party to another. This is further discussed into the following steps:

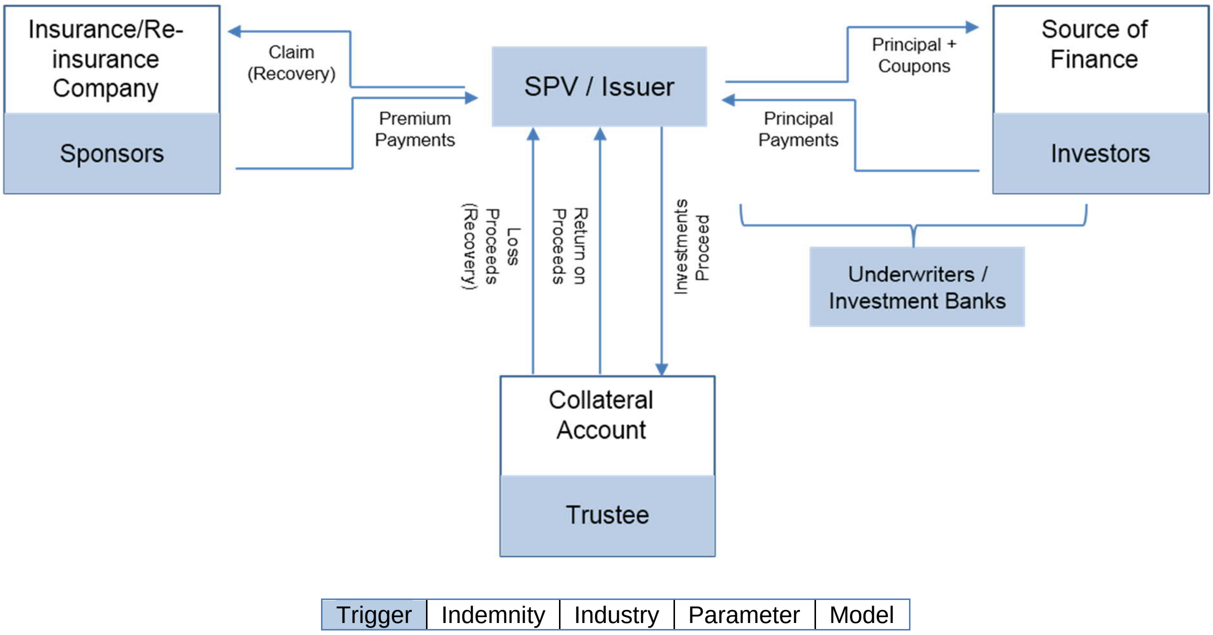

Figure 8: Catastrophe bond transactions structure and triggers

2.1.3.1 Catastrophe bond structure

The sponsor (often an insurance or reinsurance company) has provided insurance to its policyholder and are seeking to secure (re-insure) themselves for the arrival of claims due to unexpected natural disasters. This has been explained in previous sections that these natural disasters have very high economic consequences and therefore the protection is of utmost importance.

The catastrophe bonds are issued by a special purpose vehicle (SPV) rather than the sponsor itself. The sponsor constructs the SPV and SPV issue catastrophe bonds to investors. ("Special purpose vehicle is also called a special purpose entity (SPE), is a subsidiary created by a parent company to isolate financial risk"). Its legal status as a separate company makes its obligations secure even if the parent company goes bankrupt. In other words, "a special purpose vehicle/entity (SPV/SPE) is a bankruptcy-remote entity that is used to isolate or securitize assets and is often held off-balance sheet. It is also referred to as a bankruptcy-remote entity or variable interest entities since its operations are very limited". [62]–[64]

The SPV "underwrites" the issue to underwriter/investment banker rather directly issuing it to the investors, the underwriter/investment banker is held responsible for issuing catastrophe bonds to the investors. Underwriting is covered in more detail in CT2. [65, p. 6 Ch.6]

Investors pay the bond price and the money is then invested by the SPV in a collateral account, which typically invests in Treasury money market funds or supranational floating rate notes and earns a risk-free rate of return on the investment.

The sponsor and the SPV form a reinsurance contract. The sponsor is obliged to make regular premium payments to the SPV, which, in return pays regular promised coupons to its investors. The coupon payments made to the investors are generally a combination of the return received on the investments and the premium received from the sponsor.

Figure 9 shows P&C catastrophe issuance and capital outstanding for the years 1998 through year-end 2017 based on data from GC Securities* proprietary database [66].

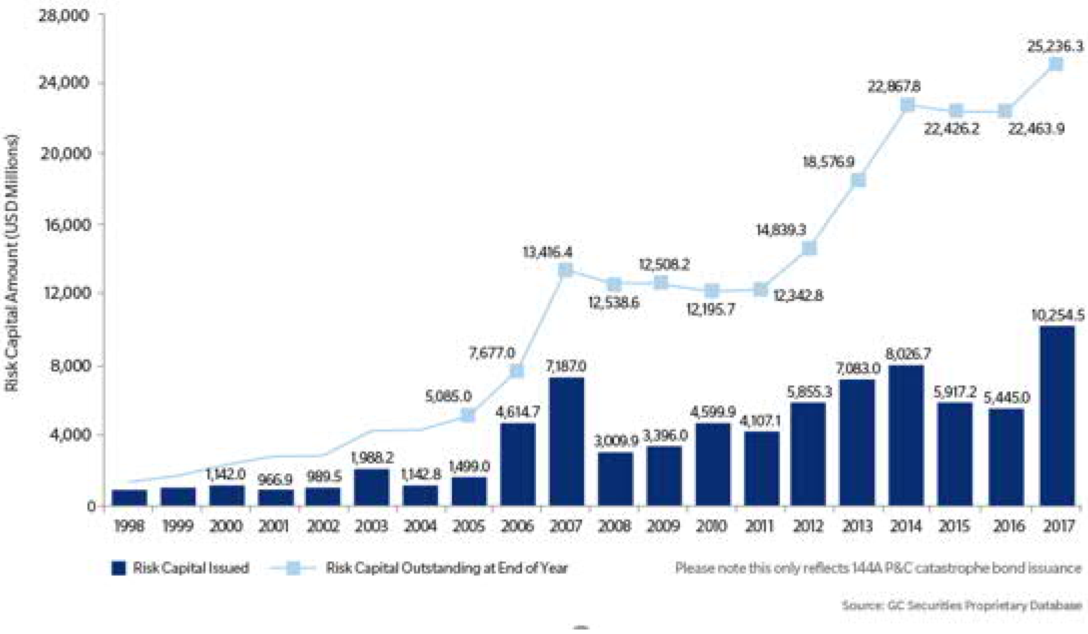

Figure 9: P&C Catastrophe bond issuance and capital outstanding 1998 – YE2017

*Guy Carpenter (GC) Securities offer securities and investments in the United States.

2.1.3.2 Catastrophe bond triggers

The key difference between an ordinary bond and catastrophe bond, that a catastrophe bond has a contractual trigger. That trigger depends on catastrophe events/event. If triggers are met, investors will suffer losses on their promised P&C payments (Principal and Coupon) to compensate the sponsors for the losses incurred by such catastrophic event. Therefore, triggers are very important.

There are majorly four common types of triggers for a catastrophe bond: indemnity, industry, parametric and model triggers. Indemnity trigger is categorised as an internal trigger, industry and parameters tend to be external, and model trigger is of a hybrid nature.

The most common triggers used are indemnity triggers, that take into consideration the actual internal losses incurred by sponsors. Industry trigger is external to the sponsors as it is calculated based on the aggregate losses of the industry and are compared to the threshold. The bond is triggered when the industry´s aggregate losses exceed a pre-determined threshold. The parameter trigger, similarly to the industry trigger, is external but is determined by the value of a certain output of the formula that is based on parameter inputs. For instance, the trigger could be based on the Richter Scale magnitude of an earthquake.

In the end, the model trigger combines a trigger event with a model of the correlation between the event and the actual losses of the sponsors (for instance, exposures by geographic area).

Figure 10 shows catastrophe bond and insurance-linked securities (ILS) by type of trigger from the Artemis Catastrophe Bond & Insurance-Linked Securities Deal Directory. With this chart you can see indemnity trigger are most prevalent in the outstanding catastrophe bond and ILS market [67].

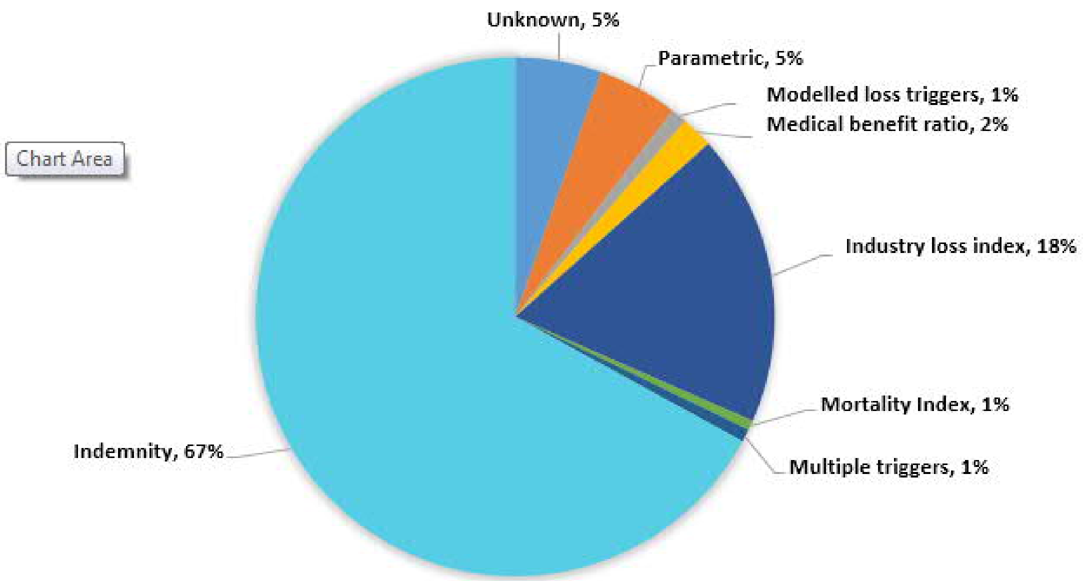

Figure 10: Catastrophe bond & ILS issuance by trigger type

In the next section, some catastrophe zero-coupon bond cash flow examples are presented.

2.1.4 Catastrophe bond: Cash flow example & credit rating methodology

Let us consider a simple catastrophe zero-coupon bond example with the following parameters assumptions:

Table 6: Parameter for zero-coupon catastrophe bond

Parameters:		
Terms of zero-coupon bond	t	**3 years**
Principal amount	€	**100**
Price of bond	€	**85**
Risk-free rate	%	**2**

Scenario 1: No triggering event occurred.

The investor pays price of the bond at the beginning of the period and receives it at the end of the period, as there is no trigger occurred. The cash flow should be as follow:

Time (Years)	0	1	2	3
Cashflow (€)	-85	0	0	100

The net present value of the proceeding cash flow is:

$$PV = -85 + 100\left(1 + \frac{2}{100}\right)^{-3} = 9.23$$

Scenario 2: A triggering event occurred, where investor's recovery rate is 0%.

Suppose the event took place in year between 1 and 2, in that the investor suffer a loss of the principal amount completely. The cash flow should be as follow:

Time (Years)	0	1	2	3
Cashflow (€)	-85	0	0	0

The net present value of the proceeding cash flow is:

$$PV = -85 + 0\left(1 + \frac{2}{100}\right)^{-3} = -85$$

Scenario 3: A triggering event occurred, where investor's recovery rate is 40%.

Suppose the event took place in year between 1 and 2, in that the investor suffer a loss of its 60% of principal amount. The cash flow should be as follow:

Time (Years)	0	1	2	3
Cashflow (€)	-85	0	0	40

The net present value of the proceeding cash flow is:

$$PV = -85 + 40\left(1 + \frac{2}{100}\right)^{-3} = -47.31$$

We can notice that the net present values vary enormously by catastrophic event scenarios, that explains the riskiness of these bonds and the importance of a pricing model. Currently, catastrophe bonds pay a higher risk premium compared to similarly rated corporate bonds.

A typical corporate bond is rated based on its probability of default due to the issuer going into bankruptcy. Essentially, catastrophe bonds (and the wider asset class Insurance-linked securities, ILS) is rated based on its probability of default due to a qualifying catastrophe triggering loss of principal. This probability is determined with the use of catastrophe models.

This use of catastrophe models, together with a detailed analysis of the transaction's legal structure and the financial strength of the various parties to the transaction, provides rating agencies with sufficient comfort that the resulting ratings adequately captures the risk to investors in these securities.

Most rated catastrophe bonds are considered “speculative” by rating agencies (Moody's: Ba, B; Standard & Poor's: BB, B), entailing that there is a “substantial” to “high” credit risk.

Additionally, the ongoing monitoring of rated catastrophe bonds, including what might trigger a rating review, is detailed in the catastrophe bond rating methodology of the agencies. Monitoring is one of the most important aspects of the rating agencies job, once the transaction has been issued and rated.

In the next Sect 2.1.5, we present some of the important literature and a theoretical foundation of the pricing the catastrophe bonds.

2.1.5 Catastrophe bond: Pricing and theoretical foundations

This section considers the theoretical foundations of catastrophe bonds. Some of the earlier and ongoing studies that are pertinent to the catastrophe bond pricing are reviewed. We consider discussions and past research works including:

- V.E. Vaugirard (2002)
- Morton N. Lane (2003)
- K. Burnecki and G. Kukla (2003)
- M. Lane and O. Mahul (2008)
- Robert A. Jarrow (2010)
- Dimitris Papachristou (2011)
- Samuel H. Cox and Hal W. Pedersen (2014)
- J. Liu, J. Xiao, L. Yan, and F. Wen (2014)
- Jaur`es NGOUFFO ZANGUE (2016)

Pricing catastrophe bonds is referenced here by V.E. Vaugirard in (2002) [68]. This paper shows the existence of a well-defined arbitrage price for catastrophe bonds as it develops a simple arbitrage approach to valuing insurance-linked securities, which accounts for catastrophic events. Under the valuation framework section, this paper underscores the binary structure of catastrophe bonds. It extends and adapts the jump-diffusion model of Merton (1976). Next, it develops a framework that allows for catastrophic events, interest rate randomness, and non-traded underlying state variables.

LFC is a consulting firm focusing on two industries; reinsurance and finance. It was also involved in natural catastrophe reinsurance brokerage and marine exposures. In a 2003 paper namely, “Rationale and results with the LFC cat bond pricing model”, Morton N. Lane, (2003) [69] primarily observes cat bond prices with the use of a several years’ experience record. In observing cat bond prices, they use a model, which is labelled as the LFC (Lane Financial) model. The paper traces the beginnings of this model’s development, its continued explanatory powers, usefulness when analysing the prices and market as well as its shortcomings.

The paper presents a theme that Cat bond pricing presents theorists with both an opportunity and a challenge. The opportunity is that for the first time ever, investors are being presented explicit probability statistics about the likelihood of full repayment at maturity. They receive these probability estimates at the time of issue. The opportunity is then to observe transaction prices and examine them relative to precise statistics provided at issue. The paper also presented a challenge of cat bonds is that by rights, the prices that are observed should be lower than they are according to the capital asset pricing theories. In the 10 years that the embryonic cat bond market has existed, observed prices have never approached low levels as per these theories. These theories suggest that any asset that diversifies an investment portfolio away from systemic market risk is desirable. Investors if they are efficient should want diversifying assets, which cat bonds represent, in large quantities. In theory, the price should be driven lower by investor demand to the point where they are just compensated for expected losses and the risk free rate.

K. Burnecki and G. Kukla (2003) [70] calculate non-arbitrage prices of a Zero-coupon and coupon CAT bond. This paper studies the 10-year catastrophe loss data provided by Property Claim Services and calibrate the pricing model.

The paper by M. Lane and O. Mahul in (2008) [71], aims to analyze the pricing of catastrophe reinsurance and to identify key components that impact catastrophe risk prices, with a particular attention to the amount of capital that is necessary to protect against extreme risks.

“A simple robust model for Cat bond valuation”: This note by Robert A. Jarrow in (2010) [72] provides a simple closed form solution for valuing Cat bonds.

“Statistical Analysis of the Spreads of Cat bonds at the time of issue”: this paper by D. Papachristou in (2011) [73], does not try to estimate what the price of a cat bond should be according to theoretical considerations, instead estimating market prices of cat bonds. The main purpose of this paper is to examine the factors that affect cat bond prices and measure the effect of these factors on the bond prices, using statistical models.

Catastrophe risk bonds provide a mechanism for direct transfer of catastrophe risk to capital markets, in contrast to transfer through a traditional reinsurance company. The article by S. H. Cox and H. W. Pedersen in 2014 [74], examines the pricing of catastrophe risk bonds.

The article briefly discusses the financial economics involved in the pricing of catastrophe risk bonds. The pricing of catastrophe risk bonds requires an incomplete markets setting, and this creates special difficulties in the pricing methodology. The use of a probability structure for the catastrophe risk, together with an equilibrium pricing theory, is used to develop a pricing method based on a model of the term structure of interest rates.

The (2014) [75] research article by J. Liu, J. Xiao, L. Yan, and F. Wen, listed the main pricing models of CAT bonds, such as Kreps model, LFC model, Christofides model, and others including Wang twofactor model. They are all using the quantitative methods to estimate essential factors of the price and then price the CAT bonds. This article highlights that there may be a credit risk attached to the catastrophe bond. This article uses the Jarrow and Turnbull method to model the credit risk, which is a different method from the other literature.

It is evident from the previous experience that natural disasters cause a significant number of damages. The study by Jaur`es NGOUFFO ZANGUE in 2016 [76], majorly highlighted that an understanding of catastrophe risk is very important, if we want to prevent and mitigate the effects of a natural disaster, both for the insurers and the insureds. Insuring companies are launching new products called Insurance-Linked-Securities (ILS) to seek new funding and avoid being insolvent. The primary purpose of this study is to find a way to evaluate catastrophic risk, and to review the instruments used to hedge catastrophe risk such as CAT bonds and their pricing.

In the next Sect 2.1.6, we use the Jarrow-Lando-Turnbull (JLT) model (1997) for the pricing of the catastrophe bond with a simple example of a zero-coupon catastrophe bond and its assumed credit ratings. We present how a reduced-form approach can be adopted to pricing such risky bonds.

2.1.6 Example: Zero-coupon catastrophe bond: Jarrow-Lando-Turnbull (JLT) model (1997)

Let us assume a zero-coupon catastrophe bond with two states. We have defined the states as credit ratings, that a ZCB can possibly take such as; A and BB after a rating review if triggered.

The transition intensities between these credit rating states and assumed to be constant as:

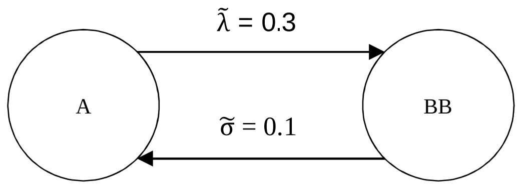

Figure 11: Two states catastrophe zero-coupon bond diagram

The matrix of constant transition intensities (the "generator" matrix) under the risk-neutral world measures be:

$$\tilde{\Lambda}(t) = \begin{array}{c} \\ A \\ BB \end{array} \begin{array}{c} \begin{array}{cc} A & BB \end{array} \\ \begin{bmatrix} -0.3 & 0.3 \\ 0.1 & -0.1 \end{bmatrix} \end{array}$$

Let $M(t)$ = denote the matrix $\int_0^t \tilde{\Lambda}(s)\,\mathbf{ds}$, and since the transition intensity is constant:

$$M(t) = \begin{array}{c} \\ A \\ BB \end{array} \begin{array}{c} \begin{array}{cc} A & BB \end{array} \\ \begin{bmatrix} -0.3t & 0.3t \\ 0.1t & -0.1t \end{bmatrix} \end{array}$$

$$M(t)^2 = \begin{bmatrix} -0.3t & 0.3t \\ 0.1t & -0.1t \end{bmatrix} \text{x} \begin{bmatrix} -0.3t & 0.3t \\ 0.1t & -0.1t \end{bmatrix}$$

$$M(t)^2 = \begin{bmatrix} 0.12t^2 & -0.12t^2 \\ -0.04t^2 & 0.04t^2 \end{bmatrix}$$

The relation between $M(t)^2$ and $M(t)$ is therefore:

$$M(t)^2 = \; -0.4\,t \text{ x } M(t)$$

Therefore $\widetilde{\Pi}(t)$ the matrix of transition probabilities, where I is an identity matrix as:

$$\widetilde{\Pi}(t) = \text{I} + \frac{M(t)}{1!} + \frac{M(t)^2}{2!} + \frac{M(t)^3}{3!} + \cdots$$

Moreover, using the relationship derived above between $M(t)^2$ and $M(t)$ as:

$$\widetilde{\Pi}(t) = \text{I} + M(t) + \frac{-0.4t \text{ x } M(t)}{2} + \frac{(-0.4t)^2 \text{x } M(t)}{6} + \cdots$$

$$\widetilde{\Pi}(t) = \text{I} + M(t)(1 + \frac{-0.4t}{2} + \frac{(-0.4t)^2}{6} + \ldots)$$

$$\widetilde{\Pi}(\boldsymbol{t}) = \mathbf{I} + \frac{\boldsymbol{M(t)}}{\boldsymbol{-0.4t}}\,(\boldsymbol{e^{-0.4t}} - \mathbf{1})$$

Let us assume the numerical parameters for zero-coupon catastrophe bond, and continue to use the JLT model for pricing of a corporate zero-coupon catastrophe bond, this is explained below in two steps as in first step, the probability that the ZCB does not default is explained. This will be used for the pricing of ZCB in the second step.

- **Step 1: Estimation of the probability that the bond does not default in time t:**

Let us deduce the matrix of transition probabilities $\widetilde{\Pi}(t)$:

Table 7: Numerical zero-coupon catastrophe bond parameters for default probability

Parameters:		
Terms of zero-coupon bond	t	**5 years**
Current state of bond		**A**
Last state of bond		**BB**

$\widetilde{\Pi}(5)$:

$$= \text{I} + \frac{M(t)}{-0.4t}\,(e^{-0.4t} - 1)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\begin{bmatrix} -0.3t & 0.3t \\ 0.1t & -0.1t \end{bmatrix}}{-0.4xt} (e^{-0.4xt} - 1)$$

$$= \begin{bmatrix} 1 & \mathbf{0} \\ 0 & 1 \end{bmatrix} + \frac{\begin{bmatrix} -0.3x5 & \mathbf{0.3x5} \\ 0.1x5 & -0.1x5 \end{bmatrix}}{-0.4x5} (e^{-0.4x5} - 1)$$

$\{\tilde{\Pi}(5)\}_{A, BB}$:

$$= \mathbf{0} + \frac{\mathbf{0.3\ x\ 5}}{-0.4x5} (e^{-0.4x5} - 1) \ = \mathbf{0.6485}$$

$\{\tilde{\Pi}(5)\}_{A, BB}$: This therefore explains that there is 64.85% probability that the zero-coupon catastrophe bond would move from investment grade credit rating state A to junk grade credit rating state BB in 5 years.

- **Step 2: Thus, ZCB Pricing formula: A is a current state:**

$$\textbf{B (t, A)} \ \ \textbf{= exp} \ (-\mathbf{r}(\boldsymbol{t})) \ \{\mathbf{1 - (1 - \alpha)} \times \{\tilde{\Pi}(\mathbf{5})\}_{\mathbf{A, BB}}\}$$

Let us assume the further parameters for pricing the ZCB:

Table 8: Numerical zero-coupon catastrophe bond parameters for pricing the bond

Parameters:		
Recovery rate	α	**30%**
Risk-free rate	r	**1%**
Terms of zero-coupon bond	t	**5 Years**

Therefore:

$$\textbf{B (t, A)} \ \ \textbf{= exp} \ (-\mathbf{0.01(5)}) \ \{\mathbf{1 - (1 - 0.3)} \times \mathbf{0.6485}\}$$

This expression therefore finally provides the discounted expected value of expected overall payoff on a zero-coupon catastrophe bond i.e. equal to a price of that bond with maturity time of 5 years:

$\textbf{B (t, A)}$ = exp (– 0.01(5)) {1 – (1 – 0.3) x 0.6485}

= exp (– 0.05) {1 – 0.7 x 0.6485}

= exp (– 0.05) {0.5461}

= €0.5195

2.1.7 Catastrophe bond investment: There is a credit risk

When investing in a bond, one of the risks is credit risk. Chances being that the issuer will default on its obligation, one way to assess the situation is to examine the credit rating of the issuer. In the case of catastrophe bond, it means the credit rating of the sponsor. Rating agencies such as Standard & Poor's, Moody's, or Fitch Ratings often evaluate bond issuers and publish ratings. These agencies assign bond ratings based upon several factors that reflect upon the financial stability of the issuer. These factors include the nature of the issuer’s debt, reliability of cash flows, ability to make interest payments and entity management.

3 – STRUCTURE: CONCLUSION, REMARKS AND EXTENSION

3.1 Conclusion remarks

In this study, we introduced the quantitative credit risk models and put effort to present the two-credit risk modelling approaches: structural and reduced-form approach. The overall study emphasizes and outlines the importance, strengths and weaknesses of different types of credit risk models.

Unlike structural models, reduced-form models do not condition default explicitly on the value of the firm, and parameters related to the firm's value need not be estimated to implement the model. Reduced-form models, such as the Jarrow-Lando-Turnbull model, consider market and credit risk. They can be calibrated using observable data and consequently incorporate market information. For pricing, hedging and risk management, reduced-form models are the preferred methodology.

The idea is to include a defaultable instrument in the study and we therefore incorporated catastrophe bond in order to show how reduced-form methodology can be utilised in practice with a very simple example of a zero-coupon catastrophe bond.

In my view, there is considerable opportunity for additional work along the lines that we have studied here. For the ease of exposition, we have only sketched the possibilities of doing statistical inference in these models for an ordinary zero-coupon bond instrument. It is also clear from the study that there are studies developed to account for other financial instruments. One can modify our results and apply these studies for such financial instruments and credit derivatives.

Credit derivatives as driven by the need to hedge and manage credit risks in a flexible way, new derivative securities have been developed to fulfil this need. Definition of Credit Derivative is as:

"Credit derivatives transfer default risk from one party to another and are used to transfer credit risk. The basic example of a credit risk derivative is a credit default swap".

The pricing and management of these credit derivatives requires more flexible and sophisticated credit risk models.

In the next section, we briefly include some information on catastrophe bonds and pandemic bonds: their history of issuance, theoretical view and applicable triggers in practice, to understand how triggers function in real world scenarios in case of pandemic bonds. We therefore extend our study by introducing such innovative bonds issued by World Bank and propose to have a detailed review for the followings studies.

3.2 Further extension

Over the past 25 years, the world has registered a significant increase in the number of natural disasters, coupled with the effect of climate change and the increase of the population in regions with high threats of natural catastrophe. This situation increases the insolvency risk for insurance companies due to the huge amount of losses that can be caused the occurrence of a catastrophe event. The insuring companies and government seek new way of insuring themselves and to avoid being insolvent in such a situation. The World Bank[7] (International Bank for Reconstruction and Development, IBRD) has introduced innovative catastrophe and other pandemic securities to provide financial support to such nations. Some of the examples are as detailed:

2014 West Africa Ebola Crisis: The financial support to West Africa to contain a pandemic outbreak.

In 2016, PEF housed (PEF stands, the Pandemic Emergency Financing Facility (PEF), a facility created by the World Bank to channel surge funding to developing countries facing the risk of a pandemic) – at the World Bank was launched to provide financial support for nations in such situations. PEF was financed by some developed countries together with a catastrophe bond issued to capital market investors, as well as swaps issued by the World Bank to insurance companies. [77]

2017 World Bank Pandemic Bond:

In 2017, the World Bank launched its first-ever pandemic bonds aimed at providing financial support to the Pandemic Emergency Financing Facility (PEF), a facility created by the World Bank to channel surge funding to developing countries facing the risk of a pandemic. It was the first time that World Bank bonds were being used to finance efforts against infectious diseases. [78]

Regarding infectious diseases, Covid-19 is one of the viruses covered by the PEF insurance window. The triggers were defined based on outbreak size (the number of cases of infections and fatalities), outbreak growth (over a defined time period) and outbreak spread (with two or more IBRD/IDA countries affected by the outbreak).

It was confirmed that one of the trigger requirements under the terms of PEF bonds and swaps (namely an exponential growth rate in IDA/IBRD countries) was met on March 31, 2020. As per the report on 17 July 2020, all activation criteria including outbreak size, spread and growth were met.

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[7] The World Bank (International Bank for Reconstruction and Development, IBRD), rated Aaa/AAA (Moody`s/S&P), is an international organization created in 1944 and the original member of the World Bank Group. It operates as a global development cooperative owned by 189 nations. It provides its members with financing, expertise and coordination services so they can achieve equitable and sustainable economic growth in their national economies and find effective solutions to pressing regional and global economic and environmental problems.

The world bank has two main goals: to end extreme poverty and promote shared prosperity. [78]

2018 World Bank Earthquake Bond: Chile, Colombia, Mexico and Peru

In 2018, the World Bank issued sustainable development bonds that collectively provide $1.36 billion in earthquake protection to Chile, Colombia, Mexico and Peru. This is the largest sovereign risk insurance transaction ever and the second largest issuance in the history of the catastrophe bond market. It is the first time that Chile, Colombia and Peru, access the capital markets to obtain insurance for natural disasters [79]

The World Bank has developed some of the most innovative catastrophe risk insurance instruments in the market to help developing nation manage risk. In the past ten years, the institution has executed approximately $1.6 billion in catastrophe risk transactions.

So suggest further studies of such risk-linked securities and the pricing models used.

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