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Linear stability for differential equations with infinite delay via semigroup theory

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Abstract

In this dissertation, we provide a proof of a Principle of Linearized Stability for a class of autonomous differential equations with infinite delay. This is done via techniques from functional analysis, namely duality theory for semigroups of bounded linear operators, following the approach of O. Diekmann and M. Gyllenberg in [12]. First, we make a detailed study of some aspects of the theory of strongly continuous semigroups (also called C_0 semigroups) of linear operators in Banach spaces. In particular, we prove the classical theorem of Hille-Yosida, characterizing infinitesimal generators of C_0 semigroups, and define the adjoint of a strongly continuous semigroup. Since the adjoint semigroup is not necessarily strongly continuous, we study whether it can be restricted to some subdomain where strong continuity holds. This is the starting point for the sun-star calculus, of which we make use throughout the remaining chapters. We introduce some elements of the sun-star theory for linear operators and give meaning to an abstract integral equation, for which we prove existence, uniqueness, continuation and regularity of solutions. We then consider, in a suitable (weighted) space of continuous functions on $(-\infty, 0]$ that vanish at $-\infty$, an initial value problem for a differential equation with infinite delay and prove an equivalence result between solutions of such equation and the solution semigroup of an abstract integral equation. After that, we study the characteristic equation of the linearized problem, and prove that the roots of this equation are precisely the eigenvalues of the infinitesimal generator of the solution semigroup of the linear equation. Moreover, we show that, on a fixed half-space, there are only finitely many such roots. Consequently, the spectral projection of the resolvent operator induces a decomposition of the phase space as the direct sum of two invariant subspaces - one with finite dimension, and the other where the semigroup is exponentially stable -, to which we can apply a theorem by Desch and Schappacher. As a result, we obtain a proof of the Principle of Linearized Stability, generalizing for this case the well-known result for ordinary and finite-delay differential equations.

Keywords: differential equations, infinite delay, linear stability, semigroup theory, sun-star theory

Resumo

O Princípio da Estabilidade Linear é um resultado bem conhecido da teoria das equações diferenciais ordinárias (EDOs) autónomas em $\mathbb{R}^n$, que afirma que a estabilidade (local) de um equilíbrio hiperbólico x^* de uma EDO $\dot{x} = f(x)$, onde f é de classe C^1, é equivalente à estabilidade da solução nula do problema linearizado, que por sua vez é determinada pela posição, no plano complexo, dos valores próprios da matriz derivada $Df(x^*)$. Mais precisamente, para EDOs lineares autónomas $\dot{x} = Ax$, onde A é uma matriz $n \times n$, é sabido que, se todos os valores próprios da matriz A têm parte real negativa, então o equilíbrio trivial é assimptoticamente estável e que, se existe pelo menos um valor próprio com parte real positiva, então é instável. Um resultado semelhante é válido para equações diferenciais com atraso finito. Neste caso, uma possível demonstração envolve teoria de C_0 semigrupos e depende fortemente do facto de os operadores $T(t)$, $t \geq 0$, que constituem o semigrupo de soluções serem "eventualmente" compactos, o que implica em particular que os valores espectrais não nulos destes operadores (a partir de certo $t > 0$) e do gerador infinitesimal do semigrupo são apenas valores próprios. Novamente, prova-se que a estabilidade da solução nula depende da posição destes valores próprios no plano complexo, como no caso anterior. Para mais detalhes relativamente à estabilidade linear para equações diferenciais autónomas com atraso finito, ver [15]. Quando se consideram equações com atraso infinito, a mesma abordagem não funciona, já que não se tem compacidade dos operadores que formam o semigrupo. O objectivo desta dissertação é então apresentar uma demonstração de um Princípio de Estabilidade Linear para uma classe de equações diferenciais autónomas com atraso infinito, que generalize os resultados anteriores para equações diferenciais ordinárias e com atraso finito. A principal referência para este trabalho é o artigo [12], de O. Diekmann e M. Gyllenberg.

No primeiro capítulo, estudamos, de forma bastante completa, alguns elementos da teoria de semigrupos fortemente contínuos de operadores lineares (também chamados C_0 semigrupos) num espaço de Banach X. Após apresentarmos, na primeira secção, os resultados básicos da teoria de C_0 semigrupos, enunciamos e provamos o conhecido teorema de Hille-Yosida, que caracteriza os geradores infinitesimais de C_0 semigrupos. A demonstração é dividida em três passos - provamos primeiro o resultado para semigrupos de contracções, de seguida para semigrupos uniformemente limitados e, finalmente, o caso geral. Como consequência da prova, obtemos uma representação para a resolvente do gerador infinitesimal como a transformada de Laplace do semigrupo. Na secção seguinte, definimos o adjunto de um C_0 semigrupo $\{T(t)\}_{t\geq 0}$, que é um novo semigrupo $\{T(t)^*\}_{t\geq 0}$, não necessariamente fortemente contínuo, no espaço dual X^*. É então relevante saber se é possível restringir o semigrupo adjunto a um sub-domínio onde se tenha continuidade forte. A resposta a esta questão é positiva e este é o resultado principal desta secção. Obtemos então, associado a X e ao C_0 semigrupo inicial $\{T(t)\}_{t\geq 0}$, um novo espaço $X^{\odot} \subset X^*$, que definimos como sendo o subespaço maximal de X^* onde $\{T(t)^*\}_{t\geq 0}$ é fortemente contínuo. Denotemos a restrição de $\{T(t)^*\}_{t\geq 0}$ a $X^{\odot}$ por $\{T(t)^{\odot}\}_{t\geq 0}$. Repetindo o processo de tomar adjuntos, obtemos um semigrupo $\{T(t)^{\odot *}\}_{t\geq 0}$ definido num espaço $X^{\odot *}$ em que X está mergulhado de forma natural. O novo semigrupo é simplesmente $\{T(t)\}_{t\geq 0}$ quando restringimos a X. Trabalhar em

$X^{\odot *}$ é vantajoso; note-se que este é um espaço dual e, como tal, podemos utilizar as ferramentas da Análise Funcional para estudar os operadores definidos neste espaço. Há, contudo, um preço a pagar: o semigrupo $\{T(t)^{\odot *}\}_{t\geq 0}$ não é necessariamente fortemente contínuo. Por este motivo, trabalhamos em X ou $X^{\odot *}$ conforme o nosso objectivo. Este é o ponto de partida da teoria *sun-star*, uma teoria de dualidade para semigrupos fortemente contínuos, desenvolvida por O. Diekmann e co-autores em [3, 4, 5, 6, 10], que usamos nos capítulos seguintes desta dissertação. Na última secção do Capítulo 1, definimos a projecção espectral associada a um operador fechado $A\colon D(A)\to X$ e enunciamos um resultado, de tipo alternativa de Fredholm, que mostra que a projecção espectral de A induz uma decomposição do espaço X em soma directa de dois subespaços.

No segundo capítulo, começamos por dar sentido a uma equação da forma

$$u(t)=T(t)\psi+j^{-1}\int_0^t T^{\odot *}(t-s)G(u(s))ds,\quad t\geq 0. \tag{AIE}$$

a que chamaremos equação integral abstracta com condição inicial ψ, onde $\{T(t)\}_{t\geq 0}$ é um C_0 semigrupo em X, $\{T(t)^{\odot *}\}_{t\geq 0}$ é definido como no parágrafo anterior, j é o mergulho $X\to X^{\odot *}$ e $G\colon X\to\mathbb{R}^m$ é uma função dada. O integral é um integral fraco-$*$ e é um elemento de $X^{\odot *}$. Na secção seguinte, estudamos existência, unicidade, continuação e regularidade de soluções desta equação, o que fazemos adaptando à presente situação a prova conhecida do teorema de existência e unicidade (de Picard) para equações diferenciais ordinárias. Segue, como consequência, que a solução da equação define, em X, um semigrupo fortemente contínuo. Após este resultado, introduzimos o problema de valores iniciais que nos propomos a estudar nesta dissertação, que é uma equação diferencial autónoma com atraso infinito da forma

$$\begin{cases}\dot{y}(t)=F(y_t), & t>0\\ y(t)=\psi(t), & t\leq 0,\end{cases} \tag{DDE}$$

no espaço de fase $X=C_{0,\rho}(\mathbb{R}_-;\mathbb{R}^n)$ das funções contínuas $f\colon\mathbb{R}_-\to\mathbb{R}^n$ tais que $\lim_{\theta\to-\infty}e^{\rho\theta}|f(\theta)|$, onde $\rho>0$ está fixo, a condição inicial $\psi\in X$, $y_t(\theta)=y(t+\theta)$, para $\theta\leq 0$, e $F\colon D\subset X\to\mathbb{R}^m$ é Fréchet diferenciável com derivada contínua. Com o objectivo de obter uma equivalência entre as soluções desta equação e o semigrupo-solução de uma equação integral abstracta da forma (AIE), definimos um semigrupo fortemente contínuo $\{T_0(t)\}_{t\geq 0}$, o mergulho $j\colon X\to X^{\odot *}$ e uma aplicação linear $\ell\colon\mathbb{R}^m\to X^{\odot *}$, e provamos que, de facto, existe uma equivalência entre soluções de (DDE) e da equação abstracta integral

$$u(t)=T_0(t)\psi+j^{-1}\int_0^t T_0^{\odot *}(t-s)(\ell\circ F)(u(s))\,ds,\quad t\geq 0.$$

Este é o resultado principal deste capítulo. Em particular, considerando o problema linearizado

$$\begin{cases}\dot{y}(t)=DF(\overline{\psi})y_t, & t>0\\ y(t)=\psi(t), & t\leq 0\end{cases}, \tag{LDDE}$$

onde $DF(\overline{\psi})$ é a derivada (de Fréchet) de F num equilíbrio $\overline{\psi}$, obtemos uma equivalência entre a solução $y=y(t)$ do problema acima e o semigrupo solução $\{T(t)\}_{t\geq 0}$ da equação integral *linearizada*

$$u(t)=T_0(t)\psi+j^{-1}\int_0^t T_0'^{*}(t-s)\ell DF(\overline{\psi})u(s)\,ds,\quad t\geq 0, \tag{LAIE}$$

no sentido em que:

- se $\{T(t)\}_{t\geq 0}$ é o C_0 semigrupo definido pela solução $y = y(\theta)$ da equação (LDDE) com condição inicial ψ, $T(t)\psi = y_t$, para $t \geq 0$, então $u(t) = T(t)\psi$ resolve a equação integral (LAIE) com condição inicial ψ;

- se $u = u(t;\psi)$ denota a solução da equação abstracta (LAIE) com condição inicial $\psi \in X$, então $T(t)\psi := u(t;\psi)$, $t \geq 0$, define um C_0 semigrupo $\{T(t)\}_{t\geq 0}$ em X e a função

$$y(t) = \begin{cases} \psi(t) & \text{se } t \leq 0 \\ (T(t)\psi)(0) & \text{se } t > 0 \end{cases}$$

  resolve a equação diferencial (LDDE).

O terceiro e último capítulo desta dissertação culmina com a demonstração de um Princípio de Estabilidade Linear para a equação (DDE). Na primeira secção, estudamos a equação característica associada ao problema linear (LDDE), que é definida por

$$\det[\lambda I - \widehat{\mu}(\lambda)] = 0, \quad \text{Re}\,(\lambda) > -\rho, \tag{EC}$$

onde μ é a medida que representa $DF(\overline{\psi})$ em $M_\rho(\mathbb{R}_+;\mathbb{R}^{m\times m})$, o espaço das medidas (matriciais) de Radon tais que $\int_0^\infty e^{\rho\theta}\, d|\mu|(\theta) < \infty$; $|\mu|$ e $\widehat{\mu}$ denotam, respectivamente, a variação total e a transformada de Laplace de μ. Começamos por estudar a forma da resolvente do gerador infinitesimal A do semigrupo linearizado $\{T(t)\}_{t\geq 0}$, de modo a identificar o espectro de A. Provamos que, no semi-plano $\Pi_+^\rho = \{\lambda \in \mathbb{C}\colon \text{Re}\,(\lambda) > -\rho\}$, o espectro de A coincide com o espectro pontual, isto é, que os únicos valores espectrais neste semi-plano são valores próprios. Mais do que isso, mostramos que esses valores próprios são precisamente as raízes características do problema e que, em Π_+^ρ, elas são em número finito. Isso permite-nos definir uma projecção espectral

$$P_+\psi = \int_\Gamma R(\lambda : A)\psi\, d\lambda,$$

onde Γ é uma curva contida em Π_ρ^+ contendo os valores próprios de A e $R(\lambda : A) = (\lambda I - A)^{-1}$ é a resolvente de A. Usando resultados do primeiro capítulo sobre a projecção espectral, obtemos uma decomposição do espaço de fase X em soma directa de dois subespaços X_+ e X_-, invariantes sob o semigrupo, tais que X_+ tem dimensão finita e a restrição de $\{T(t)\}_{t\geq 0}$ a X_- é exponencialmente estável. Mais precisamente, X_+ é a imagem da projecção espectral P_+, que coincide com a soma directa dos espaços próprios generalizados de A associados aos valores próprios no interior de Γ, e X_- é o subespaço complementar. Com estes resultados estabelecidos, apresentamos finalmente a prova de um Princípio de Estabilidade Linear para o problema (DDE), usando um teorema geral de Desch e Schappacher, generalizando o resultado conhecido para equações diferenciais ordinárias retardadas, para o caso em que o atraso é finito. Podemos enunciar este princípio da seguinte forma. Seja $\overline{\psi} \in X = C_{0,\rho}(\mathbb{R}_-;\mathbb{R}^m)$ um equilíbrio de (DDE) e $\mu \in M_\rho(\mathbb{R}_+;\mathbb{R}^{m\times m})$ a representação de $DF(\overline{\psi})$. Então:

(i) se todas as soluções de (EC) em Π_+^ρ têm parte real negativa, o equilíbrio $\overline{\psi}$ é exponencialmente estável;

(ii) se existe pelo menos uma solução de (EC) em Π_+^ρ com parte real positiva, o equilíbrio $\overline{\psi}$ é instável.

Palavras-chave: equações diferenciais, atraso infinito, estabilidade linear, teoria de semigrupos, teoria *sun-star*

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Introduction

The Principle of Linearized Stability is a classical result in the theory of ordinary differential equations. It states that the (local) stability of a hyperbolic equilibrium x^* of an ODE $\dot{x} = f(x)$, where f is a C^1 vector field, is equivalent to the stability of the trivial solution of the linearized problem, which is, in its turn, determined by the position of the eigenvalues of $Df(x^*)$ on the complex plane. To be more precise: if all the eigenvalues of $Df(x^*)$ have negative real part, then x^* is a stable equilibrium and, if there exists one eigenvalue with positive real part, then it is unstable.

A similar result holds for equations with finite delays. For such equations, the phase space is infinite-dimensional, and more sophisticated tools are needed. Let $C = C([-h,0];\mathbb{R}^n)$ be the space of $\mathbb{R}^n$-valued continuous functions on $[-h,0]$, endowed with the supremum norm. Consider the linear problem

$$\dot{x}(t) = Lx_t, \quad t > 0 \tag{0.1}$$

where $L\colon C \to \mathbb{R}^n$ is a bounded linear operator and $x_t(\theta) = x(t+\theta)$, for $\theta \in [-h,0]$. It is known that, given an initial condition $\varphi \in C$, the initial value problem

$$\dot{x}(t) = Lx_t, \text{ for } t > 0, \quad x(t) = \varphi(t), \text{ for } -h \leq t \leq 0,$$

admits a unique solution $x = x(t;\varphi)$, defined for all $t \geq -h$. Defining, for $\varphi \in C$, $T(t)\varphi = x_t(\cdot;\varphi)$, we obtain a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on C, which can be proved to be compact for $t \geq h$. Therefore, for such t, the nonzero points of the spectrum of $T(t)$ reduce to eigenvalues, and one can use the theory of C_0 semigroups to conclude that the spectrum of the infinitesimal generator A of $\{T(t)\}_{t\geq 0}$ consists of eigenvalues only, and that it coincides with the roots of the characteristic equation of (0.1). It can finally be seen that, if all the eigenvalues of A have negative real part, then $x = 0$ is a stable solution of (0.1) and, if one of them has positive real part, then the trivial solution is unstable. A good reference for these results is [15]. This provides us again with a Principle of Linearized Stability for DDEs of the form $\dot{x} = F(x_t)$, for F a continuously Fréchet differentiable function, by linearizing about a steady-state and applying the previous result.

This problem is significantly more complicated when we consider differential equations with infinite delay. For these equations, even the choice of the correct spaces in which to work is a difficult problem; in [14], for example, an axiomatic approach is proposed to deal with the choice of a suitable phase space. In addition, existence, uniqueness, continuation and stability results are more technical and harder to obtain. In particular, as far as linear stability is concerned, note that there is no hope to extend the argument of the finite delay equations to the present situation, as the crucial compactness property of the semigroup no longer holds. So different techniques need to be applied. Equations with infinite delay are deeply studied in [16].

One possible approach is the use of duality theory of linear operators, which already appears for

example in [16] and was further developed by O. Diekmann and co-authors (under the name of sun-star calculus) in the series of papers [3, 4, 5, 6, 10]. With this approach one obtains a new problem in a space $X^{\odot *}$, which is larger than the initial space X, but where X is embedded, with a new semigroup $\{T(t)^{\odot *}\}_{t\geq 0}$ that is simply $\{T(t)\}_{t\geq 0}$ when restricted to X. One clear advantage of working in $X^{\odot *}$ is that this is a dual space, and so one can apply the tools of functional analysis to obtain results for the equations. There is a price to pay, however, because the extension of the semigroup to this space is not necessarily strongly continuous on $X^{\odot *}$. So we need to work on both X and $X^{\odot *}$ according to our needs. This construction is deeply explored, for instance, in [11, Chapters 2, 3], to which we refer for more details.

In this dissertation, we present a detailed exposition of the results in [12] concerning a class of differential equations with infinite delay of the form

$$\begin{cases} \dot{y}(t) = F(y_t), & t > 0 \\ y(t) = \psi(t), & t \leq 0, \end{cases} \tag{0.2}$$

where F is continuously Fréchet differentiable and the initial data ψ is in the phase space $C_{0,\rho}(\mathbb{R}_-;\mathbb{R}^m)$ of continuous functions $f\colon \mathbb{R}_- \to \mathbb{R}^m$ such that $\lim_{\theta\to-\infty} e^{\rho\theta}|f(\theta)| = 0$, with the weighted supremum norm $||f||_{\infty,\rho} = \sup_{\theta\leq 0} e^{\rho\theta}|f(\theta)|$. Here $\rho > 0$ is a fixed positive number. The main result of this thesis is a principle of linearized stability for the initial-value problem (0.2). We show that the stability of an equilibrium of (0.2) can be deduced from the study of the linearized version of (0.2), and is determined by the position of the eigenvalues of some linear operator associated with the solution of the linear problem. In [12], this is done for a different class of equations (called renewal equations) with infinite delay, and a big part of the differential case is omitted; this dissertation fills in these details.

We now briefly explain the content of the chapters of this dissertation. The first chapter is devoted to some aspects of the theory of strongly continuous semigroups of linear operators, usually called C_0 semigroups. First, we introduce the basic definitions and results for C_0 semigroups, and then prove the general case of the classic theorem of Hille-Yosida, which provides a characterization of infinitesimal generators of strongly continuous semigroups. After that, we introduce the notion of the adjoint of a semigroup and prove one important result concerning the domain of strong continuity of the adjoint semigroup. This will be important for the definition of the bigger space $X^{\odot *}$. We also introduce the spectral projection of the resolvent of a closed operator, and state a Fredholm alternative-like result which provides us with a suitable decomposition of the phase space X.

We begin the second chapter by introducing a general integral equation, for which we study existence, uniqueness, continuation and regularity of solutions, adapting the arguments of the classical proof of Peano's theorem for ODEs. After this is done, we prove an equivalence between the problem (0.2) and one of these abstract integral equations.

On the final chapter, our first concern is to understand the position of the eigenvalues of the infinitesimal generator of the semigroup associated to the linearized equation. We start by proving that, in a fixed half-plane, the only spectral values of the generator are eigenvalues, and that these are precisely the characteristic roots of the linear problem. After this is established, we use the spectral projection to find a decomposition of the phase space as a direct sum of two suitable invariant subspaces, and apply a result of Desch and Schappacher [9] to provide a proof of the Principle of Linearized Stability for (0.2).

Finally, the Appendix contains some results from the theory of Volterra integro-differential equations, which we include as a complement to the exposition. These results are used in some proofs in Chapter 3.

Chapter 1

Semigroups of bounded linear operators

As mentioned in the Introduction, in this chapter we present some results from the theory of semigroups of bounded linear operators, which will be needed for the study of differential equations with infinite delay on Chapter 2. Our main source for this chapter is [22, Chapter 1].

In what follows, X will always denote an infinite dimensional Banach space (over $\mathbb{R}$ or $\mathbb{C}$) and, as usual, $\mathscr{L}(X)$ denotes the space of bounded linear operators on X, which is itself a Banach space with the norm

$$||T||_{\mathscr{L}(X)} = \sup_{||x||_X \leq 1} ||Tx||_X = \sup_{x \neq 0} \frac{||Tx||_X}{||x||_X}.$$

Whenever there is no danger of confusion, we omit the indices in the norms.

1.1 Strongly continuous semigroups

We first introduce the basic definitions.

Definition 1.1.1. A one parameter family $\{T(t)\}_{t\geq 0}$ of bounded linear operators on X is said to be a *semigroup of bounded linear operators on X* if

(i) $T(0) = I$, the identity operator on X;

(ii) $T(s+t) = T(s)T(t)$, for every $s,t \geq 0$.

The linear operator A defined by

$$D(A) = \left\{ x \in X : \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}, \quad Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t}, \quad \text{for } x \in D(A)$$

is called the *infinitesimal generator* of the semigroup $\{T(t)\}_{t\geq 0}$.

The majority of the semigroups that appear in this dissertation are semigroups of linear operators and, as so, we will simply refer to these as semigroups on X. However, in some occasions we will need to use nonlinear semigroups; whenever this happens, we emphasize the fact that the operators are nonlinear. The definition of a nonlinear semigroup is analogous to the definition above:

Definition 1.1.2. A one parameter family $\{T(t)\}_{t\geq 0}$ of continuous operators on X is said to be a *(nonlinear) semigroup of continuous operators on* X if

(i) $T(0) = I$, the identity operator on X;

(ii) $T(s+t) = T(s)T(t)$, for every $s,t \geq 0$.

Definition 1.1.3. Let $\{T(t)\}_{t\geq 0}$ be a semigroup (linear or nonlinear) on X. $\{T(t)\}_{t\geq 0}$ is said to be

(i) *uniformly continuous* if

$$\lim_{t\to 0^+} ||T(t) - I|| = 0.$$

(ii) *strongly continuous*, or a C_0 *semigroup,* if

$$\lim_{t\to 0^+} T(t)x = x, \quad \forall x \in X.$$

Obviously, a uniformly continuous semigroup is strongly continuous. It is also a consequence of the definition that, if $\{T(t)\}_{t\geq 0}$ is a uniformly continuous semigroup on X, then

$$\lim_{t\to s} ||T(t) - T(s)|| = 0;$$

in other words, the map $t \mapsto T(t)$ is a continuous function $[0,+\infty) \to \mathscr{L}(X)$.

It is natural to ask whether one can identify precisely which linear operators on X are the infinitesimal generators of semigroups $\{T(t)\}_{t\geq 0}$. This will be our central question throughout the next section, where we will adress the problem of characterizing the generators of strongly continuous semigroups. When the semigroup is uniformly continuous, this question has a relatively simple answer. In fact, it can be seen (see Theorem 1.2 in [22, Chapter 1]) that the infinitesimal generator of a uniformly continuous semigroup is a bounded linear operator and, conversely, that any bounded linear operator A on X is the infinitesimal generator of a uniformly continuous semigroup, which is given by the formula

$$T(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}, \quad t \geq 0.$$

Requiring uniform continuity of the semigroup is not, however, broad enough to tackle the problems we will encounter in the next section of the text. In fact, the remarks above show that infinitesimal generators of uniformly continuous semigroups are restricted to the familty of bounded linear operators, and we would like to consider more general classes of operators. This will be achieved with strongly continuous semigroups, as we shall see. First, we establish an important estimate on the norm of a C_0 semigroup.

Theorem 1.1.4. *Let $\{T(t)\}_{t\geq 0}$ be a C_0 semigroup. There exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$||T(t)|| \leq Me^{\omega t}, \quad t \geq 0.$$

Proof. We first show that there exists $\tau > 0$ such that $||T(t)||$ is bounded for $0 \leq t \leq \tau$. If such τ does not exist, we can find a sequence $\{t_n\}_{n\in\mathbb{N}} \subset [0,+\infty)$ such that $t_n \to 0$ and $||T(t_n)|| \geq n$. But then, by the uniform boundedness theorem, there exists $x \in X$ such that $||T(t_n)x||$ is unbounded, and this cannot happen because the semigroup is strongly continuous. So there exist $\tau > 0$ and $M \geq 0$ such that

$$||T(t)|| \leq M, \quad 0 \leq t \leq \tau.$$

Since $||T(0)|| = ||I|| = 1$, actually $M \geq 1$.

Now choose $\omega = \tau^{-1} \log M$. Given $t \geq 0$, we can find $n \in \mathbb{N}$ and $0 \leq \delta < \tau$ such that $t = n\tau + \delta$. Consequently,

$$||T(t)|| = ||T(\delta)T(\tau)^n|| \leq M^{n+1} \leq MM^{t/\tau} = Me^{\omega t},$$

and this proves the desired. □

Definition 1.1.5. Let $\{T(t)\}_{t\geq 0}$ be a C_0 semigroup. The number

$$\omega_0 = \inf\{\omega \in \mathbb{R} : \text{ there exists } M \geq 1 \text{ such that } ||T(t)|| \leq Me^{\omega t},\ t \geq 0\}$$

is called the *growth bound* of the semigroup $\{T(t)\}_{t\geq 0}$.

A first example of application of the estimate in Theorem 1.1.4 is the following continuity result.

Corollary 1.1.6. *If $\{T(t)\}_{t\geq 0}$ is a C_0 semigroup on X, then, for every $x \in X$, the function $t \mapsto T(t)x$ is a continuous function $[0,+\infty) \to X$.*

Proof. Let $t, h \geq 0$. The result follows from the estimates

$$||T(t+h)x - T(t)x|| \leq ||T(t)||\,||T(h)x - x|| \leq Me^{\omega t}\,||T(h)x - x||$$

and, if $t - h \geq 0$,

$$||T(t-h)x - T(t)x|| \leq ||T(t-h)||\,||x - T(h)x|| \leq Me^{\omega t}\,||x - T(h)x||.$$

□

We now prove some important properties of C_0 semigroups and their infinitesimal generators. For the definition and some properties of the integral of functions taking values in a Banach space (the so called Bochner integral), we refer for instance to [18, 26].

Proposition 1.1.7. *Let $\{T(t)\}_{t\geq 0}$ be a C_0 semigroup on X and let A be its infinitesimal generator.*

(i) For $x \in X$ and $t \geq 0$,

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} T(s)x\,ds = T(t)x.$$

(ii) For $x \in X$ and $t \geq 0$,

$$\int_0^t T(s)x\,ds \in D(A) \quad \text{and} \quad A\left(\int_0^t T(s)x\,ds\right) = T(t)x - x.$$

(iii) For $x \in D(A)$ and $t \geq 0$, $T(t)x \in D(A)$ and $AT(t)x = T(t)Ax$.

(iv) For $x \in D(A)$ and $t_1, t_2 \geq 0$,

$$T(t_1)x - T(t_2)x = \int_{t_2}^{t_1} T(s)Ax\,ds = \int_{t_2}^{t_1} AT(s)x\,ds.$$

Proof. Part *(i)* is classical and follows from the continuity of $t \mapsto T(t)x$.

To see *(ii)*, let $x \in X$, $t \geq 0$ and $h > 0$. Then

$$\begin{aligned}
\frac{T(h)-I}{h} \int_0^t T(s)x\,ds &= \frac{1}{h} \int_0^t T(s+h)x - T(s)x\,ds \\
&= \frac{1}{h} \int_h^{t+h} T(s)x\,ds - \frac{1}{h} \int_0^t T(s)x\,ds \\
&= \frac{1}{h} \int_t^{t+h} T(s)x\,ds - \frac{1}{h} \int_0^h T(s)x\,ds.
\end{aligned}$$

Letting $h \to 0$, the right-hand side converges to $T(t)x - x$, and the conclusion follows.

Now let $x \in D(A)$ and $h > 0$. Then

$$\frac{T(h)-I}{h} T(t)x = T(t)\left(\frac{T(h)-I}{h}\right)x \to T(t)Ax$$

as $h \searrow 0$. Hence $T(t)x \in D(A)$, $AT(t)x = T(t)Ax$ and the right derivative of $T(t)x$ is $T(t)Ax$. It still remains to see that the left derivative of $T(t)x$ is also $T(t)Ax$. But note that, for $0 < h \leq t$,

$$\begin{aligned}
\frac{T(t)x - T(t-h)x}{h} - T(t)Ax &= T(t-h)\left(\frac{T(h)x - x}{h} - Ax\right) + (T(t-h)Ax - T(t)Ax) \\
&= T(t-h)\left(\frac{T(h)x - x}{h} - Ax\right) + T(t-h)\left(Ax - T(h)Ax\right).
\end{aligned}$$

As $h \searrow 0$, the first term on the right-hand side converges to 0, because $||T(t-h)||$ is bounded and $x \in D(A)$, and the second one also vanishes, by strong continuity of the semigroup and again because $||T(t-h)||$ is bounded. Therefore the left derivative of $T(t)x$ is equal to $T(t)Ax$, and this proves *(iii)*.

Finally, *(iv)* follows from integrating the equalities in *(iii)* from s to t.

□

Using these results, we are now able to prove that, even though the infinitesimal generator is not in general bounded, it is nonetheless fairly well behaved.

Corollary 1.1.8. *If A is the infinitesimal generator of a C_0 semigroup $T(t)$, then $D(A)$ is dense in X and A is a closed linear operator.*

Proof. For each $x \in X$, define

$$x_t = \frac{1}{t}\int_0^t T(s)x\,ds, \quad t > 0.$$

By parts *(i)* and *(ii)* on the previous result, $x_t \in D(A)$, for $t > 0$, and $x_t \to x$ as $t \to 0^+$, which implies that $D(A)$ is in fact dense in X.

To see that A is closed, let $x_n \in D(A)$, $x_n \to x \in X$, and $Ax_n \to y \in X$. Our goal is to prove that $x \in D(A)$ and $Ax = y$. By part *(iv)* above,

$$T(t)x_n - x_n = \int_0^t T(s)Ax_n\,ds, \quad t \geq 0. \tag{1.1}$$

On each bounded interval $[0,t]$, the convergence $T(s)Ax_n \to T(s)y$ is uniform in s, since, if $s \in [0,t]$,

$$||T(s)Ax_n - T(s)y|| \leq Me^{\omega t}||Ax_n - y||.$$

We can therefore let $n \to \infty$ on (1.1) to get

$$T(t)x - x = \int_0^t T(s)y\,ds,$$

so that

$$\frac{T(t)x - x}{t} = \frac{1}{t}\int_0^t T(s)y\,ds \to T(0)y = y, \quad t \to 0^+.$$

Hence $x \in D(A)$ and $Ax = y$, as desired. □

Before studying the form of the infinitesimal generators of C_0 semigroups, we show that an infinitesimal generator uniquely determines the associated semigroup.

Theorem 1.1.9. *Let $\{T(t)\}_{t\geq 0}$ and $\{S(t)\}_{t\geq 0}$ be C_0 semigroups and A, B the corresponding generators. If $A = B$, then*

$$T(t) = S(t), \quad t \geq 0.$$

Proof. It suffices to see that the semigroups coincide on $D(A) = D(B)$, as this is a dense subspace of X. Let $x \in D(A)$ and $t \geq 0$. From Proposition 1.1.7, it follows that the function $s \mapsto T(t-s)S(s)x$ is differentiable and, for every $s \in [0,t]$,

$$\begin{aligned}\frac{d}{ds}T(t-s)S(s)x &= -AT(t-s)S(s)x + T(t-s)BS(s)x \\ &= -T(t-s)AS(s)x + T(t-s)BS(s)x \\ &= 0,\end{aligned}$$

which implies that the function is constant. In particular, its values at $s = 0$ and $s = t$ are the same, i.e. $T(t)x = S(t)x$. Since x and t are arbitrary, we conclude that the equality holds for all $t \geq 0$ and $x \in D(A)$, as desired. □

1.2 The Hille-Yosida theorem

As we have already mentioned, our goal in this subsection will be to characterize the infinitesimal generator of an arbitrary C_0 semigroup. This will be done in two steps. First, we prove a version of the result for a class of semigroups which will be called semigroups of contractions. After that, we extend the result for uniformly bounded semigroups, and from this the general result will follow.

Let $\{T(t)\}_{t\geq 0}$ be a C_0 semigroup on X. We have seen in Theorem 1.1.4 that there exist constants $\omega \geq 0$ and $M \geq 1$ such that

$$||T(t)|| \leq Me^{\omega t}, \quad t \geq 0. \tag{1.2}$$

Definition 1.2.1. The C_0 semigroup $\{T(t)\}_{t\geq 0}$ is said to be

(i) a *semigroup of contractions* or a *contractive semigroup* if $||T(t)|| \leq 1$, for all $t \geq 0$.

(ii) a *uniformly bounded semigroup* if there exists $M \geq 1$ such that $||T(t)|| \leq M$, for all $t \geq 0$.

In other words, a semigroup is said to be uniformly bounded if one can take $\omega = 0$ in (1.2) and contractive if we can additionally choose $M = 1$. Observe that this may be different from what is sometimes considered in the literature; indeed, some authors define a semigroup to be contractive if the norm of each operator is strictly less than 1.

Before stating the first version of the theorem, we recall some notions from basic functional analysis. For the proofs and the details, we refer for instance to [7]. Given a linear operator A on X, we denote, for $\lambda \in \mathbb{C}$, $A_\lambda = \lambda I - A$ and $R(\lambda : A) = (\lambda I - A)^{-1}$ (when it exists).

Definition 1.2.2. Given $A\colon D(A) \subset X \to X$ a linear operator on a Banach space X, $\lambda \in \mathbb{C}$ is said to be a *regular value* of A if the following three conditions are satisfied:

(R1) $R(\lambda : A)$ exists (i.e. A_λ is injective);

(R2) $R(\lambda : A)$ is defined on a dense subset of X (i.e. the image of A_λ is dense in X);

(R3) $R(\lambda : A)$ is bounded.

The *resolvent set* of A, which we denote by $\rho(A)$, is the set of regular values of A. More precisely,

$$\rho(A) = \{\lambda \in \mathbb{C}\colon A_\lambda \text{ is injective, } \overline{\operatorname{Im} A_\lambda} = X, \text{ and } R(\lambda : A) \text{ is bounded}\}.$$

The *spectrum* of A, $\sigma(A)$, is defined to be the complement in $\mathbb{C}$ of the resolvent of A,

$$\sigma(A) = \mathbb{C} \setminus \rho(A).$$

Observe that the spectrum of a linear operator A can be split as the disjoint union of three subsets, $\sigma(A) = \sigma_p(A) \cup \sigma_r(A) \cup \sigma_c(A)$, where:

(i) $\sigma_p(A) = \{\lambda \in \mathbb{C} \colon A_\lambda \text{ is not injective}\}$, the *point spectrum* of A;

(ii) $\sigma_r(A) = \{\lambda \in \mathbb{C} \colon A_\lambda \text{ is injective, } \overline{\operatorname{Im} A_\lambda} \neq X\}$, the *residual spectrum* of A;

(iii) $\sigma_c(A) = \{\lambda \in \mathbb{C} \colon A_\lambda \text{ is injective, } \overline{\operatorname{Im} A_\lambda} = X,\ R(\lambda : A) \text{ is not bounded}\}$, the *continuous spectrum* of A.

The elements of the point spectrum of A are usually called *eigenvalues* of A. It is clear that, whenever X is finite-dimensional, then the spectrum of A reduces to the point spectrum. In general, this is not the case in infinite-dimensional spaces. However, in Banach spaces, when A is a closed operator, one obtains an easier characterization of the resolvent set, due to the closed graph theorem.

Lemma 1.2.3. *Let X be a Banach space. If $A \colon D(A) \subset X \to X$ is a closed linear operator, then $\lambda \in \rho(A)$ if and only if $A_\lambda : D(A) \to X$ is bijective.*

Two other important results on resolvents are the following, which we will constantly use throughout this chapter.

Lemma 1.2.4. *Let $A \colon D(A) \subset X \to X$ be a linear operator. Given $\lambda, \mu \in \rho(A)$, we have:*

(i) $R(\lambda : A) R(\mu : A) = R(\mu : A) R(\lambda : A)$.

(ii) (Hilbert's resolvent identity)

$$R(\lambda : A) - R(\mu : A) = (\mu - \lambda) R(\lambda : A) R(\mu : A). \tag{1.3}$$

We now have all the tools from functional analysis needed to prove the well-known Hille-Yosida theorem. First, we consider the case of semigroups of contractions.

1.2.1 The Hille-Yosida theorem for semigroups of contractions

Theorem 1.2.5. *A linear operator A is the infinitesimal generator of a C_0 semigroup of contractions $T(t)$, $t \geq 0$, if and only if*

(i) A is closed and $\overline{D(A)} = X$

(ii) $(0, +\infty) \subset \rho(A)$ and, for every $\lambda > 0$,

$$||R(\lambda : A)|| \leq \frac{1}{\lambda}.$$

Whenever a linear operator A satisfies conditions *(i)* and *(ii)* we say that A satisfies the *Hille-Yosida conditions*. The proof of the theorem will be divided into several steps. To begin with, we show that generators of C_0 semigroups satisfy the Hille-Yosida conditions.

Proof. (1.2.5, $\Longrightarrow$) Suppose A is the infinitesimal generator of a semigroup of contractions $\{T(t)\}_{t\geq 0}$. From Corollary 1.1.8, A is closed and densely defined, and so *(i)* holds.

To prove the second statement, let $\lambda > 0$ and define, for $x \in X$,

$$R(\lambda)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt.$$

Observe that the function $t \mapsto T(t)x$ is continuous and uniformly bounded, so the integral is well defined. Moreover, it defines a bounded linear operator on X, as

$$||R(\lambda)x|| \leq \int_0^\infty e^{-\lambda t} ||T(t)x|| \, dt \leq ||x|| \int_0^\infty e^{-\lambda t} \, dt \leq \frac{1}{\lambda} ||x||, \quad x \in X,$$

and so

$$||R(\lambda)|| \leq \frac{1}{\lambda}. \tag{1.4}$$

Now observe that, if $h > 0$,

$$\begin{aligned} \frac{T(h)-I}{h} R(\lambda)x &= \frac{1}{h} \int_0^\infty e^{-\lambda t} \left(T(t+h) - T(t)x\right) dt \\ &= \frac{e^{\lambda h}-1}{h} \int_h^\infty e^{-\lambda t} T(t)x \, dt - \frac{1}{h} \int_0^h e^{-\lambda t} T(t)x \, dt \end{aligned}$$

and letting $h \to 0^+$ we obtain

$$\frac{T(h)-I}{h} R(\lambda)x \to \lambda R(\lambda)x - x, \quad \text{as } h \to 0^+.$$

Consequently, for every $x \in X$ and $\lambda > 0$,

$$R(\lambda)x \in D(A) \quad \text{and} \quad AR(\lambda)x = \lambda R(\lambda)x - x,$$

and it follows that

$$(\lambda I - A)R(\lambda) = I. \tag{1.5}$$

Also, for $x \in D(A)$,

$$\begin{aligned} R(\lambda)Ax = \int_0^\infty e^{-\lambda t} T(t)Ax \, dt &= \int_0^\infty e^{-\lambda t} AT(t)x \, dt \\ &= A\left(\int_0^\infty e^{-\lambda t} T(t)x \, dt \right) \\ &= AR(\lambda)x, \end{aligned} \tag{1.6}$$

where the third equality holds because A is closed. Hence

$$R(\lambda)(\lambda I - A) = I. \tag{1.7}$$

Therefore, it follows from (1.5) and (1.7) that $\lambda \in \rho(A)$ and that $R(\lambda : A) = R(\lambda)$. Since $\lambda > 0$ was

arbitrary, we obtain $(0,\infty) \subset \rho(A)$, proving the first claim in *(ii)*. The second statement in *(ii)* is simply (1.4). This concludes the proof. □

We learn some important facts from the proof. First, we have shown that $R(\lambda)$, defined as the Laplace transfom of the semigroup, is the inverse of $\lambda I - A$ and so we have an explicit formula for the resolvent function $R(\lambda : A)$. Also, (1.6) shows that $R(\lambda : A)$ and A commute.

One of the implications of Theorem 1.2.5 is then verified. To prove the converse statement, we first need to establish some auxiliary results.

Lemma 1.2.6. *Suppose A is a linear operator that satisfies the Hille-Yosida conditions.*

(i) For any $x \in X$,

$$\lim_{\lambda \to \infty} \lambda R(\lambda : A)x = x.$$

(ii) Define $A_\lambda := \lambda AR(\lambda : A) = \lambda^2 R(\lambda : A) - \lambda I$, for $\lambda > 0$. Then, for any $x \in D(A)$,

$$\lim_{\lambda \to \infty} A_\lambda x = Ax.$$

Proof. We start by proving *(i)*. Suppose first that $x \in D(A)$. Then

$$||\lambda R(\lambda : A)x - x|| = ||AR(\lambda : A)x|| = ||R(\lambda : A)Ax|| \leq \frac{1}{\lambda}||Ax|| \to 0, \quad \text{as} \quad \lambda \to \infty.$$

But $D(A)$ is dense in X and $\lambda R(\lambda : A)$ is a bounded operator, so the above can be extended to $x \in X$.

To see *(ii)*, note that if $x \in D(A)$, we have

$$A_\lambda x = \lambda AR(\lambda : A)x = \lambda R(\lambda : A)Ax \to Ax,$$

by part *(i)*. □

The previous lemma justifies the following definition.

Definition 1.2.7. We define the *Yosida approximation* of a linear operator A satisfying the Hille-Yosida conditions as the operator

$$A_\lambda = \lambda AR(\lambda : A) = \lambda^2 R(\lambda : A) - \lambda I, \quad \lambda > 0.$$

Observe that, even if A is not bounded, its Yosida approximation A_λ is a bounded linear operator, as $\lambda > 0$ implies that $\lambda \in \rho(A)$. But then, as previously mentioned, it must be the generator of a uniformly continuous semigroup $T_\lambda(t) = e^{tA_\lambda}$. The following result shows that this turns out to be a contractive semigroup and provides us with a useful estimate.

Lemma 1.2.8. *If A satisfies the Hille-Yosida conditions and A_λ is its Yosida approximation, then A_λ is the infinitesimal generator of a uniformly continuous semigroup of contractions e^{tA_λ}. Furthermore, for every $x \in X$ and $\lambda, \mu > 0$, we have*

$$||e^{tA_\lambda}x - e^{tA_\mu}x|| \leq t||A_\lambda x - A_\mu x||, \quad t \geq 0.$$

Proof. We know that A_λ is the generator of the uniformly continuous semigroup $T_\lambda(t) = e^{tA_\lambda}$. Note that

$$||e^{tA_\lambda}|| = e^{-t\lambda}||e^{t\lambda^2 R(\lambda:A)}|| \leq e^{-t\lambda}e^{t\lambda^2||R(\lambda:A)||} \leq 1,$$

and so $T(t)$ is indeed a semigroup of contractions.

Now let $\lambda, \mu > 0$. Since all the operators A, A_λ, A_μ commute with each other (see the observations following the first part of the proof of 1.2.5), we have, for $t \geq 0$,

$$\begin{aligned} ||e^{tA_\lambda}x - e^{tA_\mu}x|| &\leq \int_0^1 \left|\left| \frac{d}{ds}\left(e^{tsA_\lambda}e^{t(1-s)A_\mu}x\right)\right|\right| ds \\ &= \int_0^1 t\,||e^{tsA_\lambda}e^{t(1-s)A_\mu}(A_\lambda x - A_\mu x)||\,ds \\ &\leq t\,||A_\lambda x - A_\mu x||, \end{aligned}$$

concluding the proof. □

We can finally finish the proof of the Hille-Yosida theorem for contractive semigroups. Recall that it remains to prove that if A satisfies the Hille-Yosida conditions, then it is the infinitesimal generator of a semigroup of contractions.

Proof. (1.2.5, $\Longleftarrow$) Let A be a linear operator under the Hille-Yosida conditions and fix $x \in D(A)$. Then, if $\lambda, \mu > 0$, using (1.2.8) we obtain

$$||e^{tA_\lambda}x - e^{tA_\mu}x|| \leq t||A_\lambda x - A_\mu x|| \leq t||A_\lambda x - Ax|| + t||Ax - A_\mu x||,$$

which implies $e^{tA_\lambda}x$ converges as $\lambda \to \infty$ (and the convergence is uniform on bounded intervals). Since $D(A)$ is dense in X and the sequence is bounded, it follows that

$$T(t)x = \lim_{\lambda\to\infty} e^{tA_\lambda}x, \quad x \in X$$

exists for every x and this limit is uniform for t in compact intervals. It is clear that $T(t)$ is a semigroup on X, with $||T(t)|| \leq 1$. Also, since it is a uniform limit of bounded operators it is itself bounded, so $\{T(t)\}_{t\geq 0}$ is in fact a C_0 semigroup of contractions on X. To conclude, we now need to show that A is the infinitesimal generator of $T(t)$.

We start by noting that, if $x \in D(A)$, we have, for $t \geq 0$

$$T(t)x - x = \lim_{\lambda\to\infty} e^{tA_\lambda}x - x = \lim_{\lambda\to\infty}\int_0^t e^{sA_\lambda}A_\lambda x\,ds = \int_0^t T(s)Ax\,ds,$$

because of uniform convergence on $[0,t]$. Let B be the infinitesimal generator of $\{T(t)\}_{t\geq 0}$ and let $t > 0$.

Then

$$\lim_{t\to 0^+} \frac{T(t)x-x}{t} = \lim_{t\to 0^+} \frac{1}{t}\int_0^t T(s)Axds = Ax,$$

so $x \in D(B)$ and $Bx = Ax$. Since B is the generator of a semigroup of contractions, by the converse direction of the theorem we know that $1 \in \rho(B)$. By assumption, we also have $1 \in \rho(A)$. Then,

$$(I-B)D(A) = (I-A)D(A) = X,$$

and so $D(B) = (I-B)^{-1}X = D(A)$. This proves that $A = B$, and concludes the proof of the theorem. □

From the proof, we obtain the following interesting consequences:

Corollary 1.2.9. *Let A be the infinitesimal generator of a C_0 semigroup of contractions $\{T(t)\}_{t\geq 0}$. If A_λ is the Yosida approximation of A, then*

$$T(t)x = \lim_{\lambda\to\infty} e^{tA_\lambda}x, \quad t \geq 0.$$

Proof. From the proof of the theorem, the right-hand side of the equality defines a C_0 semigroup of contractions $S(t)$ whose infinitesimal generator is A. By uniqueness, $T(t) = S(t)$ must hold for all $t \geq 0$. □

Corollary 1.2.10. *Let A be the infinitesimal generator of a C_0 semigroup of contractions $\{T(t)\}_{t\geq 0}$. The resolvent set of A contains the open right half-plane,*

$$\{\lambda : \mathrm{Re}(\lambda) > 0\} \subset \rho(A)$$

and for such λ we have

$$||R(\lambda : A)|| \leq \frac{1}{\mathrm{Re}(\lambda)}.$$

Proof. The operator

$$R(\lambda)x = \int_0^\infty e^{-\lambda t}T(t)x\, dt$$

is well-defined for λ with $\mathrm{Re}(\lambda) > 0$ and we have seen in the proof that

$$R(\lambda) = (\lambda I - A)^{-1}.$$

So $\{\lambda : \mathrm{Re}(\lambda) > 0\} \subset \rho(A)$ and

$$||(\lambda I - A)^{-1}x|| \leq \int_0^\infty e^{-\mathrm{Re}(\lambda)t}||T(t)||\,||x||\, dt \leq \frac{1}{\mathrm{Re}(\lambda)}||x||.$$

□

There is an easy generalization of this version of the Hille-Yosida theorem for a C_0 semigroup $\{T(t)\}_{t\geq 0}$ satisfying

$$||T(t)|| \leq e^{\omega t}, \quad \text{for } t \geq 0.$$

Such semigroups are called ω-*contractive semigroups*. Define $S(t) = e^{-\omega t}T(t)$, for $t \geq 0$. It is clear that $\{S(t)\}_{t\geq 0}$ is also a C_0 semigroup and a simple computation yields $||T(t)|| \leq 1$; in other words, $S(t)$ is a contractive semigroup, and so Theorem 1.2.5 applies. But note that, if A is the infinitesimal generator of $\{S(t)\}_{t\geq 0}$, then $A+\omega I$ is the generator of $\{T(t)\}_{t\geq 0}$. Conversely, if A is the generator of $\{T(t)\}_{t\geq 0}$, then $A-\omega I$ is the generator of $\{S(t)\}_{t\geq 0}$. This leads us to the following characterization of the generators of ω-contractive semigroups, which generalizes Theorem 1.2.5.

Theorem 1.2.11. *A linear operator A is the infinitesimal generator of an ω-contractive C_0 semigroup $\{T(t)\}_{t\geq 0}$ if and only if*

(i) A is closed and $\overline{D(A)} = X$

(ii) $(\omega, +\infty) \subset \rho(A)$ and, for every $\lambda > \omega$,

$$||R(\lambda : A)|| \leq \frac{1}{\omega - \lambda}.$$

1.2.2 The Hille-Yosida theorem for uniformly bounded semigroups

In this section we extend the version of Hille-Yosida's theorem for contractive semigroups to the more general case of uniformly bounded semigroups. Recall that a semigroup $\{T(t)\}_{t\geq 0}$ is said to be uniformly bounded if there exists $M \geq 1$ such that $||T(t)|| \leq M$, for all $t \geq 0$. Our strategy will be to define a new norm on our Banach space X, equivalent to the original norm, under which the semigroup becomes a contractive semigroup. At this step, the previous section is applicable, and we can characterize its generator. All there will be left to do in the end is to go back to the original norm, and compare the conclusions.

To start this approach, we need the following lemma, which tells us that it does in fact exist a new norm on the space satisfying our requirements. We denote the initial norm in X by $||\cdot||$.

Lemma 1.2.12. *Let A be a linear operator on X for which $(0,\infty) \subset \rho(A)$. If there exists $M \geq 0$ such that*

$$||\lambda^n R(\lambda : A)|| \leq M, \quad \text{for all } n \in \mathbb{N}, \lambda > 0$$

then there exists a norm $|\cdot|$ on X which satisfies

$$||x|| \leq |x| \leq M||x||, \quad |\lambda R(\lambda : A)x| \leq |x|,$$

for all $x \in X$ and $\lambda > 0$.

Proof. Let $\mu > 0$ and define

$$||x||_\mu = \sup_{n\geq 0} ||\mu^n R(\mu : A)^n x||, \quad x \in X.$$

Since $x = \mu^0 (R\mu : A)^0 x$ and $||\lambda^n R(\lambda : A)|| \leq M$, it follows that

$$||x|| \leq ||x||_\mu \leq M||x||. \tag{1.8}$$

Moreover, for any $x \in X$,

$$||\mu R(\mu : A)x||_\mu = \sup_{n\geq 0} ||\mu^{n+1} R(\mu : A)^{n+1} x|| \leq \sup_{n\geq 0} ||\mu^n R(\mu : A)^n x|| = ||x||_\mu,$$

and hence

$$||\mu R(\mu : A)||_\mu \leq 1.$$

We claim that also $||\lambda R(\lambda : A)||_\mu \leq 1$, for all $0 < \lambda \leq \mu$. Indeed, let $y = R(\lambda : A)x$. Using Hilbert's identity

$$R(\lambda : A) - R(\mu : A) = (\mu - \lambda)R(\lambda : A)R(\mu : A).$$

we can write

$$y = R(\mu : A)(x + (\mu - \lambda)y)$$

and so

$$||y||_\mu \leq \frac{1}{\mu}||x||_\mu + \left(1 - \frac{\lambda}{\mu}\right)||y||_\mu,$$

which is equivalent to

$$||\lambda R(\lambda : A)x||_\mu \leq ||x||_\mu$$

and this implies the claim.

Now from this and (1.8) we obtain, for every $x \in X$ and $0 < \lambda \leq \mu$,

$$||\lambda^n R(\lambda : A)^n x|| \leq ||\lambda^n R(\lambda : A)^n x||_\mu \leq ||x||_\mu. \tag{1.9}$$

Taking the supremum over $n \geq 0$ on both members above we get

$$||x||_\lambda \leq ||x||_\mu$$

and this holds for every $0 < \lambda \leq \mu$.

Finally, define $|x| = \lim_{\mu\to\infty} ||x||_\mu$. It is clear that this is a non-negative, homogeneous function satisfying the triangle inequality. Furthermore, from (1.8) it follows that $|x| = 0$ if and only if $x = 0$ and hence $|\cdot|$ defines a norm in X. Note that (1.8) implies that $|\cdot|$ is equivalent to the original norm $||\cdot||$; in fact, it holds that

$$||x|| \leq |x| \leq M||x||$$

holds for every $x \in X$. Moreover, taking $n = 1$ in (1.9) yields

$$||\lambda R(\lambda : A)x||_\mu \leq ||x||_\mu$$

and so, taking the limit when $\mu \to \infty$, we obtain

$$|\lambda R(\lambda : A)x| \leq |x|,$$

for every $x \in X$ and $\lambda > 0$. This concludes the proof. □

Before moving on, observe that the strategy that we are employing, of alternating between norms, could cause some problems, namely as far as strong continuity of the semigroup or the Hille-Yosida conditions for the generators are concerned. However, this is not an issue here, as passing to an equivalent norm does not change neither the fact that the generator is closed and densely defined nor the strong continuity of the semigroup. All these properties are topological, and do not change when an equivalent norm is considered on the space. Therefore, we need not worry with this approach, and we can finally state and prove the version of Hille-Yosida's theorem for uniformly bounded semigroups.

Theorem 1.2.13. *A linear operator A is the infinitesimal generator of a uniformly bounded C_0 semigroup $\{T(t)\}_{t\geq 0}$ satisfying*

$$||T(t)|| \leq M, \quad t \geq 0$$

for some $M \geq 1$ if and only if

(i) A is closed and $\overline{D(A)} = X$

(ii) $(0, +\infty) \subset \rho(A)$ and, for every $\lambda > 0$ and $n \in \mathbb{N}$,

$$||R(\lambda : A)^n|| \leq \frac{M}{\lambda^n}.$$

Proof. Let A be the infinitesimal generator of a C_0 semigroup satisfying $||T(t)|| \leq M$, for all $t \geq 0$. Define

$$|x| = \sup_{t\geq 0} ||T(t)x||.$$

Then

$$||x|| \leq |x| \leq M||x||$$

for every $x \in X$, which implies that $|\cdot|$ is a norm on X which is equivalent to $||\cdot||$. Furthermore

$$|T(t)x| = \sup_{s\geq 0} ||T(s)T(t)x|| = \sup_{s\geq 0} ||T(t+s)x|| \leq ||T(t)x|| = |x|,$$

and consequently $\{T(t)\}_{t\geq 0}$ is a contractive semigroup when X is endowed with the norm $|\cdot|$. By the version of the theorem for semigroups of contractions, we conclude that A is closed and densely defined (for both $|\cdot|$ and $||\cdot||$) and that

$$|R(\lambda : A)| \leq \frac{1}{\lambda},$$

for every $\lambda > 0$. Consequently, given $x \in X$ and $\lambda > 0$,

$$||R(\lambda : A)^n x|| \leq |R(\lambda : A)^n x| \leq \frac{|x|}{\lambda^n} \leq \frac{M}{\lambda^n}||x||,$$

which implies that

$$||R(\lambda : A)^n|| \leq \frac{M}{\lambda^n},$$

as desired.

For the other direction, suppose A satisfies Hille-Yosida conditions. By Lemma 1.2.12, there exists a norm $|\cdot|$ on X which is equivalent to $||\cdot||$ and satisfies

$$||x|| \leq |x| \leq M||x||, \quad |\lambda R(\lambda : A)x| \leq |x|.$$

Considering X endowed with this norm, A remains a closed, densely defined operator with $(0,\infty) \subset \rho(A)$ but now satisfies

$$|R(\lambda : A)x| \leq \frac{1}{\lambda}|x|,$$

for every $x \in X$ and $\lambda > 0$, so that

$$|R(\lambda : A)| \leq \frac{1}{\lambda}$$

holds true for every $\lambda > 0$. By the contractive version of the Hille-Yosida theorem, A is the infinitesimal generator of a C_0 semigroup of contractions $\{T(t)\}_{t\geq 0}$ on X, when X is considered with the norm $|\cdot|$. But note that

$$||T(t)x|| \leq |T(t)x| \leq |x| \leq M||x||$$

for every $x \in X$ and $t \geq 0$, which implies that

$$||T(t)|| \leq M, \quad t \geq 0.$$

In other words, $\{T(t)\}_{t\geq 0}$ is a uniformly bounded C_0 semigroup on X (with the original norm), which is exactly what we wanted to prove. This then concludes the proof ot this version of the Hille-Yosida theorem. □

1.2.3 The general version of the Hille-Yosida theorem

The proof of the general version of Hille-Yosida's theorem will now follow very easily from the previous cases, by an argument we have already encountered when we extended the contractive case to the ω-contractive version.

In fact, suppose $\{T(t)\}_{t\geq 0}$ is a C_0 semigroup, satisfying

$$||T(t)|| \leq Me^{\omega t}$$

for every $t \geq 0$. Define, for each $t \geq 0$, $S(t) = e^{-\omega t}T(t)$. It is clear that $\{S(t)\}_{t\geq 0}$ is now a strongly continuous uniformly bounded semigroup on X and, as before, it is easy to compare the infinitesimal generators of $\{T(t)\}_{t\geq 0}$ and $\{S(t)\}_{t\geq 0}$. Namely, if A is the generator of $T(t)$, then $A - \omega I$ is the generator of $S(t)$, and, if A is the generator of $S(t)$, then $A + \omega I$ is the generator of $T(t)$. In conclusion, we obtain:

Theorem 1.2.14. (Hille-Yosida) *A linear operator A is the infinitesimal generator of a C_0 semigroup $\{T(t)\}_{t\geq 0}$ satisfying*

$$||T(t)|| \leq Me^{\omega t}, \quad t \geq 0$$

for some $M \geq 1$ and $\omega \geq 0$ if and only if

(i) A is closed and $\overline{D(A)} = X$

(ii) $(\omega, +\infty) \subset \rho(A)$ and, for every $\lambda > \omega$ and $n \in \mathbb{N}$,

$$||R(\lambda : A)^n|| \leq \frac{M}{(\lambda - \omega)^n}.$$

From now on, the term 'Hille-Yosida conditions' refers to the conditions *(i)* and *(ii)* above, instead of those in the contractive version of the Hille-Yosida theorem. We now have a full characterization of the generators of strongly continuous semigroups. It is worth mentioning that the Hille-Yosida conditions imply a stronger result on the resolvent of A, similar to what happened on Corollary 1.2.10. In fact, not only all real numbers $\lambda > \omega$ are in $\rho(A)$, but all complex numbers λ with $\operatorname{Re}\lambda > \omega$ are in $\rho(A)$, and for such λ the estimate

$$||R(\lambda : A)^n|| \leq \frac{M}{(\operatorname{Re}\lambda - \omega)^n}. \tag{1.10}$$

holds. It follows, as it did on the proof of the first part of Theorem 1.2.5, that

$$R(\lambda)x = \int_0^\infty e^{-\lambda t} T(t)x\,dt, \quad x \in X, \tag{1.11}$$

is well defined for $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > \omega$ and that $R(\lambda) = R(\lambda : A)$. In particular, $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\} \subset \rho(A)$. This proves our first claim. The estimate (1.10) is not so simple to establish. We start by observing that, on the one hand,

$$\frac{d}{d\lambda} R(\lambda : A)x = \frac{d}{d\lambda} \int_0^\infty e^{-\lambda t} T(t)x\,dt = -\int_0^\infty t e^{-\lambda t} T(t)x\,dt,$$

where we have used Lebesgue's dominated convergence to differentiate under the integral sign. Proceeding by induction,

$$\frac{d^n}{d\lambda^n} R(\lambda : A)x = (-1)^n \int_0^\infty t^n e^{-\lambda t} T(t)x\,dt, \quad n \in \mathbb{N} \tag{1.12}$$

On the other hand, dividing both members on Hilbert's identity

$$R(\lambda : A) - R(\mu : A) = (\mu - \lambda) R(\lambda : A) R(\mu : A)$$

by $\lambda - \mu$ and taking the limit when $\mu \to \lambda$, we conclude that $R(\lambda : A)$ is holomorphic at every $\lambda \in \rho(A)$ with

$$\frac{d}{d\lambda} R(\lambda : A) = -R(\lambda : A)^2.$$

Arguing again by induction, this implies that

$$\frac{d^n}{d\lambda^n} R(\lambda : A) = (-1)^n n! R(\lambda : A)^{n+1}, \quad n \in \mathbb{N}. \tag{1.13}$$

Combining (1.12) and (1.13) we obtain, for every $x \in X$,

$$R(\lambda : A)^n x = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-\lambda t} T(t)x\,dt,$$

so that

$$\begin{aligned}||R(\lambda : A)^n x|| &\leq \frac{M}{(n-1)!}\int_0^\infty t^{n-1}e^{(\omega-\operatorname{Re}\lambda)t}\,||x||\,dt \\ &= \frac{M}{(n-1)!}\,\frac{1}{(\operatorname{Re}\lambda-\omega)^n}\int_0^\infty s^{n-1}e^{-s}\,||x||\,ds \\ &= \frac{M}{(n-1)!}\,\frac{1}{(\operatorname{Re}\lambda-\omega)^n}\,\Gamma(n)\,||x||\,ds \\ &= \frac{M}{(\operatorname{Re}\lambda-\omega)^n}||x||.\end{aligned}$$

Consequently,

$$||R(\lambda : A)^n|| \leq \frac{M}{(\operatorname{Re}\lambda-\omega)^n},$$

and this proves our claim. We have thus achieved our goal for this section.

1.3 Some further results

Before we begin the study of adjoint semigroups, we present some results which relate to the previous topics and will be necessary later in the text. Here we only prove Datko-Pazy's theorem, but provide references for the details that are omitted.

Suppose a linear operator A in X is given, with A the infinitesimal generator of some C_0 semigroup $\{T(t)\}_{t\geq 0}$. We already know that A uniquely determines $T(t)$, but can we find an explicit formula for the semigroup in terms of A? We have seen that, if A is the generator of a contractive semigroup, then the answer is positive; this is the content of Corollary 1.2.9, which involves the Yosida approximation of A. This is also true in the general case, as the first part of the next result states. The second part of the theorem provides us with an alternative exponential-like formula for the semigroup.

Theorem 1.3.1. *Let A be the infinitesimal generator of a C_0 semigroup $T(t)$ on X.*

(i) For any $x \in X$, $\lim_{\lambda\to\infty} \lambda R(\lambda : A)x = x$.

(ii) If $A_\lambda = \lambda AR(\lambda : A)$ is the Yosida approximation of A, then, for all $x \in X$

$$T(t)x = \lim_{\lambda\to\infty} e^{tA_\lambda}x.$$

(iii) For any $x \in X$,

$$T(t)x = \lim_{n\to\infty}\left(I-\frac{t}{n}A\right)^{-n}x = \lim_{n\to\infty}\left[\frac{n}{t}R\left(\frac{n}{t} : A\right)\right]^n x,$$

and the limit is uniform in t on any bounded interval.

For the proofs, we refer to Theorems 5.5 and 8.3 in [22, Chapter 1].

Another important question is the relation between the spectra of the semigroup and its generator. We refer to Theorems 2.3, 2.4 and 2.5 in [22, Chapter 2] for results concerning some relations between the point, residual and continuous spectrum of the semigroup and its generator, respectively. We state only the first one.

Theorem 1.3.2. *Let $\{T(t)\}_{t\geq 0}$ be a C_0 semigroup and let A be its infinitesimal generator. Then*

$$e^{t\sigma_p(A)} \subset \sigma_p(T(t)) \subset e^{t\sigma_p(A)} \cup \{0\},$$

for every $t \geq 0$.

To be more precise, this means that, if $\lambda \in \sigma_p(A)$, then $e^{\lambda t} \in \sigma_p(T(t))$ and, conversely, if $e^{\lambda t} \in \sigma_p(T(t))$, there exists $k \in \mathbb{N}$ such that $\lambda_k = \lambda + 2\pi ik/t \in \sigma_p(A)$. This result plays an important role in the case of equations with finite delay. As mentioned in the Introduction, it can be seen that the solution semigroup of the linearized problem is eventually compact and, consequently, its spectrum (except perhaps the 0) reduces to point spectrum; using Theorem 1.3.2, one can prove that the same happens to its infinitesimal generator.

The next result is concerned with obtaining sufficient conditions to characterize the resolvent of a densely defined closed operator. Recall that, if A is a linear operator and λ, $\mu \in \rho(A)$, then Hilbert's resolvent identity (1.3) holds. Moreover, we also know that

$$\lim_{\lambda\to\infty} \lambda R(\lambda : A)x = x,$$

for any $x \in X$, where A is a linear operator satisfying the Hille-Yosida conditions. These are particularly important features of the resolvent function. The theorem below states that they are also in some sense sufficient for a given function to be the resolvent of a closed densely defined operator.

Theorem 1.3.3. *Let $S \subset \mathbb{C}$ be an unbounded subset of $\mathbb{C}$ and let, for $\lambda \in S$, $J(\lambda) : X \to X$ be a linear operator satisfying Hilbert's identity (1.3) on S. If there exists a sequence $(\lambda_n) \subset S$ such that $|\lambda_n| \to \infty$, as $n \to \infty$, and*

$$\lim_{n\to\infty} \lambda_n J(\lambda_n)x = x, \quad \text{for all } x \in X,$$

then $J(\lambda)$ is the resolvent of a unique densely defined closed operator A.

This is Corollary 9.5 in [22, Chapter 1].

To conclude this section, we now finally state and prove one interesting theorem due to Datko and Pazy, that provides us with conditions under which we can take a negative ω in (1.2).

Theorem 1.3.4. (Datko-Pazy) *Let $\{T(t)\}_{t\geq 0}$ be a C_0 semigroup on X. If, for some $p \in [1,\infty)$,*

$$\int_0^\infty ||T(t)x||^p \, dt < \infty, \quad x \in X, \tag{1.14}$$

then there are constants $M' \geq 1$ and $\mu > 0$ such that $||T(t)|| \leq M'e^{-\mu t}$, for $t \geq 0$.

Proof. We start by proving that (1.14) implies that the mapping $t \mapsto ||T(t)||$ is bounded. Choose $M \geq 1$ and $\omega \geq 0$ such that

$$||T(t)|| \leq Me^{\omega t}, \quad t \geq 0.$$

If $\omega = 0$ there is nothing to prove, so we assume $\omega > 0$.

We claim that $T(t)x \to 0$ as $t \to \infty$, for every $x \in X$. In fact, if this is false, there exist $x \in X$, $\delta > 0$ and a sequence $t_n \to \infty$ such that

$$||T(t_n)x|| \geq \delta.$$

Assume without loss of generality that $t_{n+1} - t_n > \omega^{-1}$, for all $n \in \mathbb{N}$, and define $\Delta_n = [t_n - \omega^{-1}, t_n]$. Then, the intervals Δ_n do not overlap and, for $t \in \Delta_n$,

$$\delta \leq ||T(t_n)x|| \leq ||T(t_n - t)||\,||T(t)x|| \leq Me^{\omega(t_n - t)}||T(t)x|| \leq Me||T(t)x||,$$

the last inequality because $t_n - t \leq \omega^{-1}$. Consequently,

$$\int_0^\infty ||T(t)x||^p\,dt \geq \sum_{n=1}^\infty \int_{\Delta_n} ||T(t)x||^p\,dt \geq \left(\frac{\delta}{Me}\right)^p \sum_{n=1}^\infty m(\Delta_n) = \infty,$$

and this contradicts (1.14). Therefore, $T(t)x \to 0$ as $t \to \infty$, for every $x \in X$, as claimed. By the uniform boundedness theorem, this implies that there exists $K \geq 0$ such that, for all $t \geq 0$,

$$||T(t)|| \leq K. \tag{1.15}$$

Since $||T(0)|| = 1$, actually $K \geq 1$.

Now consider the mapping

$$S\colon X \to L^p(\mathbb{R}_+), \quad (Sx)(t) = T(t)x.$$

This is well defined on the whole X, by assumption (1.14). Moreover, S is a closed operator. To see this, we prove that, if $x_n \to 0$ in X and $Sx_n \to \varphi$ in L^p, then $\varphi = 0$. Since convergence in L^p implies convergence a.e. up to a subsequence, we can choose x_{n_k} such that $(Sx_{n_k})(t) \to \varphi(t)$, as $k \to \infty$, for a.e. $t \geq 0$. But $(Sx_{n_k})(\cdot) = T(\cdot)x_{n_k}$, by definition, and we have

$$||T(\cdot)x_{n_k}||_X \leq K||x_{n_k}||_X \to 0, \quad k \to \infty.$$

Then it must be that $\varphi = 0$, proving that S is closed. By the Closed Graph Theorem, S is bounded, and so there exists $C \geq 0$ such that

$$\int_0^\infty ||T(t)x||^p\,dt \leq C^p\,||x||^p,$$

for all $x \in X$.

Let $0 < \rho < K^{-1}$, where K is as in (1.15), and define

$$t_x(\rho) = \sup\{t\colon ||T(s)x|| \geq \rho||x||, \text{ for } 0 \leq s \leq t\}, \quad x \in X$$

Since $||T(t)x|| \to 0$, it is clear that $t_x(\rho)$ is finite. It must also be positive; since the inequality holds at 0, it must also hold on a small neighbourhood of 0, by continuity of $t \mapsto ||T(t)x||$ (note that $\rho < 1$). Furthermore,

$$t_x(\rho)\rho^p||x||^p \leq \int_0^{t_x(\rho)} ||T(t)x||^p\,dt \leq \int_0^\infty ||T(t)x||^p\,dt \leq C^p||x||^p$$

and hence

$$t_x(\rho) \leq \left(\frac{C}{\rho}\right)^p := t_0, \quad \text{for all } x \in X$$

On the other hand, from the definition of $t_x(\rho)$ it follows that

$$||T(t_x(\rho))x|| = \rho||x||.$$

So, if $t > t_0$, we have

$$||T(t)x|| \leq ||T(t - t_x(\rho))||\,||T(t_x(\rho))x|| \leq K\rho||x||.$$

Define $\beta = K\rho < 1$ and fix any $t_1 > t_0$. Given $t \geq 0$, there exist $n \in \mathbb{N}$ and $0 \leq s < t_1$ such that $t = nt_1 + s$. Then

$$||T(t)|| = ||T(s)||\,||T(nt_1)|| \leq K\,||T(t_1)||^n \leq K\beta^n \leq M'e^{-\mu t},$$

where

$$M' = K\beta^{-1} \geq 1, \quad \mu = -\frac{1}{t_1}\log\beta > 0.$$

This concludes the proof. □

The case $p = 2$ was first proved by Datko in [8], and the proof strongly relies on the fact that L^2 is a Hilbert space. It was later generalized by Pazy [22], for any $p \in [1, \infty)$. This result is of course of major interest in applications, as it implies exponential decay of the semigroup at infinity.

1.4 The adjoint semigroup

In this section we define the adjoint or "dual" of a semigroup of linear operators. We show, via an example, that strong continuity is not preserved by passing to the adjoint semigroup and a natural question arises: can we restrict the adjoint semigroup to a non-trivial subspace where it is strongly continuous? The answer is yes, and this is the content of the main result of this section. Before proving it, we recall some classical results in functional analysis concerning the adjoint of linear operators. The proofs in this first subsection are omitted, and we refer to [1, 25] for the details and some more results.

1.4.1 The adjoint of a linear operator

As usual, denote by X^* the dual space of X, i.e. the space of bounded linear functionals $x^* : X \to \mathbb{C}$. It is well known that X^* is a Banach space, when endowed with the operator norm

$$||x^*|| = \sup\{\langle x, x^*\rangle : ||x|| \leq 1\}.$$

The notation $\langle \cdot, \cdot \rangle$ denotes, as usual, the duality pairing $\langle x, x^*\rangle := x^*(x)$, for $x^* \in X^*$, $x \in X$. We start by recalling some facts about adjoint operators in Banach spaces.

Definition 1.4.1. Let $S : D(S) \subset X \to X$ be a densely defined linear operator on X. The adjoint operator S^* of S is the operator $S^* : D(S^*) \subset X^* \to X^*$ defined as follows:

- $D(S^*)$ is the set of $x^* \in X^*$ for which there exists $y^* \in X^*$ such that

$$\langle Sx, x^*\rangle = \langle x, y^*\rangle, \quad x \in D(S).$$

- for $x^* \in D(S^*)$, we define $S^*x^* = y^*$.

Observe that the adjoint operator is well-defined; since $D(S)$ is dense in X, there exists at most one $y^* \in X^*$ such that the relation above holds. It is also well known that, if the initial operator is bounded, then so is its adjoint and their norms are the same. On the other hand, if S is a densely defined operator, then S^* need not be densely defined, as the following example shows.

Example 1.4.2. Let $X = \ell^1$, so that $X^* = \ell^\infty$, and define $A : D(A) \subset X \to X$ as

$$D(A) = \{x = (x_n) \in \ell^1 : (nx_n) \in \ell^1\}, \qquad Ax = (nx_n).$$

By standard arguments, one can show that A is closed and densely defined, and that the adjoint operator A^* is given by

$$D(A^*) = \{y = (y_n) \in \ell^\infty : (ny_n) \in \ell^\infty\}, \qquad A^*y = (ny_n).$$

We claim that A^* is not densely defined, i.e. that $\overline{D(A^*)} \neq \ell^\infty$. To see this, note that $y = (y_n) \in \ell^\infty$ implies that $|y_n| \to 0$, so $D(A^*) \subset c_0$, the space of sequences converging to 0. But c_0 is a proper closed subspace of ℓ^∞, so also $\overline{D(A^*)} \subset c_0$, and c_0 is a proper subspace of ℓ^∞. Consequently, $\overline{D(A^*)} \neq \ell^\infty$, and so A^* is not densely defined (in fact, it is simple to prove that actually $\overline{D(A^*)} = c_0$).

This also suggests that, a priori, one might have $D(S^*) = \{0\}$. This can in fact occur, but not if S is closed. In fact, the following result is true.

Lemma 1.4.3. *If S is a closed linear operator on X, then $D(S^*)$ is weak-$*$ dense in X^*.*

For the proof, see [25]. This implies that $D(S^*) \neq \{0\}$, for if it were then also $X^* = \{0\}$, and consequently $X = \{0\}$. The final result that we need to recall is the following relation between the resolvent of a densely defined operator and its adjoint.

Lemma 1.4.4. *Let A be a densely defined linear operator in X. If $\lambda \in \rho(A)$, then $\lambda \in \rho(A^*)$ and*

$$R(\lambda : A^*) = R(\lambda : A)^*.$$

The converse inclusion is also true if A is a closed operator. In particular, whenever A is closed, then $\sigma(A) = \sigma(A^*)$.

1.4.2 The adjoint of a semigroup of linear operators

We can now finally move on to the definition of the adjoint semigroup. Let $\{T(t)\}_{t\geq 0}$ be a semigroup on X and, for $t \geq 0$, denote by $T(t)^*$ the adjoint operator of $T(t)$. Using the definition of the adjoint operator, it is easy to see that the family $\{T(t)^*\}_{t\geq 0}$ is also a semigroup on X^*.

Definition 1.4.5. The family of operators $\{T(t)^*\}_{t\geq 0}$ defined above is called the *adjoint semigroup* of $\{T(t)\}_{t\geq 0}$.

We would like that the adjoint semigroup of a C_0 semigroup were itself strongly continuous. This (unfortunately!) is not always the case. See the example below.

Example 1.4.6. Take $X = L^1(\mathbb{R})$, so that $X^* = L^\infty(\mathbb{R})$, and define, for $t \geq 0$,

$$T(t) : L^1(\mathbb{R}) \to L^1(\mathbb{R}), \quad (T(t)f)(x) = f(x+t).$$

It is clear that the family $\{T(t)\}_{t\geq 0}$ is a semigroup on $L^1(\mathbb{R})$, and we claim that it is strongly continuous in $L^1(\mathbb{R})$. To see this, we have to prove that, if $f \in L^1(\mathbb{R})$, then $||T(t)f - f||_1 \to 0$, as $t \to 0^+$. This is easy to check if $f \in C_c(\mathbb{R})$, i.e. if f is a continuous function of compact support, by Lebesgue's dominated convergence. Since this is a dense subset of $L^1(\mathbb{R})$ and $||T(t)|| = 1$, for all $t \geq 0$, the same conclusion holds in $L^1(\mathbb{R})$. So $\{T(t)\}_{t\geq 0}$ is in fact strongly continuous. Now observe that the dual semigroup is given by

$$T(t)^* : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}), \quad (T(t)^* g)(x) = g(x-t),$$

which is not strongly continuous in $L^\infty(\mathbb{R})$. For example, consider

$$g(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

It is clear that $g \in L^\infty(\mathbb{R})$, but

$$||T(t)^* g - g||_\infty = \sup_{x\in\mathbb{R}} |g(x-t) - g(x)| = 1, \quad t > 0,$$

which proves that $\{T(t)^*\}_{t\geq 0}$ is not strongly continuous.

This example motivates the following question: if one starts off with a strongly continuous semigroup on X, is it always possible to find a (non-trivial) subset of X^* on which the corresponding adjoint semigroup is strongly continuous? And if so, can we find the largest of those sets? We will see that the answer to both of these questions is positive, and this will be crucial for establishing the sun-star theory for semigroups of operators on the next chapter. We first need a definition.

Definition 1.4.7. Let S be a linear operator on X and let Y be a subspace of X. The operator $\tilde{S}$ defined by

$$D(\tilde{S}) = \{x \in D(S) \cap Y : Sx \in Y\}, \qquad \tilde{S}x = Sx, \quad \text{for } x \in D(\tilde{S}),$$

is called the *part of S in Y*.

Theorem 1.4.8. *Let $\{T(t)\}_{t\geq 0}$ be a C_0 semigroup on X with generator A, $\{T(t)^*\}_{t\geq 0}$ be its adjoint semigroup and A^* the adjoint operator of A. Define $Y^* = \overline{D(A^*)}$ and, for each $t \geq 0$, denote by $T(t)^+$ the restriction of $T(t)^*$ to Y^*. Then $Y^* \neq \{0\}$ and*

(i) $\{T(t)^+\}_{t\geq 0}$ is a C_0 semigroup on Y^*.

(ii) The generator A^+ of $\{T(t)^+\}_{t\geq 0}$ is the part of A^* in Y^*.

Proof. We start by observing that, since A is the infinitesimal generator of the C_0 semigroup $\{T(t)\}_{t\geq 0}$, there exist constants $\omega \geq 0$ and $M \geq 1$ such that, if $\lambda > \omega$, then $\lambda \in \rho(A)$ and

$$||R(\lambda : A)^n|| \leq \frac{M}{(\lambda - \omega)^n}, \quad n \in \mathbb{N}.$$

Consequently,

$$||R(\lambda : A^*)^n|| \leq \frac{M}{(\lambda - \omega)^n}, \quad n \in \mathbb{N}$$

Now let $Y^* = \overline{D(A^*)}$. Note that Lemma 1.4.3 implies that $Y^* \neq \{0\}$, because A is closed (by the Hille-Yosida theorem). Define, for $\lambda > \omega$, an operator $J(\lambda)\colon Y^* \to X^*$ by $J(\lambda) = R(\lambda : A^*)|_{Y^*}$. Then, for any $n \in \mathbb{N}$ and $\lambda, \mu > \omega$, it follows from the definition of $J(\lambda)$ that

$$||J(\lambda)^n|| \leq \frac{M}{(\lambda - \omega)^n}, \quad J(\lambda) - J(\mu) = (\mu - \lambda)J(\lambda)J(\mu).$$

Moreover, it follows from (i) in Theorem 1.3.1 that, for any $x^* \in X^*$,

$$\lim_{\lambda\to\infty} \lambda J(\lambda)x^* = \lim_{\lambda\to\infty} \lambda R(\lambda : A^*)x^* = x^*,$$

since

$$\langle x, \lambda J(\lambda)x^*\rangle = \langle \lambda R(\lambda : A)x, x^*\rangle = \langle x, x^*\rangle,$$

for every $x \in X$. Hence, by Theorem 1.3.3, $J(\lambda)$ is the resolvent of a closed, densely defined operator A^+ on Y^* with $(\omega, +\infty) \subset \rho(A^+)$ and then, by the Hille-Yosida theorem, $J(\lambda)$ is the resolvent of the infinitesimal generator of some C_0 semigroup $\{T(t)^+\}_{t\geq 0}$ on Y^*.

We now prove that $T(t)^+$ is the restriction of $T(t)^*$ to Y^*, which gives *(i)*. Let $x \in X$, $x^* \in D(A^*)$ and $t \geq 0$. We have

$$\left\langle \left(I - \frac{t}{n}A\right)^{-n} x, x^* \right\rangle = \left\langle x, \left(I - \frac{t}{n}A^+\right)^{-n} x^* \right\rangle, \quad n \in \mathbb{N},$$

and letting $n \to \infty$ we obtain, by *(iii)* in Theorem 1.3.1,

$$\langle T(t)x, x^*\rangle = \langle x, T^*(t)x^*\rangle, \quad t \geq 0.$$

Since $D(A^*)$ is dense in Y^*, this means precisely that $T(t)^+$ is the restriction of $T(t)^*$ to the subspace Y^*. This proves the first statement.

Finally, to prove *(ii)*, it remains to see that A^+ is the part of A^* in Y^*. Let $x^* \in D(A^*)$ be such that

$A^*x^* \in Y^*$. Then, $(\lambda I^* - A^*)x^* \in Y^*$ and we have

$$(\lambda I^* - A^+)^{-1}(\lambda I^* - A^*)x^* = x^*.$$

Hence, $x^* \in D(A^+)$. Applying $\lambda I^* - A^*$ to both sides of the equality above we obtain

$$(\lambda I^* - A^*)x^* = (\lambda I^* - A^+)x^*,$$

which implies that $A^*x^* = A^+x^*$. Therefore A^+ is indeed the part of A^* in Y^*. □

The previous theorem already provides an answer to one of the questions we had raised, as it states that there does indeed always exist a subspace of the dual space where the adjoint semigroup is strongly continuous. In other words, this means that, given a C_0 semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space X, the set

$$\mathscr{A} = \{Y^* \subset X^* : \{T(t)^*|_{Y^*}\}_{t\geq 0} \text{ is a } C_0 \text{ semigroup on } Y^*\}$$

is non-empty. Also, it is clear that if the dual semigroup is strongly continuous on a chain $\{Y^*_\lambda\}_{\lambda\in\Lambda}$, where Λ is some index set and $Y^*_\lambda \in \mathscr{A}$, for all $\lambda \in \Lambda$, then the same happens on the union of the sets, i.e.

$$\bigcup_{\lambda\in\Lambda} Y^*_\lambda \in \mathscr{A}.$$

Therefore, by Zorn's lemma, we conclude that there always exists a non-zero maximal subspace of X^* on which the dual semigroup is a C_0 semigroup.

Definition 1.4.9. Given a Banach space X and a C_0 semigroup $\{T(t)\}_{t\geq 0}$ on X, we denote by $X^\odot$ the maximal subspace of X^* where the adjoint semigroup $\{T(t)^*\}_{t\geq 0}$ is strongly continuous. We denote the restriction of $\{T(t)^*\}_{t\geq 0}$ to $X^\odot$ by $\{T(t)^\odot\}_{t\geq 0}$.

The symbol $\odot$ is the chinese symbol for 'sun', and so $X^\odot$ reads as X *sun*.

In summary, starting with a Banach space X and a C_0 semigroup $\{T(t)\}_{t\geq 0}$ on X, we have obtained a new strongly continuous semigroup $\{T(t)^\odot\}_{t\geq 0}$ on the space $X^\odot$. Repeating the process of taking duals and restrictions to maximal invariant subspaces, we obtain the semigroups $\{T(t)^{\odot *}\}_{t\geq 0}$ - the adjoint of $\{T(t)^\odot\}_{t\geq 0}$, defined on $X^{\odot *}$ - and $\{T(t)^{\odot\odot}\}_{t\geq 0}$ - the restriction of $\{T(t)^{\odot *}\}_{t\geq 0}$ to the maximal invariant subspace $X^{\odot\odot}$ of $X^{\odot *}$ where strong continuity holds. These definitions are the starting point for the sun-star theory of linear operators.

Remark 1.4.10. Even though this does not play a role in the dissertation, it can actually be seen that $X^\odot = \overline{D(A^*)}$, i.e. $\overline{D(A^*)}$ is precisely the maximal domain of strong continuity of the adjoint semigroup (here, we are using the notation of Theorem 1.4.8). For details, see for example the Introduction in [23] and Chapter 1 of [21].

1.5 The spectral projection

In this short section, we introduce the spectral projection of the resolvent of a closed linear operator and establish some well-known results which will be used in Chapter 3. We consider analytic functions $f\colon \Lambda \subset \mathbb{C} \to \mathscr{L}(X)$, where, for each $\lambda \in \Lambda$, $f(\lambda)$ denotes an element of the operator space $\mathscr{L}(X)$. The definition of this operational calculus can be found on [25, Chapter V], to which we refer for the details. We stress, however, that this extension is done in such a way that the usual results of complex analysis - Cauchy's theorem, the Residue theorem, Laurent expansions, etc. - carry over to this new class of analytic functions. We omit the proofs, but refer to [25, Section V.10] for the details.

Let A be a closed linear operator and suppose that $\rho(A) \neq \emptyset$. It can be seen that the resolvent function

$$\rho(A) \to \mathscr{L}(X), \quad \lambda \mapsto R(\lambda : A)$$

is analytic on $\rho(A)$ and that isolated points $\lambda_0 \in \sigma(A)$ are isolated singularities of $R(\lambda : A)$. Consequently, there exists a Laurent expansion of $R(\lambda : A)$ in powers of $\lambda - \lambda_0$, which we write as

$$R(\lambda : A) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n A_n + \sum_{n=1}^{\infty} (\lambda - \lambda_0)^{-n} B_n.$$

In the expression above, the coefficients A_n and B_n are in $\mathscr{L}(X)$ and this representation is valid in a punctered disc

$$D_{\lambda_0} = \{\lambda \in \mathbb{C}\colon 0 < ||\lambda - \lambda_0|| < r\}$$

of radius r and centred at λ_0 containing no other spectral values of A. The coefficients are given by the expressions

$$A_n = \frac{1}{2\pi i} \int_C (\lambda - \lambda_0)^{-n-1} R(\lambda : A)\, d\lambda,$$

$$B_n = \frac{1}{2\pi i} \int_C (\lambda - \lambda_0)^{n-1} R(\lambda : A)\, d\lambda,$$

where C is any counterclockwise curve contained on D_{λ_0}. The operator B_1 will play a special role in the structure of the space X, so we provide it with a different name; in what follows, denote $P_+ = B_1$.

Proposition 1.5.1. *The coefficient operator P_+, defined as*

$$P_+ = \frac{1}{2\pi i} \int_C R(\lambda : A)\, d\lambda,$$

is a bounded projection onto a closed subspace X_+ of X.

Definition 1.5.2. P_+ is called the *spectral projection* of $R(\lambda : A)$.

The next theorem is a kind of a Fredholm alternative result for closed operators and shows that the spectral projection induces a decomposition of the space X. First we introduce a definition. Given a linear operator A, denote by $\mathscr{N}(A)$ and $\mathscr{R}(A)$ the null-space and the image of A, respectively. It is clear

that

$$\{0\} = \mathscr{N}(A^0) \subset \mathscr{N}(A) \subset \mathscr{N}(A^2) \subset \ldots,$$

and it is easy to see that, if for some $n \in \mathbb{N}$ we have $\mathscr{N}(A^n) = \mathscr{N}(A^{n+1})$, then we also have $\mathscr{N}(A^n) = \mathscr{N}(A^k)$, for $k > n$. Analogously, we have

$$X = \mathscr{R}(A^0) \supset \mathscr{R}(A) \supset \mathscr{R}(A^2) \supset \ldots$$

and, again, if $\mathscr{R}(A^n) = \mathscr{R}(A^{n+1})$, for some $n \in \mathbb{N}$, then $\mathscr{R}(A^n) = \mathscr{R}(A^k)$, for all $k > n$.

Definition 1.5.3. Let A be a linear operator.

(i) The smallest nonnegative integer $p \in \mathbb{N}$ such that $\mathscr{N}(A^p) = \mathscr{N}(A^{p+1})$ is called the *ascent of* A. If no such p exists, then the ascent of A is said to be ∞.

(ii) The smallest nonnegative integer $q \in \mathbb{N}$ such that $\mathscr{R}(A^q) = \mathscr{R}(A^{q+1})$ is called the *descent of* A. If no such q exists, then the descent of A is said to be ∞.

Theorem 1.5.4. *Let A be a closed linear operator with non-empty resolvent.*

(i) If λ_0 is a pole of $R(\lambda : A)$ of order p, then the ascent and the descent of $\lambda_0 I - A$ are both equal to p.

(ii) The range of the spectral projection P_+ is $\mathscr{N}[(\lambda_0 I - A)^p]$, the null space of $(\lambda_0 I - A)^p$, and the range of the complementary projection $I - P_+$ is $\mathscr{R}[(\lambda_0 I - A)^p]$, the range of $(\lambda_0 I - A)^p$.

(iii) We have the decomposition $X = \mathscr{N}[(\lambda_0 I - A)^p] \oplus \mathscr{R}[(\lambda_0 I - A)^p]$.

In particular, if A is the infinitesimal generator of a C_0 semigroup, then it is closed, and the previous results apply. This is the setting that we will encounter in Chapter 3, and Theorem 1.5.4 will be used to obtain a decomposition of the phase space as the direct sum of two invariant subspaces.

Chapter 2

An abstract formulation for differential equations with infinite delay

In this chapter, we introduce some notions of the sun-star theory for linear operators and study existence, uniqueness and continuation of solutions for integral equations of the form

$$u(t) = T_0(t)\psi + j^{-1}\int_0^t T_0^{\odot *}(t-s)F(u(s))ds, \quad t \geq 0, \tag{2.1}$$

where $\{T_0(t)\}_{t\geq 0}$ is a strongly continuous semigroup on a Banach space X, $\psi \in X$, F is a given function and $j\colon X \to X^{\odot *}$ is an embedding defined below. We then consider an initial value problem for an autonomous differential equation with infinite delay, and formulate this problem as an abstract equation of the form above. We conclude by proving an equivalence result between these two formulations.

2.1 Sun-star theory

In this section, we give meaning to a formula of the type (2.1). When studying a concrete differential equation, we want to work indistinctly with the equation and an equivalent integral formulation on the phase space X, and the formula above will provide us exactly with that equivalent expression. As we have mentioned in the end of Section 1.4, the symbol $\odot$ is the chinese symbol for 'sun', so that $T^{\odot}$ and $T^{\odot *}$ read as 'T sun', 'T sun star', respectively. This explains the name of the section.

A short remark on notation is timely: throughout the text, given $\varphi \in X$, $\varphi^* \in X^*$ and $\varphi^{**} \in X^{**}$, we employ the duality notation

$$\langle \varphi, \varphi^* \rangle := \varphi^*(\varphi), \qquad \langle \varphi^{**}, \varphi^* \rangle := \varphi^{**}(\varphi^*).$$

To be more precise, every time the duality pairing $\langle\ ,\ \rangle$ is used, the element on the right always is an element of the first dual space X^*.

We now define the mapping j and the integral term in (2.1), following the construction in [3]. Define an embedding of X into $X^{\odot *}$ by

$$j : X \to X^{\odot *}, \quad \langle j\psi, \psi^{\odot} \rangle = \langle \psi, \psi^{\odot} \rangle, \qquad \text{for } \psi \in X,\ \psi^{\odot} \in X^{\odot} \subset X^*. \tag{2.2}$$

The mapping j defined above is a bounded linear function between the Banach spaces X and $X^{\odot *}$. Two other properties of j are the following:

Lemma 2.1.1. *Let $j : X \to X^{\odot *}$ be as above. Then*

$$T^{\odot *}(t)j = jT(t), \quad \text{and} \quad j(X) \subset X^{\odot\odot}$$

for all $t \geq 0$.

Proof. The first claim is simple to check; indeed, if $\psi \in X$, $\psi^{\odot} \in X^{\odot}$, then, for all $t \geq 0$,

$$\langle T^{\odot *}(t)j\psi, \psi^{\odot}\rangle = \langle j\psi, T^{\odot}(t)\psi^{\odot}\rangle = \langle \psi, T^{\odot}(t)\psi^{\odot}\rangle = \langle T(t)\psi, \psi^{\odot}\rangle = \langle j(T(t)\psi), \psi^{\odot}\rangle.$$

Hence $T^{\odot *}(t)j\psi = jT(t)\psi$, for all $\psi \in X$, which means precisely that $T^{\odot *}(t)j = jT(t)$, $t \geq 0$.

To see that $j(X) \subset X^{\odot\odot}$, let $j\psi \in j(X)$. From the previous paragraph, $T^{\odot *}j = jT(t)$, for every $t \geq 0$, and so we have

$$||T^{\odot *}(t)(j\psi) - j\psi|| = ||j(T(t)\psi) - j\psi|| \leq ||j||\,||T(t)\varphi - \varphi|| \to 0, \quad t \to 0^{+},$$

because $\{T(t)\}_{t\geq 0}$ is strongly continuous. □

In [3], only the case $j(X) = X^{\odot\odot}$ is considered and, when this happens, we say that X is sun-reflexive with respect to the semigroup $\{T(t)\}_{t\geq 0}$. We do not make this assumption in this text, and instead replace sun-reflexivity by two other weaker conditions. We lose the identification between X and $X^{\odot\odot}$ and have to work with the inverse of the embedding j, but this approach allows for the treatment of a more general situation. For a deeper understanding of the role of sun-reflexivity in these types of problems, we refer to [12].

We now define the integral term in (2.1); it is an expression of the form

$$\int_0^t T^{\odot *}(t-s)f(s)ds, \quad t \geq 0, \tag{2.3}$$

where $f : \mathbb{R}_+ \to X^{\odot *}$ is assumed to be continuous. This could be done in more generality, but continuity is sufficient for the concrete problem we address in this dissertation. In fact, in [3] this is defined for a function f which is only weak-$*$ continuous. For generality, we consider Z to be any Banach space and define the integral for a continuous function function $z^* \colon [a,b] \to Z^*$. Note that, for any $z \in Z$, the integral

$$\int_a^b \langle z, z^*(t)\rangle_{Z\times Z^*}\, dt$$

is well defined (as a Riemann integral) and we have the estimate

$$\left|\int_a^b \langle z, z^*(t)\rangle_{Z\times Z^*}\, dt\right| \leq (b-a)\,||z||\sup_{t\in[a,b]}||z^*(t)||.$$

From now on, we omit the index on the duality pairing.

Definition 2.1.2. Given a continuous function $z^*\colon [a,b] \to Z^*$, we define the integral

$$\int_a^b z^*(t)dt$$

as the linear functional

$$z \mapsto \int_a^b \langle z, z^*(t)\rangle dt.$$

The considerations above show that $\int_a^b z^*(t)dt$ is a bounded linear functional on Z, with

$$\left|\left|\int_a^b z^*(t)\,dt\right|\right| \leq (b-a) \sup_{t\in[a,b]} ||z^*(t)||,$$

hence an element of Z^*. So, by definition,

$$\left\langle z, \int_a^b z^*(t)dt \right\rangle := \int_a^b \langle z, z^*(t)\rangle dt,$$

and with this definition the integral behaves very nicely when composed with adjoints of bounded linear functionals; in fact, if L is a bounded linear functional on Z, then $t \mapsto L^*z^*(t)$ is also continuous, and we have

$$\int_a^b L^*z^*(t)dt = L^*\int_a^b z^*(t)dt,$$

since, given $z \in Z$,

$$\left\langle z, \int_a^b L^*z^*(t)dt \right\rangle = \int_a^b \langle z, L^*z^*(t)\rangle dt = \int_a^b \langle Lz, z^*(t)\rangle dt = \left\langle z, L^*\int_a^b z^*(t)dt \right\rangle.$$

We now go back to the integral expression in (2.3), where f is continuous. Note that, for fixed $t \geq 0$, the function

$$[0,t] \ni s \mapsto T^{\odot *}(t-s)f(s) \tag{2.4}$$

is continuous. This follows from noting that, for every $s \in [0,t]$, $T^{\odot *}(t-s)f(s) = f(s)T^{\odot}(t-s)$, and the function

$$[0,t] \ni s \mapsto f(s)T^{\odot}(t-s)$$

is in fact continuous, because $\{T^{\odot}(t)\}_{t\geq 0}$ is a C_0 semigroup and so Corollary 1.1.6 applies. Recall that $\{T^{\odot *}(t)\}_{t\geq 0}$ is not necessarily strongly continuous, so this argument could not have been applied directly to the expression (2.4). As a consequence, we can define the integral of (2.4) in the sense of Definition 2.1.2.

Now choose $M \geq 1$, $\omega \in \mathbb{R}$ such that $||T(t)|| \leq Me^{\omega t}$, for $t \geq 0$. The next result provides an upper bound for the norm of the integral as an element of $Z^* = X^{**}$.

Lemma 2.1.3. *With the notation above,*

$$\left|\left|\int_0^t T^{\odot *}(t-s)f(s)ds\right|\right| \leq M\frac{e^{\omega t}-1}{\omega} \sup_{0\leq s\leq t} ||f(s)||,$$

for every $t \geq 0$.

Proof. Fix $t \geq 0$ and let $x^* \in X^\odot$ be such that $||x^*|| \leq 1$. Let $M_f := \sup_{0\leq s\leq t} ||f(s)||$. We have

$$\begin{aligned}
\left|\left\langle \int_0^t T^{\odot *}(t-s)f(s)ds, x^* \right\rangle\right| &\leq \int_0^t |\langle T^{\odot *}(t-s)f(s), x^*\rangle| ds = \int_0^t |\langle f(s), T^\odot(t-s)x^*\rangle| ds \\
&\leq M_f \int_0^t ||T^\odot(t-s)x^*|| ds \\
&\leq M_f \int_0^t \sup_{||x||\leq 1} |\langle x, T^\odot(t-s)x^*\rangle| ds \\
&= M_f \int_0^t \sup_{||x||\leq 1} |\langle T(t-s)x, x^*\rangle| ds \\
&\leq M_f \int_0^t ||T(t-s)|| ds \\
&\leq M_f \int_0^t M e^{\omega(t-s)} ds = M\frac{e^{\omega t}-1}{\omega} M_f,
\end{aligned}$$

as desired. One should understand the quotient $(e^{\omega t}-1)/\omega$ as t when $\omega = 0$. □

Besides the estimates in the previous lemma, we can actually prove that the integral operator is an element of $X^{\odot\odot}$ which is continuous as a function of t.

Theorem 2.1.4. *Let $f : [0,+\infty) \to X^{\odot *}$ be continuous. Then*

$$[0,\infty) \ni t \mapsto \int_0^t T^{\odot *}(t-s)f(s)ds$$

is a continuous $X^{\odot\odot}-$valued function.

Proof. For fixed $t \geq 0$, define

$$F(t) := \int_0^t T^{\odot *}(t-s)f(s)ds.$$

We start by proving that $F(t) \in X^{\odot\odot}$, for all $t \geq 0$. If $t = 0$, the result is immediate. So assume $t > 0$ and let $0 < h < t$. Then

$$\begin{aligned}
T^{\odot *}(h)F(t) - F(t) &= \int_0^t T^{\odot *}(t+h-s)f(s)ds - \int_0^t T^{\odot *}(t-s)f(s)ds \\
&= \int_h^{t+h} T^{\odot *}(s)f(t+h-s)ds - \int_0^t T^{\odot *}(s)f(t-s)ds \\
&= -\int_0^h T^{\odot *}(s)f(t-s)ds + \int_t^{t+h} T^{\odot *}(s)f(t+h-s)ds \\
&\quad + \int_h^t T^{\odot *}(s)\left(f(t+h-s) - f(t-s)\right)ds.
\end{aligned}$$

By Lemma 2.1.3, we can control the norms of the first two terms by

$$Kh \sup_{0\leq s\leq t+h} ||f(s)||,$$

for a constant $K > 0$ sufficiently large. The last term is bounded above by

$$K \sup_{0\leq s\leq t} ||f(s+h) - f(s)||,$$

and so also tends uniformly to zero when $h \to 0^+$, by uniform continuity of f on compact intervals. We then conclude that

$$T^{\odot *}(h)F(t) - F(t) \to 0, \quad h \to 0^+,$$

which means that $F(t) \in X^{\odot\odot}$.

We now prove that F is continuous. Let $h > 0$. We have

$$F(t+h) - F(t) = T^{\odot *}(h)F(t) - F(t) + \int_0^h T^{\odot *}(s)f(t+h-s)ds$$

and

$$F(t-h) - F(t) = \int_0^h T^{\odot *}(t-s)f(s)ds + \int_h^t T^{\odot *}(t-s)\left(f(s-h) - f(s)\right)ds,$$

and the factors on the right hand-side of both equations converge to 0 when $h \to 0^+$. This implies continuity of F, proving the theorem. □

Now that some continuity results and estimates have been established, we focus on studying some properties of the solutions of (2.1).

2.2 Abstract setting

In this section we prove existence, uniqueness and continuation of solutions of the abstract integral equation (2.1), where all the terms are as in the previous section.

2.2.1 Abstract integral equations

Let X be a Banach space and $\{T_0(t)\}_{t\geq 0}$ a strongly continuous semigroup on X, with infinitesimal generator A_0. Fix $M \geq 1$ and $\omega \geq 0$ such that

$$||T_0(t)|| \leq Me^{\omega t}, \quad t \geq 0.$$

Let X' be a subspace of $X^{\odot}$ such that:

(i) X' is closed;

(ii) X' is invariant under $\{T_0^{\odot}(t)\}_{t\geq 0}$;

(iii) X' separates points in X, i.e. given $x, y \in X$, $x \neq y$, there exists $x' \in X'$ such that $\langle x, x'\rangle \neq \langle y, x'\rangle$.

Finally, define $T_0'(t) = T_0^{\odot}(t)|_{X'}$, for every $t \geq 0$.

Lemma 2.2.1. *With the notation introduced above, $\{T_0'(t)\}_{t\geq 0}$ is a strongly continuous semigroup on X' and its infinitesimal generator A_0' is the part of $A_0^{\odot}$ in X', where $A_0^{\odot}$ is the infinitesimal generator of $\{T_0^{\odot}(t)\}_{t\geq 0}$.*

Proof. It is clear that $\{T_0'(t)\}_{t\geq 0}$ is strongly continuous on X'. To see that A_0' is the part of $A_0^{\odot}$ in X', we have to check that

$$D(A_0') = \{x \in D(A_0^{\odot}) \cap X' : A_0^{\odot} x \in X'\} \quad \text{and} \quad A_0' x = A_0^{\odot} x, \;\; x \in D(A_0').$$

The second equality is immediate from the fact that A_0' is simply a restriction of $A_0^{\odot}$, so let us check the first equality.

To see the first inclusion, if $x \in D(A_0')$, then $x \in X'$ by definition and there exists

$$A_0' x = A_0^{\odot} x = \lim_{t\to 0^+} \frac{T_0^{\odot}(t)x - x}{t} = \lim_{t\to 0^+} \frac{T_0'(t)x - x}{t},$$

so in fact $x \in D(A_0^{\odot})$. Moreover, $A_0^{\odot} x \in X'$ because X' is closed and $A_0^{\odot} x$ is a limit of elements of X'. The other inclusion is immediate. □

Define the linear mapping $j : X \to X'^*$ by

$$\langle jx, x' \rangle := \langle x, x' \rangle, \quad x \in X, \;\; x' \in X' \subset X^*.$$

Observe that j is bounded, since $x' \in X^*$, and that j is an injection, because X' separates points.

We now consider the abstract integral equation (AIE)

$$u(t) = T_0(t)\varphi + j^{-1} \int_0^t T_0'^*(t-s)(\ell \circ F)(u(s))\, ds, \quad t \geq 0, \tag{AIE}$$

where $\varphi \in X$, $\ell : \mathbb{R}^m \to X'^*$ is a given bounded linear injection, $j : X \to X'^*$ is the canonical injection defined in (2.2), and $F : X \to \mathbb{R}^m$ is a given (in general, nonlinear) mapping. We will refer to the equation above as the (AIE) with initial condition φ. When dealing with concrete equations, the functions ℓ, F are determined by the problem. In order for this equation to make sense, we need the integral term to belong to the range of j. Here, we shall assume this fact, together with a suitable estimate on the pre-image of the integral term:

(H) There exist $K \geq 1$ and $\xi \in \mathbb{R}$ such that, for all continuous functions $h : \mathbb{R} \to \mathbb{R}^m$ and all $t \geq 0$, one has

$$\int_0^t T_0'^*(t-s)\ell h(s) ds \in j(X) \tag{H1}$$

$$\left\| j^{-1} \int_0^t T_0'^*(t-s)\ell h(s) ds \right\| \leq K \int_0^t e^{\xi(t-s)} |h(s)| ds \tag{H2}$$

Hypothesis **(H)** relaxes the assumption in [3] that X is sun-reflexive, i.e. that $j(X) = X^{\odot\odot}$. In this case, **(H)** automatically holds. We do not assume sun-reflexivity in this dissertation, but the weaker assumption **(H)** is crucial for establishing existence and uniqueness for the integral equation. This is the topic of the next subsection.

2.2.2 Existence and uniqueness of solutions

Our goal is to prove the following result.

Theorem 2.2.2. *Assume **(H)** and that $F : X \to \mathbb{R}^m$ is Lipschitz continuous. Then, for all $\varphi \in X$, the abstract integral equation (AIE) has a unique solution defined on $[0,\infty)$, which has the form*

$$u(t) = \Sigma(t)\varphi, \quad t \geq 0,$$

where $\{\Sigma(t)\}_{t\geq 0}$ is a nonlinear semigroup on X.

The proof will be divided into several steps. We start by proving (local) existence of solutions, adapting the classical proof of Picard's existence theorem for ordinary differential equations.

Lemma 2.2.3. *Assume **(H)** and that $F : X \to \mathbb{R}^m$ is Lipschitz continuous. Given $\varphi \in X$, there exists a solution $u\colon [0,\alpha] \to X$, $\alpha > 0$, of the abstract integral equation (AIE) in $[0,\alpha]$ with initial condition φ.*

Proof. Choose $\alpha, r > 0$ and define

$$\mathscr{D} = \{u \in C\left([0,\alpha], \overline{B}_r(\varphi)\right) : u(0) = \varphi\}.$$

Let $C > 0$ be such that $|F(\psi)| \leq C$ in $\overline{B}_r(\varphi)$ and denote by L the (global) Lipschitz constant of F. Define, for $u \in \mathscr{D}$, the operator $A\colon \mathscr{D} \to C([0,\alpha], X)$ by

$$(Au)(t) = T_0(t)\varphi + j^{-1}\int_0^t T_0'^*(t-s)(\ell \circ F)(u(s))\,ds, \quad 0 \leq t \leq \alpha$$

Observe that finding a solution of (AIE) on $[0,\alpha]$ is equivalent to finding a fixed point of A.

Note that, for $u \in \mathscr{D}$ and $t \in [0,\alpha]$,

$$\begin{aligned} ||(Au)(t) - \varphi|| &\leq ||T_0(t)\varphi - \varphi|| + K\int_0^t e^{\xi(t-s)}|F(u(s))||\,ds \\ &\leq ||T_0(t)\varphi - \varphi|| + KC\,\frac{e^{\xi t}-1}{\xi}, \end{aligned}$$

where we have used the estimate (H2). It is obvious that, when $t \to 0$, the second term of the right-hand side converges to 0, and so does the first one, due to strong continuity of the semigroup. Consequently, we can choose $0 < t_0 \leq \alpha$ sufficiently small so that

$$||Au(t) - \varphi|| \leq ||T_0(t)\varphi - \varphi|| + KC\,\frac{e^{\xi t}-1}{\xi} \leq r, \quad \text{for } t \leq t_0.$$

Moreover, if $u, v \in \mathscr{D}$ and $t \in [0, \alpha]$,

$$\begin{aligned}
||(Au)(t) - (Av)(t)|| &= \left|\left| j^{-1} \int_0^t T_0'^*(t-s)\ell\left(F(u(s)) - F(v(s))\right) ds \right|\right| \\
&\leq K \int_0^t e^{\xi(t-s)} |F(u(s)) - F(v(s))| \, ds \\
&\leq KL \int_0^t e^{\xi(t-s)} |u(s) - v(s)| \, ds \\
&\leq 2KLr \int_0^t e^{\xi(t-s)} \, ds \\
&= 2KLr \frac{e^{\xi t} - 1}{\xi},
\end{aligned}$$

which again converges to zero as $t \to 0$. So choose $t_1 > 0$ such that

$$||(Au)(t) - (Av)(t)|| \leq 2KLr \frac{e^{\xi t} - 1}{\xi} < 1, \qquad \text{for } t \leq t_1.$$

So re-define $\alpha = \min\{t_0, t_1\}$. The computations above show that, for this new choice of α, $A(\mathscr{D}) \subset \mathscr{D}$ and that A is a contraction. By Banach's contraction principle, A has a unique fixed point in $\mathscr{D}$. But a fixed point of A is precisely a solution of (AIE), so this proves that (AIE), with initial condition φ, has a unique solution u defined on $[0, \alpha]$. □

Now that we have local existence of a unique solution for the equation, we aim to prove that we can extend it to the whole interval $[0, \infty)$. We start by noting that maximal solutions (or non-continuable solutions) must be defined on a half-open interval $[0, \alpha)$, with $\alpha \leq \infty$. In fact, if such a solution is defined on a closed interval $[0, \alpha]$, then the argument of the previous theorem can be used to extend it to a larger interval $[0, \alpha + \delta]$, which contradicts maximality. To see that α must in fact be ∞, we need the following two auxilliary results.

Lemma 2.2.4. *Let $u\colon [0, \alpha) \to X$ be the unique maximal solution of (AIE) with initial condition $\varphi \in X$. If $\alpha < \infty$, then u is bounded.*

Proof. Suppose u is the maximal solution of the equation, defined on $[0, \alpha)$ with $\alpha < \infty$. Then

$$\begin{aligned}
||u(t)|| &\leq ||T_0(t)\varphi|| + K \int_0^t e^{\xi(t-s)} |F(u(s))| \, ds \\
&\leq ||T_0(t)\varphi|| + K \int_0^t e^{\xi(t-s)} \left(|F(u(s)) - F(0)| + |F(0)|\right) ds \\
&\leq Me^{\omega t} ||\varphi|| + K|F(0)| \frac{e^{\xi t} - 1}{\xi} + K \int_0^t e^{\xi(t-s)} |F(u(s)) - F(0)| \, ds \\
&\leq C(t) + KL \int_0^t e^{\xi(t-s)} ||u(s)|| \, ds,
\end{aligned}$$

where

$$C(t) = Me^{\omega t} ||\varphi|| + K|F(0)| \frac{e^{\xi t} - 1}{\xi}.$$

Observe that, since $t \in [0,\alpha)$ and $\alpha < \infty$, $C(t)$ is bounded on $[0,\alpha)$. Using the estimate above it follows from Gronwall's inequality that

$$||u(t)|| \leq C(t) + \int_0^t C(t)KLe^{\xi(t-s)}e^{KL(t-s)}\,ds,$$

which again is bounded since t lies in a bounded interval. This concludes the proof. □

Lemma 2.2.5. *Let $u\colon [0,\alpha) \to X$ be the unique maximal solution of (AIE) with initial condition $\varphi \in X$. If $\alpha < \infty$, then the limit $u(\alpha^-)$ exists.*

Proof. Suppose $\alpha < \infty$. We start by noting that, by the previous lemma, u must be bounded in $[0,\alpha)$. Define

$$u_\alpha = T_0(\alpha)\varphi + j^{-1}\int_0^\alpha T_0'^*(\alpha - s)(\ell \circ F)(u(s))\,ds$$

and let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence such that $t_n \nearrow \alpha$. We will prove that $u(t_n) \to u_\alpha$.

We have the obvious estimate

$$||u(\alpha) - u(t_n)|| \leq A_n + B_n,$$

where

$$A_n = ||T_0(\alpha)\varphi - T_0(t_n)\varphi||$$

and

$$B_n = \left|\left| j^{-1}\left\{\int_0^\alpha T_0'^*(\alpha - s)(\ell \circ F)(u(s))\,ds - \int_0^{t_n} T_0'^*(t_n - s)(\ell \circ F)(u(s))\,ds\right\}\right|\right|.$$

By strong continuity, $A_n \to 0$ as $n \to \infty$.

Define

$$\psi_n = j^{-1}\int_0^{t_n} T_0'^*(t_n - s)(\ell \circ F)(u(s))\,ds, \quad \psi_\infty = j^{-1}\int_0^\alpha T_0'^*(\alpha - s)(\ell \circ F)(u(s))\,ds.$$

To prove that also $B_n \to 0$, we show that $\psi_n \to \psi_\infty$ as $n \to \infty$. Since j^{-1} is bounded, it suffices to prove that

$$\int_0^{t_n} T_0'^*(t_n - s)(\ell \circ F)(u(s))\,ds \to \int_0^\alpha T_0'^*(\alpha - s)(\ell \circ F)(u(s))\,ds, \quad n \to \infty.$$

By definition of convergence in X'^*, we need to see that, for every $\varphi' \in X'$ with $||\varphi'|| \leq 1$,

$$\int_0^{t_n} \langle T_0'^*(t_n - s)(\ell \circ F)(u(s)), \varphi'\rangle\,ds \to \int_0^\alpha \langle T_0'^*(\alpha - s)(\ell \circ F)(u(s)), \varphi'\rangle\,ds, \quad n \to \infty.$$

For every $n \in \mathbb{N}$ and $\varphi' \in X'$ with $||\varphi'|| \leq 1$, we have

$$\begin{aligned}
&\int_0^\alpha \langle T_0'^*(\alpha - s)(\ell \circ F)(u(s)), \varphi'\rangle\,ds - \int_0^{t_n} \langle T_0'^*(t_n - s)(\ell \circ F)(u(s)), \varphi'\rangle\,ds \\
&\quad = \int_{t_n}^\alpha \langle T_0'^*(\alpha - s)(\ell \circ F)(u(s)), \varphi'\rangle\,ds + \int_0^{t_n} \langle [T_0'^*(\alpha - s) - T_0'^*(t_n - s)](\ell \circ F)(u(s)), \varphi'\rangle\,ds.
\end{aligned}$$

We claim that both these integral terms converge to zero. Note that, for every $s \in [0,\alpha]$,

$$\langle T_0'^*(\alpha - s)(\ell \circ F)(u(s)), \varphi'\rangle \mathbb{1}_{[t_n,\alpha]}(s) \to 0, \quad n \to \infty$$

and

$$\begin{aligned}|\langle T_0'^*(\alpha-s)(\ell\circ F)(u(s)),\varphi'\rangle \mathbb{1}_{[t_n,\alpha]}(s)| &\leq |\langle T_0'^*(\alpha-s)(\ell\circ F)(u(s)),\varphi'\rangle| \\ &= |\langle (\ell\circ F)(u(s)), T_0'(\alpha-s)\varphi'\rangle| \\ &\leq ||\ell||\,|F(u(s))|\,||T_0'(\alpha-s)||\,||\varphi'|| \\ &\leq ||\ell||\,C\,||u(s)||\,M'\,e^{\omega'(\alpha-s)} \\ &\leq ||\ell||\,C\,K\,M' e^{\omega'(\alpha-s)} \in L^1(0,\alpha),\end{aligned}$$

where C is such that $|F(u(s))| \leq C\,||u(s)||$, K is an upper bound on the solution and M', ω' are such that $||T_0'(t)|| \leq M' e^{\omega' t}$, for all $t \geq 0$. By Lebesgue's dominated convergence theorem, the first integral tends to zero, as $n \to \infty$. The reasoning is analogous for the second integral. For $s \in [0,\alpha]$, an estimate as above yields

$$\begin{aligned}|\langle [T_0'^*(\alpha-s) - T_0'^*(t_n-s)](\ell\circ F)(u(s)),\varphi'\rangle \mathbb{1}_{[0,t_n]}(s)| &\leq ||\ell||\,C\,K\,||T_0'(t_n-s)||\,||T_0'(\alpha-t_n)\varphi' - \varphi'|| \\ &\leq ||\ell||\,C\,K\,M'\,e^{\omega'(\alpha-s)}\,||T_0'(\alpha-t_n)\varphi' - \varphi'|| \to 0,\end{aligned}$$

by strong continuity, and

$$\begin{aligned}|\langle [T_0'^*(\alpha-s) - T_0'^*(t_n-s)](\ell\circ F)(u(s)),\varphi'\rangle| &\leq ||\ell||\,C\,K\,(M')^2\,(e^{\omega'(\alpha-s)} + e^{\omega'(t_n-s)}) \\ &\leq 2||\ell||\,C\,K\,(M')^2\,e^{\omega'(\alpha-s)} \in L^1(0,\alpha),\end{aligned}$$

so that (again by dominated convergence) the second integral also vanishes when $n \to \infty$. Consequently, $B_n \to 0$, concluding the proof. □

The fact that the solution must be defined on the entire interval $[0,\infty)$ follows now from the previous result and from the remarks preceding Lemma 2.2.4. Indeed, if α were not infinity, then from the previous lemma the limit $u(\alpha^-)$ exists, and so $u(\alpha)$ makes sense. However, note that taking the limit when $t \to \alpha$ in (AIE) shows that u can be extended to a continuous function defined on $[0,\alpha]$ and satisfying the equation on the closed interval $[0,\alpha]$. This contradicts the fact that u was a maximal solution, since these are defined on half-open intervals. In conclusion:

Lemma 2.2.6. *The unique maximal solution of* (AIE) *with initial condition $\varphi \in X$ is defined on the whole half-line* $[0,+\infty)$.

To conclude the proof of Theorem 2.2.2, it remains to see that the solution has the stated form. Denote the unique solution u of (AIE) with initial condition φ by

$$\Sigma(t)\varphi := u(t), \quad t \geq 0.$$

Theorem 2.2.7. *The family $\{\Sigma(t)\}_{t\geq 0}$ defines a strongly continuous (nonlinear) semigroup on X.*

Proof. Let $\varphi \in X$ and let u be the solution of (AIE) with initial condition φ. It is clear thar $\Sigma(0)\varphi = I$.

Let $t_1, t_2 \geq 0$. By definition, we have

$$\Sigma(t_1+t_2)\varphi = u(t_1+t_2) \quad \text{and} \quad \Sigma(t_1)\Sigma(t_2)\varphi = \Sigma(t_1)u(t_2) = v(t_1),$$

where v is the solution of (AIE) with initial condition $u(t_2)$, i.e. v solves

$$v(s) = T_0(s)u(t_2) + j^{-1}\int_0^s T_0'^*(s-r)(\ell \circ F)(v(r))\,dr.$$

Define $w(s) = u(s+t_2)$, for $s \geq 0$. Note that

$$\begin{aligned} w(s) &= u(s+t_2) \\ &= T_0(s+t_2)\varphi + j^{-1}\int_0^{t_2+s} T_0'^*(t_2+s-r)(\ell \circ F)(u(r))\,dr \\ &= T_0(s)\left(T_0(t_2)\varphi + j^{-1}\int_0^{t_2} T_0'^*(t_2-r)(\ell \circ F)(u(r))\,dr\right) + j^{-1}\int_0^s T_0'^*(s-r)(\ell \circ F)(w(r))\,dr \\ &= T_0(s)u(t_2) + j^{-1}\int_0^s T_0'^*(s-r)(\ell \circ F)(w(r))\,dr. \end{aligned}$$

This means that w is a solution of (AIE) with initial condition $u(t_2)$. By uniqueness, we must have

$$\Sigma(s)u(t_2) = w(s), \quad s \geq 0.$$

In particular, for $s = t_1$, we obtain

$$\Sigma(t_1)\Sigma(t_2)\varphi = \Sigma(t_1)u(t_2)\varphi = w(t_1) = u(t_1+t_2) = \Sigma(t_1+t_2)\varphi.$$

Since φ is arbitrary, the semigroup property follows. It is clear from the equation that the solution semigroup is strongly continuous. □

2.2.3 Differentiability of the solution semigroup

We now aim to be able to differentiate the solution semigroup of (AIE) at steady-states, which we now define.

Definition 2.2.8. A *steady-state* or an *equilibrium* of a semigroup $\{\Sigma(t)\}_{t\geq 0}$ on X is an element $\bar{\varphi} \in X$ such that

$$\Sigma(t)\bar{\varphi} = \bar{\varphi}, \quad t \geq 0.$$

We expect that, under some regularity assumptions on F, the semigroup $\{\Sigma(t)\}_{t\geq 0}$ is differentiable at a steady-state $\bar{\varphi} \in X$, so that we could talk about a linearization of the equation at a steady-state. As the next theorem shows, this is in fact true, and the derivative of $\{\Sigma(t)\}_{t\geq 0}$ will be a semigroup of linear operators on X solving an integral equation of the type (AIE).

Theorem 2.2.9. *Let F be continuously Fréchet differentiable, $\{\Sigma(t)\}_{t\geq 0}$ the (nonlinear) semigroup defined in Theorem 2.2.2 and $\bar{\varphi}$ a steady-state of $\{\Sigma(t)\}_{t\geq 0}$. Then, the nonlinear operator $\Sigma(t)$ has a*

Fréchet derivative

$$T(t) := D\Sigma(t)(\bar{\varphi})$$

at $\bar{\varphi}$. *Furthermore, for every* $\varphi \in X$, $T(t)\varphi$ *is the unique solution of the linear integral equation*

$$T(t)\varphi = T_0(t)\varphi + j^{-1}\int_0^t T_0'^*(t-s)\ell DF(\bar{\varphi})T(s)\varphi\, ds, \quad t \geq 0. \tag{2.5}$$

Proof. We can assume, without loss of generality, that $\bar{\varphi} = 0$ (otherwise, we could perform the change of variables $\psi = \varphi - \bar{\varphi}$ and get an equation with 0 as its steady-state). The proof follows in several steps.

Step 1. $F(0) = 0$.

To prove this, it suffices to show that $(\ell \circ F)(0) = 0$, because ℓ is an injection. Note that $\bar{\varphi} = 0$ solves the equation

$$0 = j^{-1}\int_0^t T_0'^*(t-s)(\ell \circ F)(0)ds, \quad t \geq 0$$

which implies that

$$0 = \int_0^t \langle T_0'^*(t-s)(\ell \circ F)(0), \varphi'\rangle ds = \int_0^t \langle (\ell \circ F)(0), T_0'(s)\varphi'\rangle ds, \quad \varphi' \in X', \ t \geq 0$$

Differentiating with respect to t yields

$$0 = \langle (\ell \circ F)(0), T_0'(t)\varphi'\rangle = \langle T_0'^*(t)(\ell \circ F)(0), \varphi'\rangle, \quad \varphi' \in X', \ t \geq 0.$$

Then we must have $T_0'^*(t)(\ell \circ F)(0) = 0$, for all $t \geq 0$, from where we conclude that $(\ell \circ F)(0) = 0$.

Step 2. $||\Sigma(t)\varphi - T(t)\varphi|| \leq K \sup_{0\leq s\leq t} |F(\Sigma(s)\varphi) - DF(0)\Sigma(s)\varphi|$, *where* $K = K(t)$ *is finite*

Subtracting (2.5) from (AIE) we get, for every $t \geq 0$,

$$\begin{aligned}\Sigma(t)\varphi - T(t)\varphi = j^{-1}&\int_0^t T_0'^*(t-s)\ell\left(F\left(\Sigma(s)\varphi\right) - DF(0)\Sigma(s)\varphi\right) ds \\ &+ j^{-1}\int_0^t T_0'^*(t-s)\ell DF(0)\left(\Sigma(s)\varphi - T(s)\varphi\right) ds.\end{aligned}$$

Now define

$$u(t) := ||\Sigma(t)\varphi - T(t)\varphi||, \qquad f(t) := |F\left(\Sigma(t)\varphi\right) - DF(0)\Sigma(t)\varphi|, \quad t \geq 0.$$

Using (H2), we have the estimate

$$u(t) \leq K\int_0^t e^{\xi(t-s)}f(s)ds + K\int_0^t e^{\xi(t-s)}||DF(0)||u(s)ds, \quad t \geq 0.$$

Multiplying by $e^{-\xi t}$ we obtain

$$v(t) := e^{-\xi t}u(t) \leq K\int_0^t e^{-\xi s}f(s)ds + K\int_0^t ||DF(0)||v(s)ds, \quad t \geq 0.$$

Hence, if w is the solution of the equation

$$w(t) = K\int_0^t e^{-\xi s} f(s)ds + K\int_0^t ||DF(0)||w(s)ds, \quad t \geq 0,$$

it follows from applying Gronwall's inequality to the function $v - w$ that $v(t) \leq w(t)$, for $t \geq 0$. But note that w solves the differential equation

$$w'(t) = Ke^{-\xi t} f(t) + K||DF(0)||w(t), \quad t \geq 0,$$

so

$$w(t) = K\int_0^t e^{K||DF(0)||(t-s)} e^{-\xi s} f(s)ds, \quad t \geq 0,$$

which implies then that, for all $t \geq 0$,

$$u(t) = e^{\xi t} v(t) \leq e^{\xi t} w(t) = K\int_0^t e^{(K||DF(0)||+\xi)(t-s)} f(s)ds.$$

Then, for each $t \geq 0$,

$$||\Sigma(t)\varphi - T(t)\varphi|| \leq K(t) \sup_{0\leq s\leq t} |F(\Sigma(s)\varphi) - DF(0)\Sigma(s)\varphi|,$$

where

$$K(t) = K\int_0^t e^{(K||DF(0)||+\xi)(t-s)}\, ds$$

is a finite bound depending on t.

Step 3. $||\Sigma(t)\varphi|| \leq C(t)\,||\varphi||$ *where* $C = C(t)$ *is finite*

Fix $\varphi \in X$, $t \geq 0$. Observe that $\Sigma(t)\overline{\varphi} = \Sigma(t)0 = 0$, by definition of steady-state. Now note that, because F is Fréchet differentiable, there exists a function $\alpha(s) \to 0$ as $s \to 0$ such that

$$|F(\Sigma(s)\varphi) - DF(0)\Sigma(s)\varphi| = \alpha(s)||\Sigma(s)\varphi||.$$

Consequently,

$$\begin{aligned}
||\Sigma(t)\varphi|| &\leq ||T_0(t)||\,||\varphi|| + K\int_0^t e^{\xi(t-s)} |F(\Sigma(s)\varphi)|\, ds \\
&\leq ||T_0(t)||\,||\varphi|| + K\int_0^t \alpha(s) e^{\xi(t-s)} ||\Sigma(s)\varphi||\, ds + K||DF(0)||\int_0^t e^{\xi(t-s)} ||\Sigma(s)\varphi||\, ds, \\
&= ||T_0(t)||\,||\varphi|| + K\int_0^t (\alpha(s) + ||DF(0)||)\, e^{\xi(t-s)}\, ||\Sigma(s)\varphi||\, ds.
\end{aligned}$$

Hence, if $u(t) = e^{-\xi t}||\Sigma(t)\varphi||$, the last equality above reads as

$$u(t) \leq e^{-\xi t}||T_0(t)||\,||\varphi|| + K\int_0^t (\alpha(s) + ||DF(0)||)\, u(s)\, ds,$$

and so, by Gronwall's inequality,

$$u(t) \leq e^{-\xi t}||T_0(t)||\,||\varphi|| + K\int_0^t e^{-\xi s}||T_0(s)||\,||\varphi||\, (\alpha(s) + ||DF(0)||)\, e^{\int_s^t \alpha(s) + ||DF(0)||\, dr}\, ds \tag{2.6}$$

which implies

$$||\Sigma(t)\varphi|| \leq C(t)\,||\varphi||,$$

where

$$C = C(t) = ||T_0(t)|| + K\int_0^t e^{\xi(t-s)}||T_0(s)||(\alpha(s) + ||DF(0)||)\,e^{\int_s^t \alpha(s)+||DF(0)||\,dr}\,ds.$$

Step 4. $\Sigma(t)$ *is Fréchet differentiable at* $\overline{\psi} = 0$.

Finally, using the fact that F is continuously Fréchet differentiable at 0 and $F(0) = 0$ together with Step 3 we obtain

$$|F(\Sigma(s)\varphi) - DF(0)\Sigma(s)\varphi| \leq ||\Sigma(s)\varphi||\,\alpha(||\Sigma(s)\varphi||) \leq C(t)\,||\varphi||\,\alpha(||\Sigma(s)\varphi||),$$

where $\alpha(s) \to 0$ as $s \to 0$. Consequently, by Step 2,

$$||\Sigma(t)\varphi - T(t)\varphi|| \leq M(t)\,||\varphi||\,\alpha(||\Sigma(s)\varphi||),$$

which proves that $\Sigma(t)$ is indeed Fréchet differentiable at $\bar{\varphi} = 0$ and that $T(t) = D\Sigma(t)(\bar{\varphi})$. □

As a consequence,

Corollary 2.2.10. *With the notation of Theorem 2.2.9,* $\{T(t)\}_{t\geq 0}$ *is a* C_0 *semigroup on* X.

Proof. This follows from Theorem 2.2.7, because, by Theorem 2.2.9, the semigroup $\{T(t)\}_{t\geq 0}$ solves the abstract integral equation (2.5). □

Essentially, Theorem 2.2.9 above says that we can linearize the abstract integral equation (AIE) about a steady-state $\bar{\varphi}$; more precisely, it states that, if F is C^1, the solution of (AIE) is Fréchet differentiable at $\bar{\varphi}$, and that this derivative also solves an integral equation. This is important because, as we shall see, it provides us with the basic assumption of a theorem by Desch and Schappacher [9], which is crucial for the proof of the principle of linearized stability.

2.3 Autonomous differential equations with infinite delay

In this section, we consider the initial-value problem (IVP)

$$\begin{cases} \dot{y}(t) = F(y_t), & t > 0 \\ y(t) = \psi(t), & t \leq 0, \end{cases} \tag{DDE}$$

where ψ lives on an appropriate Banach space X and $F : X \to \mathbb{R}^m$ is continuously Fréchet differentiable on X. Our goal for this section is to obtain an equivalence between solutions of (DDE) and solutions of an abstract integral equation (AIE). More precisely, we prove that (DDE) is equivalent to the abstract equation

$$u(t) = T_0(t)\varphi + j^{-1}\int_0^t T_0'^*(t-s)(\ell \circ F)(u(s))\,ds, \quad t \geq 0, \tag{AIE}$$

where X, X', the semigroup $\{T_0(t)\}_{t\geq 0}$ and the linear injection ℓ are to be defined.

We start, of course, with the choice of the phase space. Let $\rho > 0$. As phase space X, we choose the Banach space $X = C_{0,\rho}(\mathbb{R}_-;\mathbb{R}^m)$ of continuous $\mathbb{R}^m$-valued functions ψ on $\mathbb{R}_- := (-\infty, 0]$ such that

$$\theta \mapsto e^{\rho\theta}\psi(\theta)$$

is continuous and vanishes at $-\infty$, i.e

$$\lim_{\theta\to-\infty} e^{\rho\theta}\psi(\theta) = 0,$$

endowed with the norm

$$||\psi||_{\infty,\rho} = \sup_{\theta\leq 0} e^{\rho\theta}|\psi(\theta)|.$$

As before, whenever there is no ambiguity, we omit the indices of the norm, and write $||\cdot||$ instead of $||\cdot||_{\infty,\rho}$.

In order to linearize the equation (DDE) in the neighbourhood of an equilibrium, we need a representation theorem for bounded linear operators $L : X \to \mathbb{R}^m$. We will thus focus on providing an answer to this question.

First, consider the case $m = 1$. Observe that the space X is simply a weighted version of the usual space of continuous functions that vanish at $-\infty$, so the arguments employed in [24] to study the dual space of $C_0(\mathbb{R}_-;\mathbb{R})$ carry over to X in a straightforward way. For this reason, these details are omitted, and we identify X^* with the space $M_\rho := M_\rho(\mathbb{R}^+;\mathbb{R})$ of Radon measures on $\mathbb{R}^+$ such that

$$||\mu||_\rho := \int_0^\infty e^{\rho\theta}\, d|\mu|(\theta) < \infty,$$

where $|\mu|$ is the total variation of μ. This identification is given as follows. Given any bounded linear functional $\Phi : X \to \mathbb{R}$, there exists a unique $\mu \in M_\rho$ that represents Φ, i.e. such that

$$\Phi(\varphi) = \int_0^\infty \varphi(-s)\, d\mu(s),$$

for every $\varphi \in X$, and this correspondence is an isometry, meaning that $||\Phi||_{X^*} = ||\mu||_\rho$. Associated with this identification, we obtain the duality pairing

$$\langle \varphi, \mu\rangle_{X\times X^*} := \int_0^\infty \varphi(-s)\, d\mu(s).$$

To extend this identification to $\mathbb{R}^m$, consider a linear operator $\Phi = (\Phi_1, \ldots, \Phi_m) : C_{0,\rho}(\mathbb{R}_-;\mathbb{R}^m) \to \mathbb{R}^m$. Each linear operator $\Phi_i : C_{0,\rho}(\mathbb{R}_-;\mathbb{R}^m) \to \mathbb{R}$ acts on an m-tuple $f = (f_1, \ldots, f_m)$ and can be decomposed as

$$\Phi_i(f) = \sum_{j=1}^m \Phi_{ij} f_j,$$

where each Φ_{ij} is a map $C_{0,\rho}(\mathbb{R}_-;\mathbb{R}) \to \mathbb{R}$, which is represented by a Radon measure μ_{ij} on $\mathbb{R}^+$, as in the previous paragraph. We conclude that each bounded linear operator $\Phi = (\Phi_1, \ldots, \Phi_m) : C_{0,\rho}(\mathbb{R}_-;\mathbb{R}^m) \to \mathbb{R}^m$ can be represented by a matrix $\mu = [\mu_{ij}]_{1\leq i,\, j\leq m}$, and we identify the space of bounded linear operators $X \to \mathbb{R}^m$ with the space of matrix-valued measures $M_\rho(\mathbb{R}_+;\mathbb{R}^{m\times m})$ such that

$$||\mu||_\rho = \int_0^\infty e^{\rho\theta} d|\mu|(\theta) < \infty.$$

Here, $|\mu|$ is the matrix version of the total variation measure of μ, defined as

$$|\mu|(E) = \sup \sum_{i=1}^{N} \mu(E_i), \tag{2.7}$$

where the supremum is taken over all N and all (finite) partitions $\{E_i\}$ of E. For some results on matrix measures, see for example [13, Section 3.5]. The action of this space on X is given via the pairing

$$\langle \psi, \mu \rangle = \int_0^\infty [d\mu(\theta)]\ \psi(-\theta), \tag{2.8}$$

where $\Phi(\psi) = \langle \psi, \mu \rangle$, so that the integral above denotes a vector whose i-th entry is

$$\sum_{j=1}^{m} \int_0^\infty \psi_j(-\theta)\, d\mu_{ij}(\theta).$$

Note that, for $m > 1$, the pairing (2.8) above is not the duality pairing between X and its dual X^*. By definition, the result of the action of the dual space on X is a real number, and not a vector. Reasoning as in the one-dimensional case, the dual space of X is identified with the space of vector-valued measures $\mu = (\mu_1, \ldots, \mu_m) \in M_\rho(\mathbb{R}_+; \mathbb{R}^m)$, with the action

$$\langle \psi, \mu \rangle_{X \times X^*} = \int_0^\infty \psi(-\theta)\, d\mu(\theta) := \sum_{i=1}^{m} \int_0^\infty \psi_i(-\theta)\, d\mu_i(\theta).$$

Remark 2.3.1. Let $\rho_0 := \rho$ be the positive number fixed in the beginning of this section. In Chapter 3, it will be necessary to consider $0 < \rho_1 < \rho_0$ and work in the different space C_{0,ρ_1}. Observe that, in this case, it holds that $C_{0,\rho_1} \subset C_{0,\rho_0}$ and $M_{\rho_0} \subset M_{\rho_1}$, with continuous injections. Consequently, F is still Fréchet differentiable viewed as a function on C_{0,ρ_1}, and its Fréchet derivative is represented by a measure $\mu_1 \in M_{\rho_1}$ that coincides, on the smaller space M_{ρ_0}, with the initial representation μ_0 of the derivative of F as a function on C_{0,ρ_0}.

To write (DDE) in the form (AIE), we now choose the semigroup $\{T_0(t)\}_{t\geq 0}$. We define it as being a semigroup of shift operators and with extension to the left by the value at zero: given $\psi \in X$, define

$$(T_0(t)\psi)(\theta) = \begin{cases} \psi(t+\theta), & \text{if } \theta \leq -t \\ \psi(0), & \text{if } -t < \theta \leq 0 \end{cases}.$$

Lemma 2.3.2. *Define $\{T_0(t)\}_{t\geq 0}$ as above. Then:*

(i) $\{T_0(t)\}_{t\geq 0}$ is a strongly continuous semigroup on X.

(ii) The infinitesimal generator A_0 of $\{T(t)\}_{t\geq 0}$ is defined by

$$D(A_0) = \{\psi \in X :\ \psi \text{ is differentiable, } \dot{\psi} \in X \text{ and } \dot{\psi}(0) = 0\}$$

and, for $\psi \in D(A_0)$, $A_0\psi = \dot{\psi}$.

Proof. It is clear that, for each $t \geq 0$, $T_0(t)$ defines a bounded linear operator $X \to X$ and the first two properties of the definition follow from a simple calculation.

To prove that the semigroup is strongly continuous, let $\psi \in X$ and $\varepsilon > 0$. By definition of X, there exists $M > 0$ such that

$$e^{\rho\theta}|\psi(\theta)| < \frac{\varepsilon}{2},$$

for all $\theta \leq -M+1$. Now, if $\theta \leq -M$, then, for any $t \in [0,1]$, we have

$$e^{\rho\theta}|\psi(t+\theta) - \psi(\theta)| \leq e^{\rho(t+\theta)}|\psi(t+\theta)| + e^{\rho\theta}|\psi(\theta)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

On the other hand, note that ψ is uniformly continuous on the compact $[-M,0]$, and so there exists $\delta > 0$ such that, for any $\theta_1, \theta_2 \in [-M,0]$ with $|\theta_1 - \theta_2| < \delta$, it holds that

$$|\psi(\theta_1) - \psi(\theta_2)| < \varepsilon.$$

So let $\theta \in [-M,0]$ and assume $0 < t < \delta$. If $-t < \theta \leq 0$, then also $|\theta| < \delta$, and so

$$e^{\rho\theta}|\psi(0) - \psi(\theta)| \leq |\psi(0) - \psi(\theta)| < \varepsilon.$$

If $\theta \leq -t$, then $\theta, t+\theta \in [-M,0]$ and

$$e^{\rho\theta}|\psi(t+\theta) - \psi(\theta)| \leq |\psi(t+\theta) - \psi(\theta)| < \varepsilon.$$

Consequently,

$$\lim_{t\to 0^+} ||T_0(t)\psi - \psi|| = 0,$$

which means that $\{T_0(t)\}_{t\geq 0}$ is strongly continuous. This proves (i).

We now show that A_0 defined as in the statement of the lemma is indeed the generator of $\{T_0(t)\}_{t\geq 0}$. First, suppose that $\varphi \in D(A_0)$ and let $\psi = A_0\varphi$. Then

$$0 = \lim_{t\to 0^+} \left|\left| \frac{T_0(t)\varphi - \varphi}{t} - \psi \right|\right| = \lim_{t\to 0^+} \sup_{\theta\leq 0} e^{\rho\theta} \left| \frac{(T_0(t)\varphi)(\theta) - \varphi(\theta)}{t} - \psi(\theta) \right|. \tag{2.9}$$

Let $\theta \leq 0$. For $t > 0$ such that $t + \theta \leq 0$, we have

$$e^{\rho\theta} \left| \frac{\varphi(t+\theta) - \varphi(\theta)}{t} - \psi(\theta) \right| \leq \left|\left| \frac{T_0(t)\varphi - \varphi}{t} - \psi \right|\right| \to 0, \quad t \searrow 0,$$

hence

$$\left| \frac{\varphi(t+\theta) - \varphi(\theta)}{t} - \psi(\theta) \right| \to 0, \quad t \searrow 0,$$

which means that φ is right-differentiable on $(-\infty, 0)$ with right-derivative ψ. Note that this convergence is uniform for $\theta \leq 0$. The argument for the left-derivative is similar, but some caution is needed. In fact, let $\varepsilon > 0$ and $t > 0$ be such that $\theta - t \leq 0$ and $|t| < 1$. Then,

$$e^{\rho\theta} \left| \frac{\varphi(\theta - t) - \varphi(\theta)}{-t} - \psi(\theta) \right| \leq e^{\rho\theta} \left| \frac{\varphi(s+t) - \varphi(s)}{t} - \psi(s) \right| + e^{\rho\theta}|\psi(s) - \psi(s+t)|, \tag{2.10}$$

where we let $s = \theta - t$. By the previous paragraph, the first term on the right-hand side goes to 0 as $t \searrow 0$,

uniformly for $s \leq 0$. Choose $M > 1$ such that $e^{\rho\theta}|\psi(\theta)| < \varepsilon$, for $\theta \leq -M+1$. For such θ, the second term in (2.10) is bounded above by 2ε, hence also tends to 0 as $t \searrow 0$. For $\theta \geq -M+1$, the same is true by uniform continuity of g on $[-M+1, 0]$. In conclusion,

$$\left| \frac{\varphi(\theta - t) - \varphi(\theta)}{-t} - \psi(\theta) \right| \to 0, \quad t \searrow 0,$$

which proves that φ is also left-differentiable with left-derivative ψ. Hence we conclude that φ is differentiable with derivative $\psi \in X$. The fact that $\dot{\varphi}(0) = \psi(0) = 0$ follows from setting $\theta = 0$ in (2.9).

Conversely, assume $\varphi \in X$ is such that $\dot{\varphi} \in X$ and $\dot{\varphi}(0) = 0$. Then

$$\left| \frac{\varphi(t+\theta) - \varphi(\theta)}{t} - \dot{\varphi}(\theta) \right| = \left| \frac{1}{t} \int_0^t \dot{\varphi}(\theta + s) - \dot{\varphi}(\theta)\, ds \right| \to 0, \quad t \searrow 0,$$

uniformly for $\theta \leq 0$. Thus $\varphi \in D(A_0)$ and $A_0\varphi = \dot{\varphi}$. □

For the subspace X' of X^*, we choose $X' = \mathbb{R}^m \times L^1_\rho(\mathbb{R}_+; \mathbb{R}^m)$, where $L^1_\rho(\mathbb{R}_+; \mathbb{R}^m)$ is the space of (equivalence classes) of measurable functions $f \colon \mathbb{R}_+ \to \mathbb{R}^m$ such that

$$||f|| := \int_0^\infty e^{\rho\theta}|f(\theta)|\, dt < \infty$$

A pair $(c, g) \in X'$ naturally induces a measure

$$\mu = c\delta_0 + g\, dx,$$

where δ_0 denotes the Dirac delta centred at 0, and with this identification one can view X' as a subspace of X^*. This space inherits from X^* the pairing

$$\langle \psi, (c, g) \rangle_{X \times X'} = \psi(0) \cdot c + \int_0^\infty \psi(-\theta) \cdot g(\theta)\, d\theta, \quad (c, g) \in X', \ \ \psi \in X$$

and the norm

$$||(c, g)||_{X'} = |c| + ||g||_{1,\rho},$$

and so X' is a closed subspace of X^* (as it is complete). Furthermore, if $(c, g) \in X'$, then, for all $\psi \in X$ and $t \geq 0$ we have

$$\begin{aligned}
\langle T_0(t)\psi, (c, g) \rangle_{X \times X'} &= c \cdot (T_0(t)\psi)(0) + \int_0^\infty g(\theta) \cdot (T_0(t)\psi)(-\theta)\, d\theta \\
&= c \cdot \psi(0) + \int_{-\infty}^0 g(-\theta)(T_0(t)\psi)(\theta)\, d\theta \\
&= c \cdot \psi(0) + \int_{-\infty}^{-t} g(-\theta) \cdot \psi(t + \theta)\, d\theta + \int_{-t}^0 g(-\theta) \cdot \psi(0)\, d\theta \\
&= \left(c + \int_0^t g(\theta)\, d\theta \right) \cdot \psi(0) + \int_{-\infty}^0 g(t - \theta) \cdot \psi(\theta)\, d\theta \\
&= \left(c + \int_0^t g(\theta)\, d\theta \right) \cdot \psi(0) + \int_{-\infty}^0 g_t(-\theta) \cdot \psi(\theta)\, d\theta \\
&= \left\langle \psi, \left(c + \int_0^t g(\theta)\, d\theta, g_t \right) \right\rangle,
\end{aligned}$$

where $g_t \in L^1_\rho(\mathbb{R}_+;\mathbb{R}^m)$ is defined as the function

$$g_t(\theta) = g(t+\theta), \quad \theta \geq 0.$$

In other words, the action of the adjoint semigroup on X' is given by

$$T_0'(t)(c,g) = \left(c + \int_0^t g(\theta)d\theta, g_t\right), \quad (c,g) \in X', \quad t \geq 0. \tag{2.11}$$

Now we check that X' satisfies the properties needed for the application of the abstract framework developed in Section 2.2.

Lemma 2.3.3. *$X' = \mathbb{R}^m \times L^1_\rho(\mathbb{R}_+;\mathbb{R}^m)$ is a closed subset of $X^\odot$ which is invariant under $\{T_0'(t)\}_{t\geq 0}$ and separates points.*

Proof. It is clear from the expression in (2.11) that X' is invariant under $\{T_0'(t)\}_{t\geq 0}$. In fact, if $g \in L^1_\rho$, then also $g_t \in L^1_\rho$, for all $t \geq 0$, with

$$||g_t||_{1,\rho} \leq e^{-\rho t}||g||_{1,\rho}.$$

To see that the semigroup is strongly continuous on X', let $(c,g) \in X'$. Observe that

$$||T_0'(t)(c,g) - (c,g)|| = \left|\int_0^t g(s)\,ds\right| + ||g_t - g||_{1,\rho} \leq \int_0^t |g(s)|\,ds + \int_0^\infty |g(t+s) - g(s)|e^{\rho\theta}\,ds.$$

We now need to prove that both terms on the right-hand side above converge to 0 as $t \to 0^+$. This is clear for the second term; in fact, if $g \in C_c^\infty(\mathbb{R}_+)$, then the result follows from uniform continuity of g on compact sets (similar to what we did in the proof of Lemma 2.3.2), and since $C_c^\infty(\mathbb{R}_+)$ is dense in L^1_ρ the conclusion follows. As far as the first term is concerned, note that

$$\int_0^t |g(s)|ds = \int_0^\infty |g(s)|\mathbb{1}_{[0,t]}(s)ds$$

and

$$|g(s)|\mathbb{1}_{[0,t]}(s) = e^{\rho s}|g(s)|e^{-\rho s}\mathbb{1}_{[0,t]}(s) \leq e^{\rho s}|g(s)| \in L^1,$$

so the conclusion follows from Lebesgue's dominated convergence theorem. Therefore, $T_0'(t)$ is strongly continuous on X'.

Finally, let us prove that X' separates points. Let $\psi_1, \psi_2 \in X$, $\psi_1 \neq \psi_2$, and choose $t_0 \in \mathbb{R}_-$ such that $\psi_1(t_0) \neq \psi_2(t_0)$. Then, there exists $i \in \{1,\ldots,m\}$ such that $\psi_1^i(t_0) \neq \psi_2^i(t_0)$, where ψ_j^i denotes the $i-$th component of $\psi_j = (\psi_j^1,\ldots,\psi_j^m)$, for $j = 1,2$. Since ψ_j^i are continuous functions, there exists $\varepsilon > 0$ such that

$$\psi_1^i(t) - \psi_2^i(t) \neq 0, \quad t \in (t_0 - \varepsilon, t_0 + \varepsilon).$$

Assume, without loss of generality, that

$$\psi_1^i(t) - \psi_2^i(t) > 0, \quad t \in (t_0 - \varepsilon, t_0 + \varepsilon).$$

Now let $c = 0$ and define $g = (g_1, ..., g_m)$ where

$$g_k = \begin{cases} 0 & \text{if } k \neq i \\ \mathbb{1}_{(t_0-\varepsilon,\, t_0+\varepsilon)} & \text{if } k = i \end{cases}, \quad k = 1, \dots, m$$

Then, clearly $(c,g) \in X'$ and we have

$$\langle (c,g), \psi_1 - \psi_2 \rangle = \int_{t_0-\varepsilon}^{t_0+\varepsilon} \psi_1^i(t) - \psi_2^i(t) dt > 0,$$

which implies that $\langle (c,g), \psi_1 \rangle \neq \langle (c,g), \psi_2 \rangle$. So in fact X' separates points, and this completes the proof. □

So in fact this is an adequate choice for X'. The dual of this space is the space $X'^* = \mathbb{R}^m \times L^\infty_\rho(\mathbb{R}_-; \mathbb{R}^m)$, where $L^\infty_\rho := L^\infty_\rho(\mathbb{R}_-; \mathbb{R}^m)$ is defined as the set of (equivalence classes of) measurable functions $\varphi : \mathbb{R}_- \to \mathbb{R}^m$ such that

$$\operatorname*{ess\,sup}_{\theta \leq 0} e^{\rho\theta} |\varphi(\theta)| < \infty,$$

and the duality pairing is given by

$$\langle (c,\varphi), (d,\psi) \rangle_{(X')^* \times X'} = c \cdot d + \int_0^\infty \varphi(-\theta) \cdot \psi(\theta)\, d\theta, \quad (c,\varphi) \in (X')^* = \mathbb{R}^m \times L^\infty_\rho, \ \ (d,\psi) \in X'$$

Finally, for the linear map $\ell : \mathbb{R}^m \to X'^*$ we choose the natural injection

$$\ell(c) = (c, 0), \quad c \in \mathbb{R}^m.$$

Now that this has been established, in order to guarantee existence and uniqueness of solutions for the integral equation we have to prove that (H1) and (H2) hold. The following series of lemmas will show that this is in fact true, and therefore we can apply the results of the previous section.

Lemma 2.3.4. *For every $z \in \mathbb{R}^m$, $t \geq 0$ and $(c,g) \in X'$,*

$$\langle T_0'^*(t) \ell z, (c,g) \rangle = z \cdot \left(c + \int_0^t g(\theta) d\theta \right), \quad t \geq 0$$

Proof. Let $z \in \mathbb{R}^m, (c,g) \in X'$ and fix $t \geq 0$. We have

$$\langle T_0'^*(t) \ell z, (c,g) \rangle = \langle \ell z, T_0'(t)(c,g) \rangle = \left\langle (z,0), \left(c + \int_0^t g(\theta) d\theta, g_t \right) \right\rangle = z \cdot \left(c + \int_0^t g(\theta) d\theta \right),$$

as desired. □

Lemma 2.3.5. *Let $h : \mathbb{R}_+ \to \mathbb{R}^m$ be a continuous function. Then, for every $t \geq 0$ and $(c,g) \in X'$, we have*

$$\left\langle \int_0^t T_0'^*(t-\tau) \ell h(\tau)\, d\tau, (c,g) \right\rangle = c \cdot \int_0^t h(\tau) d\tau + \int_0^t \int_0^{t-s} g(s) \cdot h(\sigma)\, d\sigma\, ds.$$

Proof. Let $(c,g) \in X'$. Using the previous lemma, we have, for every $t \geq 0$,

$$\begin{aligned}\left\langle \int_0^t T_0'^*(t-\tau)\ell h(\tau)\,d\tau, (c,g) \right\rangle &= \int_0^t \langle T_0'^*(t-\tau)\ell h(\tau), (c,g)\rangle\, d\tau \\ &= \int_0^t h(\tau)\cdot\left(c+\int_0^{t-\tau} g(s)ds\right) d\tau \\ &= c\cdot\int_0^t h(\tau)\,d\tau + \int_0^t h(\tau)\cdot\left(\int_0^{t-\tau} g(s)ds\right) d\tau \\ &= c\cdot\int_0^t h(\tau)\,d\tau + \int_0^t \int_0^{t-s} h(\tau)\cdot g(s)\,d\tau\, ds,\end{aligned}$$

which proves the result. □

Observe that, for this choice of X and X', the linear injection $j\colon X \to X'^*$ is given by

$$\langle j\psi, (c,g)\rangle = \langle \psi, (c,g)\rangle = \psi(0)\cdot c + \int_0^\infty \psi(-\theta)\cdot g(\theta)\,d\theta, \quad \psi \in X,\ (c,g) \in X'^*.$$

Lemma 2.3.6. *Let $t \geq 0$ and $h : \mathbb{R}_+ \to \mathbb{R}^m$ be a continuous function. Define $\psi \in X$ by*

$$\psi(\theta) = \begin{cases} \int_0^{t+\theta} h(s)ds & \text{if } -t \leq \theta \leq 0 \\ 0 & \text{if } \theta < -t \end{cases}.$$

Then, for any $t \geq 0$,

$$\int_0^t T_0'^*(t-\tau)\ell h(\tau)d\tau = j\psi.$$

In particular,

$$\int_0^t T_0'^*(t-\tau)\ell h(\tau)d\tau \in j(X)$$

and

$$\left\| j^{-1}\int_0^t T_0'^*(t-\tau)\ell h(\tau)d\tau \right\|_{\infty,\rho} \leq \int_0^t |h(s)|ds, \quad t \geq 0.$$

Proof. For every $(c,g) \in X'$, we have

$$\begin{aligned}\langle \psi, (c,g)\rangle &= c\cdot\psi(0) + \int_0^\infty \psi(-\theta)\cdot g(\theta)d\theta \\ &= c\cdot\int_0^t h(s)ds + \int_0^t \left(\int_0^{t-\theta} h(s)ds\right)\cdot g(\theta)d\theta \\ &= c\cdot\int_0^t h(s)ds + \int_0^t \int_0^{t-\theta} h(s)\cdot g(\theta)ds d\theta \\ &= \left\langle \int_0^t T_0'^*(t-\tau)\ell h(\tau)d\tau, (c,g)\right\rangle,\end{aligned}$$

by the previous lemma. This implies that

$$\int_0^t T_0'^*(t-\tau)\ell h(\tau)d\tau = j\psi,$$

for all $t \geq 0$, and the estimate now follows from

$$||\psi||_X = \sup_{\theta \leq 0} e^{\rho\theta}|\psi(\theta)| \leq \sup_{-t \leq \theta \leq 0} e^{\rho\theta} \int_0^{t+\theta} |h(s)|ds = \int_0^t |h(s)|ds.$$

□

In other words, hypothesis **(H)** holds, and we can take $M = 1$ and $\omega = 0$ in (H2), so that the results from Section 2.2 can be applied to our integral equation.

Before we state the main result, we remark that, if the solution $u : \mathbb{R}_+ \to X$ of (AIE) and F are continuous, then as a particular case of Lemma 2.3.6 we obtain

$$\left(j^{-1}\int_0^t T_0'^*(t-s)\ell F(u(s))ds\right)(\theta) = \begin{cases} \int_0^{t+\theta} F(u(s))ds, & \text{if } -t \leq \theta \leq 0 \\ 0 & \text{if } \theta < -t \end{cases}.$$

We can now prove that, given $\psi \in X$, there is an equivalence between solutions of the delay differential equation

$$\begin{cases} \dot{y}(t) = F(y_t) & t > 0 \\ y(t) = \psi(t) & t \leq 0 \end{cases} \tag{DDE}$$

and the abstract integral equation

$$u(t) = T_0(t)\psi + j^{-1}\int_0^t T_0'^*(t-s)\ell F(u(s))\,ds, \quad t \geq 0. \tag{AIE}$$

Theorem 2.3.7. *Let $\psi \in X = C_{0,\rho}(\mathbb{R}_-;\mathbb{R}^m)$.*

(i) Suppose that $y : \mathbb{R} \to \mathbb{R}^m$ is continuous and satisfies (DDE). Then the function $u : [0,\infty) \to X$ defined by $u(t) = y_t$, for $t \geq 0$, is continuous and satisfies (AIE).

(ii) If $u : [0,\infty) \to X$ is continuous and satisfies (AIE), then the function $y : \mathbb{R} \to \mathbb{R}^m$ defined by

$$y(t) = \begin{cases} \psi(t) & \text{if } -\infty < t < 0 \\ u(t)(0) & \text{if } t \geq 0 \end{cases}$$

is continuous and satisfies (DDE).

Proof. To see *(i)*, suppose y is a solution of (DDE). The fact that u is continuous follows from continuity of translation in $C_{0,\rho}(\mathbb{R}_-;\mathbb{R}^m)$. If $\theta \in [-t,0]$, we have

$$y(t+\theta) = \psi(0) + \int_0^{t+\theta} F(y_s)ds,$$

because y is a solution of the differential equation. Consequently, for such θ,

$$u(t)(\theta) - (T_0(t)\psi)(\theta) = y(t+\theta) - \psi(0) = \int_0^{t+\theta} F(u(s))ds$$

by the remark just before the theorem. On the other hand, if $\theta < -t$,

$$u(t)(\theta) - (T_0(t)\psi)(\theta) = y(t+\theta) - \psi(t+\theta) = \psi(t+\theta) - \psi(t+\theta) = 0.$$

By the same remark,

$$u(t)(\theta) - (T_0(t)\psi)(\theta) = \begin{cases} \int_0^{t+\theta} F(u(s))ds & \text{if } -t \le \theta \le 0 \\ 0 & \text{if } \theta < -t \end{cases} = \left(j^{-1} \int_0^t T_0'^*(t-s)\ell F(u(s))ds \right)(\theta), \tag{2.12}$$

which means precisely that u is a solution of (AIE).

To prove *(ii)*, let u be a continuous solution of the integral equation and define

$$y(t) = \begin{cases} \psi(t) & \text{if } -\infty < t < 0 \\ u(t)(0) & \text{if } t \ge 0 \end{cases}.$$

The initial condition is clearly satisfied. Let $t \ge 0, h > 0$. Then

$$\begin{aligned} \frac{y(t+h) - y(t)}{h} &= \left(\frac{T_0(t+h)\psi - T_0(t)\psi}{h} \right)(0) + \frac{1}{h}\int_h^{t+h} F(u(s))ds \\ &\to (A_0\psi)(0) + F(u(t)) = F(u(t)), \end{aligned}$$

as $h \to 0^+$. Recall that A_0 is the infinitesimal generator of the semigroup T_0. If $0 < h \le t$, then as in the proof of Lemma 2.3.2 the left derivative of y is $F(u(t))$. Consequently, y is differentiable and $\dot{y}(t) = F(u(t))$, for $t \ge 0$. The proof is now finished if we show that $u(t) = y_t$ for all $t \ge 0$. But this follows from (2.12), since, if $\theta \in (-t, 0]$, then

$$y_t(\theta) = \psi(0) + \int_0^{t+\theta} F(u(s))ds = \psi(0) + u(t)(\theta) - (T_0(t)\psi)(\theta) = u(t)(\theta),$$

and, if $\theta \le -t$,

$$y_t(\theta) = y(t+\theta) = \psi(t+\theta) = \psi(t+\theta) + u(t)(\theta) - (T_0(t)\psi)(\theta) = u(t)(\theta),$$

since $(T_0(t)\psi)(\theta) = \psi(t+\theta)$ for $\theta \le -t$. This shows that in fact $u(t) = y_t$, and concludes the proof of the theorem. □

In particular, the theorem above provides us with a correspondence between solutions of the linearized problems. Let us first describe the linearization of the equation (DDE) about an equilibrium, i.e. a constant solution of the equation. Let $\bar{\psi}$ be an equilibrium of (DDE). Since F is continuously Fréchet differentiable, we may expand $F(\bar{\psi} + x_t)$ about $\bar{\psi}$. Neglecting higher order terms, we obtain the linearized equation

$$\dot{y}(t) = DF(\bar{\psi})y_t.$$

Now by definition of the Fréchet derivative, the function $DF(\bar{\psi}) : X \to \mathbb{R}^m$ is a bounded, linear operator on $C_{0,\rho}(\mathbb{R}_-; \mathbb{R}^m)$, and so there exists a measure $\mu \in M_\rho(\mathbb{R}_+; \mathbb{R}^{m\times m})$ such that

$$DF(\bar{\psi})y_t = \langle y_t, \mu \rangle.$$

Consequently, the linearized version of (DDE) about $\bar{\psi}$ reads as

$$\begin{cases} \dot{y}(t) = \langle y_t, \mu \rangle, & t > 0 \\ y(t) = \psi(t), & t \leq 0 \end{cases} \quad \text{or} \quad \begin{cases} \dot{y}(t) = DF(\overline{\psi}) y_t, & t > 0 \\ y(t) = \psi(t), & t \leq 0 \end{cases}. \tag{LDDE}$$

By Theorem 2.3.7, this problem is equivalent to the linearized abstract integral equation

$$u(t) = T_0(t)\psi + j^{-1} \int_0^t T_0'^*(t-s)\ell\langle u(s), \mu \rangle \, ds, \quad t \geq 0. \tag{LAIE}$$

To be more precise:

- if $\{T(t)\}_{t\geq 0}$ is the C_0 semigroup defined by the solution of the (LDDE), $T(t)\psi = y_t$, where y solves the equation with initial condition ψ, then $u(t) = T(t)\psi$ solves the linearized abstract integral equation (LAIE);
- if u is the solution of (LAIE) with initial condition $\psi \in X$, then by Theorem 2.2.2 u defines a semigroup $\{T(t)\}_{t\geq 0}$ of operators on X and the function

$$y(t) = \begin{cases} \psi(t) & \text{if } t \leq 0 \\ (T(t)\psi)(0) & \text{if } t > 0 \end{cases} \tag{2.13}$$

  is a solution of the IVP (LDDE).

From now on, we refer to the semigroup associated to the solution of (LDDE) as the linearized semigroup. This equivalence is the starting point for the next chapter.

Chapter 3

Principle of Linearized Stability

In this chapter, we finally state and prove a Principle of Linearized Stability for the equation (DDE). First, we study the characteristic equation of the linearized problem (LDDE) associated with a steady-state $\overline{\psi}$, and prove that, if $\{T(t)\}_{t\geq 0}$ is the semigroup determined by the solution of (LDDE) and A is its infinitesimal generator, then the only spectral values of A in the half-space $\{\mathrm{Re}\,(\lambda) > -\rho\}$ are eigenvalues, and coincide exactly with the characteristic roots. We then use a theorem by Desch and Schappacher [9] to prove the Principle of Linearized Stability for (DDE).

3.1 The characteristic equation

Let $\{T(t)\}_{t\geq 0}$ be the linearized semigroup, i.e. the semigroup associated with the solution of the linearized equation (LDDE), defined by

$$T(t)\psi = y_t \tag{3.1}$$

where $y = y(\theta)$ solves (LDDE) with initial condition $\psi \in X$. In this section, we are interested in describing the spectrum of its infinitesimal generator, which will be denoted by A. First, we need to determine A.

Lemma 3.1.1. *The infinitesimal generator A of the linearized semigroup $\{T(t)\}_{t\geq 0}$ is given by*

$$D(A) = \left\{ \psi \in X : \psi \text{ is differentiable, } \dot{\psi} \in X \text{ and } \dot{\psi}(0) = \int_0^\infty [d\mu(\theta)]\psi(-\theta) \right\}$$

and, for $\psi \in D(A)$,

$$A\psi = \dot{\psi}.$$

Proof. The proof is very similar to that of part *(ii)* in Lemma 2.3.2, so we omit the details. □

Before introducing the characteristic equation, recall that, given a measure $\mu \in M_\rho(\mathbb{R}_+;\mathbb{R}^{m\times m})$, the Laplace transform of μ is defined by

$$\hat{\mu}(\lambda) = \int_0^\infty [d\mu(\theta)]\, e^{-\lambda\theta},$$

which is well defined for all λ such that Re $(\lambda) \geq -\rho$ (see the beginning of Section 2.3 and the Appendix). Define

$$\Pi_+^\rho := \{\lambda \in \mathbb{C} \colon \mathrm{Re}(\lambda) > -\rho\}.$$

Definition 3.1.2. For $\lambda \in \Pi_+^\rho$, consider the matrix $\Delta(\lambda) = \lambda I - \hat{\mu}(\lambda)$. The equation

$$\det \Delta(\lambda) = 0 \tag{CE}$$

is called the *characteristic equation* of problem (LDDE). The solutions of the characteristic equation are called *characteristic roots*.

Let $L = DF(\overline{\psi})$, where $\overline{\psi}$ is a steady-state of (DDE). Identifying L with a measure $\mu \in M_\rho(\mathbb{R}_-; \mathbb{R}^{m\times m})$ so that

$$L(\psi) = \int_0^\infty [d\mu(\theta)]\, \psi(-\theta), \quad \psi \in X,$$

observe that the characteristic equation (CE) reads as

$$\det[\lambda I - L(e^{\lambda \cdot} I)] = 0, \tag{3.2}$$

for $\lambda \in \Pi_\rho^+$, which is the usual form of the characteristic equation for delay differential equations. Note also that, even though the characteristic equation (CE) still makes sense on the line Re $(\lambda) = -\rho$, (3.2) is not defined for such λ; indeed, $e^{\lambda \cdot} I \notin X$ if Re $(\lambda) = -\rho$.

We now prove that, on the half-space Π_+^ρ, there is only a finite number of spectral values of A, which turn out to be eigenvalues of A, and that they coincide with the characteristic roots. To see this, we start by establishing an explicit expression for the resolvent, which will reveal part of its properties.

Recall that, A being the generator of the C_0 semigroup $\{T(t)\}_{t\geq 0}$, it is a closed operator, by Hille-Yosida's theorem. Consequently, $\lambda I - A$ is also closed, for $\lambda \in \mathbb{C}$. Our aim is to find $\rho(A) \cap \Pi_+^\rho$. By Lemma 1.2.3, the elements of this set are precisely those $\lambda \in \Pi_+^\rho$ such that $\lambda I - A \colon D(A) \to X$ is bijective.

Let $\lambda \in \Pi_+^\rho$ and $\psi \in X$. Observe that $\varphi \in D(A)$ is such that $\psi = (\lambda I - A)\varphi$ if and only if

$$\text{(i) } \lambda\varphi - \dot{\varphi} = \psi, \quad \text{(ii) } \dot{\varphi} \in X, \quad \text{(iii) } \lambda\varphi(0) - \int_0^\infty [d\mu(\theta)]\varphi(-\theta) = \psi(0). \tag{3.3}$$

The condition in (i) is a first order ODE, and it is satisfied if and only if, for $\theta \leq 0$,

$$\varphi(\theta) = e^{\lambda\theta}\varphi(0) + \int_\theta^0 e^{\lambda(\theta - s)}\psi(s)\, ds. \tag{3.4}$$

Condition (iii) now reads as

$$\lambda\varphi(0) - \int_0^\infty [d\mu(\theta)] \left(e^{-\lambda\theta}\varphi(0) + \int_{-\theta}^0 e^{-\lambda(\theta+s)}\psi(s)\, ds \right) = \psi(0),$$

which is itself equivalent to

$$\Delta(\lambda)\varphi(0) = \psi(0) + \int_0^\infty [d\mu(\theta)] \left(\int_0^\theta e^{-\lambda s}\psi(s-\theta) \right) ds. \tag{3.5}$$

Inserting (3.5) into (3.4), it follows that, if λ is such that $\det \Delta(\lambda) \neq 0$, then

$$\varphi(\theta) = e^{\lambda\theta}\left\{\Delta(\lambda)^{-1}\left[\psi(0) + \int_0^\infty [d\mu(r)]\left(\int_0^r e^{-\lambda s}\psi(s-r)\,ds\right)\right] + \int_\theta^0 e^{-\lambda s}\psi(s)\,ds\right\}. \tag{3.6}$$

It will follow that $\lambda I - A$ is bijective once we prove that φ defined above is an element of X.

Lemma 3.1.3. *Suppose $\lambda \in \Pi_+^\rho$ is such that $\det \Delta(\lambda) \neq 0$. The function φ defined in (3.6) is an element of X.*

Proof. For $\theta \leq 0$, write

$$\varphi(\theta) = e^{\lambda\theta}\Delta(\lambda)^{-1}\psi(0) + \Delta(\lambda)^{-1}\varphi_1(\theta) + \varphi_2(\theta),$$

where

$$\varphi_1(\theta) = e^{\lambda\theta}\int_0^\infty [d\mu(r)]\left(\int_0^r e^{-\lambda s}\psi(s-r)\,ds\right), \quad \varphi_2(\theta) = e^{\lambda\theta}\int_\theta^0 e^{-\lambda s}\psi(s)\,ds.$$

It is clear that $\varphi \in X$ if $\varphi_1, \varphi_2 \in X$.

We first see that φ_2 is in X. Let $\varepsilon > 0$ and define $\rho_0 = \operatorname{Re}(\lambda) + \rho$. Then ρ_0 is positive, because $\lambda \in \Pi_+^\rho$. Choose $M_1 > 0$ such that

$$\theta \leq -M_1 \Longrightarrow e^{\rho\theta}|\psi(\theta)| < \frac{\rho_0\varepsilon}{2}$$

and $M_2 > 0$ such that

$$\theta \leq -M_2 \Longrightarrow e^{\rho_0\theta}\int_{-M_1}^0 e^{-\rho_0 s}e^{\rho s}|\psi(s)|\,ds < \frac{\varepsilon}{2}.$$

Then, for $\theta \leq \min\{-M_1, -M_2\}$,

$$\begin{aligned} e^{\rho\theta}|\varphi_2(\theta)| &= e^{\rho_0\theta}\int_\theta^0 e^{-\rho_0 s}e^{\rho s}|\psi(s)|\,ds \\ &= e^{\rho_0\theta}\int_\theta^{-M_1} e^{-\rho_0 s}\frac{\rho_0\varepsilon}{2}\,ds + e^{\rho_0\theta}\int_{-M_1}^0 e^{-\rho_0 s}e^{\rho s}|\psi(s)|\,ds \\ &< \frac{\rho_0\varepsilon}{2}\frac{1}{\rho_0}\left(1 - e^{\rho_0(\theta+M_1)}\right) + \frac{\varepsilon}{2} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This proves that

$$\lim_{\theta\to-\infty} e^{\rho\theta}|\varphi_2(\theta)| = 0,$$

that is, that $\varphi_2 \in X$.

To see that $\varphi_1 \in X$, first note that

$$r \mapsto \int_0^r e^{-\lambda s}\psi(s-r)\,ds$$

is simply the function $r \mapsto \varphi_2(-r)$, and hence an element of X, by the previous paragraph. In fact,

$$\int_0^r e^{-\lambda s}\psi(s-r)\,ds = \int_{-r}^0 e^{-\lambda(s+r)}\psi(s)\,ds = e^{-\lambda r}\int_{-r}^0 e^{-\lambda s}\psi(s)\,ds = \varphi_2(-r).$$

Consequently,

$$\left|\int_0^\infty [d\mu(r)]\left(\int_0^r e^{-\lambda s}\psi(s-r)\,ds\right)\right| = |\langle\varphi_2,\mu\rangle|$$

is bounded, and hence

$$e^{\rho\theta}|\varphi_1(\theta)| = e^{(\rho+\mathrm{Re}\,(\lambda))\theta}\left|\int_0^\infty [d\mu(r)]\left(\int_0^r e^{-\lambda s}\psi(s-r)\,ds\right)\right| \to 0, \quad \theta\to-\infty,$$

because $\lambda \in \Pi_+^\rho$.

□

Since $\varphi \in X$, also $\dot\varphi = \lambda\varphi - \psi \in X$, and so condition (ii) is satisfied. In conclusion, this proves that if $\lambda \in \Pi_+^\rho$ is not a characteristic root, then $\lambda \in \rho(A)$. The converse is also true, because characteristic roots are eigenvalues of A. Indeed, if $\lambda \in \Pi_+^\rho$ is such that $\det\Delta(\lambda) = 0$, then there exists $\gamma \neq 0$ such that $\Delta(\lambda)\gamma = \lambda\gamma$ and the function $\psi(\theta) = e^{\lambda\theta}\gamma$ is an element of X satisfying conditions (3.3) with $\psi \equiv 0$. We have thus seen that, for $\lambda \in \Pi_+^\rho$,

$$\text{(i) if } \det\Delta(\lambda) \neq 0, \text{ then } \lambda \in \rho(A); \qquad \text{(ii) if } \det\Delta(\lambda) = 0, \text{ then } \lambda \in \sigma_p(A).$$

Actually, it follows from the same computation that there are no eigenvalues of A outside Π_+^ρ, for if $\varphi(t) = e^{\lambda t}\gamma$, for some non-zero vector γ, and $\mathrm{Re}\,(\lambda) \leq -\rho$, then $\varphi \notin X$. Therefore,

$$\sigma_p(A) = \sigma_p(A)\cap\Pi_+^\rho = \sigma(A)\cap\Pi_+^\rho = \{\lambda \in \Pi_\rho^+ :\ \det\Delta(\lambda) = 0\}.$$

This shows that, on the half-plane Π_+^ρ, we recover the behaviour of differential equations with finite delay, in the sense that the spectrum of the infinitesimal generator (in the half-plane Π_ρ^+) is, as in that case, only formed by eigenvalues. We have furthermore obtained a representation for the resolvent function. For $\lambda \in \rho(A)$, the computations above show that, given $\psi \in X$,

$$(R(\lambda : A)\psi)(\theta) = e^{\lambda\theta}\left\{\Delta(\lambda)^{-1}\left[\psi(0) + \int_0^\infty [d\mu(r)]\left(\int_0^r e^{-\lambda s}\psi(s-r)\,ds\right)\right] + \int_\theta^0 e^{-\lambda s}\psi(s)\,ds\right\}, \tag{3.7}$$

for any $\theta \leq 0$.

We now collect the results we have just proved and show some other properties of the characteristic roots in the following theorem:

Theorem 3.1.4. *Let $\Pi_+^\rho = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > -\rho\}$ and consider the notation above.*

(i) A point $\lambda \in \Pi_+^\rho$ belongs to the spectrum of A if and only if it is a root of the characteristic equation

$$\det(\lambda I - \widehat{\mu}(\lambda)) = 0. \tag{3.8}$$

(ii) Every root of (3.8) in Π_+^ρ is an eigenvalue of A, and there are at most finitely many such roots in Π_+^ρ.

(iii) The mapping $\lambda \mapsto R(\lambda, A)$ is meromorphic in Π_+^ρ; more precisely, it is analytic in Π_+^ρ except at the characteristic roots λ_ℓ, where it has a pole of order at most equal to the multiplicity of λ_ℓ as a root of (3.8).

Proof. Part *(i)* is a consequence of our previous discussion.

The first statement of *(ii)* was also seen above, so we need only prove that there are at most finitely many characteristic roots in Π_+^ρ. The function

$$p(\lambda) := \det(\lambda I - \hat{\mu}(\lambda)), \quad \lambda \in \Pi_+^\rho$$

is analytic and not identically zero, which implies that its zeros are isolated points in $\mathbb{C}$, that the set of zeros is countable (possibly finite) and, in case they are in infinite number, say $\{\lambda_n\}_{n\in\mathbb{N}}$, then necessarily $|\lambda_n| \to \infty$. We claim that there must be only finitely many characteristic roots.

Indeed, suppose there is an infinite number of characteristic roots $\{\lambda_n\}_{n\in\mathbb{N}}$ in Π_+^ρ. By the first paragraph, necessarily $|\lambda_n| \to \infty$. Then, there exist vectors $\gamma_n \neq 0$, which we can assume to satisfy $||\gamma_n|| = 1$, for all $n \in \mathbb{N}$, such that

$$\lambda_n \gamma_n = \int_0^\infty [d\mu(s)]\, e^{-\lambda_n s}\gamma_n, \quad n \in \mathbb{N}.$$

Consequently, for any $n \in \mathbb{N}$,

$$1 = \left| \frac{1}{\lambda_n} \int_0^\infty [d\mu(s)] e^{-\lambda_n s}\gamma_n \right| \leq \frac{1}{|\lambda_n|} \int_0^\infty ||e^{-\lambda_n s}\gamma_n||\, d|\mu|(s) = \frac{1}{|\lambda_n|} \int_0^\infty e^{-(\mathrm{Re}(\lambda_n)+\rho)s} e^{\rho s}\, d|\mu|(s).$$

Choose n large enough so that $|\lambda_n| > ||\mu||$. For such n, the estimate above yields

$$1 \leq \frac{1}{|\lambda_n|} \int_0^\infty e^{-(\mathrm{Re}(\lambda_n)+\rho)s} e^{\rho s}\, d|\mu|(s) \leq \frac{1}{|\lambda_n|} \int_0^\infty e^{\rho s}\, d|\mu|(s) = \frac{||\mu||}{|\lambda_n|} < 1,$$

and this is a contradiction. Therefore there can only exist finitely many characteristic roots on Π_+^ρ.

Finally, to prove *(iii)*, observe that the function $\lambda \mapsto \lambda I - \hat{\mu}(\lambda)$ is analytic, for Π_+^ρ. By the well-known Cramer's rule, the inverse $(\lambda I - \hat{\mu}(\lambda))^{-1}$ can be expressed as an analytic matrix-valued function divided by the scalar function $\det(\lambda I - \hat{\mu}(\lambda))$, and so the statements in *(iii)* are true for $\lambda \mapsto (\lambda I - \hat{\mu}(\lambda))^{-1}$. By the representation in (3.6), the same is true for the resolvent function, and this finishes the proof of the theorem. For results on a spectral analysis for linear operators, namely an operational calculus for the representation of the resolvent operator, see [25]. □

It is worth mentioning that the approach described above is quite different from the one in [12], the main reference for this dissertation. In fact, the authors in [12] obtain a representation of the same form as the one in (3.7) by writing the resolvent $R(\lambda : A)$ as the Laplace transform of the linearized semigroup $\{T(t)\}_{t\geq 0}$,

$$R(\lambda : A)\psi = \int_0^\infty e^{-\lambda t} T(t)\psi\, dt, \quad \lambda \in \Pi_+^\rho,\ \psi \in X,$$

which is well defined for λ with real part greater than the growth bound ω of the semigroup (see Definition 1.1.5, the proof of Hille-Yosida's Theorem 1.2.14 and the expression (1.11)). So one needs to guarantee that the growth bound ω of the semigroup is such that $\mathrm{Re}\,(\lambda) > \omega$, for $\lambda \in \Pi_+^\rho$, i.e. that $\omega + \rho \leq 0$. This is not explicitly done in [12], nor is it mentioned. As a consequence, it is not so clear to

us how the statements in Theorem 3.1.4 were obtained in [12], and so here the arguments in [12] had to be modified.

To conclude this section, we establish two criteria for exponential stability of the linearized semigroup. We denote by $L^\infty_\rho(\mathbb{R}_+;\mathbb{R}^m)$ the (weighted) space of function $f\colon \mathbb{R}_+ \to \mathbb{R}^m$ such that

$$\sup_{t\geq 0} e^{\rho t}|f(t)| < \infty.$$

Theorem 3.1.5. *Assume that the solution $y = y(\theta)$ of (LDDE) with initial condition $\psi \in X$ lies in $L^\infty_\rho(\mathbb{R}_+;\mathbb{R}^m)$, for all initial conditions ψ. Then the solution semigroup T is uniformly exponentially stable, that is, there exist $\varepsilon > 0$ and $M \geq 1$ such that*

$$||T(t)|| \leq Me^{-\varepsilon t}, \quad t \geq 0.$$

Proof. By Datko-Pazy's Theorem 1.3.4, it suffices to see that

$$\int_0^\infty ||T(t)\psi||dt < \infty,$$

for all $\psi \in X$. So let $\psi \in X$ be arbitrary and denote $K := \sup_{\theta\geq 0} e^{\rho\theta}|y(\theta)| < \infty$. We have

$$\begin{aligned}
\int_0^\infty ||T(t)\psi||\,dt &= \int_0^\infty \sup_{\theta\leq 0} e^{\rho\theta}\,|(T(t)\psi)(\theta)|\,dt \\
&= \int_0^\infty e^{-\rho t} \sup_{\theta\leq 0} e^{\rho(\theta+t)}|y(\theta+t)|\,dt \\
&\leq \int_0^\infty e^{-\rho t} \max\left\{ \sup_{\theta+t\leq 0} e^{\rho(\theta+t)}|\psi(\theta+t)|,\ \sup_{\theta+t>0} e^{\rho(\theta+t)}|y(\theta+t)| \right\} dt \\
&\leq \int_0^\infty e^{-\rho t} \sup_{\theta+t\leq 0} e^{\rho(\theta+t)}|\psi(\theta+t)|\,dt + \int_0^\infty e^{-\rho t} \sup_{\theta+t>0} e^{\rho(\theta+t)}|y(\theta+t)|\,dt \\
&\leq (||\psi|| + K)\int_0^\infty e^{-\rho t}\,dt < \infty,
\end{aligned}$$

and the conclusion follows. □

For the proof of the second result, we need the theory of Volterra integro-differential equations, which we briefly explore in the Appendix. For convenience of the reader, we recall some of these results applied to our concrete situation. Let $\mu \in M(\mathbb{R}^+;\mathbb{R}^{m\times m})$, the Banach space of matrix-valued Radon measures on J with bounded total variation (see (2.7)). The solution of an integro-differential linear equation of the form

$$\begin{cases} \dot{y}(t) = \displaystyle\int_0^t [d\mu(s)]\,y(t-s) + f(t), & t > 0 \\ y(0) = y_0 \end{cases} \tag{3.9}$$

can be expressed by the formula

$$y(t) = r(t)y_0 + \int_0^t r(t-s)f(s)ds, \quad t \geq 0, \tag{3.10}$$

where the $m \times m$ matrix-valued function r is the solution of the problem

$$\begin{cases} \dot{r}(t) = \int_0^t [d\mu(s)]\, r(t-s) & t > 0 \\ r(0) = I \end{cases}. \tag{3.11}$$

Such function r is called the *differential resolvent kernel* of the measure μ. By Grossman-Miller's theorem (see Theorem A.8), it holds that $r \in L^1(\mathbb{R}_+;\mathbb{R}^{m\times m})$ if and only if the characteristic equation (3.8) (with $\rho = 0$) has no roots with nonnegative real part. Moreover, this result also states that, if $r \in L^1(\mathbb{R}_+;\mathbb{R}^{m\times m})$, then also $\dot{r} \in L^1(\mathbb{R}_+;\mathbb{R}^{m\times m})$, and as a consequence r is also in $L^\infty(\mathbb{R}_+;\mathbb{R}^{m\times m})$.

Theorem 3.1.6. *Let $\mu \in M_\rho(\mathbb{R}_+;\mathbb{R}^{m\times m})$. If the characteristic equation (3.8) has no roots with $\mathrm{Re}(\lambda) \geq -\rho$, then the solution semigroup $\{T(t)\}_{t\geq 0}$ of the linearized problem is uniformly exponentially stable. More precisely, there exists $M \geq 1$ such that*

$$||T(t)|| \leq Me^{-\rho t}, \quad t \geq 0.$$

Proof. Let $y = y(\theta)$ be the solution of the linearized problem (LDDE). Observe that

$$\begin{aligned} ||T(t)\psi|| = ||y_t|| &= \sup_{\theta\leq 0} e^{\rho\theta}|y(t+\theta)| = e^{-\rho t}\sup_{\theta\leq 0} e^{\rho(t+\theta)}|y(t+\theta)| \\ &= e^{-\rho t}\max\left\{\sup_{\theta\leq 0}|\psi(\theta)|,\ \sup_{0<s\leq t} e^{\rho s}|y(s)|\right\} \\ &= e^{-\rho t}\max\left\{||\psi||,\ \sup_{0<s\leq t} e^{\rho s}|y(s)|\right\}. \end{aligned}$$

It is then enough to prove that there exists $M \geq 1$ such that, for all $\psi \in X$,

$$\sup_{s\in\mathbb{R}_+} e^{\rho s}|y(s)| \leq M||\psi||.$$

Note that, since $y = y(t)$ is a solution of (LDDE), we have, for all $t \geq 0$,

$$\dot{y}(t) = \int_0^\infty [d\mu(\theta)]\, y(t-\theta) = \int_0^t [d\mu(\theta)]\, y(t-\theta) + \int_t^\infty [d\mu(\theta)]\, \psi(t-\theta), \tag{3.12}$$

so that $y = y(t)$ solves the Volterra equation (3.9) with

$$f(t) = \int_t^\infty [d\mu(s)]\, \psi(t-s), \quad y_0 = \psi(0) \tag{3.13}$$

(see the considerations after Theorem 2.3.7). Let r be the differential resolvent kernel of μ (i.e. the solution of (3.11)). Multiplying (3.10) by $e^{\rho t}$ we get

$$e^{\rho t}y(t) = r^\rho(t)\psi(0) + \int_0^t r^\rho(t-s)f^\rho(s)ds, \tag{3.14}$$

where $r^\rho(t) = e^{\rho t}r(t)$, and $f^\rho(t) = e^{\rho t}f(t)$. Now, in order to apply Grossman-Miller's Theorem A.8, we

need to prove that $r^\rho \in L^1(\mathbb{R}_+;\mathbb{R}^{m\times m})$. Clearly, $r^\rho(0) = I$ and

$$\dot{r}^\rho(t) = \rho r^\rho(t) + e^{\rho t}\dot{r}(t) = \rho r^\rho(t) + \int_0^t e^{\rho s}[d\mu(s)]\, r^\rho(t-s),$$

which means that r^ρ is the differential resolvent kernel of the measure

$$\mu^\rho := \rho I\, \delta_0 + e^{\rho s}\,\mu,$$

where δ_0 denotes the Dirac unit point mass at 0 and I is the identity matrix. The Laplace transform of μ^ρ is given by

$$\widehat{\mu}^\rho(\lambda) = \int_0^\infty [\rho I\, d\delta_0(s)]\, e^{-\lambda s} + \int_0^\infty [d\mu(s)]\, e^{(\rho-\lambda)s} = \rho I + \widehat{\mu}(\lambda-\rho),$$

and so the corresponding characteristic equation is

$$\det\left((\lambda-\rho)I - \widehat{\mu}(\lambda-\rho)\right) = 0. \tag{3.15}$$

Since by assumption there are no roots of (3.8) with $\mathrm{Re}(\lambda) \geq -\rho$, we conclude that there are no roots of (3.15) with $\mathrm{Re}(\lambda) \geq 0$. By Grossman-Miller's Theorem A.8,

$$r^\rho \in L^1(\mathbb{R}_+;\mathbb{R}^{m\times m}) \cap L^\infty(\mathbb{R}_+;\mathbb{R}^{m\times m}).$$

The fact that $r^\rho \in L^\infty$ implies that the first term on the right-hand side of (3.14) is bounded by $M_1||\psi||$, for some M_1. Moreover, because $r^\rho \in L^1$ and

$$|f^\rho(t)| \leq e^{\rho t}\int_t^\infty |\psi(t-s)|\, d|\mu|(s) \leq \int_t^\infty e^{\rho s}e^{\rho(t-s)}|\psi(t-s)|\, d|\mu|(s) \leq |\mu|\,||\psi||,$$

also the second term is bounded by $M_2||\psi||$ for some M_2. This finishes the proof. □

In particular, from Theorem 3.1.6 we obtain the following result:

Theorem 3.1.7. *Under the assumptions of Theorem 3.1.6,*

$$\omega \leq -\rho,$$

where ω is the growth bound of the semigroup $\{T(t)\}_{t\geq 0}$.

3.2 The Principle of Linearized Stability

We now state the main result of this chapter.

Theorem 3.2.1. (Principle of Linearized Stability) *Let $\rho > 0$ and assume that $F : C_{0,\rho}(\mathbb{R}_-;\mathbb{R}^m) \to \mathbb{R}^m$ is continuously Fréchet differentiable. Let $\overline{\psi}$ be a steady-state of (DDE). Let $\mu \in M_\rho(\mathbb{R}_+;\mathbb{R}^{m\times m})$*

represent $DF(\overline{\psi})$,

$$DF(\overline{\psi})\psi = \langle \mu, \psi \rangle = \int_0^\infty [d\mu(s)]\,\psi(-s), \quad \psi \in C_{0,\rho}(\mathbb{R}_-;\mathbb{R}^m).$$

(i) If all the characteristic roots of (CE) *in* Π_+^ρ *have negative real part, then the steady-state* $\overline{\psi}$ *is exponentially stable.*

(ii) If there exists at least one characteristic root of (CE) *in* Π_+^ρ *with positive real part, then the steady-state* $\overline{\psi}$ *is unstable.*

To prove this theorem, there still is some work to be done. As we have already mentioned, our ultimate tool for the proof will be a theorem by Desch and Schappacher (in [9]), which is a stability result for a strongly continuous semigroup depending on the position (on the complex plane) of the eigenvalues of its infinitesimal generator. Their result is stated below. For a proof, we refer to [9] or alternatively to [12, Corollary 5.2]

Theorem 3.2.2. (Desch and Schappacher, [9]) *Let* $\{\Sigma(t)\}_{t\geq 0}$ *be a strongly continuous nonlinear semigroup. Let* $\bar{\psi}$ *be a steady-state of* $\{\Sigma(t)\}_{t\geq 0}$. *Assume that*

(h1) for each $t \geq 0$, $\Sigma(t)$ *has a Fréchet derivative* $T(t)$ *at* $\bar{\psi}$;

(h2) $\{T(t)\}_{t\geq 0}$ *constitutes a strongly continuous semigroup on* X;

(h3) there exists a decomposition $X = X_+ \oplus X_-$ *into* $T(t)$*-invariant subspaces* X_- *and* X_+, *for all* $t \geq 0$, *such that* X_+ *is finite dimensional and the restriction of* $T(t)$ *to* X_- *is uniformly exponentially stable.*

Let A *be the infinitesimal generator of* $\{T(t)\}_{t\geq 0}$. *Then:*

(i) $\bar{\psi}$ *is (locally) exponentially stable if* $\mathrm{Re}(\lambda) < 0$, *for all* $\lambda \in \sigma(A|_{X_+})$.

(ii) $\bar{\psi}$ *is unstable if there exists* $\lambda \in \sigma(A|_{X_+})$ *with* $\mathrm{Re}(\lambda) > 0$.

In order to apply the theorem of Desch and Schappacher, we use the results of Section 1.5 to obtain a desirable decomposition of the phase space X. Consider the spectral projection

$$P_+\psi = \frac{1}{2\pi i}\int_{\Gamma_+} R(\lambda : A)\psi\, d\lambda, \quad \psi \in X,$$

where Γ_+ is a counterclockwise countour in the open right half-plane Π_+^ρ surrounding all the eigenvalues of A in Π_+^ρ. In this setting, Theorem 1.5.4 implies the following:

Proposition 3.2.3. P_+ *is a projection onto a closed subspace* X_+ *of* X, *which is the direct sum of the generalized eigenspaces of* A *associated with the eigenvalues in the interior of* Γ_+.

Proof. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A in Π_+^ρ. By Cauchy's theorem for multiply connected domains, we have

$$P_+ = \sum_{i=1}^{n} P_+^i,$$

where P_+^i is the spectral projection of the resolvent onto the generalized eigenspace of λ_i, i.e.

$$P_+^i = \int_{C_i} R(\lambda : A)\, d\lambda,$$

with C_i is a counterclockwise path surrounding λ_i and no other eigenvalue. The conclusion now follows from applying Theorem 1.5.4 to each P_+^i. □

Moreover, it is clear that X_+ is invariant under the semigroup $\{T(t)\}_{t\geq 0}$. Indeed, since $T(t)$ commutes with A (see *(iii)* in Proposition 1.1.7), it follows that, if $(A-\lambda I)^p \varphi = 0$, then

$$(A-\lambda I)^p T(t)\varphi = T(t)(A-\lambda I)^p \varphi = T(t)0 = 0,$$

for every $t \geq 0$. Thus $T(t)(X_+) \subset X+$, for every $t \geq 0$. Now let X_- be the complementary subspace of X_+ in X, i.e. $X_- := (I - P_+)X$, which exists because X_+ is closed. It is clear that X_- is also invariant under $\{T(t)\}_{t\geq 0}$, and that we have

$$X = X_+ \oplus X_-. \tag{3.16}$$

To apply Desch and Schappacher's theorem, it remains to prove that X_+ is finite dimensional, and that the restriction of the semigroup $\{T(t)\}_{t\geq 0}$ to X_- is uniformly exponentially stable.

Proposition 3.2.4. *With the notation above, $X_+ := P_+X$ is finite dimensional.*

Proof. Using the formula in (3.6) we obtain, for $\theta \leq 0$,

$$(P_+\psi)(\theta) = \frac{1}{2\pi i}\int_{\Gamma_+} e^{\lambda\theta}(\lambda I - \widehat{\mu}(\lambda))^{-1}\psi(0)\, d\lambda + \frac{1}{2\pi i}\int_{\Gamma_+} e^{\lambda\theta}(\lambda I - \widehat{\mu}(\lambda))^{-1}(P(\lambda)\psi)(\theta)\, d\lambda, \tag{3.17}$$

where

$$(P(\lambda)\psi)(\theta) = e^{\lambda\theta}\int_0^\infty [d\mu(r)]\left(\int_0^r e^{-\lambda s}\psi(s-r)\, ds\right), \quad \theta \leq 0.$$

Note that, since the third term on the right-hand side of (3.6) is a holomorphic function of λ, its integral over Γ_+ is zero, by Cauchy's theorem.

First we see that $P(\lambda) : X \to X$ is a bounded linear operator, for $\lambda \in \Pi_+^\rho$. The fact that the operator is linear is obvious. Now, given $\psi \in X$ and $\theta \leq 0$, we have

$$e^{\rho\theta}|(P(\lambda)\psi)(\theta)| \leq e^{(\rho+\mathrm{Re}(\lambda))\theta}||\psi||\int_0^\infty e^{\rho\theta}\left(\int_0^\infty e^{-(\mathrm{Re}(\lambda)+\rho)s}\, ds\right) d|\mu|(\theta) \leq \frac{|\mu|}{\mathrm{Re}(\lambda)+\rho}||\varphi||,$$

which proves that the operator is bounded. Finally, the proof that $P(\lambda)$ sends X to X was done in Lemma 3.1.3.

Now consider the second term in (3.17). By Theorem 3.1.4, there are only finitely many eigenvalues λ of A with $\mathrm{Re}\,\lambda > -\rho$, say $\lambda_1, \ldots, \lambda_p$. We can write the Laurent series of the meromorphic function

$(\lambda I - \widehat{\mu}(\lambda))^{-1}$ around the eigenvalue λ_ℓ, $\ell = 1, \ldots, p$,

$$(\lambda I - \widehat{\mu}(\lambda))^{-1} = \sum_{j=0}^{p_\ell - 1} \frac{j! K_{\ell,j}}{(\lambda - \lambda_\ell)^{j+1}} + g_\ell(\lambda)(\psi),$$

where $g_\ell(\cdot)(\psi)$ is holomorphic on a neighbourhood of λ_ℓ and p_ℓ is the order of λ_ℓ as a pole of $(\lambda I - \widehat{\mu}(\lambda))^{-1}$. For clarity of the argument, assume for the moment that $P(\lambda)$ is the identity operator. In this case, writing the integrand as a power series around λ_ℓ, we obtain

$$\begin{aligned} e^{\lambda\theta}(\lambda I - \widehat{\mu}(\lambda))^{-1}\psi &= e^{\lambda_\ell\theta}\, e^{(\lambda-\lambda_\ell)\theta} \left(\sum_{j=0}^{p_\ell-1} \frac{j! K_{\ell,j}(\psi)}{(\lambda-\lambda_\ell)^{j+1}} + g_\ell(\lambda)(\psi) \right) \\ &= e^{\lambda_\ell\theta} \left(\sum_{k=0}^{\infty} \frac{(\lambda-\lambda_\ell)^k t^k}{k!} \right) \left(\sum_{j=0}^{p_\ell-1} \frac{j! K_{\ell,j}(\psi)}{(\lambda-\lambda_\ell)^{j+1}} + g_\ell(\lambda)(\psi) \right) \end{aligned}$$

and it now follows that the residue of the integrand at λ_ℓ is given by $\sum_{j=0}^{p_\ell-1} K_{\ell,j}(\psi)\,\theta^j e^{\lambda_\ell\theta}$. By the Residue Theorem,

$$(P_+\psi)(\theta) = \frac{1}{2\pi i}\int_{\Gamma_+} e^{\lambda\theta}(\lambda I - \widehat{\mu}(\lambda))^{-1}\psi\, d\lambda = \sum_{\ell=1}^{n}\sum_{j=0}^{p_\ell-1} K_{\ell,j}(\psi)\,\theta^j e^{\lambda_\ell\theta}.$$

For the general case, the argument is similar to the one above; the residue of the integrand is now given by an expression of the same form $\sum_{j=0}^{p_\ell-1} \alpha_{\ell,j}(\psi) t^j e^{\lambda_\ell t}$, where the new coefficients $\alpha_{\ell,j}(\psi)$ now include the contribution from the expansion of $P(\lambda)$ around the eigenvalues. As a consequence, we obtain

$$(P_+\psi)(\theta) = \frac{1}{2\pi i}\int_{\Gamma_+} e^{\lambda\theta}(\lambda I - \widehat{\mu}(\lambda))^{-1}P(\lambda)\psi\, d\lambda = \sum_{\ell=1}^{n}\sum_{j=0}^{p_\ell-1} \alpha_{\ell,j}(\psi)\,\theta^j e^{\lambda_\ell\theta}.$$

The same reasoning applies to the first term in the right-hand side of (3.17). In conclusion, $X^+ = P_+X$ is a finite dimensional subspace of X. □

We now provide two different proofs for uniform exponential stability of the semigroup on X_-. First, note that we can assume that there are no characteristic roots on the vertical line $\{\lambda : \mathrm{Re}\,(\lambda) = -\rho\}$. Indeed, let $\rho_0 = \rho$. By part *(ii)* in Theorem 3.1.4, we can choose $0 < \rho_1 < \rho_0$ such that there are no characteristic roots on the vertical line $\{\lambda : \mathrm{Re}\,(\lambda) = -\rho_1\}$. Exactly as explained in Remark 2.3.1, we have $C_{0,\rho_1} \subset C_{0,\rho_0}$, with a continuous injection, so that F is still Fréchet differentiable as a function in C_{0,ρ_1}, with the same Fréchet derivative $DF(\overline{\psi})$ (in rigour, it is the restriction to C_{0,ρ_1} of the Fréchet derivative of F as a function in C_{0,ρ_0}). Consequently, we obtain the same characteristic equation, now well-defined for every $\lambda \in \Pi_+^{\rho_1}$, and Theorem 3.1.4 can be proved exactly in the same way. Then we may work with $\rho = \rho_1$, and assume without loss of generality that there are no characteristic roots λ with $\mathrm{Re}\,(\lambda) = -\rho$.

Proposition 3.2.5. *The restriction of the solution semigroup to X_- is uniformly exponentially stable.*

The first proof uses the stability criterion given by Theorem 3.1.5, and is an adaptation of the arguments in [12] for the same result in the case of renewal equations.

Proof. Using the decomposition $X = X_+ \oplus X_-$, the solution y of (LDDE) has the unique decomposition

$$y(t) = y^+(t) + y^-(t), \quad t \in \mathbb{R},$$

where $y^\pm$ is the solution of (LDDE) with initial condition $y^\pm = P_\pm \psi$. By Lemma 3.1.5, it is enough to prove that y^- satisfies

$$\sup_{\theta \geq 0} e^{\rho\theta} |y^-(t)| < \infty.$$

We now use Theorem A.10, with $\varphi(t) = e^{\rho t}$. In this case, $\omega_\varphi = \alpha_\varphi = -\rho$. From (3.9), $y = y(t)$ solves the Volterra integro-differential equation (3.12), i.e.

$$\dot{y}(t) = (\mu * y)(t) + f(t), \quad t \geq 0,$$

where f is given by

$$f(t) = \int_t^\infty [d\mu(\theta)]\, \psi(t-\theta).$$

For $t \geq 0$,

$$e^{\rho t}|f(t)| \leq e^{\rho t} \int_t^\infty |\psi(t-\theta)|\, d|\mu|(\theta) = \int_t^\infty e^{\rho(t-\theta)} |\psi(t-\theta)|\, e^{\rho\theta}\, d|\mu|(\theta) \leq ||\psi||_{\infty,\rho}\, ||\mu||_\rho,$$

which implies that $f \in L^\infty_\rho(\mathbb{R}_-;\mathbb{R}^m)$. Since, by assumption, there are no characteristic roots λ with $\mathrm{Re}\,(\lambda) = -\rho$, by Theorem A.10 the solution $y = y(t)$ of (LDDE) is of the form

$$y(t) = \sum_{\ell=1}^{n} \sum_{j=0}^{p_\ell - 1} \alpha_{\ell j}(\psi) t^j e^{\lambda_\ell t} + \xi(\psi; t),$$

where λ_ℓ are the characteristic roots in Π^ρ_+, $\alpha_{\ell,j}$ are obtained exactly as in the proof of Proposition 3.2.4 and $\xi(\psi;\cdot)$ can be chosen in $L^\infty_\rho(\mathbb{R}_+;\mathbb{R}^m)$. Now note that the expression above must exactly be the decomposition of y as the sum of an element in X_+ and X_-; indeed, the first summand is an element of X_+, by the proof of Theorem 3.2.4, and of course

$$L^\infty_\rho(\mathbb{R}_+;\mathbb{R}^m) \cap X_+ = \{0\}.$$

Consequently, it must be that

$$y^-(\cdot) = \xi(\psi;\cdot) \in L^\infty_\rho(\mathbb{R}_+;\mathbb{R}^m),$$

as desired. This concludes the proof. □

An alternative proof of Proposition 3.2.5 uses the second stability criterion, Theorem 3.1.6.

Proof. By Theorem 3.1.6, it suffices to prove that the corresponding characteristic equation for X_- has no roots λ with $\mathrm{Re}\,(\lambda) \geq -\rho$. It is clear that $A_- = A|_{X_-}$ is the infinitesimal generator of the restricted semigroup $\{T(t)|_{X_-}\}_{t\geq 0}$. By Theorem 3.1.4, we conclude that

$$\sigma(A_-) \cap \Pi^\rho_+ = \sigma_p(A_-) \cap \Pi^\rho_+ = \{\lambda \in \Pi^\rho_+ :\ \det \Delta_-(\lambda) = 0\},$$

where

$$\det \Delta_-(\lambda) = 0$$

is the characteristic equation associated with the infinitesimal generator A_- as in Section 3.1.

We claim that there are no characteristic roots for X_-. Indeed, if λ is such a root, then it must be an eigenvalue of A_-, and so $A_-\psi = \lambda\psi$ for some $\psi \in X_-$, $\psi \neq 0$. But then λ is an eigenvalue of A, and by Proposition 3.2.3 also $\psi \in X_+$. This is a contradiction. Furthermore, ρ is such that there are no characteristic roots λ with $\mathrm{Re}\,(\lambda) = -\rho$. Hence Proposition 3.1.6 can be applied, and the conclusion follows. □

The proof of the Principle of Linearized Stability now follows directly from Proposition 3.2.4 and 3.2.5, using the theorem of Desch and Schappacher 3.2.2.

Before presenting an example of application of the Principle of Linearized Stability, we observe that differential equations with delays arise very naturally in applications as models (for example) for the dynamics of some population. In some situations, although the life of an organism is never infinite, it is useful not to establish a bound on the age of the population, and so the system is modelled by an equation with infinite delay. To study the asymptotic behaviour of solutions of these systems, it is therefore relevant to obtain criteria for stability of equilibria. In some classical references, e.g. [13, 16], only stability results have been established and there were no instability criteria. As such, the results in [12], namely Theorem 3.2.1, the Principle of Linearized Stability, were, at the time of publication, new in the literature.

3.3 Example: Volterra population's equation

We conclude with an application of the Principle of Linearized Stability to a simple model from population dynamics.

We consider a simple Volterra population's equation of logistic type with infinite delay, of the form

$$\dot{x}(t) = x(t)\left(b - ax(t) - \int_{-\infty}^{t} k(t-s)x(s)\,ds\right), \quad t \geq 0, \tag{3.18}$$

where $a, b > 0$ and $k \geq 0$ is a non-negative integrable kernel. The equation above has been studied by many authors, e.g. see [19, 20], and serves as a model for a batch fermentation process in microbiology.

First, consider the linear equation

$$\dot{x}(t) = -ax(t) + \int_{-\infty}^{t} k(t-s)x(s)\,ds, \quad t \geq 0, \tag{3.19}$$

where $a > 0$, $k\colon [0,\infty) \to \mathbb{R}$ is continuous and satisfies

$$\int_0^\infty e^{\rho s}|k(s)|\,ds < \infty, \text{ for some } \rho > 0. \tag{h1}$$

Let $X = C_{0,\rho}(\mathbb{R}_-;\mathbb{R})$. Note that we can write (3.19) as

$$\dot{x}(t) = -ax(t) + \int_0^\infty k(s)x(t-s)\,ds, \quad t \geq 0, \tag{3.20}$$

so, if $L\colon X \to \mathbb{R}$ is the operator

$$L\varphi = -a\varphi(0) + \int_0^\infty k(s)\varphi(-s)\,ds,$$

then (3.19) is simply

$$\dot{x}(t) = Lx_t,$$

where as usual $x_t(s) = x(t+s)$, for $s \in (-\infty, 0]$. The characteristic roots of the problem in Π_+^ρ are the solutions of

$$\lambda - L(e^{\lambda \cdot}) = 0 \Longleftrightarrow \lambda + a - \int_0^\infty k(s)e^{-\lambda s}\,ds = 0. \tag{3.21}$$

We claim that, if (h1) and

$$\int_0^\infty |k(s)|\,ds < a, \tag{h2}$$

hold, then there are no characteristic roots with nonnegative real part. In fact, suppose λ is such that $\mathrm{Re}\,(\lambda) \geq 0$ and satisfies (3.21). Then

$$0 \leq \mathrm{Re}\,(\lambda) \leq -a + \int_0^\infty |k(s)|e^{-\mathrm{Re}\,(\lambda)s}\,ds \leq -a + \int_0^\infty |k(s)|\,ds < 0,$$

which is a contradiction. Consequently, every characteristic root in Π_+^ρ has negative real part, and therefore 0 is an exponentially asymptotically stable solution of (3.19).

Now assume (h1) and

$$\int_0^\infty k(s)\,ds > a. \tag{h3}$$

Define, for $\lambda \in \mathbb{R}$ with $\lambda > -\rho$,

$$h(\lambda) = \lambda + a - \int_0^\infty k(s)e^{-\lambda s}\,ds.$$

Observe that

$$h(0) = a - \int_0^\infty k(s)\,ds < 0, \tag{3.22}$$

by (h3). Moreover, note that we clearly have, for any $s \geq 0$,

$$\lim_{\lambda \to \infty} k(s)e^{-\lambda s} = 0,$$

and

$$|k(s)e^{-\lambda s}| = e^{\rho s}|k(s)|\,e^{-(\rho+\lambda)s} \leq e^{\rho s}|k(s)| \in L^1(\mathbb{R}_+).$$

By Lebesgue's dominated convergence theorem, it follows that

$$\lim_{\lambda \to \infty} \int_0^\infty k(s)e^{-\lambda s}\,ds = 0,$$

and hence

$$\lim_{\lambda \to \infty} h(\lambda) = \infty. \tag{3.23}$$

Consequently, (3.22) and (3.23) imply that there exists $\lambda_0 > 0$ such that $h(\lambda_0) = 0$. In other words, there exists a characteristic root in Π_+^ρ with positive real part. Hence, under (h1) and (h3), 0 is an unstable equilibrium of (3.19).

We now consider Volterra's equation (3.18), where we assume that the coefficients $a, b > 0$ and that the kernel $k \geq 0$ satisfies (h1). Again, we work on the phase space $X = C_{0,\rho}(\mathbb{R}_-;\mathbb{R})$. It is easy to see that

(3.18) has two equilibria,

$$x \equiv 0, \quad \text{and} \quad x \equiv b\left(a + \int_0^\infty k(s)\,ds\right)^{-1} =: x^*.$$

Changing the variable $y(t) = x(t) - x^*$, we obtain the equation

$$\begin{aligned} y'(t) &= (y(t)+x^*)\left(b - ay(t) - ax^* - \int_{-\infty}^t k(t-s)y(s)\,ds - \int_{-\infty}^t k(t-s)x^*\,ds\right) \\ &= (y(t)+x^*)\left(-ay(t) - \int_{-\infty}^t k(t-s)y(s)\,ds\right), \end{aligned}$$

so that the linearization about x^* is the equation

$$\dot{y}(t) = -ax^*y(t) - x^*\int_{-\infty}^t k(t-s)y(s)\,ds,$$

which is of the form (3.19) with a new coefficient $a_1 := ax^*$ and a new kernel $k_1 := -x^*k$. In this case, the characteristic equation reads as

$$\lambda + ax^* + x^*\int_0^\infty k(s)e^{-\lambda s}\,ds = 0,$$

for $\lambda \in \Pi_+^\rho$. In particular, observe that, if (h1) and (h2) hold, then we have seen in the previous case that there are no characteristic roots in Π_+^ρ with non-negative real part. By the Principle of Linearized Stability 3.2.1, it follows that x^* is a (locally) exponentially asymptotically stable equilibrium of (3.18). The instability criterion, however, cannot be applied to this equation, as (h3) does not hold in this situation.

Appendix: Volterra integro-differential equations

In this appendix, we introduce some basic results and definitions from the theory of Volterra integro-differential equations which are used in Chapter 3. For simplicity of the exposition, we do not present the proofs, but they can be found on [13, Chapters 3, 4, 7]; after each result, we indicate the corresponding theorem in [13].

Given $J \subset \mathbb{R}$ we denote by $M(J) = M(J;\mathbb{R}^{m\times m})$ the Banach space of matrix-valued Radon measures on J with bounded total variation, with the norm given by the total variation (see (2.7)).

Definition A. 1. For $J = [0,T]$, $T > 0$, or $J = \mathbb{R}$, the *convolution* of a measure $\mu \in M(J)$ and a Lebesgue measurable $\mathbb{R}^n$-valued function a, defined on J, is the function

$$(\mu * a)(t) = \int_J [d\mu(s)]\, a(t-s),$$

defined for those $t \in J$ for which the function $s \mapsto a(t-s)$ is $|\mu|$-integrable.

In order to study the features of integrodifferential equations, we need to introduce the notion of a Laplace transform for a measure.

Definition A. 2. The *Laplace transform* of a locally finite measure μ on $J = \mathbb{R}^+$ or $J = \mathbb{R}$ is the function

$$\widehat{\mu}(z) = \int_0^\infty [d\mu(t)]\, e^{-zt},$$

defined for those $z \in \mathbb{C}$ for which this integral converges absolutely.

The next result shows that the Laplace transform of a measure shares identic properties with the Laplace transform for integrable functions.

Theorem A. 3. *Let $\mu \in M(\mathbb{R}^+)$.*

(i) The Laplace transform $\widehat{\mu}(z)$ is defined, bounded and uniformly continuous in the closed half-plane $\operatorname{Re} z \geq 0$*, and analytic on the open half-plane* $\operatorname{Re} z > 0$*. Moreover, $\mu = 0$ if and only if $\widehat{\mu}(z) = 0$, for all z in the closed half-plane* $\operatorname{Re} z \geq 0$*;*

(ii) If $\widehat{\mu}(z_0)$ is defined for some $z_0 \in \mathbb{C}$, then $\widehat{\mu}(z)$ is defined on the entire vertical line $\operatorname{Re} z \geq \operatorname{Re} z_0$*;*

(iii) If $a \in L^1_{\mathrm{loc}}(\mathbb{R})$, then $(\widehat{\mu * a})(z) = \widehat{\mu}(z)\widehat{a}(z)$ for all those $z \in \mathbb{C}$ where the transforms are defined.

This is Theorem 8.2 in [13, Chapter 3].

We now focus on the the initial-value problem (IVP)

$$x'(t) = (\mu * x)(t) + f(t), \quad t \in \mathbb{R}^+; \quad x(0) = x_0. \tag{IDE}$$

We first have to say what we mean by a solution of (IDE). In general, x cannot satisfy the equation everywhere because, even if the function f is continuous, the convolution $\mu * f$ will have jump discontinuities at the points where μ has point masses. The notion of solution is introduced below.

Definition A. 4. A function x is a *solution of* (IDE) *on an interval* $[0,T)$ if x is locally absolutely continuous, $x(0) = x_0$, and the equation

$$x'(t) = (\mu * x)(t) + f(t)$$

is satisfied for almost all $t \in [0,T)$.

The next theorem gives existence and uniqueness for the homogeneous problem associated with (IDE).

Theorem A. 5. *Let $\mu \in M(\mathbb{R}_+; \mathbb{R}^{n\times n})$. Then, there is a unique locally absolutely continuous $\mathbb{R}^{n\times n}$-valued function r on $\mathbb{R}^+$, with $r(0) = I$, the identity matrix, that satisfies the equations*

$$r'(t) = (\mu * r)(t) = (r * \mu)(t) \tag{A.2}$$

for almost all $t \in \mathbb{R}_+$. Furthermore, the derivative r' is equal a.e. to a function which is locally of bounded variation.

This result is Theorem 3.1 in [13, Chapter 3]. The function r on the theorem above is, in some sense, what replaces the notion of a fundamental solution for systems of ordinary differential equations. As so, it deserves a special name.

Definition A. 6. The solution of the homogeneous problem r of (A.2) is called the *differential resolvent of μ*.

Using the differential resolvent, we obtain existence and uniqueness for the more general problem (IDE).

Theorem A. 7. *Let $\mu \in M(\mathbb{R}^+;\mathbb{R}^{n\times n})$. Then, for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^+;\mathbb{R}^{n\times n})$, the equation (IDE) has a unique solution x, which is given by the variation of constants formula*

$$x(t) = r(t)x_0 + (r * f)(t), \quad t \in \mathbb{R}^+,$$

where r is the differential resolvent of μ.

For the proof, see Theorem 3.3 in [13, Chapter 3].

In Chapter 3 of this dissertation, these types of equations appear as a linearization of a delay differential equation and one problem that we will face is the invertibility of the matrix $\lambda I - \widehat{\mu}(\lambda)$, i.e. we will be interested in finding λ such that

$$\det[\lambda I - \widehat{\mu}(\lambda)] \neq 0. \tag{A.3}$$

We can describe exactly when (A.3) holds.

Theorem A. 8. (Grossman-Miller) *Let $\mu \in M(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and r be the differential resolvent of μ. Then*

$$\det[\lambda I - \widehat{\mu}(\lambda)] \neq 0, \quad \mathrm{Re}\,(\lambda) \geq 0$$

if, and only if, $r \in L^1(\mathbb{R}^+;\mathbb{R}^{n\times n})$. Moreover, if $r \in L^1(\mathbb{R}^+;\mathbb{R}^{n\times n})$, then $r' \in L^1(\mathbb{R}^+;\mathbb{R}^{n\times n})$, and r' is equal a.e. to a function of bounded variation.

This is Theorem 3.5 in [13, Chapter 3]. To conclude the appendix, we now consider integro-differential equations (IDE) where the measure and the functions live in weighted spaces, and state a result concerning the form of solutions of these equations.

Definition A. 9. Let $\varphi\colon J \subset \mathbb{R} \to \mathbb{R}$ be a function. We say

(i) φ is *submultiplicative* on J if $\varphi(0) = 1$ and $\varphi(s+t) \leq \varphi(s)\varphi(t)$, for all $s,t \in J$;

(ii) $\varphi\colon J \subset \mathbb{R} \to \mathbb{R}$ is a *weight function* on J if it is positive, Borel measurable, locally bounded and locally bounded away from zero on J.

Given a function φ, we define the two limits

$$\alpha_\varphi = -\lim_{t\to-\infty} \frac{\log\varphi(t)}{t}, \quad \omega_\varphi = -\lim_{t\to\infty} \frac{\log\varphi(t)}{t}.$$

It can be seen that, if φ is a submultiplicative weight function, then both limits α_φ, ω_φ exist and we have that $-\infty < \alpha_\varphi \leq \omega_\varphi < \infty$ (see Lemma 4.1 in [13, Chapter 4]). Finally, consider the following weighted spaces:

(i) $M_\varphi(\mathbb{R}_+;\mathbb{R}^{m\times m})$, the space of measures $\mu \in M(\mathbb{R}_+;\mathbb{R}^{m\times m})$ such that $\varphi\mu \in M(\mathbb{R}_+;\mathbb{R}^{m\times m})$, i.e. such that

$$||\mu||_\varphi = \int_0^\infty \varphi(t)\, d|\mu|(t) < \infty,$$

with the norm $||\cdot||_\varphi$;

(ii) $L^\infty_\varphi(\mathbb{R}_+;\mathbb{R}^m)$, the space of functions $f\colon \mathbb{R}_+ \to \mathbb{R}^m$ such that

$$||f||_{\infty,\varphi} = \sup_{t\geq 0} \varphi(t)|f(t)| < \infty,$$

with the norm $||\cdot||_{\infty,\varphi}$.

Theorem A. 10. *Let φ be a submultiplicative weight function defined on $\mathbb{R}$ such that $\alpha_\varphi = \omega_\varphi$ and suppose that, for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,(\lambda) = \omega_\varphi$,*

$$\det[\lambda I - \widehat{\mu}(\lambda)] \neq 0. \tag{3.24}$$

Let $f \in L^\infty_\varphi(\mathbb{R}_+;\mathbb{R}^m)$. Then, the solution $x = x(t)$ of (IDE) is of the form

$$x(t) = \sum_{\ell=1}^{n} \sum_{j=0}^{p_\ell - 1} \alpha_{\ell,j}(\psi)\, t^j e^{z_\ell t} + y(t),$$

where $z_\ell \in \mathbb{C}$ are such that $\mathrm{Re}\,(z_\ell) > \omega_\varphi$, the $\alpha_{\ell,j}$ are vectors and the function y is in $L^\infty_\varphi(\mathbb{R}_+;\mathbb{R}^m)$. More specifically, the complex numbers z_ℓ are the roots of (3.24) in $\{\lambda : \mathrm{Re}\,(\lambda) > \omega_\varphi\}$ and the $\alpha_{\ell,j}$ are obtained with the Residue theorem (exactly as in the proof of Proposition 3.2.4).

Observe that, in particular, the function $\varphi(t) = e^{\rho t}$, where $\rho > 0$, satisfies the assumptions of the theorem. This is a particular case of Theorem 2.3 in [13, Chapter 7].

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