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Stochastic Volatility Jump-diffusion Models As Time-Changed Lévy Processes

Ricardo Matos

Dissertation

Master's Degree in Financial Mathematics

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Acknowledgments

"Não sou nada.
Nunca serei nada.
Não posso querer ser nada.
À parte isso,
tenho em mim todos os sonhos do mundo."
Álvaro de Campos

"As nossas dúvidas são traidoras,
e fazem-nos perder o que, com frequência,
poderíamos ganhar,
por simples medo de tentar."
William Shakespeare

"O carácter do homem é o seu destino."
Heraclitus

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Abstract

This thesis focuses on applying the time-changed Lévy processes technique firstly presented by Carr and Wu (2004) in order to deduce the Bakshi et al. (1997) model with a general jump size distribution. The second goal is reach a full correlation scheme, after reaching the fundamental theorem, where we show how to compute the joint characteristic function of a finite number of time-changed Lévy processes under the leverage-neutral measure, we obtain an exact characteristic function for an asset price with stochastic volatility, stochastic interest rates, jumps and a full correlation scheme. As far as we know this is the first time that the exact characteristic function of an model that take into account stochastic volatility, stochastic interest rates, jumps and a full correlation scheme is achieved.

Keywords: Time-changed Lévy process. European standard call and put. Stochastic volatility. Stochastic interest rate. Jump diffusion. Bakshi, Cao and Chen model. Full correlation. General jump size distribution.

Resumo

Esta tese foca-se na aplicação da técnica de *time-changed Lévy processes*, apresentada em primeiro lugar por Carr and Wu (2004), a fim de deduzir o modelo de Bakshi et al. (1997) com uma distribuição arbitrária do tamanho do salto. O segundo objectivo passa por obter um modelo com correlação total, depois de deduzir o teorema fundamental onde se obtém a função característica conjunta de um número finito de *time-changed Lévy processes* sob a medida de alavancagem neutra. Posteriormente, obtivémos a função característica exacta para o preço de um activo com volatilidade estocástica, taxas de juros estocásticas, saltos e correlação total. Tanto quanto sabemos, foi a primeira vez que se obteve a função característica exacta de um modelo com volatilidade estocástica, taxas de juros estocásticas, saltos e correlação total.

Palavras-chave: Time-changed Lévy process. Opção Europeia standard de compra e venda. Volatilidade estocástica. Taxas de juro estocásticas. Difusão com saltos. Modelo de Bakshi, Cao and Chen. Correlação total. Tamanho do salto com distribuição arbitrária.

Contents

Introduction

We have focused on two fundamental goals during this thesis. The first is to obtain the Bakshi et al. (1997) model with a general jump size distribution as a time-changed Lévy process. This technique can greatly simplify the computation of the characteristic function of the asset price through a complex-valued change of measure on one part of the time-changed Lévy process. Later on this thesis we will see that this change of measure affects only the instantaneous activity rate. The second goal is to preserve the stochastic volatility, stochastic interest rates and jumps presented by the Bakshi et al. (1997) model, while adding an exact full correlation scheme. Our technique can incorporate an finite number of time-changed Lévy processes and take into account their correlation. With this technique we may capture some characteristics of the asset price that otherwise we would not be able to. It could be of main importance for a large range of derivatives to have a joint characteristic function since almost all of them depend on more than one source of uncertain.
The paper that we use as starting point is Carr and Wu (2004). Nevertheless, a great variety of theoretical results is obtained from Tankov and Cont (2004), Chesney et al. (2009), Pascucci (2011), Privault (2013), Shreve (2004), Bjork (1998) and Brigo and Mercurio (2006). Since one complex-valued change of measure is needed, the work of Dellacherie et al. (1992) is essential to the progress on this thesis.
This thesis is organized as follows. In the first chapter, we provide the fundamental tools for the understanding of our work. On the second chapter we present the time-changed Lévy process technique, and the fundamental theorem needed for writing the characteristic function of the time-changed Lévy process as a Laplace transform of business time — could be, for example, an

integrated CIR process. On the third chapter we develop the technique for pricing options within a scenario of constant interest rates using the above method. In the fourth chapter we make a generalization by allowing the interest rate to be itself a stochastic process, and we deduce our general model that nests all the models that we have deduced before. The fifth chapter contains the numerical implementation of our general method with two kinds of jump size distributions: The normal distribution and the double exponential distribution. We use a Gauss-Kronrod quadrature for the inversion of the characteristic function.
Finally, in chapter six a full correlation scheme is deduced and an exact characteristic function is computed.

Chapter 1

Mathematical tools

1.1 Signed and complex measures

In this chapter we will provide the fundamental tools that we will use in this thesis.

Definition 1.1. A Banach space is a vector space S which is equipped with a norm $\| . \|$ and which is complete with respect to that norm, i.e., every Cauchy sequence in S converges to a point in S. Let (x_n) be a Cauchy sequence, then there exists an element x in S such that

$$\lim_{n\to\infty} \| x_n - x \| = 0.$$

Definition 1.2. Let $(X, \mathscr{N})$ be a measurable space and S be a Banach space. A function $\mu : \mathscr{N} \to S$ is a vector measure or a S-measure if:
i) $\mu(\emptyset) = 0$;
ii) Countably additive : For any countable family $(A_j)_{j\in J}$ of disjoint sets in $\mathscr{N}$ we have $\mu\left(\bigcup_{j\in J} A_j\right) = \sum_{j\in J} \mu(A_j)$.

When $S = \mathbb{R}$ or $S = \mathbb{C}$ we have a signed measure and a complex measure, respectively — see Machado (2011, p. 363). A signed measure results if in the "usual" definition of measure the requirement of non-negativity is removed.

Corollary 1.1. ***(Jordan decomposition theorem)***
Every signed measure is the difference of two positive measures, at least one of which is finite.

Proof. See Cohn (1980, p. 125). ■

The representation $\mu = \mu^+ - \mu^-$ is called the Jordan decomposition of μ. The measures μ^+ and μ^- are called the positive part and the negative part of μ, respectively.

Definition 1.3. Let $(X, \mathcal{N})$ be a measurable space. A complex measure on $(X, \mathcal{N})$ is a function $\mu : \mathcal{N} \to \mathbb{C}$ that satisfies:
i) $\mu(\emptyset) = 0$;
ii) Countably additive : For any countable sequence (A_n) of disjoint sets in $\mathcal{N}$ we have $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$.

Each complex measure μ on $(X, \mathcal{N})$ can be written as $\mu = \mu^1 + \imath \mu^2$, where μ^1 and μ^2 are finite signed measures on $(X, \mathcal{N})$. Then, the Jordan decomposition theorem implies that each complex measure μ can be written as $\mu = \mu^1_+ - \mu^1_- + \imath \mu^2_+ - \imath \mu^2_-$ where $\mu^1_+, \mu^1_-, \mu^2_+, \mu^2_-$ are finite positive "usual" measures on $(X, \mathcal{N})$.

Definition 1.4. Let $(X, \mathcal{N})$ be a measurable space and f be a $\mathcal{N}$-measurable function. Then, we define the integral of f with respect to a complex measure μ as follows:

$$\int f \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu^1_+ - \int f \,\mathrm{d}\mu^1_- + \imath \int f \,\mathrm{d}\mu^2_+ - \imath \int f \,\mathrm{d}\mu^2_-.$$

Definition 1.5. A measure μ is said to be absolutely continuous with respect to ν if, for any measurable set A , $\nu(A) = 0 \Rightarrow \mu(A) = 0$.
If μ is absolutely continuous with respect to ν and ν is absolutely continuous with respect to μ then μ and ν are said to be equivalent measures which is denoted by $\mu \sim \nu$.

Theorem 1.1. ***Radon-Nikodym theorem***

Let $(X, \mathcal{N})$ be a measurable space, let μ be a σ-finite[1] positive measure on $(X, \mathcal{N})$ and ν be a complex measure on $(X, \mathcal{N})$. If ν is absolutely continuous with respect to μ, then there exists a function h that satisfies $\nu(A) = \int_A h \, \mathrm{d}\mu$ for each $A \in \mathcal{N}$. The function h is unique μ-almost everywhere.

Proof. See Cohn (1980, p. 135). ■

Usually the function h is called the Radon-Nikodym derivative and is denoted by $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} := h$.

1.2 Basic Tools

Given a filtration $(\mathcal{F}_t)_{t \in [0,T]}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, τ is called a $\mathcal{F}_t$-stopping time if $\forall t \geq 0, \{\tau \leq t\} \in \mathcal{F}_t$.

Proposition 1.1. ***Sampling Theorem***

If $(M_t)_{t \in [0,T]}$ is a martingale and $\{\tau, \eta\}$ is a set of stopping times with $0 \leq \tau \leq \eta \leq T$ then

$$E[M_\eta / \mathcal{F}_\tau] = M_\tau.$$

[1] A measure is called σ-finite if a set X is a countable union of measurable sets with finite measure.

Proof. See Doob (1990, p. 370). ■

In particular, a martingale stopped at a stopping time is still a martingale.

Definition 1.6. A function $f : [0,T] \to \mathbb{R}$ is cadlag if it is right-continuous with left limits, i.e., $\forall t \in [0,T]$ the following limits exist,

$$
\begin{aligned}
f(t-) &= \lim_{s\to t, s<t} f(s),\\
f(t+) &= \lim_{s\to t, s>t} f(s),
\end{aligned}
$$

and $f(t) = f(t+)$.
We will denote $f(t) - f(t-)$ by $\triangle f(t)$. Of course $\triangle f(t)$ will be different from 0 only when f jumps. A cadlag function may have at most a countable number of jumps or discontinuities and $\forall \epsilon > 0$ the number of jumps such that $\triangle f(t) > \epsilon$ should be finite.

Definition 1.7. Lévy process

A Lévy process is a stochastic process $(X_t)_{t\geq 0}$ on $(\Omega, \mathscr{F}, \mathbb{P})$ which is cadlag and has the following properties:
i) $X_0 = 0$;
ii) for every increasing sequence of times $t_0, ..., t_n$, $X_{t_0}, X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$ are independent;
iii) $X_{t+h} - X_t$ has the same law of X_h;
iv) $\forall \epsilon > 0, \lim_{h\to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \epsilon) = 0$.

The last condition states that the process has at most a countable number of jumps.
Examples of Lévy processes are the Poisson process and the Wiener process (Brownian Motion). In the first case, and conditional on $\mathscr{F}_s \subseteq \mathscr{F}_t$ for all $s \leq t$, the increments of the process follow a Poisson distribution with expected value $\lambda(t-s)$, where $\lambda > 0$ is the intensity of the process. For the second case, the increments of the process follow a normal distribution with mean 0 and variance $t - s$.

Definition 1.8. A Poisson process $(N_t)_{t\geq 0}$ is a Lévy process that is piecewise constant with jump size 1, where the occurrence of jumps has a Poisson distribution with parameter λt, i.e.

$$\forall n \in \mathbb{N}, \mathbb{P}(N_t = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!},$$

and the characteristic function is given by:

$$E\left[e^{\imath u N_t}\right] = e^{\lambda t(e^{\imath u}-1)}\ , u \in \mathbb{R}.$$

Definition 1.9. A compound Poisson process is a stochastic process $(Z_t)_{t\geq 0}$ with intensity $\lambda > 0$ and jump size distribution f such that

$$Z_t = \sum_{i=1}^{N_t} Y_i,$$

where Y_i are i.i.d. random variables that represent the jump size and follow a distribution f while N_t is a Poisson process with intensity λ that is independent from Y_i.

Definition 1.10. Lévy measure
If $(X_t)_{t\geq 0}$ is a Lévy process, the Lévy measure ν of X is defined by:

$$\nu(A) = E[\#\{t \in [0,1] : \triangle X_t \neq 0, \triangle X_t \in A\}], A \text{ is Borelian.}$$

$\nu(A)$ is the expected number, per unit time, of jumps whose size belongs to A. Furthermore if $(X_t)_{t\geq 0}$ is a compound Poisson process with intensity λ and a law of jump size f we have, $\nu(\mathrm{dx})\,\mathrm{dt} = \lambda f(\mathrm{dx})\,\mathrm{dt}$.

Definition 1.11. Quadratic variation

If $(X_t)_{t\geq 0}$ is a Lévy process,[2] its quadratic variation is the adapted cadlag process defined by:

$$[X,X]_t = |X_t|^2 - 2\int_0^t X_{u-}\,\mathrm{dX_u}.$$

Definition 1.12. Quadratic covariation

Given two Lévy processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$, the Lévy process defined by

$$[X,Y]_t = X_tY_t - X_0Y_0 - \int_0^t X_{u-}\,\mathrm{dY_u} - \int_0^t Y_{u-}\,\mathrm{dX_u},$$

is called the quadratic covariation of X and Y.

The quadratic covariation has the following important properties:

i) If X,Y are Lévy processes and φ_1,φ_2 are integrable predictable processes, then

$$\left[\int \varphi_1\,\mathrm{dX}, \int \varphi_2\,\mathrm{dY}\right]_t = \int_0^t \varphi_1\varphi_2\,\mathrm{d[X,Y]};$$

ii) The quadratic covariation is not affected by the drift term in X or Y, it is only sensitive to the martingale term.

Example 1.1. *If we consider the Brownian motion defined by $Z_t = \sigma W_t$, by Definition 1.11 its quadratic variation is:*

$$[Z,Z]_t = \sigma^2 W_t^2 - 2\sigma^2 \int_0^t W_u\,\mathrm{dW_u}.$$

[2]A more general definition can be obtained if we consider a semi-martingale instead of a Lévy process, but for our purposes we will just consider Lévy processes. Particular cases of semi-martingales are the Wiener process, Poisson process and all Lévy processes.

Applying Ito's Lemma to $f(x) = \sigma^2 x^2$, we have:

$$\sigma^2 W_t^2 = 2\sigma^2 \int_0^t W_u \, \mathrm{dW_u} + \sigma^2 t,$$

so we conclude that $[Z,Z]_t = \sigma^2 t$.

1.2.1 Cholesky Theorem

Theorem 1.2. ***Cholesky Theorem***
If A is a real, symmetric and a positive definite matrix, then it has a unique factorization, $A = L.L^T$, where L is a lower triangular matrix with positive diagonal.[3]

Proof. See Kincaid and Cheney (2001, p. 157). ■

Remark 1.1. *Consider two correlated Brownian Motions, $\widetilde{W_1}$ and $\widetilde{W_2}$, and a third one uncorrelated with the other two $\widetilde{W_3}$, with a correlation matrix*

$$C = \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho \in [-1,1],$$

C is real and symmetric. It is also definite positive because its eigenvalues are positive in the case of $\rho \in (-1,1)$; even if $\rho = -1$ or $\rho = 1$ it is easy to show that the Cholesky factorization is still verified.
Its Cholesky factorization is $C = L.L^T$, where

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \rho & \sqrt{1-\rho^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho \in [-1,1].$$

[3]The matrix L can be found via a Cholesky factorization algorithm — see Kincaid and Cheney (2001, p. 158).

Moreover, if we consider three independent Brownian motions, W_1, W_2 and W_3, it can be shown — see Korn et al. (2010, p. 113) — that:

$$\begin{pmatrix} \widetilde{W_1} \\ \widetilde{W_2} \\ \widetilde{W_3} \end{pmatrix} = L. \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix}.$$

1.2.2 Lévy-Khinchin representation

Theorem 1.3. ***Lévy-Khinchin representation***
If $(X_t)_{t\geq 0}$ is a Lévy process with characteristic triplet (μ, σ, ν), then

$$\phi_{X_t}(\theta) \equiv E\left[e^{\imath\theta X_t}\right] = e^{-t\Psi_x(\theta)}, \qquad \theta \in \mathbb{R},$$

and the characteristic exponent $\Psi_x(\theta)$ is given by:[4]

$$\Psi_x(\theta) = \frac{1}{2}\sigma\theta^2 - \imath\mu\theta - \int_{\mathbb{R}} \left(e^{\imath\theta x} - 1 - \imath\theta x \Pi_{|x|\leq 1}\right) \nu(\mathrm{dx}).$$

Proof. See Tankov and Cont (2004, p. 83). ∎

Remark 1.2. *The Lévy process is specified by the Lévy triplet (μ, σ, ν). Intuitively, μ can be seen has the drift of the continuous part of the Lévy process, σ is the variance of the Brownian motion and ν is the Lévy measure.*
In Theorem 1.3 we truncate the jumps larger than 1, but there are other formulations of the theorem, where we use a function $z(x)$ that obeys to certain regularity conditions instead of $\Pi_{|x|\leq 1}$; $z(x)$ is called the truncation function. Different choices of $z(x)$ do not affect σ and ν but μ is affected by $z(x)$. If the Lévy measure satisfies the condition $\int_{|x|\geq 1} |x|\nu(\mathrm{dx}) < \infty$, we can use a simpler form for the expression of the characteristic exponent $\Psi_x(\theta)$:

[4] $\Pi_A(x)$ is the indicator function: it is 1 if x belong to A and 0 otherwise.

$$\Psi_x(\theta) = \tfrac{1}{2}\sigma\theta^2 - \imath\mu\theta - \int_{\mathbb{R}} (e^{\imath\theta x} - 1 - \imath\theta x)\nu(\mathrm{dx}).$$

Proposition 1.2. *A Lévy process has piecewise constant trajectories iff its Lévy triplet has the form $(\mu, 0, \nu)$ with $\mu = \int_{|x|<1} x\nu(\mathrm{dx})$ that satisfies $\int_{\mathbb{R}} \nu(\mathrm{dx}) < \infty$. In this case, its characteristic exponent has the form:*

$$\Psi_x(\theta) = \int_{\mathbb{R}} (1 - e^{\imath\theta x})\nu(\mathrm{dx}). \tag{1.2.1}$$

Proof.

By Theorem 1.3 we know that the characteristic exponent has the form:

$$\Psi_x(\theta) = -\imath\mu\theta - \int_{\mathbb{R}} (e^{\imath\theta x} - 1 - \imath\theta x \Pi_{|x|\le 1})\nu(\mathrm{dx}),$$

that is:

$$\Psi_x(\theta) = -\imath\mu\theta + \imath\theta \int_{|x|<1} x\nu(\mathrm{dx}) - \int_{\mathbb{R}} (e^{\imath\theta x} - 1)\nu(\mathrm{dx}). \tag{1.2.2}$$

Since

$$\int_{|x|<1} x\nu(\mathrm{dx}) = \mu,$$

equation (1.2.1), follows immediately from equation (1.2.2). ■

Example 1.2. *Consider the compound Poisson process $(Z_t)_{t\ge 0}$ with intensity λ. Since this process has piecewise constant trajectories, the Lévy triplet is $(b, 0, \nu)$ with $b = \int_{|x|<1} x\nu(\mathrm{dx})$, and assume that the jump size distribution is Gaussian with mean μ and variance σ^2, that is:*

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Since $\nu(\mathrm{dx}) = \lambda f(\mathrm{dx})$*, we have:*

$$\nu(dx) = \lambda \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \,\mathrm{dx}.$$

Then the characteristic exponent $\psi_x(\theta)$ *of such process is:*

$$\Psi_x(\theta) = \int_{\mathbb{R}} (1 - e^{\imath\theta x})\lambda \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \,\mathrm{dx},$$

that is:

$$\Psi_x(\theta) = \lambda \left(-\int_{\mathbb{R}} e^{\imath\theta x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \,\mathrm{dx} + \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \,\mathrm{dx} \right). \tag{1.2.3}$$

The second term on the right-hand of the equation (1.2.3) *is one by definition of a probability density function and the first term is the characteristic function of a normal random variable. Therefore:*

$$\Psi_x(\theta) = \lambda \left[1 - e^{\imath\theta\mu - \frac{1}{2}\sigma^2\theta^2} \right]. \tag{1.2.4}$$

1.2.3 Dubins-Schwarz′s Theorem

Theorem 1.4. ***Dubins-Schwarz′s Theorem***

A martingale M *such that* $[M,M]_\infty = \infty$ *is a time changed Brownian motion, i.e., there exists a Brownian motion* W *such that:*

$$M_t = W_{[M,M]_t}.$$

Proof. See Chesney et al. (2009, p. 210). ■

In particular, if ξ satisfies the usual conditions for the Itô's integral to be well defined, then $X = \int \xi \,\mathrm{dW}$ is a martingale because is the Itô's integral and its quadratic variation is $[X,X]_t = \int_0^t \xi_s^2 \,\mathrm{ds}$.[5] Therefore,

$$\int_0^t \xi \,\mathrm{dW} = W_{\int_0^t \xi_s^2 \,\mathrm{ds}},$$

for a Brownian motion W.

1.2.4 Girsanov's Theorem

Theorem 1.5. ***Girsanov's Theorem***

Let $\mathbb{P}$ and $\mathbb{Q}$ be equivalent measures and $(\Omega, \mathscr{F}, \mathbb{P})$ be a measurable space. Consider the following Itô's process :

$$\mathrm{dX_t} = a(t,X)\,\mathrm{dt} + b(t,X)\,\mathrm{dW_t}, \tag{1.2.5}$$

and consider the process $(L_t)_{t\in[0,T]}$ that satisfies the Novikov's condition[6] *and is defined as*

$$L_t = e^{\int_0^t \theta_s \,\mathrm{dW^M_s} - \frac{1}{2}\int_0^t \theta_s^2 \,\mathrm{ds}} \qquad \textit{with} \qquad E^{\mathbb{P}}[L_t] = 1. \tag{1.2.6}$$

Consider also that the Brownian motions W and W^M are correlated with correlation $\mathrm{d} < \mathrm{W}, \mathrm{W^M} >_\mathrm{t} = \rho \,\mathrm{dt}$. *Then, we have the following results:*

[5] It is simply necessary to apply Itô's lemma to $f(x) = x^2$.

[6] $E^{\mathbb{P}}\left(e^{\frac{1}{2}\int_0^T \theta_s^2 \,\mathrm{ds}}\right) < \infty$. This condition is sufficient to ensure that $(L_t)_{t\in[0,T]}$ is a $\mathbb{P}$-martingale.

i) *L_T defines a Radon-Nikodym derivative*[7]

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = L_T, \tag{1.2.7}$$

since L_T is a martingale, we have:

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|\mathscr{F}_t = \frac{L_T}{L_t}; \tag{1.2.8}$$

ii) *$W^{\mathbb{Q}}$ is a new Brownian motion in $(\Omega, \mathscr{F}, \mathbb{Q})$ and is defined by:*

$$\mathrm{dW}_\mathrm{t}^{\mathbb{Q}} = \mathrm{dW_t} - \theta_t \, \mathrm{d} < \mathrm{W}, \mathrm{W}^\mathrm{M} >_\mathrm{t} = \mathrm{dW_t} - \rho\theta_t \, \mathrm{dt}; \tag{1.2.9}$$

iii) *The process X_t takes the following form under $\mathbb{Q}$:*

$$\mathrm{dX_t} = a(t,X)\,\mathrm{dt} + b(t,X)\,\mathrm{dW_t} = a(t,X)\,\mathrm{dt} + b(t,X)(\mathrm{dW}_\mathrm{t}^{\mathbb{Q}} + \rho\theta_t\,\mathrm{dt}) \tag{1.2.10}$$

$$= (a(t,X) + b(t,X)\rho\theta_t)\,\mathrm{dt} + b(t,X)\,\mathrm{dW}_\mathrm{t}^{\mathbb{Q}}. \tag{1.2.11}$$

Proof. See Zhu (2010, p. 13). ■

1.2.5 Girsanov's Theorem for Jump-Diffusion processes

Theorem 1.6. ***Girsanov's Theorem for Jump-Diffusion processes***
Let $\widetilde{\mathbb{Q}}$ and $\mathbb{Q}$ be equivalent measures, and $(\Omega, \mathscr{F}, \mathbb{Q})$ a measurable space. Let $Z_t = \sum_{i=1}^{N_t} Y_i$ be a compound Poisson process with intensity λ and a jump size distribution f.
Consider the following Radon-Nikodym derivative

$$\frac{\mathrm{d}\widetilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}} = e^{\int_0^T \theta_s \, \mathrm{dW}_\mathrm{s}^\mathrm{M} - \frac{1}{2}\int_0^T \theta_s^2 \, \mathrm{ds}} e^{(\lambda - \widetilde{\lambda})T} \prod_{i=1}^{N_T} \frac{\widetilde{\lambda}\widetilde{f}(Y_i)}{\lambda f(Y_i)}. \tag{1.2.12}$$

[7] Usually in the literature, L_t is strictly positive and $\mathbb{Q}$ is the "usual" measure but L_t can be real or complex valued — see Beghdadi-Sakrani (2002, p. 375) and Dellacherie et al. (1992, p. 350) — and $\mathbb{Q}$ a complex measure.

Consider also that the Brownian motions $\mathrm{W}^{\mathbb{Q}}$ *and* W^M *are correlated with correlation* $\mathrm{d} < \mathrm{W}^{\mathbb{Q}}, \mathrm{W}^{\mathrm{M}} >_{\mathrm{t}} = \rho\, \mathrm{dt}$. *Then under the new measure* $\widetilde{\mathbb{Q}}$, *the process*

$$\mathrm{dW}_{\mathrm{t}}^{\widetilde{\mathbb{Q}}} = \mathrm{dW}_{\mathrm{t}}^{\mathbb{Q}} - \theta_t\, \mathrm{d} < \mathrm{W}^{\mathbb{Q}}, \mathrm{W}^{\mathrm{M}} >_{\mathrm{t}} = \mathrm{dW}_{\mathrm{t}}^{\mathbb{Q}} - \rho\theta_t\, \mathrm{dt}, \tag{1.2.13}$$

is a Brownian motion in $(\Omega, \mathscr{F}, \widetilde{\mathbb{Q}})$.

Furthermore, under the new measure $\widetilde{\mathbb{Q}}$, *the compound Poisson process has the following parameters:*

$$\widetilde{\lambda} = \lambda E\left(e^{Y}\right), \tag{1.2.14}$$

and

$$\widetilde{f}(Y_i) = \frac{e^{Y_i} f(Y_i)}{E\left(e^{Y}\right)}. \tag{1.2.15}$$

And Y *follows the same distribution,* f, *as the i.i.d. random variables* Y_i.

Proof. See Shreve (2004, p. 502). ■

Remark 1.3. *Using equations* (1.2.14) *and* (1.2.15), *equation* (1.2.12) *can be rewritten as:*

$$\frac{\mathrm{d}\widetilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}} = e^{\int_0^T \theta_s\, \mathrm{dW}_{\mathrm{s}}^{\mathrm{M}} - \frac{1}{2}\int_0^T \theta_s^2\, \mathrm{ds}}\, e^{(\lambda - \lambda E(e^{Y}))T + \sum_{i=1}^{N_T} Y_i}. \tag{1.2.16}$$

Since the right-hand side of equation (1.2.16) *is a martingale, we have:*

$$\left.\frac{\mathrm{d}\widetilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}\right|\mathscr{F}_t = e^{\int_t^T \theta_s\, \mathrm{dW}_{\mathrm{s}}^{\mathrm{M}} - \frac{1}{2}\int_t^T \theta_s^2\, \mathrm{ds}}\, e^{(\lambda - \lambda E(e^{Y}))\tau + \sum_{i=1}^{N_T} Y_i - \sum_{i=1}^{N_t} Y_i}, \qquad \tau := T - t.$$

1.2.6 Fourier Inversion Theorem

Theorem 1.7. ***Fourier Inversion Theorem***

Let $f_X(x)$ be the probability density function of the random variable X and $F_X(x) = \mathbb{P}(X \leq x)$ be the distribution function. We define

$$\phi_X(\xi) \equiv E\left[e^{\imath\xi X}\right] = \int_{\mathbb{R}} e^{\imath\xi x} f_X(x)\,\mathrm{dx}$$

as the characteristic function of X. Then,

$$f_X(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\imath\xi x}\phi_X(\xi)\,\mathrm{d}\xi, \tag{1.2.17}$$

and

$$F_X(x) = \frac{1}{2} + \frac{1}{2\pi}\int_0^{+\infty} \frac{e^{\imath\xi x}\phi_X(-\xi) - e^{-\imath\xi x}\phi_X(\xi)}{\imath\xi}\,\mathrm{d}\xi. \tag{1.2.18}$$

Proof. See de Figueiredo (2012, p. 203) for equation (1.2.17) and Kendall and Stuart (1977, p. 98) or Gil-Pelaez (1951) for equation (1.2.18). ■

Remark 1.4. *Equation* (1.2.18) *can be restated as:*

$$F_X(x) = \frac{1}{2} + \frac{1}{2\pi}\int_0^{+\infty} \frac{e^{\imath\xi x}\phi_X(-\xi)}{\imath\xi} + \frac{e^{-\imath\xi x}\phi_X(\xi)}{-\imath\xi}\,\mathrm{d}\xi.$$

Since $e^{\imath\xi x}$ is the complex conjugate of $e^{-\imath\xi x}$ and $\phi_X(-\xi)$ is the complex conjugate of $\phi_X(\xi)$, then $e^{\imath\xi x}\phi_X(-\xi)$ is the complex conjugate of $e^{-\imath\xi x}\phi_X(\xi)$, consequently $\frac{e^{\imath\xi x}\phi_X(-\xi)}{\imath\xi}$ is the complex conjugate of $\frac{e^{-\imath\xi x}\phi_X(\xi)}{-\imath\xi}$. Because $z + \overline{z} = 2Re(z), \forall z \in \mathbb{C}$, we have:

$$F_X(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^{+\infty} Re\left[\frac{e^{-\imath\xi x}\phi_X(\xi)}{\imath\xi}\right]\mathrm{d}\xi. \tag{1.2.19}$$

1.2.7 Change of Numeráire

A numeráire is a strictly positive stochastic process $(N_t)_{t\in\mathbb{R}^+}$, that is adapted to $\mathscr{F}_t$. The relative price of an asset S_t in terms of the numeráire N_t is given by

$$\hat{S}_t := \frac{S_t}{N_t}, \qquad t \in \mathbb{R}^+.$$

Consider that $(r_t)_{t\in\mathbb{R}^+}$ denotes an $\mathscr{F}_t$-adapted interest rate process. The discounted price, $\hat{S}_t$, is the price S_t, expressed in terms of the numeráire

$$N_t = e^{\int_0^t r_s \, \mathrm{d}s}.$$

The risk neutral measure, $\mathbb{Q}$, is a measure under which the discounted price process

$$\hat{S}_t = \frac{S_t}{N_t} = e^{-\int_0^t r_s \, \mathrm{d}s} S_t, \qquad t \in \mathbb{R}^+,$$

is a martingale.

Assumption 1.1. *Consider a generic numeráire $(N_t)_{t\in\mathbb{R}^+}$, the discounted numeráire*

$$\hat{N}_t = e^{-\int_0^t r_s \, \mathrm{d}s} N_t,$$

is a martingale under the risk neutral measure $\mathbb{Q}$.

Definition 1.13. Taking the process $(N_t)_{t\in\mathbb{R}^+}$ as the numeráire, the forward measure $\mathbb{Q}^T$ is defined by the following Radon-Nikodym derivative:

$$\frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{Q}} = e^{-\int_0^T r_s \, \mathrm{d}s} \frac{N_T}{N_0}.$$

Remark 1.5. *Note that Definition 1.13 is equivalent to stating*

$$E^{\mathbb{Q}^T}(\xi) = E^{\mathbb{Q}}\left(e^{-\int_0^T r_s\,\mathrm{ds}}\frac{N_T}{N_0}\xi\right),$$

for every integrable $\mathscr{F}_T$-measurable random variable ξ.

Theorem 1.8. *We have*

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\bigg|\mathscr{F}_t = e^{-\int_t^T r_s\,\mathrm{ds}}\frac{N_T}{N_t}, \qquad 0 \le t \le T.$$

Proof.

We want to proof that

$$E^{\mathbb{Q}^T}(\xi|\mathscr{F}_t) = E^{\mathbb{Q}}\left(e^{-\int_t^T r_s\,\mathrm{ds}}\frac{N_T}{N_t}\xi\bigg|\mathscr{F}_t\right).$$

Consider that ξ is integrable and $\mathscr{F}_T$-measurable. Then, for every integrable and $\mathscr{F}_t$-measurable random variable κ, we have:

$$E^{\mathbb{Q}^T}(\kappa\xi|\mathscr{F}_t) = \kappa E^{\mathbb{Q}^T}(\xi|\mathscr{F}_t).$$

Using Remark 1.5, we know that:

$$\begin{aligned}
E^{\mathbb{Q}^T}\left[\kappa E^{\mathbb{Q}^T}(\xi|\mathscr{F}_t)\right] &= E^{\mathbb{Q}^T}\left[E^{\mathbb{Q}^T}(\kappa\xi|\mathscr{F}_t)\right] \\
&= E^{\mathbb{Q}^T}[\kappa\xi] \\
&= E^{\mathbb{Q}}\left[e^{-\int_0^T r_s\,\mathrm{ds}}\frac{N_T}{N_0}\kappa\xi\right] \\
&= E^{\mathbb{Q}}\left[e^{-\int_0^t r_s\,\mathrm{ds}}\frac{N_t}{N_0}\kappa E^{\mathbb{Q}}\left(e^{-\int_t^T r_s\,\mathrm{ds}}\frac{N_T}{N_t}\xi\bigg|\mathscr{F}_t\right)\right] \\
&= E^{\mathbb{Q}^T}\left[\kappa E^{\mathbb{Q}}\left(e^{-\int_t^T r_s\,\mathrm{ds}}\frac{N_T}{N_t}\xi\bigg|\mathscr{F}_t\right)\right].
\end{aligned}$$

Finally, it follows that

$$E^{\mathbb{Q}^T}(\xi|\mathscr{F}_t) = E^{\mathbb{Q}}\left(e^{-\int_t^T r_s\,\mathrm{ds}}\frac{N_T}{N_t}\xi\middle|\mathscr{F}_t\right).$$

■

Proposition 1.3. *The price at time t, of an option with an $\mathscr{F}_T$-measurable payoff ξ is:*

$$O_t = E^{\mathbb{Q}}\left(e^{-\int_t^T r_s\,\mathrm{ds}}\xi\middle|\mathscr{F}_t\right) = N_t E^{\mathbb{Q}^T}\left(\frac{\xi}{N_T}\middle|\mathscr{F}_t\right).$$

Proof.

From Theorem 1.8, we have:

$$\begin{aligned}
O_t &= E^{\mathbb{Q}}\left(e^{-\int_t^T r_s\,\mathrm{ds}}\xi\middle|\mathscr{F}_t\right)\\
&= E^{\mathbb{Q}}\left(e^{-\int_t^T r_s\,\mathrm{ds}}\frac{N_T}{N_t}\frac{N_t}{N_T}\xi\middle|\mathscr{F}_t\right)\\
&= E^{\mathbb{Q}^T}\left(\frac{N_t}{N_T}\xi\middle|\mathscr{F}_t\right)\\
&= N_t E^{\mathbb{Q}^T}\left(\frac{\xi}{N_T}\middle|\mathscr{F}_t\right), \qquad 0 \le t \le T.
\end{aligned}$$

■

Theorem 1.9. *Given*

$$\frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{Q}} = e^{-\int_0^T r_s\,\mathrm{ds}}\frac{N_T}{N_0},$$

and a Brownian motion W_t under $\mathbb{Q}$, we have:

$$\mathrm{dW_t^{\mathbb{Q}^T}} = \mathrm{dW_t^{\mathbb{Q}}} - \frac{1}{N_t}\,\mathrm{dN_t\,dW_t^{\mathbb{Q}}},$$

where $\mathrm{W}_t^{\mathbb{Q}^T}$ is a Brownian motion under $\mathbb{Q}^T$.

Proof.

We define Φ_t as:

$$\begin{aligned}\Phi_t &= E^{\mathbb{Q}}\left(\frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{Q}}\bigg|\mathscr{F}_t\right)\\ &= E^{\mathbb{Q}}\left(e^{-\int_0^T r_s\,\mathrm{ds}}\frac{N_T}{N_0}\bigg|\mathscr{F}_t\right), \qquad t \in [0,T].\end{aligned} \tag{1.2.20}$$

The right-hand side of equation (1.2.20) is a martingale under Assumption 1.1. Then, using a more general version Girsanov's Theorem — see Privault (2013, p. 276) and Protter (2004, p. 132) — we have:

$$\mathrm{dW}_{\mathrm{t}}^{\mathbb{Q}^{\mathrm{T}}} = \mathrm{dW}_{\mathrm{t}}^{\mathbb{Q}} - \frac{1}{\Phi_t}\,\mathrm{d}\Phi_{\mathrm{t}}\,\mathrm{dW}_{\mathrm{t}}^{\mathbb{Q}}, \tag{1.2.21}$$

where $W_t^{\mathbb{Q}^T}$ is a $\mathbb{Q}^T$-Brownian motion.
Using Assumption 1.1, equation (1.2.20) becomes:

$$\Phi_t = e^{-\int_0^t r_s\,\mathrm{ds}}\frac{N_t}{N_0}, \qquad 0 \le t \le T.$$

Applying Ito's Lemma to

$$f(t,x) = \frac{x}{N_0}e^{-\int_0^t r_s\,\mathrm{ds}},$$

and since

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{e^{-\int_0^t r_s\,\mathrm{ds}}}{N_0},\\ \frac{\partial f}{\partial t} &= -\frac{x}{N_0}r_t e^{-\int_0^t r_s\,\mathrm{ds}},\end{aligned}$$

we have:

$$\mathrm{d}f(t,N_t) = -\frac{N_t}{N_0}r_t e^{-\int_0^t r_s\,\mathrm{ds}}\,\mathrm{dt} + \frac{e^{-\int_0^t r_s\,\mathrm{ds}}}{N_0}\,\mathrm{dN_t},$$

which is the same as:

$$\mathrm{d}\Phi_{\mathrm{t}} = -r_t \Phi_t \,\mathrm{dt} + \frac{\Phi_t}{N_t}\,\mathrm{dN_t}. \tag{1.2.22}$$

Taking equation (1.2.21), and using equation (1.2.22), we have:

$$\mathrm{dW}_{\mathrm{t}}^{\mathbb{Q}^{\mathrm{T}}} = \mathrm{dW}_{\mathrm{t}}^{\mathbb{Q}} - \frac{1}{N_t}\,\mathrm{dN_t}\,\mathrm{dW}_{\mathrm{t}}^{\mathbb{Q}} + r_t\,\mathrm{dt}\,\mathrm{dW}_{\mathrm{t}}^{\mathbb{Q}},$$

since $\mathrm{dt}\,\mathrm{dW}_{\mathrm{t}}^{\mathbb{Q}} = 0$ we have:

$$\mathrm{dW}_{\mathrm{t}}^{\mathbb{Q}^{\mathrm{T}}} = \mathrm{dW}_{\mathrm{t}}^{\mathbb{Q}} - \frac{1}{N_t}\,\mathrm{dN_t}\,\mathrm{dW}_{\mathrm{t}}^{\mathbb{Q}}.$$

■

Remark 1.6. *Consider the forward numeráire, $N_t = P(t,T)$, that is the price of a bond at time t, with maturity T:*

$$P(t,T) = E^{\mathbb{Q}}\left(e^{-\int_t^T r_s\,\mathrm{ds}}\middle|\mathscr{F}_t\right), \qquad P(T,T) = 1.$$

Then,

$$\left.\frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{Q}}\right|\mathscr{F}_t = e^{-\int_t^T r_s\,\mathrm{ds}}\frac{P(T,T)}{P(t,T)} = e^{-\int_t^T r_s\,\mathrm{ds}} E^{\mathbb{Q}}\left(e^{\int_t^T r_s\,\mathrm{ds}}\middle|\mathscr{F}_t\right).$$

The discounted bond price process $\left(e^{-\int_0^t r_s\,\mathrm{ds}} P(t,T)\right)_{t\in[0,T]}$*, is a $\mathbb{Q}$-martingale, because:*

$$\begin{aligned}
E^{\mathbb{Q}}\left(e^{-\int_0^t r_s\,\mathrm{ds}} P(t,T)\middle|\mathscr{F}_u\right) &= E^{\mathbb{Q}}\left(e^{-\int_0^t r_s\,\mathrm{ds}} E^{\mathbb{Q}}\left(e^{-\int_t^T r_s\,\mathrm{ds}}\middle|\mathscr{F}_t\right)\middle|\mathscr{F}_u\right)\\
&= E^{\mathbb{Q}}\left(E^{\mathbb{Q}}\left(e^{-\int_0^T r_s\,\mathrm{ds}}\middle|\mathscr{F}_t\right)\middle|\mathscr{F}_u\right)\\
&= E^{\mathbb{Q}}\left(e^{-\int_0^T r_s\,\mathrm{ds}}\middle|\mathscr{F}_u\right)
\end{aligned}$$

$$= e^{-\int_0^u r_s\,\mathrm{ds}} E^{\mathbb{Q}}\left(e^{-\int_u^T r_s\,\mathrm{ds}} \middle| \mathscr{F}_u \right)$$

$$= e^{-\int_0^u r_s\,\mathrm{ds}} P(u,T), \qquad u \le t.$$

From Proposition 1.3, we have:

$$O_t = E^{\mathbb{Q}}\left(e^{-\int_t^T r_s\,\mathrm{ds}} \xi \middle| \mathscr{F}_t \right) = P(t,T) E^{\mathbb{Q}^T}\left(\xi \middle| \mathscr{F}_t \right), \qquad 0 \le t \le T.$$

Moreover, from Theorem 1.9, we have:

$$\mathrm{dW}_{\mathrm{t}}^{\mathbb{Q}^{\mathrm{T}}} = \mathrm{dW}_{\mathrm{t}}^{\mathbb{Q}} - \frac{1}{P(t,T)}\,\mathrm{dP(t,T)}\,\mathrm{dW}_{\mathrm{t}}^{\mathbb{Q}}. \tag{1.2.23}$$

Remark 1.7. *Consider the numeráire, $N_t = S_t e^{qt}$, where q is the dividend yield, then:*

$$\frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{Q}}\bigg| \mathscr{F}_t = e^{-\int_t^T r_s\,\mathrm{ds}} \frac{S_T e^{qT}}{S_t e^{qt}} = \frac{S_T}{S_t} e^{-\int_t^T r_s\,\mathrm{ds}} e^{q\tau}, \qquad \tau := T - t. \tag{1.2.24}$$

Here the discounted asset price process $\left(e^{-\int_0^t r_s\,\mathrm{ds}} S_t e^{qt} \right)_{t \in [0,T]}$ is a $\mathbb{Q}$-martingale, by the very definition of the process S_t, under the measure $\mathbb{Q}$, with a dividend yield q.
From Proposition 1.3, and using $\xi = S_T \xi'$, where ξ' is $\mathscr{F}_T$-measurable, we have:

$$O_t = E^{\mathbb{Q}}\left(e^{-\int_t^T r_s\,\mathrm{ds}} S_T \xi' \middle| \mathscr{F}_t \right)$$

$$= S_t e^{qt} E^{\mathbb{Q}^T}\left(\frac{S_T \xi'}{S_T e^{qT}} \middle| \mathscr{F}_t \right)$$

$$= S_t e^{-q\tau} E^{\mathbb{Q}^T}\left(\xi' \middle| \mathscr{F}_t \right).$$

Furthermore, from Theorem 1.9, we have:

$$\mathrm{dW}_{\mathrm{t}}^{\mathbb{Q}^{\mathrm{T}}} = \mathrm{dW}_{\mathrm{t}}^{\mathbb{Q}} - \frac{1}{S_t e^{qt}}\,\mathrm{dS_t e^{qt}}\,\mathrm{dW}_{\mathrm{t}}^{\mathbb{Q}}.$$

Later in this thesis we will denote this measure, $\mathbb{Q}^T$, by $\mathbb{Q}^S$.

1.2.8 Ito's Lemma for jump-diffusion processes

Lemma 1.1. ***Ito's Lemma for jump-diffusion processes***
Let X be a diffusion process with jumps defined by:

$$X_t = X_0 + \int_0^t \mu(s, X_s)\,\mathrm{ds} + \int_0^t \sigma(s, X_s)\,\mathrm{dW_s} + \sum_{i=1}^{N_t} \Delta X_i,$$

where $\mu(t, X_t)$ and $\sigma(t, X_t)$ are continuous adapted processes with

$$E\left[\int_0^T \sigma^2(s, X_s)\,\mathrm{ds}\right] < \infty.$$

Then, for any $C^{1,2}$ function, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$, the process $Y_t = f(t, X_t)$ can be represented as:

$$\begin{aligned} f(t, X_t) = f(0, X_0) &+ \int_0^t \left[\frac{\partial f}{\partial s}(s, X_s) + \mu(s, X_s)\frac{\partial f}{\partial x}(s, X_s)\right]\mathrm{ds} \\ &+ \frac{1}{2}\int_0^t \sigma^2(s, X_s)\frac{\partial^2 f}{\partial x^2}(s, X_s)\,\mathrm{ds} + \int_0^t \sigma(s, X_s)\frac{\partial f}{\partial x}(s, X_s)\,\mathrm{dW_s} \\ &+ \sum_{\{i \geq 1, T_i \leq t\}} \left[f(X_{T_{i-}} + \Delta X_i) - f(X_{T_{i-}})\right]. \end{aligned}$$

In differential notation we have:

$$\begin{aligned} \mathrm{dY_t} = \frac{\partial f}{\partial t}(t, X_t)\,\mathrm{dt} + \mu(t, X_t)\frac{\partial f}{\partial x}(t, X_t)\,\mathrm{dt} + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2 f}{\partial x^2}(t, X_t)\,\mathrm{dt} \\ + \sigma(t, X_t)\frac{\partial f}{\partial x}(t, X_t)\,\mathrm{dW_t} + [f(X_{t-} + \Delta X_t) - f(X_{t-})]. \end{aligned}$$

Proof. See Tankov and Cont (2004, p. 276). ■

1.2.9 The Feynman-Kac Theorem

Theorem 1.10. ***The Feynman-Kac Theorem***
Assume that x_t is an Ito's process that follows the stochastic process

$$\mathrm{dx_t} = \mu(x_t,t)\,\mathrm{dt} + \sigma(x_t,t)\,\mathrm{dW_t}.$$

Let $V(x_t,t) \in C^{2,1}[\mathbb{R}\times[0,\infty) \to \mathbb{R}]$, and suppose that $V(x_t,t)$ is the solution of the following PDE

$$\frac{\partial V}{\partial t} + \mu(x_t,t)\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2(x_t,t)\frac{\partial^2 V}{\partial x^2} - r(x_t,t)V(x_t,t) = 0,$$

with boundary condition $V(X_T,T)$, and $r(x_t,t) \in C(\mathbb{R},[0,\infty))$. Then $V(x_t,t)$ is given by

$$V(x_t,t) = E\left[e^{-\int_t^T r(x_u,u)\,\mathrm{du}} V(X_T,T) \middle| \mathscr{F}_t \right].$$

Note that the Feynman-Kac theorem can be used in both directions.

Proof. See Zhu (2010, p. 8). ■

For a Feynman-Kac Theorem for jump-diffusion processes see Chesney et al. (2009, p. 557).

Chapter 2

Time-changed Lévy processes

2.1 The idea

Since the pioneering work of Black and Scholes (1973), many studies of times series of asset returns and derivatives prices have been made, and the conclusion usually obtained is that the classic Black-Scholes option pricing model fails to explain at least three facts:

i) Asset prices have discontinuities, or commonly said, asset prices jump. This feature was firstly discussed by Merton (1976).
ii) The volatility of asset returns varies stochastically over time. Heston (1993) used a mean reverting square root process to model volatility.
iii) Asset returns and their volatility are correlated. This was first discussed by Black (1976) as the "leverage effect".

Time-changed Lévy processes can capture all these three empirical findings. To capture the stochastic volatility, a stochastic time change to the Lévy process is made. This stochastic time change has to obey to certain conditions: It has to be positive and non-decreasing since it is describing a clock on which the Lévy process is running. The difference to the "original clock" is that randomness in business activity generates randomness in volatility, so in periods of high business activity the volatility tends to be also higher. In order to capture the correlation in asset returns and their volatility, the Lévy process is allowed to be correlated with the stochastic

time change. The jumps in asset prices are easily introduced in these models and it is easy to determine the characteristic function of such processes because the time-changed Lévy process and their jumps are uncorrelated. With this feature, these models become much more realistic because heavy tails and sudden movements in asset prices are generic properties of these models. The markets in such scenario are incomplete, which can be seen as an advantage if we want a more realistic model, because some risks can not be hedged and perfect hedge does not exist.
In order to describe these models, we will follow the approach of Carr and Wu (2004).

Consider a Lévy process $(X_t)_{t\geq 0}$ and let $\mathscr{F}$ be the σ-algebra generated by the past values of the process $(X_t)_{t\geq 0}$, completed by the null sets $\mathscr{N}$,[1] i.e.

$$\mathscr{F} = \sigma(X_t, t \geq 0) \bigvee \mathscr{N},$$

and let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a filtration $(\mathscr{F}_t)_{t\geq 0}$.

Let T_t be an increasing cadlag process such that for each fixed t, $\{T_t \leq t\} \in \mathscr{F}_t$, this is, T_t is a stopping time with respect to $\mathscr{F}$. Consider that $T_t \underset{t\to\infty}{\to} \infty$ and is finite $\mathbb{P} - a.s.$. We define

$$T_t := \int_0^t v(s)\,\mathrm{ds} \tag{2.1.1}$$

as the business time where $v(t)$ is the instantaneous activity rate, which is a positive cadlag process. The family of stopping times $(T_t)_{t\geq 0}$ defines a random time change.

If we evaluate the Lévy process $(X_t)_{t\geq 0}$ at the random time $(T_t)_{t\geq 0}$, we obtain a time-changed Lévy process $(Y_t)_{t\geq 0}$ denoted by

$$Y_t \equiv X_{T_t}, \tag{2.1.2}$$

and we define its characteristic function as

[1]Notice that null sets are stuffed into $\mathscr{F}_0$, meaning that if a certain evolution of X is impossible, its impossibility is already known at $t = 0$

$$\phi_{Y_t}(\xi) \equiv E\left[e^{\imath\xi Y_t}\right] = E\left[e^{\imath\xi X_{T_t}}\right]. \tag{2.1.3}$$

When we consider a parameter ξ that belongs to the complex plane, $\phi_{Y_t}(\xi)$ is called the generalized Fourier transform — see Titchmarsh (1948, p. 4–44).

This kind of process has already been studied in the literature — see Tankov and Cont (2004, p. 108–113) and Pascucci (2011, p. 471). The technique used is called *subordination* and like the technique above it simply makes a random time change in a Lévy process. In this context, the business time is called the subordinator and has the same characteristics as T_t. Usually, the random time change is made on a Brownian motion and the subordinators are processes like the Gamma process, Inverse Gaussian process, Variance Gamma or Normal inverse Gaussian. The subordination technique assumes that the subordinator and the Lévy process are *independent processes*. The subordinator can be any Lévy process; it is a pure jump process of possibly infinite activity plus a deterministic linear drift. Therefore the time change can have jumps and not be absolutely continuous. Under our approach, the time change will always be absolutely continuous; $v(s)$ can exhibit jumps, but T_t is always continuous. We assume that the time change and the Lévy process are correlated. However, if the subordinator and the Lévy process are independent, the time changed process still remains a Lévy process — see Pascucci (2011, p. 472).

In this thesis, we will focus on an integrated CIR — see Tankov and Cont (2004, p. 476) — process for the business time; some details about this process are given later.

2.2 Affine activity rate

In this section we will follow Carr and Wu (2004).

Let X_t be a Markov process that starts at X_0 and satisfies the following stochastic differential equation (SDE):

$$\mathrm{dX_t} = \mu(X_t)\,\mathrm{dt} + \sigma(X_t)\,\mathrm{dW_t} + q\,\mathrm{dJ}(\gamma(\mathrm{X_t})). \tag{2.2.1}$$

Here, W is a Brownian motion and J is a Poisson process with intensity

$\gamma(X_t)$ and jump size distribution q. Some technical conditions are required for $\mu(X_t)$ and $\sigma(X_t)$, in order to the SDE (2.2.1) to have a strong solution — see *Proposition 1* of Duffie et al. (2000, p. 1351).

Definition 2.1. The Laplace transform of the stopping time T_t is:

$$\mathscr{L}_{T_t}(\lambda) \equiv E\left[e^{-\lambda T_t}\right]. \tag{2.2.2}$$

Proposition 2.1. *If the instantaneous activity rate $v(s)$, the drift $\mu(X_t)$, the diffusion variance $\sigma(X_t)^2$ and the intensity $\gamma(X_t)$ are all affine in X_t and if $\mu(X_t)$ and $\sigma(X_t)$ satisfy some technical conditions stated in Proposition 1 of Duffie et al. (2000), then the Laplace transform* (2.2.3) *is exponential affine in X_0. That is, if*

1. $v(t) = b_v X_t + c_v$,
2. $\mu(X_t) = a - kX_t$,
3. $\sigma(X_t)^2 = \alpha + \beta X_t$,
4. $\gamma(X_t) = a_\gamma + b_\gamma X_t$,

then

$$\mathscr{L}_{T_t}(\lambda) = e^{-b(t)X_0 - c(t)}, \tag{2.2.3}$$

where

$$b'(t) = \lambda b_v - kb(t) - \frac{\beta b(t)^2}{2} - b_\gamma\left(\phi_q(ib(t)) - 1\right), \tag{2.2.4}$$

and

$$c'(t) = \lambda c_v + ab(t) - \frac{\alpha b(t)^2}{2} - a_\gamma\left(\phi_q(ib(t)) - 1\right), \tag{2.2.5}$$

with

$$b(0) = c(0) = 0.$$

Proof. The proof follows from applying the generalized Itô's lemma to (2.2.3) and using SDE (2.2.1). See Duffie et al. (2000, p. 1351).

■

2.3 Fundamental theorem

Theorem 2.1. ***Fundamental theorem of time-changed Lévy processes***

The generalized Fourier transform of the time-changed Lévy process $Y_T \equiv X_{T_T}$ is given by:

$$\phi_{Y_T}(\xi) \equiv E^{\mathbb{P}}\left[e^{\imath\xi Y_T}\right] = E^{\mathbb{P}}\left[e^{\imath\xi X_{T_T}}\right] = E^{\mathbb{Q}(\xi)}\left[e^{-T_T\Psi_x(\xi)}\right] \equiv \mathscr{L}^{\mathbb{Q}(\xi)}_{T_T}(\Psi_x(\xi)). \quad (2.3.1)$$

$\mathbb{Q}(\xi)$ is a complex valued measure that is absolutely continuous with respect to $\mathbb{P}$, and its Radon-Nikodym derivative is given by:[2]

$$\frac{\mathrm{d}\mathbb{Q}(\xi)}{\mathrm{d}\mathbb{P}} \equiv M_T(\xi) = e^{\imath\xi Y_T + T_T\Psi_x(\xi)}, \qquad \xi \in \mathbb{C}_\xi, \quad (2.3.2)$$

where $\mathbb{C}_\xi$ is the subset of $\mathbb{C}$ where $\phi_{Y_T}(\xi)$ is well defined.
And since $M_t(\xi)$ is a complex valued martingale, we have:

$$\left.\frac{\mathrm{d}\mathbb{Q}(\xi)}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_t} \equiv \frac{M_T(\xi)}{M_t(\xi)} = e^{\imath\xi(Y_T-Y_t)+(T_T-T_t)\Psi_x(\xi)}, \qquad \xi \in \mathbb{C}_\xi.$$

Notice that $\mathscr{L}^{\mathbb{Q}(\xi)}_{T_T}$ is not the usual Laplace transform because of the dependence of the measure $\mathbb{Q}(\xi)$ on ξ. We recall that given a process $(X_t)_{t\in[0,T]}$ its Laplace transform is given by:

$$\mathscr{L}_{X_T}(\xi) \equiv E\left[e^{-\xi X_T}\right] = \phi_{X_T}(\imath\xi), \qquad \xi \in \mathbb{C}_\xi, \quad (2.3.3)$$

where $\mathbb{C}_\xi$ is the subset of $\mathbb{C}$ where $\mathscr{L}_{X_T}(\xi)$ is well defined — see Titchmarsh (1948, p. 6).

Proof.

i) Consider the σ-algebra generated by the past values of the processes $(Y_t)_{t\in[0,T]}$ and $(T_t)_{t\in[0,T]}$, completed by the null sets, i.e.

[2] The measure $\mathbb{Q}(\xi)$ is equivalent to $\mathbb{P}$ but is not a probability measure, i.e., $\mathbb{Q}^{(\xi)}(\Omega) \neq 1$. The measure $\mathbb{Q}(\xi)$ is a particular case of the measure $\mathbb{P}^u$ in Giesecke and Zhu (2013, p. 747).

$$\mathscr{G} = \sigma(Y_t, T_t, t \in [0,T]) \vee \mathscr{N},$$

and let $(\Omega, \mathscr{G}, \mathbb{P})$ be a probability space with a filtration $(\mathscr{G}_t)_{t\in[0,T]}$.
First we need to proof that $M_t(\xi)$ is a complex valued $\mathbb{P}$-martingale with respect to $(\mathscr{G}_t)_{t\in[0,T]}$. Let us define $M_t^1(\xi) \equiv e^{\imath\xi X_t + t\Psi_x(\xi)}$.

a) $M_t^1(\xi)$ is adapted to the filtration generated by the process $(X_t)_{t\in[0,T]}$ completed by the null sets, $(\mathscr{F}_t)_{t\in[0,T]}$.

b) $E^{\mathbb{P}}\left[\left|M_t^1(\xi)\right|\right] \le \left|e^{\imath\xi X_t}\right|\left|e^{t\Psi_x(\xi)}\right| = \left|e^{\imath(a+\imath b)X_t}\right|\left|e^{t\Psi_x(\xi)}\right| = \left|e^{\imath a X_t - bX_t}\right|\left|e^{t\Psi_x(\xi)}\right| \le$

$$\le \left|e^{\imath a X_t}\right|\left|e^{-bX_t}\right|\left|e^{t\Psi_x(\xi)}\right| = \left|e^{-bX_t}\right|\left|e^{t\Psi_x(\xi)}\right| < \infty, \qquad a, b \in \mathbb{R},$$

since $\Psi_x(\xi)$ and X_t are both finite by definition.[3]

c) For $0 \le s < t$ we have:

$$E^{\mathbb{P}}\left[\frac{M_t^1(\xi)}{M_s^1(\xi)}\middle| X_s\right] = E^{\mathbb{P}}\left[e^{\imath\xi(X_t - X_s)+(t-s)\Psi_x(\xi)}\middle| X_s\right] = e^{(t-s)\Psi_x(\xi)} E^{\mathbb{P}}\left[e^{\imath\xi(X_{t-s})}\middle| X_s\right]$$
$$= e^{(t-s)\Psi_x(\xi)} e^{-(t-s)\Psi_x(\xi)} = 1.$$

Therefore, $M_t^1(\xi)$ is a complex valued $\mathbb{P}$-martingale with respect to $(\mathscr{F}_t)_{t\in[0,T]}$. By abuse of language we denote that filtration by X_s. For each fixed $t \in [0,T]$, T_t is a stopping time that is finite $\mathbb{P}$-a.s.. Therefore, by Proposition 1.1, $M_t(\xi) = M^1_{T_t}(\xi)$ is a complex valued $\mathbb{P}$-martingale with respect to the $(\mathscr{G}_t)_{t\in[0,T]}$.

ii)

$$\phi_{Y_T}(\xi) \equiv E^{\mathbb{P}}\left[e^{\imath\xi Y_T}\right] = E^{\mathbb{P}}\left[e^{\imath\xi Y_T + T_T\Psi_x(\xi) - T_T\Psi_x(\xi)}\right] = E^{\mathbb{P}}\left[M_T(\xi) e^{-T_T\Psi_x(\xi)}\right]$$
$$= E^{\mathbb{Q}(\xi)}\left[e^{-T_T\Psi_x(\xi)}\right] = \mathscr{L}^{\mathbb{Q}(\xi)}_{T_T}(\Psi_x(\xi)). \qquad \blacksquare$$

Remark 2.1. *If the random time T_T is independent of X_T, no measure change is required. By the tower rule, we have:*

[3] A small correction is made to the original work of Carr and Wu (2004).

$$\phi_{Y_T}(\xi) = E^{\mathbb{P}}\left[e^{\imath\xi X_{T_T}}\right] = E^{\mathbb{P}}\left[E^{\mathbb{P}}\left[e^{\imath\xi X_{T_T}} \Big/ T_T = u\right]\right].$$

The outside expectation is taken on all possible values of T_T and the inside expectation is taken on X_{T_T} conditional on a fixed value of $T_T = u$. Since T_T is independent of X_T:

$$E^{\mathbb{P}}\left[e^{\imath\xi X_{T_T}} \Big/ T_T = u\right] = E^{\mathbb{P}}\left[e^{\imath\xi X_{T_T}}\right].$$

Then for all fixed T_T we have:

$$E^{\mathbb{P}}\left[E^{\mathbb{P}}\left[e^{\imath\xi X_{T_T}} \Big/ T_T = u\right]\right] = E^{\mathbb{P}}\left[E^{\mathbb{P}}\left[e^{\imath\xi X_{T_T}}\right]\right] = E^{\mathbb{P}}\left[e^{-T_T\Psi_x(\xi)}\right] = \mathscr{L}_{T_T}(\Psi_x(\xi)),$$

where $\mathscr{L}_{T_T}(\Psi_x(\xi))$ is the usual Laplace transform on T_T.

Corollary 2.1. ***Time changed Brownian motion***
Suppose that $X_t = \sigma W_t + \mu t$, and T_t verifies the usual conditions. Then,

$$\phi_{Y_T}(\xi) = \mathscr{L}_{T_T}^{\mathbb{Q}(\xi)}\left(-\imath\mu\xi + \frac{1}{2}\sigma^2\xi^2\right). \tag{2.3.4}$$

$\mathbb{Q}(\xi)$ is a complex valued measure that is absolutely continuous with respect to $\mathbb{P}$, and its Radon-Nikodym derivative is given by:

$$\frac{\mathrm{d}\mathbb{Q}(\xi)}{\mathrm{d}\mathbb{P}} \equiv M_T(\xi) = e^{\imath\xi\sigma\int_0^T \sqrt{v_s}\,\mathrm{dW_s} + \frac{1}{2}\sigma^2\xi^2\int_0^T v_s\,\mathrm{ds}}, \qquad \xi \in \mathbb{C}_\xi, \tag{2.3.5}$$

where $\mathbb{C}_\xi$ is the subset of $\mathbb{C}$ where $\phi_{Y_T}(\xi)$ is well defined, and v_s is the instantaneous activity rate satisfying the usual conditions.
Since $M_t(\xi)$ is a martingale, we have:

$$\left.\frac{\mathrm{d}\mathbb{Q}(\xi)}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_t} \equiv \frac{M_T(\xi)}{M_t(\xi)} = e^{\imath\xi\sigma\int_t^T \sqrt{v_s}\,\mathrm{dW_s} + \frac{1}{2}\sigma^2\xi^2\int_t^T v_s\,\mathrm{ds}}, \qquad \xi \in \mathbb{C}_\xi.$$

Proof.

Consider the Lévy process $X_t = \sigma W_t + \mu t$. Its characteristic triplet is $(\mu, \sigma^2, 0)$, its characteristic exponent is given by $\Psi_x(\xi) = \left(-\imath\mu\xi + \frac{1}{2}\sigma^2\xi^2\right)$ and by Theorem 2.1, we have:

$$\phi_{Y_T}(\xi) = \mathscr{L}_{T_T}^{\mathbb{Q}(\xi)}\left(-\imath\mu\xi + \frac{1}{2}\sigma^2\xi^2\right). \tag{2.3.6}$$

The Radon-Nikodym derivative is given by:

$$\frac{\mathrm{d}\mathbb{Q}(\xi)}{\mathrm{d}\mathbb{P}} \equiv M_T(\xi) = e^{\imath\xi Y_T + T_T\Psi_x(\xi)} = e^{\imath\xi X_{T_T} + T_T\Psi_x(\xi)}, \qquad \xi \in \mathbb{C}_\xi, \tag{2.3.7}$$

where $\mathbb{C}_\xi$ is the subset of $\mathbb{C}$ where $\phi_{Y_T}(\xi)$ is well defined. Since $X_t = \sigma W_t + \mu t$ and $\Psi_x(\xi) = \left(-\imath\mu\xi + \frac{1}{2}\sigma^2\xi^2\right)$, we know that $X_{T_t} = \sigma W_{T_t} + \mu T_t$. Using also equation (2.1.2), equation (2.3.7) becomes:

$$e^{\imath\xi\left(\sigma W_{T_T} + \mu T_T\right) + T_T\left(-\imath\mu\xi + \frac{1}{2}\sigma^2\xi^2\right)} = e^{\imath\xi\left(\sigma W_{\int_0^T v_s \,\mathrm{s}} + \mu\int_0^T v_s\,\mathrm{ds}\right) + \int_0^T v_s\,\mathrm{ds}\left(-\imath\mu\xi + \frac{1}{2}\sigma^2\xi^2\right)}. \tag{2.3.8}$$

Using Theorem 1.4, $W_{\int_0^T v_s\,\mathrm{ds}} = \int_0^T \sqrt{v_s}\,\mathrm{dW_s}$, and, therefore,

$$\frac{\mathrm{d}\mathbb{Q}(\xi)}{\mathrm{d}\mathbb{P}} \equiv M_T(\xi) = e^{\imath\xi\sigma\int_0^T \sqrt{v_s}\,\mathrm{dW_s} + \frac{1}{2}\sigma^2\xi^2\int_0^T v_s\,\mathrm{ds}}, \qquad \xi \in \mathbb{C}_\xi. \tag{2.3.9}$$

$\blacksquare$

Chapter 3

Asset pricing with constant interest rates

Let S_t denote the price of an asset at time t, Y_t be a time-changed Lévy process and $\widetilde{Z_t} = \sum_{i=1}^{N_t} Y_i$ a compound Poisson process which is independent from Y_t.
Consider the process defined as $G_t = Y_t + \widetilde{Z_t}$ and let $S_T = S_0 e^{(r-q)T + Y_T + \widetilde{Z_T}}$, where r is the risk free and constant interest rate and q is the dividend yield. Under no arbitrage we need to have:

$$E^{\mathbb{Q}}\left[e^{Y_T + \widetilde{Z_T}}\right] = 1. \tag{3.0.1}$$

Since the process $\widetilde{Z_T}$ is independent from Y_T we have

$$E^{\mathbb{Q}}\left[e^{Y_T + \widetilde{Z_T}}\right] = E^{\mathbb{Q}}\left[e^{Y_T}\right] E^{\mathbb{Q}}\left[e^{\widetilde{Z_T}}\right] = 1. \tag{3.0.2}$$

1. $E^{\mathbb{Q}}\left[e^{Y_T}\right] = 1$ is obtained by writing the process Y_t under the risk neutral measure with $Y_0 = 0$.

2. To see that $E^{\mathbb{Q}}\left[e^{\widetilde{Z_T}}\right] = 1$, from equation (1.2.4), we know that given a compound Poisson process Z_t,

$$\phi_{Z_T}(\xi) \equiv E^{\mathbb{Q}}\left[e^{\imath\xi Z_T}\right] = e^{-T\Psi_z(\xi)} = e^{-T\lambda(1 - E^{\mathbb{Q}}[e^{\imath\xi Y}])}.$$

Since $\phi_{Z_T}(-\imath) = E^{\mathbb{Q}}\left[e^{Z_T}\right]$, we have:

$$E^{\mathbb{Q}}\left[e^{Z_T}\right] = e^{-T\lambda(1-E^{\mathbb{Q}}[e^Y])} = e^{T\lambda(E^{\mathbb{Q}}[e^Y]-1)}. \tag{3.0.3}$$

The right hand side of equation (3.0.3) is called the cumulant exponent. Now if we consider $\widetilde{Z_t} = Z_t - t\lambda\left(E^{\mathbb{Q}}\left[e^Y\right]-1\right)$ we know that $E^{\mathbb{Q}}\left[e^{\widetilde{Z_T}}\right] = 1$, where $\lambda > 0$ is the intensity and Y is a random variable that follows a distribution f.

Finally we get

$$G_t = Y_t + Z_t - t\lambda\left(E^{\mathbb{Q}}\left[e^Y\right]-1\right),$$

and S_T is given by

$$S_T = S_0 e^{(r-q-\lambda(E^{\mathbb{Q}}[e^Y]-1))T+Y_T+Z_T}, \tag{3.0.4}$$

i.e.

$$\frac{S_T}{e^{(r-q)T}} = S_0 e^{(-\lambda(E^{\mathbb{Q}}[e^Y]-1))T+Y_T+Z_T},$$

with

$$E^{\mathbb{Q}}\left[e^{(-\lambda(E^{\mathbb{Q}}[e^Y]-1))T+Y_T+Z_T}\right] = E^{\mathbb{Q}}\left[e^{Y_T}\right]E^{\mathbb{Q}}\left[e^{(-\lambda(E^{\mathbb{Q}}[e^Y]-1))T+Z_T}\right] = 1$$

Denoting $s_T \equiv \log\frac{S_T}{S_0}$, the characteristic function of s_T is given by:

$$\begin{aligned}\phi_{s_T}(\xi) \equiv E^{\mathbb{Q}}\left[e^{\imath\xi s_T}\right] &= E^{\mathbb{Q}}\left[e^{\imath\xi(r-q-\lambda(E^{\mathbb{Q}}[e^Y]-1))T+\imath\xi Y_T+\imath\xi Z_T}\right]\\ &= e^{\imath\xi(r-q-\lambda(E^{\mathbb{Q}}[e^Y]-1))T}E^{\mathbb{Q}}\left[e^{\imath\xi Y_T}\right]E^{\mathbb{Q}}\left[e^{\imath\xi Z_T}\right].\end{aligned}$$

We conclude that:

$$\phi_{s_T}(\xi) = e^{\imath\xi(r-q-\lambda(E^{\mathbb{Q}}[e^Y]-1))T}e^{T\lambda[\phi_Y(\xi)-1]}\mathscr{L}_{T_T}^{\mathbb{Q}(\xi)}(\Psi_x(\xi)). \tag{3.0.5}$$

3.1 Heston model

The Heston model was first introduced by Heston (1993), and prescribes that the evolution of the asset price under risk neutral measure $\mathbb{Q}$ is given by:

$$\begin{aligned}
\mathrm{dS_t} &= (r-q)S_t\,\mathrm{dt} + S_t\sqrt{v_t}\,\mathrm{dZ_1^{\mathbb{Q}}(t)},\\
\mathrm{dv_t} &= k(\theta - v_t)\,\mathrm{dt} + \sigma\sqrt{v_t}\left(\rho\,\mathrm{dZ_1^{\mathbb{Q}}(t)} + \sqrt{1-\rho^2}\,\mathrm{dZ_2^{\mathbb{Q}}(t)}\right),
\end{aligned}$$

where $Z_1^{\mathbb{Q}}$ and $Z_2^{\mathbb{Q}}$ are two independent Brownian motions.
Defining $x_t := \log(S_t)$, and applying Itô's Lemma, we have:

$$\mathrm{d}\log(\mathrm{S_t}) = \frac{1}{S_t}\,\mathrm{dS_t} - \frac{1}{2S_t^2}S_t^2 v_t\,\mathrm{dt},$$

which is equivalent to

$$\begin{aligned}
\mathrm{dx_t} &= (r-q)\,\mathrm{dt} + \sqrt{v_t}\,\mathrm{dZ_1^{\mathbb{Q}}(t)} - \frac{1}{2}v_t\,\mathrm{dt}\\
&= (r-q-\frac{1}{2}v_t)\,\mathrm{dt} + \sqrt{v_t}\,\mathrm{dZ_1^{\mathbb{Q}}(t)}.
\end{aligned}$$

For purposes of option pricing we need to know the model under the measure $\mathbb{Q}^S$, where $\mathbb{Q}^S$ denotes the equivalent martingale measure associated to the numeráire $S_t e^{qt}$. Then, using Remark 1.7, and considering that the free interest rate is constant, we have:

$$\frac{\mathrm{d}\mathbb{Q}^S}{\mathrm{d}\mathbb{Q}}\bigg|\mathscr{F}_t = \frac{S_T}{S_t}e^{-(r-q)\tau}, \qquad \tau := T-t,$$

and

$$\mathrm{dZ_1^{\mathbb{Q}^S}(t)} = \mathrm{dZ_1^{\mathbb{Q}}(t)} - \frac{1}{S_t e^{qt}}\,\mathrm{dS_te^{qt}}\,\mathrm{dZ_1^{\mathbb{Q}}(t)}. \tag{3.1.1}$$

Applying Itô's Lemma

$$\mathrm{dS_te^{qt}} = qS_te^{qt}\,\mathrm{dt} + e^{qt}\,\mathrm{dS_t} \tag{3.1.2}$$

$$= qS_te^{qt}\,\mathrm{dt} + (r-q)S_te^{qt}\,\mathrm{dt} + S_te^{qt}\sqrt{v_t}\,\mathrm{dZ_1^{\mathbb{Q}}(t)}. \tag{3.1.3}$$

Then, using equation (3.1.3), equation (3.1.1) becomes:

$$
\begin{aligned}
\mathrm{dZ}_1^{\mathbb{Q}^{\mathrm{S}}}(\mathrm{t}) &= \mathrm{dZ}_1^{\mathbb{Q}}(\mathrm{t}) - \left(q\,\mathrm{dt} + (r-q)\,\mathrm{dt} + \sqrt{v_t}\,\mathrm{dZ}_1^{\mathbb{Q}}(\mathrm{t})\right)\mathrm{dZ}_1^{\mathbb{Q}}(\mathrm{t}) \\
&= \mathrm{dZ}_1^{\mathbb{Q}}(\mathrm{t}) - \sqrt{v_t}\,\mathrm{dt}.
\end{aligned}
$$

Therefore, the evolution of the asset price under measure $\mathbb{Q}^S$ is given by:

$$
\begin{aligned}
\mathrm{dx_t} &= (r - q + \frac{1}{2}v_t)\,\mathrm{dt} + \sqrt{v_t}\,\mathrm{dZ}_1^{\mathbb{Q}^{\mathrm{S}}}(\mathrm{t}) \\
\mathrm{dv_t} &= (k\theta - (k - \sigma\rho)v_t)\,\mathrm{dt} + \sigma\sqrt{v_t}\left(\rho\,\mathrm{dZ}_1^{\mathbb{Q}^{\mathrm{S}}}(\mathrm{t}) + \sqrt{1-\rho^2}\,\mathrm{dZ}_2^{\mathbb{Q}^{\mathrm{S}}}(\mathrm{t})\right).
\end{aligned}
$$

We can resume this information, by taking j=1 for measure $\mathbb{Q}^S$, and j=2 for measure $\mathbb{Q}$:

$$
\mathrm{dx_t} = (r - q + \mu_j v_t)\,\mathrm{dt} + \sqrt{v_t}\,\mathrm{dZ}_1^{\mathrm{j}}(\mathrm{t}) \tag{3.1.4}
$$

$$
\mathrm{dv_t} = (k\theta - \beta_j v_t)\,\mathrm{dt} + \sigma\sqrt{v_t}\left(\rho\,\mathrm{dZ}_1^{\mathrm{j}}(\mathrm{t}) + \sqrt{1-\rho^2}\,\mathrm{dZ}_2^{\mathrm{j}}(\mathrm{t})\right), \tag{3.1.5}
$$

where

$$
\mu_1 = \frac{1}{2}, \tag{3.1.6}
$$

$$
\mu_2 = -\frac{1}{2}, \tag{3.1.7}
$$

$$
\beta_1 = k - \sigma\rho, \tag{3.1.8}
$$

$$
\beta_2 = k. \tag{3.1.9}
$$

The characteristic function of the random variable x_T,

$$
\phi_{x_T}^j(\xi) = E^j\left[e^{\imath\xi x_T}\middle|\mathscr{F}_t\right],
$$

is of the form

$$
\phi_{x_T}^j(\xi) = e^{C_j(\tau) + D_j(\tau)v_t + \imath\xi x_t},
$$

where $C_j(\tau)$, and $D_j(\tau)$, are the solution of the following ordinary differential equations (ODEs):

$$
\begin{aligned}
\frac{\partial D_j}{\partial\tau} &= -\frac{1}{2}\xi^2 + \imath\rho\sigma\xi D_j + \frac{1}{2}\sigma^2 D_j^2 + \imath\xi\mu_j - \beta_j D_j, \\
\frac{\partial C_j}{\partial\tau} &= (r-q)\imath\xi + k\theta D_j.
\end{aligned}
$$

For a detailed explanation of this model see Heston (1993).

3.2 Heston model as a time changed Lévy process

Consider the following time changed Lévy process under the risk neutral measure $\mathbb{Q}$:

1. $Y_t = X_{T_t}$
2. $X_t = W_x^{\mathbb{Q}}(t) + \mu t$
3. $\mathrm{dv_t} = k(\theta - v_t)\,\mathrm{dt} + \sigma\sqrt{v_t}\,\mathrm{dW_v^{\mathbb{Q}}(t)}$
4. $\mathrm{d} < \mathrm{W_x^{\mathbb{Q}}}, \mathrm{W_v^{\mathbb{Q}}} >_\mathrm{t} = \rho\,\mathrm{dt}$

From Theorem 1.2, using Remark 1.1, and given two independent Brownian motions, $Z_x^{\mathbb{Q}}$ and $Z_v^{\mathbb{Q}}$, we know that:

$$W_x^{\mathbb{Q}}(t) = Z_x^{\mathbb{Q}}(t), \tag{3.2.1}$$

$$W_v^{\mathbb{Q}}(t) = \rho Z_x^{\mathbb{Q}}(t) + \sqrt{1-\rho^2} Z_v^{\mathbb{Q}}(t). \tag{3.2.2}$$

Therefore, we can rewrite the model as:

1. $Y_t = X_{T_t}$
2. $X_t = Z_x^{\mathbb{Q}}(t) + \mu t$
3. $\mathrm{dv_t} = k(\theta - v_t)\,\mathrm{dt} + \sigma\sqrt{v_t}\left(\rho\,\mathrm{dZ_x^{\mathbb{Q}}(t)} + \sqrt{1-\rho^2}\,\mathrm{dZ_v^{\mathbb{Q}}(t)}\right)$
4. $\mathrm{d} < \mathrm{Z_x^{\mathbb{Q}}}, \mathrm{Z_v^{\mathbb{Q}}} >_\mathrm{t} = 0$

In the above formulae, θ is the long term mean, $k(\geq 0)$ is the speed of mean reversion and σ is the volatility of the variance process. It is well known since Feller (1951) that, if $2k\theta \geq \sigma^2$, the process never touches zero; in the opposite case, the process will occasionally touch zero and be reflected. We will consider the case of $2k\theta \geq \sigma^2$ in order to the instantaneous activity rate to be well defined.

Under the risk neutral measure $\mathbb{Q}$, we know that the process $(e^{Y_t})_{t\geq 0}$ must be a martingale. Using Theorem 1.4, we have:

$$E^{\mathbb{Q}}\left[e^{Y_T-Y_t}\middle|\mathscr{F}_t\right] = E^{\mathbb{Q}}\left(e^{\int_0^T \sqrt{v_s}\,\mathrm{dZ}_\mathrm{x}^{\mathbb{Q}}(\mathrm{s}) + \mu\int_0^T v_s\,\mathrm{ds} - \int_0^t \sqrt{v_s}\,\mathrm{dZ}_\mathrm{x}^{\mathbb{Q}}(\mathrm{s}) - \mu\int_0^t v_s\,\mathrm{ds}}\middle|\mathscr{F}_t\right)$$

$$= e^{-\int_0^t \sqrt{v_s}\,\mathrm{dZ}_\mathrm{x}^{\mathbb{Q}}(\mathrm{s}) - \mu\int_0^t v_s\,\mathrm{ds}}\, E^{\mathbb{Q}}\left(e^{\int_0^T \sqrt{v_s}\,\mathrm{dZ}_\mathrm{x}^{\mathbb{Q}}(\mathrm{s}) + \mu\int_0^T v_s\,\mathrm{ds}}\middle|\mathscr{F}_t\right), \tag{3.2.3}$$

In order to the expectation contained in the right-hand side of equation (3.2.3) to be a martingale, we know from equation (1.2.6) and from Novikov′s condition that $\mu = -\frac{1}{2}$, and, therefore,

$$X_t = Z_x^{\mathbb{Q}}(t) - \frac{1}{2}t. \tag{3.2.4}$$

We also need to know X_t under measure $\mathbb{Q}^S$. Using the results from Section 1.2.7, we have:

$$\left.\frac{\mathrm{d}\mathbb{Q}^S}{\mathrm{d}\mathbb{Q}}\right|_{\mathscr{F}_t} \equiv \frac{S_T}{S_t}e^{-(r-q)\tau} \qquad with \qquad \tau = T - t.$$

From equation (3.0.4) and since the process S_t has no jumps in this framework, we have, $\frac{S_T}{S_t} = e^{(r-q)\tau + Y_T - Y_t}$. Therefore,

$$\left.\frac{\mathrm{d}\mathbb{Q}^S}{\mathrm{d}\mathbb{Q}}\right|_{\mathscr{F}_t} = e^{Y_T - Y_t} = e^{\int_t^T \sqrt{v_s}\,\mathrm{dZ}_\mathrm{x}^{\mathbb{Q}}(\mathrm{s}) - \frac{1}{2}\int_t^T v_s\,\mathrm{ds}}. \tag{3.2.5}$$

From Girsanov's Theorem, it follows immediately that

$$\mathrm{dZ}_\mathrm{x}^{\mathbb{Q}^\mathrm{S}}(\mathrm{t}) = \mathrm{dZ}_\mathrm{x}^{\mathbb{Q}}(\mathrm{t}) - \sqrt{v_t}\,\mathrm{dt}. \tag{3.2.6}$$

Using equation (3.2.4), we can write Y_t as:

$$Y_t = Z_x^{\mathbb{Q}}(T_t) - \frac{1}{2}T_t = \int_0^t \sqrt{v_s}\, \mathrm{d}Z_\mathrm{x}^{\mathbb{Q}}(\mathrm{s}) - \frac{1}{2}\int_0^t v_s\, \mathrm{ds}.$$

From equation (3.2.6), and under measure $\mathbb{Q}^S$, we have:

$$Y_t = \int_0^t \sqrt{v_s}\left(\mathrm{d}Z_\mathrm{x}^{\mathbb{Q}^\mathrm{S}}(\mathrm{s}) + \sqrt{v_s}\, \mathrm{ds}\right) - \frac{1}{2}\int_0^t v_s\, \mathrm{ds} = \int_0^t \sqrt{v_s}\, \mathrm{d}Z_\mathrm{x}^{\mathbb{Q}^\mathrm{S}}(\mathrm{s}) + \frac{1}{2}\int_0^t v_s\, \mathrm{ds}.$$

Therefore, it follows immediately that X_t is given under measure $\mathbb{Q}^S$ by

$$X_t = Z_x^{\mathbb{Q}^S}(t) + \frac{1}{2}t. \tag{3.2.7}$$

Now we are ready to obtain the model under measures $\mathbb{Q}$ and $\mathbb{Q}^S$. Under measure $\mathbb{Q}$, we have:

$$Y_t = X_{T_t} \tag{3.2.8}$$

$$X_t = Z_x^{\mathbb{Q}}(t) - \frac{1}{2}t \tag{3.2.9}$$

$$\mathrm{dv_t} = k(\theta - v_t)\,\mathrm{dt} + \sigma\sqrt{v_t}\left(\rho\, \mathrm{d}Z_\mathrm{x}^{\mathbb{Q}}(\mathrm{t}) + \sqrt{1-\rho^2}\, \mathrm{d}Z_\mathrm{v}^{\mathbb{Q}}(\mathrm{t})\right) \tag{3.2.10}$$

$$\mathrm{d} < Z_\mathrm{x}^{\mathbb{Q}}, Z_\mathrm{v}^{\mathbb{Q}} >_\mathrm{t} = 0 \tag{3.2.11}$$

Using equation (3.2.6), equation (3.2.10), becomes:

$$\mathrm{dv_t} = (k\theta - (k-\sigma\rho)v_t)\,\mathrm{dt} + \sigma\sqrt{v_t}\left(\rho\, \mathrm{d}Z_\mathrm{x}^{\mathbb{Q}^\mathrm{S}}(\mathrm{t}) + \sqrt{1-\rho^2}\, \mathrm{d}Z_\mathrm{v}^{\mathbb{Q}^\mathrm{S}}(\mathrm{t})\right).$$

Therefore, under measure $\mathbb{Q}^S$ we have:

$$Y_t = X_{T_t} \tag{3.2.12}$$

$$X_t = Z_x^{\mathbb{Q}^S}(t) + \frac{1}{2}t \tag{3.2.13}$$

$$\mathrm{dv_t} = (k\theta - (k-\sigma\rho)v_t)\,\mathrm{dt} + \sigma\sqrt{v_t}\left(\rho\, \mathrm{d}Z_\mathrm{x}^{\mathbb{Q}^\mathrm{S}}(\mathrm{t}) + \sqrt{1-\rho^2}\, \mathrm{d}Z_\mathrm{v}^{\mathbb{Q}^\mathrm{S}}(\mathrm{t})\right) \tag{3.2.14}$$

$$\mathrm{d} < Z_\mathrm{x}^{\mathbb{Q}^\mathrm{S}}, Z_\mathrm{v}^{\mathbb{Q}^\mathrm{S}} >_\mathrm{t} = 0 \tag{3.2.15}$$

In order to get the characteristic function of Y_T, we need to write v_t under the leverage neutral measure $\mathbb{Q}(\xi)$. Starting with the model under measure $\mathbb{Q}$, it follows from Corollary 2.1 that

$$\phi_{Y_T}(\xi) = \mathscr{L}_{T_T}^{\mathbb{Q}(\xi)}(\Psi_x(\xi)) = \mathscr{L}_{T_T}^{\mathbb{Q}(\xi)}\left(\frac{1}{2}\iota\xi + \frac{1}{2}\xi^2\right). \tag{3.2.16}$$

From equation (2.3.5) we obtain:

$$\frac{\mathrm{d}\mathbb{Q}(\xi)}{\mathrm{d}\mathbb{Q}} \equiv M_T(\xi) = e^{\iota\xi\int_0^T \sqrt{v_s}\,\mathrm{dZ_x^{\mathbb{Q}}(s)} + \frac{1}{2}\xi^2\int_0^T v_s\,\mathrm{ds}},$$

and from Girsanov's Theorem it follows that

$$\mathrm{dZ_x^{\mathbb{Q}(\xi)}(t)} = \mathrm{dZ_x^{\mathbb{Q}}(t)} - \iota\xi\sqrt{v_t}\,\mathrm{d} < \mathrm{Z_x^{\mathbb{Q}}}, \mathrm{Z_x^{\mathbb{Q}}} >_{\mathrm{t}} = \mathrm{dZ_x^{\mathbb{Q}}(t)} - \iota\xi\sqrt{v_t}\,\mathrm{dt}. \tag{3.2.17}$$

Finally, the process v_t can be written, under the measure $\mathbb{Q}(\xi)$, as:

$$\mathrm{dv_t} = (k\theta - (k - \sigma\rho\iota\xi)v_t)\,\mathrm{dt} + \sigma\sqrt{v_t}\left(\rho\,\mathrm{dZ_x^{\mathbb{Q}(\xi)}(t)} + \sqrt{1-\rho^2}\,\mathrm{dZ_v^{\mathbb{Q}(\xi)}(t)}\right).$$

Following the same procedure, the characteristic function under the measure $\mathbb{Q}^S$ is:

$$\phi_{Y_T}(\xi) = \mathscr{L}_{T_T}^{\mathbb{Q}^S(\xi)}(\Psi_x(\xi)) = \mathscr{L}_{T_T}^{\mathbb{Q}^S(\xi)}\left(-\frac{1}{2}\iota\xi + \frac{1}{2}\xi^2\right),$$

and

$$\frac{\mathrm{d}\mathbb{Q}^S(\xi)}{\mathrm{d}\mathbb{Q}^S} \equiv M_T(\xi) = e^{\iota\xi\int_0^T \sqrt{v_s}\,\mathrm{dZ_x^{\mathbb{Q}^S}(s)} + \frac{1}{2}\xi^2\int_0^T v_s\,\mathrm{ds}}.$$

Therefore, the process v_t under measure $\mathbb{Q}^S(\xi)$ can be written as:

$$\mathrm{dv_t} = (k\theta - (k - \sigma\rho - \sigma\rho\iota\xi)v_t)\,\mathrm{dt} + \sigma\sqrt{v_t}\left(\rho\,\mathrm{dZ_x^{\mathbb{Q}^S(\xi)}(t)} + \sqrt{1-\rho^2}\,\mathrm{dZ_v^{\mathbb{Q}^S(\xi)}(t)}\right).$$

In summary, under measure $\mathbb{Q}(\xi)$ we have:

$$\phi_{Y_T}(\xi) = \mathscr{L}_{T_T}^{\mathbb{Q}(\xi)}(\Psi_x(\xi)) = \mathscr{L}_{T_T}^{\mathbb{Q}(\xi)}\left(\frac{1}{2}\imath\xi + \frac{1}{2}\xi^2\right) \tag{3.2.18}$$

$$Y_t = X_{T_t} \tag{3.2.19}$$

$$X_t = Z_x^{\mathbb{Q}}(t) - \frac{1}{2}t \tag{3.2.20}$$

$$\mathrm{dv_t} = (k\theta - (k - \sigma\rho\imath\xi)v_t)\,\mathrm{dt} + \sigma\sqrt{v_t}\left(\rho\,\mathrm{dZ_x^{\mathbb{Q}(\xi)}(t)} + \sqrt{1-\rho^2}\,\mathrm{dZ_v^{\mathbb{Q}(\xi)}(t)}\right) \tag{3.2.21}$$

$$\mathrm{d} < \mathrm{Z_x^{\mathbb{Q}(\xi)}}, \mathrm{Z_v^{\mathbb{Q}(\xi)}} >_{\mathrm{t}} = 0, \tag{3.2.22}$$

and under measure $\mathbb{Q}^S(\xi)$ we have:

$$\phi_{Y_T}(\xi) = \mathscr{L}_{T_T}^{\mathbb{Q}^S(\xi)}(\Psi_x(\xi)) = \mathscr{L}_{T_T}^{\mathbb{Q}^S(\xi)}\left(-\frac{1}{2}\imath\xi + \frac{1}{2}\xi^2\right) \tag{3.2.23}$$

$$Y_t = X_{T_t} \tag{3.2.24}$$

$$X_t = Z_x^{\mathbb{Q}^S}(t) + \frac{1}{2}t \tag{3.2.25}$$

$$\mathrm{dv_t} = (k\theta - (k - \sigma\rho - \sigma\rho\imath\xi)v_t)\,\mathrm{dt} + \sigma\sqrt{v_t}\left(\rho\,\mathrm{dZ_x^{\mathbb{Q}^S(\xi)}(t)} + \sqrt{1-\rho^2}\,\mathrm{dZ_v^{\mathbb{Q}^S(\xi)}(t)}\right) \tag{3.2.26}$$

$$\mathrm{d} < \mathrm{Z_x^{\mathbb{Q}^S(\xi)}}, \mathrm{Z_v^{\mathbb{Q}^S(\xi)}} >_{\mathrm{t}} = 0. \tag{3.2.27}$$

It should be noted that the Wiener process Z_v is the same process under every measure that we have considered. Equations (3.2.22) and (3.2.27) are consequence of equation (3.2.17), and item **ii)** of Definition 1.12.

From Proposition 2.1, and taking $j = 1$ for measure $\mathbb{Q}^S(\xi)$, and $j = 2$ for measure $\mathbb{Q}(\xi)$, we have the following parameters for the process v_t:

1. $b_v = 1$, $\quad c_v = 0$, $\quad a^j = k\theta$, $\quad k^j = \beta^j - \imath\xi\rho\sigma$, $\quad \alpha = 0$,
2. $\beta = \sigma^2$, $\quad a_\gamma = b_\gamma = 0$, $\quad \lambda = \Psi_x(\xi) = \frac{1}{2}\xi^2 - \mu^j\imath\xi$,
3. $\beta^1 = k - \sigma\rho$, $\quad \beta^2 = k$, $\quad \mu^1 = \frac{1}{2}$, $\quad \mu^2 = -\frac{1}{2}$.

Furthermore, using equations (3.2.4) and (3.2.7) and Proposition 2.1, we have:

$$\phi^j_{Y_T}(\xi) = \mathscr{L}^j_{T_T}\left(\frac{1}{2}\xi^2 - \mu^j \imath\xi\right) = e^{-b_j(T)v_0 - c_j(T)} \tag{3.2.28}$$

with

$$b'_j(t) = \frac{1}{2}\xi^2 - \mu^j \imath\xi - (\beta^j - \imath\xi\rho\sigma)b_j(t) - \frac{\sigma^2}{2}b_j(t)^2, \tag{3.2.29}$$

$$c'_j(t) = k\theta b_j(t), \tag{3.2.30}$$

$$b_j(0) = c_j(0) = 0. \tag{3.2.31}$$

From Proposition D.1, and since

$$a = \frac{1}{2}\xi^2 - \mu^j \imath\xi,$$

$$c = \imath\xi\rho\sigma - \beta^j,$$

$$d = -\frac{\sigma^2}{2},$$

$$k_1 = k\theta,$$

we have the solution of such problem:

$$b_j(T) = \frac{\imath\xi\rho\sigma - \beta^j - \Delta_j}{\sigma^2}\frac{1 - e^{\Delta_j T}}{1 - e^{\Delta_j T}\Lambda_j} \tag{3.2.32}$$

and

$$c_j(T) = \frac{k\theta}{\sigma^2}\left[\left(\imath\xi\rho\sigma - \beta^j - \Delta_j\right)T + 2\log\left(\frac{1 - \Lambda_j e^{\Delta_j T}}{1 - \Lambda_j}\right)\right] \tag{3.2.33}$$

with

$$\Delta_j = \sqrt{(\imath\xi\rho\sigma - \beta^j)^2 + \sigma^2(\xi^2 - 2\mu^j \imath\xi)}, \tag{3.2.34}$$

and

$$\Lambda_j = \frac{\imath\xi\rho\sigma - \beta^j - \Delta_j}{\imath\xi\rho\sigma - \beta^j + \Delta_j}. \tag{3.2.35}$$

From equation (3.0.5), and since we are considering no jumps, we can compute, $\phi_{s_T}(\xi)$, under measures $\mathbb{Q}^S(\xi)$ and $\mathbb{Q}(\xi)$, as follows:

$$\phi^j_{s_T}(\xi) = e^{\imath\xi(r-q)T}\mathscr{L}^j_{T_T}\left(\frac{1}{2}\xi^2 - \mu^j\imath\xi\right) \tag{3.2.36}$$

$$= e^{\imath\xi(r-q)T}e^{-b_j(T)v_0-c_j(T)}, \tag{3.2.37}$$

where $b_j(T)$ and $c_j(T)$, are given by equations (3.2.32) and (3.2.33), respectively.

We can resume the information above in the following Theorem.

Theorem 3.1. ***The Heston model as a time-changed Lévy process***

Under measures $\mathbb{Q}(\xi)(j=2)$ and $\mathbb{Q}^S(\xi)(j=1)$, the Heston model can be written as a time-changed Lévy process, and its specification is as follows:

$$Y^j_t = X^j_{T_t}$$

$$X^j_t = Z^j_x(t) + \mu^j t$$

$$\mathrm{dv_t} = (k\theta - (\beta^j - \sigma\rho\imath\xi)v_t)\,\mathrm{dt} + \sigma\sqrt{v_t}\left(\rho\,\mathrm{dZ^j_x(t)} + \sqrt{1-\rho^2}\,\mathrm{dZ^j_v(t)}\right)$$

$$\mathrm{d} < \mathrm{Z^j_x}, \mathrm{Z^j_v} >_{\mathrm{t}} = 0,$$

with

$$\beta^1 = k - \sigma\rho, \qquad \beta^2 = k, \qquad \mu^1 = \frac{1}{2}, \qquad \mu^2 = -\frac{1}{2}.$$

Then, using equation (3.0.4)*, we can model the asset price as*

$$S_T = S_0 e^{(r-q)T + Y^j_T},$$

and the characteristic function of $s_T \equiv \log\frac{S_T}{S_0}$ is given by:

$$\phi^j_{s_T}(\xi) = e^{\imath\xi(r-q)T}e^{-b_j(T)v_t-c_j(T)}, \tag{3.2.38}$$

where

$$b_j(T) = \frac{\imath\xi\rho\sigma - \beta^j - \Delta_j}{\sigma^2}\frac{1-e^{\Delta_j T}}{1-e^{\Delta_j T}\Lambda_j} \tag{3.2.39}$$

$$c_j(T) = \frac{k\theta}{\sigma^2}\left[\left(\imath\xi\rho\sigma - \beta^j - \Delta_j\right)T + 2\log\left(\frac{1-\Lambda_j e^{\Delta_j T}}{1-\Lambda_j}\right)\right], \tag{3.2.40}$$

$$\Delta_j = \sqrt{(\imath\xi\rho\sigma - \beta^j)^2 + \sigma^2(\xi^2 - 2\mu^j\imath\xi)}, \tag{3.2.41}$$

and

$$\Lambda_j = \frac{\imath\xi\rho\sigma - \beta^j - \Delta_j}{\imath\xi\rho\sigma - \beta^j + \Delta_j}. \tag{3.2.42}$$

■

Equation (3.2.38), yields Heston (1993, equation 17).

3.3 Heston model with jumps as a time changed Lévy process

In the Heston model with jumps, since we are considering the discontinuous part of the process, the evolution of the asset price is given by equation (3.0.4),

$$S_T = S_0 e^{(r-q-\lambda(E^{\mathbb{Q}}[e^Y]-1))T + Y_T + Z_T}, \tag{3.3.1}$$

where $Z_t = \sum_{i=1}^{N_t} V_i$ is a compound Poisson process , and V_i are i.i.d. random variables that represent the jump size and follow a distribution f while N_t is a Poisson process with intensity λ that is independent from V_i.
As we have seen in Section 3.2, the Heston model under the risk neutral measure $\mathbb{Q}$ can be written as in equations (3.2.8) - (3.2.11).

We also need to know the model under the measure $\mathbb{Q}^S$. Using Theorem 1.8, and since the risk free interest rate is constant and we are considering dividends, we have:

$$\frac{\mathrm{d}\mathbb{Q}^S}{\mathrm{d}\mathbb{Q}}\bigg|\mathscr{F}_t = \frac{S_T}{S_t} e^{-(r-q)\tau}, \qquad \tau = T - t. \tag{3.3.2}$$

Using equation (3.0.4), $\frac{S_T}{S_t}$, can be written as:

$$\frac{S_T}{S_t} = e^{(r-q-\lambda(E^{\mathbb{Q}}[e^V]-1))\tau + Y_T - Y_t + Z_T - Z_t}. \tag{3.3.3}$$

Using equation (3.3.3), equation (3.3.2) becomes:

$$\left.\frac{\mathrm{d}\mathbb{Q}^S}{\mathrm{d}\mathbb{Q}}\right|\mathscr{F}_t = e^{(-\lambda E^{\mathbb{Q}}[e^V]+\lambda)\tau + Z_T - Z_t} e^{Y_T - Y_t}. \tag{3.3.4}$$

Using equation (3.2.5), equation (3.3.4) becomes:

$$\left.\frac{\mathrm{d}\mathbb{Q}^S}{\mathrm{d}\mathbb{Q}}\right|\mathscr{F}_t = e^{(-\lambda E^{\mathbb{Q}}[e^V]+\lambda)\tau + Z_T - Z_t} e^{\int_t^T \sqrt{v_s}\,\mathrm{dZ}_\mathrm{x}^{\mathbb{Q}}(\mathrm{s}) - \frac{1}{2}\int_t^T v_s\,\mathrm{ds}}. \tag{3.3.5}$$

Using Theorem 1.6, and Remark 1.3, it follows immediately that:

$$\begin{aligned}
&\mathrm{dZ}_\mathrm{x}^{\mathbb{Q}^\mathrm{S}}(\mathrm{t}) = \mathrm{dZ}_\mathrm{x}^{\mathbb{Q}}(\mathrm{t}) - \sqrt{v_t}\,\mathrm{dt};\\
&\widetilde{\lambda} = \lambda E^{\mathbb{Q}}\left(e^V\right);\\
&\widetilde{f}(\mathrm{V}_i) = \frac{e^{V_i} f(V_i)}{E^{\mathbb{Q}}\left(e^V\right)},
\end{aligned}$$

where $\widetilde{\lambda}$ and $\widetilde{f}(V_i)$ are the intensity and the jump size distribution, respectively, of the compound Poisson process under the measure $\mathbb{Q}^S$.
Then, and as shown in Section 3.2, the model under measure $\mathbb{Q}^S$ is given by equations (3.2.12) - (3.2.15), and the compound Poisson process has intensity $\widetilde{\lambda}$ and jump size distribution $\widetilde{f}(V_i)$.

Therefore, using Theorem 3.1, we get a generalization of the Heston (1993) model which includes the jump part of the process. Consider that for $j = 1$ we take the measure $\mathbb{Q}^S(\xi)$, and for $j = 2$ the measure $\mathbb{Q}(\xi)$.

Theorem 3.2. ***The Heston model with jumps as a time-changed Lévy process***

The Heston model with jumps can be written as a time-changed Lévy process, and its specification is as follows:

$$\begin{aligned}
&Y_t^j = X_{T_t}^j\\
&X_t^j = Z_x^j(t) + \mu^j t\\
&\mathrm{dv_t} = (k\theta - (\beta^j - \sigma\rho\imath\xi)v_t)\,\mathrm{dt} + \sigma\sqrt{v_t}\left(\rho\,\mathrm{dZ_x^j(t)} + \sqrt{1-\rho^2}\,\mathrm{dZ_v^j(t)}\right)\\
&\mathrm{d} < \mathrm{Z_x^j}, \mathrm{Z_v^j} >_\mathrm{t} = 0,
\end{aligned}$$

with

$$\beta^1 = k - \sigma\rho, \qquad \beta^2 = k, \qquad \mu^1 = \frac{1}{2}, \qquad \mu^2 = -\frac{1}{2},$$
$$\lambda_1 = \lambda E^{\mathbb{Q}}\left(e^V\right), \qquad \lambda_2 = \lambda, \qquad f_1(V_i) = \frac{e^{V_i} f(V_i)}{E^{\mathbb{Q}}\left(e^V\right)}, \qquad f_2(V_i) = f(V_i),$$

where λ is the intensity, and $f(V_i)$ is the jump size distribution of the initial compound Poisson process.
Then using equation (3.0.4), we can model the asset price as:

$$S_T = S_0\, e^{(r-q)T + Y_T^j}\, e^{(\lambda - \lambda E^{\mathbb{Q}}[e^V])T + Z_T^j}. \tag{3.3.6}$$

Denoting $s_T \equiv \log\left(\frac{S_T}{S_0}\right)$, and since the compound Poisson process is independent from the diffusion part, we can use equations (3.2.38) and (3.0.5) to obtain:

$$\phi_{s_T}^j(\xi) = e^{\imath\xi(r-q)T} e^{-b_j(T)v_0 - c_j(T)} e^{\imath\xi(\lambda - \lambda E^{\mathbb{Q}}[e^V])T} e^{T\lambda_j\left[\phi_V^j(\xi) - 1\right]}, \tag{3.3.7}$$

where $b_j(T)$ and $c_j(T)$ are given by equations (3.2.39) and (3.2.40), respectively.

■

3.3.1 Jumps with a Gaussian distribution

In this Section, we have deduced a generalization of Bates (1996), since we allowed the jump size distribution to be any distribution and not necessarily a Gaussian one. In fact, for each distribution that we chose for the jump size distribution, we will obtain a different model: Even though the diffusion part still remains the same, the discontinuous part will be different.
We will begin by deducing the Bates model. In this model, under the risk neutral measure $\mathbb{Q}$, the jump size distribution follows a normal law. Hence, for each $i \geq 1$, we have:

$$V_i \backsim \mathrm{N}\left(\log(1+\mu) - \frac{1}{2}\sigma^2; \sigma^2\right),$$

where V_i are i.i.d. random variables.

In order to compute the characteristic function in equation (3.3.7), we have to know

1. $E^{\mathbb{Q}}\left(e^V\right)$
2. $\widetilde{f}(V_i) = \frac{e^{V_i} f(V_i)}{E^{\mathbb{Q}}(e^V)}$.

In the first case, and since the characteristic function of a normal variable is well known, we have:

$$\begin{aligned} E^{\mathbb{Q}}\left(e^V\right) &= \phi_V(-\imath) \\ &= e^{\log(1+\mu)-\frac{1}{2}\sigma^2+\frac{1}{2}\sigma^2} \\ &= 1+\mu. \end{aligned}$$

In the second one, we have:

$$\widetilde{f}_{V_i}(x) = \frac{e^x f_{V_i}(x)}{E^{\mathbb{Q}}\left(e^V\right)} \tag{3.3.8}$$

$$= \frac{e^x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(x-\log(1+\mu)+\frac{1}{2}\sigma^2\right)^2}{2\sigma^2}}}{1+\mu}. \tag{3.3.9}$$

Notice that

$$-\frac{\left(x-\log(1+\mu)+\frac{1}{2}\sigma^2\right)^2}{2\sigma^2}+x$$

can be written as

$$-\frac{x^2-2x\log(1+\mu)+\sigma^2 x+\log(1+\mu)^2-\sigma^2\log(1+\mu)+\frac{1}{4}\sigma^2-2x\sigma^2}{2\sigma^2},$$

which is equivalent to

$$-\frac{x^2-2x\log(1+\mu)-\sigma^2 x+\log(1+\mu)^2+\sigma^2\log(1+\mu)+\frac{1}{4}\sigma^2-2\sigma^2\log(1+\mu)}{2\sigma^2},$$

that is the same as

$$-\frac{\left(x-\log(1+\mu)-\frac{1}{2}\sigma^2\right)^2}{2\sigma^2}+\log(1+\mu). \tag{3.3.10}$$

Using expression (3.3.10), equation (3.3.9) becomes:

$$
\begin{aligned}
\widetilde{f}_{V_i}(x) &= \frac{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(x-\log(1+\mu)-\frac{1}{2}\sigma^2\right)^2}{2\sigma^2}+\log(1+\mu)}}{1+\mu} \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(x-\log(1+\mu)-\frac{1}{2}\sigma^2\right)^2}{2\sigma^2}}.
\end{aligned}
$$

Then, we conclude that

$$
\widetilde{V_i} \backsim \mathrm{N}\left(\log(1+\mu)+\frac{1}{2}\sigma^2;\sigma^2\right).
$$

Now we are ready to compute $\phi_V^j(\xi)$, for both measures:

$$
\phi_V^{\mathbb{Q}(\xi)}(\xi) = e^{\imath\xi\log(1+\mu)+\frac{\imath\xi}{2}\sigma^2(\imath\xi-1)} = (1+\mu)^{\imath\xi} e^{\frac{\imath\xi}{2}\sigma^2(\imath\xi-1)}, \tag{3.3.11}
$$

$$
\phi_V^{\mathbb{Q}^S(\xi)}(\xi) = e^{\imath\xi\log(1+\mu)+\frac{\imath\xi}{2}\sigma^2(\imath\xi+1)} = (1+\mu)^{\imath\xi} e^{\frac{\imath\xi}{2}\sigma^2(\imath\xi+1)}. \tag{3.3.12}
$$

Therefore, we can write the characteristic function of the Heston model with Gaussian jumps, using equations (3.3.7), (3.3.11) and (3.3.12):

$$
\phi_{s_T}^j(\xi) = e^{\imath\xi(r-q)T} e^{-b_j(T)v_0-c_j(T)} e^{-\imath\xi\lambda\mu T} e^{T\lambda_j\left[(1+\mu)^{\imath\xi} e^{\frac{\imath\xi}{2}\sigma^2(\imath\xi+2\mu^j)}-1\right]}, \tag{3.3.13}
$$

where

$$
\mu^1 = \frac{1}{2}, \qquad \mu^2 = -\frac{1}{2}, \qquad \lambda_1 = \lambda(1+\mu), \qquad \lambda_2 = \lambda.
$$

$b_j(T)$ and $c_j(T)$, are still given by equations (3.2.39) and (3.2.40). Notice that equation (3.3.13) yields equations ($A12$) and ($A13$) from Bakshi et al. (1997, p. 2046), in a non-zero dividend yield scenario.

3.3.2 Jumps with a double exponential distribution

In this scenario, and under the risk neutral measure $\mathbb{Q}$, the jump size follows a double exponential distribution

$$f_V(x) = p\eta_1 e^{-\eta_1 x}\Pi_{x\geq 0} + (1-p)\eta_2 e^{\eta_2 x}\Pi_{x<0},$$

where $\eta_1 > 1$ and $\eta_2 > 0$. The requirement $\eta_1 > 1$ is needed to ensure that $E^{\mathbb{Q}}\left(e^V\right) < \infty$.
Its characteristic function is given by:

$$\begin{aligned}
\phi_V(\xi) \equiv E\left[e^{\imath\xi V}\right] &= \int_{\mathbb{R}} e^{\imath\xi x}\left(p\eta_1 e^{-\eta_1 x}\Pi_{x\geq 0} + (1-p)\eta_2 e^{\eta_2 x}\Pi_{x<0}\right)\,\mathrm{dx} \\
&= p\int_0^{+\infty} e^{\imath\xi x}\eta_1 e^{-\eta_1 x}\,\mathrm{dx} + (1-p)\int_{-\infty}^{0} e^{\imath\xi x}\eta_2 e^{\eta_2 x}\,\mathrm{dx}.
\end{aligned}$$

The first integral is the characteristic function of an exponential random variable. In the second one, a change of variables, $y = -x$, yields:

$$\begin{aligned}
\phi_V(\xi) &= p\frac{1}{1-\frac{\imath\xi}{\eta_1}} + (1-p)\int_{+\infty}^{0} -e^{-\imath\xi y}\eta_2 e^{-\eta_2 y}\,\mathrm{dy} \\
&= p\frac{\eta_1}{\eta_1 - \imath\xi} + (1-p)\int_0^{+\infty} e^{-\imath\xi y}\eta_2 e^{-\eta_2 y}\,\mathrm{dy} \\
&= p\frac{\eta_1}{\eta_1 - \imath\xi} + (1-p)\frac{1}{1+\frac{\imath\xi}{\eta_2}} \\
&= p\frac{\eta_1}{\eta_1 - \imath\xi} + (1-p)\frac{\eta_2}{\eta_2 + \imath\xi}.
\end{aligned}$$

Now we are ready to compute $E^{\mathbb{Q}}\left(e^V\right)$:

$$E^{\mathbb{Q}}\left(e^V\right) = \phi_V(-\imath) = p\frac{\eta_1}{\eta_1 - 1} + (1-p)\frac{\eta_2}{\eta_2 + 1}.$$

We can also compute $\widetilde{f}(V_i) = \frac{e^{V_i} f(V_i)}{E^{\mathbb{Q}}(e^V)}$:

$$\begin{aligned}
\widetilde{f}(V_i) &= \frac{e^{V_i} f(V_i)}{E^{\mathbb{Q}}\left(e^V\right)} \\
&= p\left(p\frac{\eta_1}{\eta_1-1}+(1-p)\frac{\eta_2}{\eta_2+1}\right)^{-1}\frac{\eta_1}{\eta_1-1}(\eta_1-1)e^{-(\eta_1-1)x}\Pi_{x\geq 0}+ \\
&+(1-p)\left(p\frac{\eta_1}{\eta_1-1}+(1-p)\frac{\eta_2}{\eta_2+1}\right)^{-1}\frac{\eta_2}{\eta_2+1}(\eta_2+1)e^{(\eta_2+1)x}\Pi_{x<0} \\
&= \widetilde{p}\widetilde{\eta_1}e^{-\widetilde{\eta_1}x}\Pi_{x\geq 0}+\widetilde{q}\widetilde{\eta_2}e^{\widetilde{\eta_2}x}\Pi_{x<0},
\end{aligned}$$

with

$$\begin{aligned}
\widetilde{p} &= p\left(p\frac{\eta_1}{\eta_1-1}+(1-p)\frac{\eta_2}{\eta_2+1}\right)^{-1}\frac{\eta_1}{\eta_1-1}, \qquad \widetilde{\eta_1}=\eta_1-1, \\
\widetilde{q} &= (1-p)\left(p\frac{\eta_1}{\eta_1-1}+(1-p)\frac{\eta_2}{\eta_2+1}\right)^{-1}\frac{\eta_2}{\eta_2+1}, \qquad \widetilde{\eta_2}=\eta_2+1,
\end{aligned}$$

and $\widetilde{p}+\widetilde{q}=1$.
Now we can compute $\phi_V^j(\xi)$, for both measures:

$$\phi_V^{\mathbb{Q}(\xi)}(\xi) = p\frac{\eta_1}{\eta_1-\iota\xi}+(1-p)\frac{\eta_2}{\eta_2+\iota\xi}, \tag{3.3.14}$$

$$\phi_V^{\mathbb{Q}^S(\xi)}(\xi) = \widetilde{p}\frac{\widetilde{\eta_1}}{\widetilde{\eta_1}-\iota\xi}+\widetilde{q}\frac{\widetilde{\eta_2}}{\widetilde{\eta_2}+\iota\xi}. \tag{3.3.15}$$

Therefore, we can write the characteristic function of the Heston model with double exponential jumps, using Theorem 3.2 and equations (3.3.14) and (3.3.15):

$$\phi_{s_T}^j(\xi) = e^{\iota\xi(r-q)T}e^{-b_j(T)v_0-c_j(T)}e^{\iota\xi\lambda\left[1-p\frac{\eta_1}{\eta_1-1}-(1-p)\frac{\eta_2}{\eta_2+1}\right]T}e^{T\lambda_j\left[\phi_V^j(\xi)-1\right]}, \tag{3.3.16}$$

where

$$\lambda_1 = \lambda\left(p\frac{\eta_1}{\eta_1-1}+(1-p)\frac{\eta_2}{\eta_2+1}\right), \qquad \lambda_2=\lambda.$$

This model is a generalization of Kou (2002), where the volatility is now a stochastic process.

Chapter 4

Asset pricing with stochastic interest rates

In this framework, the risk free interest rate is consider to be a stochastic process. We will only consider the case where the risk free interest rate follows a square root process as in Cox et al. (1985), but other processes can be considered as well — see for instance Vasicek (1977) and Hull and White (1990).
Therefore, the asset price is now a generalization of equation (3.0.4), and is given by:

$$S_T = S_0 e^{(-q-\lambda(E^{\mathbb{Q}}[e^V]-1))T+\int_0^T r_s\,\mathrm{ds}+Y_T+Z_T}, \tag{4.0.1}$$

with

$$\mathrm{dr_t} = k(\theta - r_t)\,\mathrm{dt} + \sigma\sqrt{r_t}\,\mathrm{dW_t^{\mathbb{Q}}}, \tag{4.0.2}$$

and

$$\frac{S_T}{e^{-qT+\int_0^T r_s\,\mathrm{ds}}},$$

is a martingale under the measure $\mathbb{Q}$.

We will assume that r_t is independent from Y_T and Z_T. In this case, the characteristic function of $s_T \equiv \log \frac{S_T}{S_0}$ is given by:

$$\phi_{s_T}(\xi) \equiv E^{\mathbb{Q}}\left[e^{\imath\xi s_T}\right] = E^{\mathbb{Q}}\left[e^{\imath\xi(-q-\lambda(E^{\mathbb{Q}}[e^V]-1))T+\imath\xi\int\limits_0^T r_s\,\mathrm{ds}+\imath\xi Y_T+\imath\xi Z_T}\right]$$

$$= e^{\imath\xi(-q-\lambda(E^{\mathbb{Q}}[e^V]-1))T} E^{\mathbb{Q}}\left[e^{\imath\xi\int\limits_0^T r_s\,\mathrm{ds}}\right] E^{\mathbb{Q}}\left[e^{\imath\xi Y_T}\right] E^{\mathbb{Q}}\left[e^{\imath\xi Z_T}\right].$$

We conclude that:

$$\phi_{s_T}(\xi) = e^{\imath\xi(-q-\lambda(E^{\mathbb{Q}}[e^V]-1))T} E^{\mathbb{Q}}\left[e^{\imath\xi\int\limits_0^T r_s\,\mathrm{ds}}\right] e^{T\lambda[\phi_V(\xi)-1]} \mathscr{L}_{T_T^v}^{\mathbb{Q}(\xi)}(\Psi_{x^v}(\xi)). \quad (4.0.3)$$

The expectation on the right-hand side of equation (4.0.3), can be viewed as the characteristic function of the following time-changed Lévy process:

$$\begin{aligned}
Y_t^r &= X_{T_t^r}^r, \\
T_t^r &:= \int\limits_0^t r(s)\,\mathrm{ds}, \\
X_t^r &= t, \\
\mathrm{dr_t} &= k(\theta - r_t)\,\mathrm{dt} + \sigma\sqrt{r_t}\,\mathrm{dW_t^{\mathbb{Q}}}.
\end{aligned}$$

Since T_T is independent from X_T, using Remark 2.1, we have

$$\phi_{Y_T}(\xi) \equiv E^{\mathbb{Q}}\left[e^{\imath\xi Y_T}\right] = E^{\mathbb{Q}}\left[e^{\imath\xi X_{T_T}}\right] = E^{\mathbb{Q}}\left[e^{\imath\xi\int\limits_0^T r_s\,\mathrm{ds}}\right]$$

$$= \mathscr{L}_{T_T^r}^{\mathbb{Q}}(\Psi_{x^r}(\xi)) = \mathscr{L}_{T_T^r}^{\mathbb{Q}}(-\imath\xi).$$

Therefore equation (4.0.3), can be written as:

$$\phi_{s_T}(\xi) = e^{\imath\xi(-q-\lambda(E^{\mathbb{Q}}[e^V]-1))T} \mathscr{L}_{T_T^r}^{\mathbb{Q}}(-\imath\xi) e^{T\lambda[\phi_V(\xi)-1]} \mathscr{L}_{T_T^v}^{\mathbb{Q}(\xi)}(\Psi_{x^v}(\xi)). \quad (4.0.4)$$

4.1 Zero coupon bond

We define the time-t price of a T-maturity zero coupon bond as

$$P(t,T) := E^{\mathbb{Q}}\left[e^{-\int_t^T r_s \, \mathrm{ds}} \middle| \mathscr{F}_t \right]. \tag{4.1.1}$$

Applying the Feynman-Kac Theorem 1.10, $P(t,T)$ is the solution of the following PDE:

$$\frac{\partial P(t,T)}{\partial t} + k(\theta - r_t)\frac{\partial P(t,T)}{\partial r_t} + \frac{1}{2}\sigma^2 r_t \frac{\partial^2 P(t,T)}{\partial r_t^2} - r_t P(t,T) = 0, \tag{4.1.2}$$

with $P(T,T) = 1$.

Since the drift and the instantaneous variance of the process r_t are affine functions of the state variables, the price of the zero coupon bond is of the form

$$P(t,T) = e^{A(\tau)+B(\tau)r_t}, \qquad \tau = T - t. \tag{4.1.3}$$

Using equations (4.1.3) and (4.1.2), we obtain the ODEs for $A(\tau)$ and $B(\tau)$

$$\frac{\partial B(\tau)}{\partial \tau} = -1 - kB(\tau) + \frac{1}{2}\sigma^2 B(\tau)^2 \tag{4.1.4}$$

$$\frac{\partial A(\tau)}{\partial \tau} = k\theta B(\tau), \tag{4.1.5}$$

with $A(0) = B(0) = 0$.
Using Proposition D.1 and some algebra, we have:

$$B(\tau) = \frac{2\left(1 - e^{\Delta\tau}\right)}{2\Delta + (k+\Delta)\left(e^{\Delta\tau} - 1\right)}, \tag{4.1.6}$$

$$A(\tau) = -\frac{k\theta}{\sigma^2}\left((-\Delta - k)\tau + 2\log\left[1 + \frac{(k+\Delta)\left(e^{\Delta\tau} - 1\right)}{2\Delta} \right] \right), \tag{4.1.7}$$

$$\Delta = \sqrt{k^2 + 2\sigma^2}. \tag{4.1.8}$$

The dynamics of a T-maturity zero coupon bond price can be easily obtained using equation (4.0.2) and Ito's Lemma:

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = \alpha(t,T)\,\mathrm{dt} + \sigma(t,T)\,\mathrm{dW}_{\mathrm{t}}^{\mathbb{Q}}, \tag{4.1.9}$$

where

$$\alpha(t,T) = \frac{1}{P(t,T)}\left[\frac{\partial P(t,T)}{\partial t} + k(\theta - r_t)\frac{\partial P(t,T)}{\partial r_t} + \frac{1}{2}\sigma^2 r_t \frac{\partial^2 P(t,T)}{\partial r_t^2}\right], \qquad (4.1.10)$$

and

$$\sigma(t,T) = \frac{1}{P(t,T)}\sigma\sqrt{r_t}\frac{\partial P(t,T)}{\partial r_t} = \frac{1}{P(t,T)}\sigma\sqrt{r_t}B(\tau)P(t,T) = \sigma\sqrt{r_t}B(\tau). \qquad (4.1.11)$$

It is of main importance, for the purpose of option pricing, to know the dynamics of r_t under the measure $\mathbb{Q}^T$ defined in Remark 1.6. Therefore,

Remark 4.1. *From equation* (1.2.23)*, we have*

$$\mathrm{dW_t^{\mathbb{Q}^T}} = \mathrm{dW_t^{\mathbb{Q}}} - \frac{\mathrm{dP(t,T)}}{P(t,T)}\,\mathrm{dW_t^{\mathbb{Q}}}.$$

Using equations (4.1.9) *and* (4.1.11)*, it follows that*

$$\mathrm{dW_t^{\mathbb{Q}^T}} = \mathrm{dW_t^{\mathbb{Q}}} - \sigma(t,T)\,\mathrm{dt} = \mathrm{dW_t^{\mathbb{Q}}} - \sigma\sqrt{r_t}B(\tau)\,\mathrm{dt}.$$

Therefore the stochastic differential equation followed by r_t, under the measure $\mathbb{Q}^T$ is:

$$\mathrm{dr_t} = k(\theta - r_t)\,\mathrm{dt} + \sigma\sqrt{r_t}\left(\mathrm{dW_t^{\mathbb{Q}^T}} + \sigma\sqrt{r_t}B(\tau)\,\mathrm{dt}\right) \qquad (4.1.12)$$

$$= \left(k\theta - \left(k - \sigma^2 B(\tau)\right)r_t\right)\,\mathrm{dt} + \sigma\sqrt{r_t}\,\mathrm{dW_t^{\mathbb{Q}^T}}. \qquad (4.1.13)$$

For a detailed reading about this topic see Bjork (1998) and Brigo and Mercurio (2006).

4.2 The Bakshi, Cao and Chen (1997) model as a time changed Lévy process

Consider the following time changed Lévy process under the risk neutral measure $\mathbb{Q}$:

1. $Y_t = Y_t^v + Y_t^r$,
2. $Y_t^v = X^v_{T_t{}^v}$,
3. $Y_t^r = X^r_{T_t{}^r}$,
4. $X_t^v = W_x^{\mathbb{Q}}(t) + \mu t$,
5. $X_t^r = t$,
6. $\mathrm{dv_t} = k_v(\theta_v - v_t)\,\mathrm{dt} + \sigma_v\sqrt{v_t}\,\mathrm{dW_v^{\mathbb{Q}}(t)}$,
7. $\mathrm{dr_t} = k_r(\theta_r - r_t)\,\mathrm{dt} + \sigma_r\sqrt{r_t}\,\mathrm{dW_r^{\mathbb{Q}}(t)}$,
8. $\mathrm{d} < \mathrm{W_x^{\mathbb{Q}}}, \mathrm{W_v^{\mathbb{Q}}} >_{\mathrm{t}} = \rho\,\mathrm{dt}$,
9. $\mathrm{d} < \mathrm{W_x^{\mathbb{Q}}}, \mathrm{W_r^{\mathbb{Q}}} >_{\mathrm{t}} = 0$,
10. $\mathrm{d} < \mathrm{W_v^{\mathbb{Q}}}, \mathrm{W_r^{\mathbb{Q}}} >_{\mathrm{t}} = 0$,

with

$$T_t^v := \int_0^t v(s)\,\mathrm{ds}, \tag{4.2.1}$$

$$T_t^r := \int_0^t r(s)\,\mathrm{ds}. \tag{4.2.2}$$

From Theorem 1.2, and using Remark 1.1, we know that given three independent Brownian motions, $Z_x^{\mathbb{Q}}$, $Z_v^{\mathbb{Q}}$ and $Z_r^{\mathbb{Q}}$:

$$W_x^{\mathbb{Q}}(t) = Z_x^{\mathbb{Q}}(t), \tag{4.2.3}$$

$$W_v^{\mathbb{Q}}(t) = \rho Z_x^{\mathbb{Q}}(t) + \sqrt{1-\rho^2}Z_v^{\mathbb{Q}}(t), \tag{4.2.4}$$

$$W_r^{\mathbb{Q}}(t) = Z_r^{\mathbb{Q}}(t). \tag{4.2.5}$$

Under the risk neutral measure $\mathbb{Q}$, we have

$$\begin{aligned} S_T &= S_0 e^{(-q-\lambda(E^{\mathbb{Q}}[e^V]-1))T+Y_T+Z_T} \\ &= S_0 e^{(-q-\lambda(E^{\mathbb{Q}}[e^V]-1))T+Y_t^v+Y_t^r+Z_T}, \end{aligned} \tag{4.2.6}$$

and

$$\frac{S_T}{e^{-qT+Y_t^r}}$$

is a martingale under measure $\mathbb{Q}$.
Then using equations (3.2.8) - (3.2.11), we conclude that under measure $\mathbb{Q}$, the time changed Lévy process is:

$$\begin{aligned} &Y_t = Y_t^v + Y_t^r \\ &Y_t^v = X_{T_t{}^v}^v \\ &Y_t^r = X_{T_t{}^r}^r \\ &X_t^v = Z_x^{\mathbb{Q}}(t) - \frac{1}{2}t \\ &X_t^r = t \\ &\mathrm{dv_t} = k_v(\theta_v - v_t)\,\mathrm{dt} + \sigma_v\sqrt{v_t}\left(\rho\,\mathrm{dZ_x^{\mathbb{Q}}(t)} + \sqrt{1-\rho^2}\,\mathrm{dZ_v^{\mathbb{Q}}(t)}\right) \\ &\mathrm{dr_t} = k_r(\theta_r - r_t)\,\mathrm{dt} + \sigma_r\sqrt{r_t}\,\mathrm{dZ_r^{\mathbb{Q}}(t)} \end{aligned}$$

Under the measure $\mathbb{Q}^S$, and using the fact that if S_T is given by equation (4.2.6), then equations (1.2.24) and (3.3.5) are equal, by Theorem 1.6 and Remark 1.3, it follows immediately that:

$$\begin{aligned} &\mathrm{dZ_x^{\mathbb{Q}^S}(t)} = \mathrm{dZ_x^{\mathbb{Q}}(t)} - \sqrt{v_t}\,\mathrm{dt}, \\ &\mathrm{dZ_v^{\mathbb{Q}^S}(t)} = \mathrm{dZ_v^{\mathbb{Q}}(t)}, \\ &\mathrm{dZ_r^{\mathbb{Q}^S}(t)} = \mathrm{dZ_r^{\mathbb{Q}}(t)}, \\ &\widetilde{\lambda} = \lambda E^{\mathbb{Q}}\left(e^V\right), \\ &\widetilde{f}(\mathrm{V}_i) = \frac{e^{V_i} f(\mathrm{V}_i)}{E^{\mathbb{Q}}\left(e^V\right)}, \end{aligned}$$

where $\widetilde{\lambda}$ and $\widetilde{f}(V_i)$ are the intensity and the jump size distribution, respectively, of the compound Poisson process under the measure $\mathbb{Q}^S$.

Under measure $\mathbb{Q}^T$, defined in Remark 1.6, we have already seen in Remark 4.1 that:

$$\mathrm{dZ}_\mathrm{r}^{\mathbb{Q}^\mathrm{T}}(\mathrm{t}) = \mathrm{dZ}_\mathrm{r}^{\mathbb{Q}}(\mathrm{t}) - \sigma_r \sqrt{r_t} B(\tau)\,\mathrm{dt};$$
$$\mathrm{dZ}_\mathrm{x}^{\mathbb{Q}^\mathrm{T}}(\mathrm{t}) = \mathrm{dZ}_\mathrm{x}^{\mathbb{Q}}(\mathrm{t});$$
$$\mathrm{dZ}_\mathrm{v}^{\mathbb{Q}^\mathrm{T}}(\mathrm{t}) = \mathrm{dZ}_\mathrm{v}^{\mathbb{Q}}(\mathrm{t}).$$

Using the fact that T_t^r is independent from X_t^r, we know that:

$$\mathrm{dZ}_\mathrm{r}^{\mathbb{Q}^{\mathrm{S}(\xi)}}(\mathrm{t}) = \mathrm{dZ}_\mathrm{r}^{\mathbb{Q}^\mathrm{S}}(\mathrm{t});$$
$$\mathrm{dZ}_\mathrm{r}^{\mathbb{Q}^{\mathrm{T}(\xi)}}(\mathrm{t}) = \mathrm{dZ}_\mathrm{r}^{\mathbb{Q}^\mathrm{T}}(\mathrm{t}).$$

Now we are ready to write the model under the measures $\mathbb{Q}^T(\xi)$ and $\mathbb{Q}^S(\xi)$. Using equations (3.2.18) - (3.2.27), we have, under measure $\mathbb{Q}^T(\xi)$:

$$\phi_{Y_T}(\xi) = \mathscr{L}_{T_T^v}^{\mathbb{Q}^{T(\xi)}}(\Psi_{x^v}(\xi))\,\mathscr{L}_{T_T^r}^{\mathbb{Q}^{T(\xi)}}(\Psi_{x^r}(\xi)) = \mathscr{L}_{T_T^v}^{\mathbb{Q}^{T(\xi)}}\left(\frac{1}{2}\imath\xi + \frac{1}{2}\xi^2\right)\mathscr{L}_{T_T^r}^{\mathbb{Q}^{T}}(-\imath\xi) \tag{4.2.7}$$

$$Y_t = Y_t^v + Y_t^r \tag{4.2.8}$$

$$Y_t^v = X_{T_t{}^v}^v \tag{4.2.9}$$

$$Y_t^r = X_{T_t{}^r}^r \tag{4.2.10}$$

$$X_t^v = Z_x^{\mathbb{Q}^T}(t) - \frac{1}{2}t \tag{4.2.11}$$

$$X_t^r = t \tag{4.2.12}$$

$$\mathrm{dv_t} = (k_v\theta_v - (k_v - \sigma_v\rho\imath\xi)v_t)\,\mathrm{dt} + \sigma_v\sqrt{v_t}\left(\rho\,\mathrm{dZ}_\mathrm{x}^{\mathbb{Q}^{\mathrm{T}(\xi)}}(\mathrm{t}) + \sqrt{1-\rho^2}\,\mathrm{dZ}_\mathrm{v}^{\mathbb{Q}^{\mathrm{T}(\xi)}}(\mathrm{t})\right) \tag{4.2.13}$$

$$\mathrm{dr_t} = \left(k_r\theta_r - \left(k_r - \sigma_r^2 B(\tau)\right)r_t\right)\,\mathrm{dt} + \sigma_r\sqrt{r_t}\,\mathrm{dZ}_\mathrm{r}^{\mathbb{Q}^\mathrm{T}}(\mathrm{t}). \tag{4.2.14}$$

Under measure $\mathbb{Q}^S(\xi)$:

$$\phi_{Y_T}(\xi) = \mathscr{L}_{T_T^v}^{\mathbb{Q}^{S(\xi)}}(\Psi_{x^v}(\xi))\,\mathscr{L}_{T_T^r}^{\mathbb{Q}^{S(\xi)}}(\Psi_{x^r}(\xi)) = \mathscr{L}_{T_T^v}^{\mathbb{Q}^{S(\xi)}}\left(-\frac{1}{2}\imath\xi + \frac{1}{2}\xi^2\right)\mathscr{L}_{T_T^r}^{\mathbb{Q}^{S}}(-\imath\xi) \tag{4.2.15}$$

$$Y_t = Y_t^v + Y_t^r \tag{4.2.16}$$
$$Y_t^v = X_{T_t{}^v}^v \tag{4.2.17}$$
$$Y_t^r = X_{T_t{}^r}^r \tag{4.2.18}$$
$$X_t^v = Z_x^{\mathbb{Q}^S}(t) + \frac{1}{2}t \tag{4.2.19}$$
$$X_t^r = t \tag{4.2.20}$$
$$\mathrm{dv_t} = (k_v\theta_v - (k_v - \sigma_v\rho - \sigma_v\rho\imath\xi)v_t)\,\mathrm{dt} + \sigma_v\sqrt{v_t}\left(\rho\,\mathrm{dZ_x^{\mathbb{Q}^{S(\xi)}}(t)} + \sqrt{1-\rho^2}\,\mathrm{dZ_v^{\mathbb{Q}^{S(\xi)}}(t)}\right) \tag{4.2.21}$$
$$\mathrm{dr_t} = k_r(\theta_r - r_t)\,\mathrm{dt} + \sigma_r\sqrt{r_t}\,\mathrm{dZ_r^{\mathbb{Q}^S}(t)}. \tag{4.2.22}$$

Since $\mathscr{L}_{T_T^v}^{\mathbb{Q}^{T(\xi)}}\left(\frac{1}{2}\imath\xi + \frac{1}{2}\xi^2\right)$ and $\mathscr{L}_{T_T^v}^{\mathbb{Q}^{S(\xi)}}\left(-\frac{1}{2}\imath\xi + \frac{1}{2}\xi^2\right)$ are already given by equation (3.2.28), we only need to determine $\mathscr{L}_{T_T^r}^{\mathbb{Q}^T}(-\imath\xi)$ and $\mathscr{L}_{T_T^r}^{\mathbb{Q}^S}(-\imath\xi)$.

In order to determine $\mathscr{L}_{T_T^r}^{\mathbb{Q}^S}(-\imath\xi)$, and since the dynamics of the process r_t under this measure is given by equation (4.2.22), using Proposition 2.1, we have:

$$\mathscr{L}_{T_T^r}^{\mathbb{Q}^S}(-\imath\xi) = e^{-b(T)r_0 - c(T)}, \tag{4.2.23}$$

with

$$b'(t) = -\imath\xi - k_r b(t) - \frac{1}{2}\sigma_r^2 b(t)^2,$$
$$c'(t) = k_r\theta_r b(t),$$

and

$$b(0) = c(0) = 0.$$

Using Proposition D.1, $b(t)$ and $c(t)$ are given by:

$$b(t) = -\frac{-k_r - \Delta}{-\sigma_r^2}\frac{1 - e^{\Delta t}}{1 - e^{\Delta t}\Lambda}, \tag{4.2.24}$$
$$c(t) = \frac{k_r\theta_r}{\sigma_r^2}\left[(-k_r - \Delta)t + 2\log\left(\frac{1 - \Lambda e^{\Delta t}}{1 - \Lambda}\right)\right], \tag{4.2.25}$$
$$\Delta = \sqrt{k_r^2 - 2\sigma_r^2\imath\xi}, \tag{4.2.26}$$

$$\Lambda = \frac{-k_r - \Delta}{-k_r + \Delta}. \tag{4.2.27}$$

We can simplify equation (4.2.24). Using equation (4.2.27), we obtain:

$$\begin{aligned}
-\frac{-k_r - \Delta}{-\sigma_r^2} \frac{1 - e^{\Delta t}}{1 - e^{\Delta t}\Lambda} &= -\frac{-k_r - \Delta}{-\sigma_r^2} \frac{1 - e^{\Delta t}}{1 - e^{\Delta t} \frac{-k_r - \Delta}{-k_r + \Delta}} \\
&= \frac{\left(1 - e^{\Delta t}\right)(-k_r + \Delta)(k_r + \Delta)}{-\sigma_r^2 \left(2\Delta - k_r - \Delta - e^{\Delta t}(-k_r - \Delta)\right)} \\
&= \frac{2\imath\xi \left(1 - e^{\Delta t}\right)}{2\Delta + (-k_r - \Delta)\left(1 - e^{\Delta t}\right)}.
\end{aligned}$$

Expression (4.2.25) can be also simplified, because:

$$\begin{aligned}
\frac{1 - \Lambda e^{\Delta t}}{1 - \Lambda} &= \frac{1 - \frac{-k_r - \Delta}{-k_r + \Delta} e^{\Delta t}}{1 - \frac{-k_r - \Delta}{-k_r + \Delta}} \\
&= \frac{-k_r + \Delta + (k_r + \Delta)e^{\Delta t}}{2\Delta} \\
&= \frac{2\Delta + (k_r + \Delta)\left(e^{\Delta t} - 1\right)}{2\Delta} \\
&= 1 + \frac{(k_r + \Delta)\left(e^{\Delta t} - 1\right)}{2\Delta}.
\end{aligned}$$

Therefore, $b(t)$ and $c(t)$ can be written as:

$$b(t) = \frac{2\imath\xi \left(1 - e^{\Delta t}\right)}{2\Delta + (-k_r - \Delta)\left(1 - e^{\Delta t}\right)}, \tag{4.2.28}$$

$$c(t) = \frac{k_r \theta_r}{\sigma_r^2} \left[(-k_r - \Delta)t + 2\log\left(1 + \frac{(k_r + \Delta)\left(e^{\Delta t} - 1\right)}{2\Delta}\right)\right]. \tag{4.2.29}$$

To determine $\mathscr{L}_{T^r_T}^{\mathbb{Q}^T}(-\imath\xi)$, and since the dynamics of the process r_t under this measure is given by equation (4.2.14), consider the following trial solution to the characteristic function of Y_t^r:

$$\phi_{Y_T}(\xi,t) = E^{\mathbb{Q}^T}\left[e^{\imath\xi\int_t^T r_s\,\mathrm{ds}} \middle| \mathscr{F}_t \right] = e^{[-B(\tau)+b(\tau)]r_t+a(\tau)}, \tag{4.2.30}$$

$$\phi_{Y_T}(\xi,T) = 1, \tag{4.2.31}$$

$$\phi_{Y_T}(\xi,0) = \mathscr{L}^{\mathbb{Q}^T}_{T^r_T}(-\imath\xi), \tag{4.2.32}$$

where $B(\tau)$ is given by equation (4.1.6).
Applying the Feynman-Kac Theorem to equation (4.2.30), we have:

$$\frac{\partial \phi_{Y_T}}{\partial t} + \left(k_r\theta_r - \left(k_r - \sigma_r^2 B(\tau)\right) r_t\right)\frac{\partial \phi_{Y_T}}{\partial r_t} + \frac{1}{2}\sigma_r^2 r_t \frac{\partial^2 \phi_{Y_T}}{\partial r_t^2} + \imath\xi r_t \phi_{Y_T} = 0, \tag{4.2.33}$$

which can be restated as:

$$\begin{aligned}
&\left[\left(\frac{\partial B}{\partial \tau} - \frac{\partial b}{\partial \tau}\right) r_t - \frac{\partial a}{\partial \tau}\right]\phi_{Y_T} + (b(\tau) - B(\tau))\phi_{Y_T}\left(k_r\theta_r - \left(k_r - \sigma_r^2 B(\tau)\right) r_t\right)\\
&+ \frac{1}{2}\sigma_r^2 r_t (b(\tau) - B(\tau))^2 \phi_{Y_T} + \imath\xi r_t \phi_{Y_T} = 0.
\end{aligned}$$

We can rearrange the terms of the PDE as follows:

$$\begin{aligned}
&r_t\left[\frac{\partial B}{\partial \tau} - \frac{\partial b}{\partial \tau} - (b(\tau) - B(\tau))\left(k_r - \sigma_r^2 B(\tau)\right) + \frac{1}{2}\sigma_r^2 (b(\tau) - B(\tau))^2 + \imath\xi\right]\\
&- \frac{\partial a}{\partial \tau} + (b(\tau) - B(\tau)) k_r\theta_r = 0,
\end{aligned}$$

which leads us to the following two ODEs

$$\begin{aligned}
&\frac{\partial B}{\partial \tau} - \frac{\partial b}{\partial \tau} - b(\tau)k_r + b(\tau)\sigma_r^2 B(\tau) + B(\tau)k_r - \sigma_r^2 B(\tau)^2\\
&+ \frac{1}{2}\sigma_r^2 b(\tau)^2 - \sigma_r^2 B(\tau) b(\tau) + \frac{1}{2}\sigma_r^2 B(\tau)^2 + \imath\xi = 0,\\
&\frac{\partial a}{\partial \tau} = b(\tau)k_r\theta_r - B(\tau)k_r\theta_r.
\end{aligned}$$

Using equation (4.1.4), we have:

$$\frac{\partial b}{\partial \tau} = \imath\xi - 1 - k_r b(\tau) + \frac{1}{2}\sigma_r^2 b(\tau)^2; \tag{4.2.34}$$

$$\frac{\partial a}{\partial \tau} = b(\tau)k_r\theta_r - B(\tau)k_r\theta_r, \tag{4.2.35}$$

with $a(0) = b(0) = 0$.

Using Proposition D.1, and following the same procedure we used in equation (4.2.28), $b(\tau)$ can be written as:

$$b(\tau) = \frac{2(\imath\xi - 1)\left(1 - e^{\Delta^T \tau}\right)}{-2\Delta^T - (-k_r - \Delta^T)\left(1 - e^{\Delta^T \tau}\right)}, \tag{4.2.36}$$

with

$$\Delta^T = \sqrt{k_r^2 - 2\sigma^2(\imath\xi - 1)}. \tag{4.2.37}$$

From Proposition D.1, and using the same procedure as in equation (4.2.29), we can compute $a(\tau)$ as follows:

$$a(\tau) = -\frac{k_r\theta_r}{\sigma_r^2}\left[\left(-k_r - \Delta^T\right)\tau + 2\log\left(1 + \frac{(k_r + \Delta^T)\left(e^{\Delta^T \tau} - 1\right)}{2\Delta^T}\right)\right] - A(\tau), \tag{4.2.38}$$

where $A(\tau)$ is given by equation (4.1.7).
We conclude that $\phi_{Y_T}(\xi, t)$ is given by:

$$\begin{aligned}
\phi_{Y_T}(\xi, t) &= e^{[-B(\tau) + b(\tau)]r_t + a(\tau)} && (4.2.39)\\
&= e^{b(\tau)r_t + a_1(\tau) - B(\tau)r_t - A(\tau)} && (4.2.40)\\
&= e^{b(\tau)r_t + a_1(\tau) - \log[P(t,T)]}, && (4.2.41)
\end{aligned}$$

where $P(t,T)$ is given by equation (4.1.3), and

$$a_1(\tau) = -\frac{k_r\theta_r}{\sigma_r^2}\left[\left(-k_r - \Delta^T\right)\tau + 2\log\left(1 + \frac{(k_r + \Delta^T)\left(e^{\Delta^T \tau} - 1\right)}{2\Delta^T}\right)\right]. \tag{4.2.42}$$

We are now ready to determine $\mathscr{L}^{\mathbb{Q}^T}_{T^r_T}(-\imath\xi)$, which is equal to

$$\mathscr{L}^{\mathbb{Q}^T}_{T^r_T}(-\imath\xi) = \phi_{Y_T}(\xi, 0) = e^{b(T)r_0 + a_1(T) - \log[P(0,T)]}. \tag{4.2.43}$$

We can resume all the information above, in the following theorem. Consider again that for $j = 1$ we take the measure $\mathbb{Q}^S(\xi)$, and for $j = 2$ the measure $\mathbb{Q}^T(\xi)$.

Theorem 4.1. ***The Bakshi, Cao and Chen (1997) model as a time-changed Lévy process***
The Bakshi, Cao and Chen (1997) model, with an arbitrary distribution for the jump size, can be written as a time-changed Lévy process, and its specification is as follows:

$$
\begin{aligned}
Y_t &= Y_t^v + Y_t^r \\
Y_t^v &= X^v_{T_t{}^v} \\
Y_t^r &= X^r_{T_t{}^r} \\
X_t^v &= Z_x^j(t) + \mu^j t \\
X_t^r &= t \\
\mathrm{dv_t} &= (k_v\theta_v - (\beta^j - \sigma_v\rho\imath\xi)v_t)\,\mathrm{dt} + \sigma_v\sqrt{v_t}\left(\rho\,\mathrm{dZ_x^j(t)} + \sqrt{1-\rho^2}\,\mathrm{dZ_v^j(t)}\right) \\
\mathrm{dr_t} &= \left(k_r\theta_r - \left(k_r - \alpha^j\right)r_t\right)\,\mathrm{dt} + \sigma_r\sqrt{r_t}\,\mathrm{dZ_r^j(t)}
\end{aligned}
$$

with

$$
\beta^1 = k_v - \sigma_v\rho, \qquad \beta^2 = k_v, \qquad \mu^1 = \frac{1}{2}, \qquad \mu^2 = -\frac{1}{2}, \qquad \alpha^1 = 0, \qquad \alpha^2 = \sigma_r^2 B(\tau),
$$

$$
\lambda_1 = \lambda E^{\mathbb{Q}}\left(e^V\right), \qquad \lambda_2 = \lambda, \qquad f_1(V_i) = \frac{e^{V_i} f(V_i)}{E^{\mathbb{Q}}\left(e^V\right)}, \qquad f_2(V_i) = f(V_i),
$$

where λ is the intensity and $f(V_i)$ is the jump size distribution of the initial compound Poisson process.
Then using equation (4.0.1)*, we can model the asset price as*

$$
S_T = S_0 e^{(-q-\lambda(E^{\mathbb{Q}}[e^V]-1))T + Y_T^j + Z_T^j}, \tag{4.2.44}
$$

and the characteristic function of $s_T \equiv \log\left(\frac{S_T}{S_0}\right)$ given by equation (4.0.4) *can be written as:*

$$
\phi^j_{s_T}(\xi) = e^{\imath\xi(-q-\lambda(E^{\mathbb{Q}}[e^V]-1))T}\,\mathscr{L}^j_{T_T^r}(-\imath\xi)\,e^{T\lambda\left[\phi_V^j(\xi)-1\right]}\,\mathscr{L}^j_{T_T^v}(\Psi_{x^v}(\xi)). \tag{4.2.45}
$$

Where $\mathscr{L}^j_{T_T^r}(-\imath\xi)$ is given by equations (4.2.23) *and* (4.2.43) *and $\mathscr{L}^j_{T_T^v}(\Psi_{x^v}(\xi))$ is given by equation* (3.2.28)*.*

Remark 4.2. ***The Bakshi, Cao and Chen (1997) model with Gaussian Jumps***

From Theorem 4.1, and using equations (3.3.13), (4.2.45) *and* (4.2.23), *the characteristic function of* s_T *under measure* $\mathbb{Q}^S(\xi)$ *is given by:*

$$\begin{aligned}
\phi_{s_T}^{\mathbb{Q}^S(\xi)}(\xi) = \exp\Bigg[& -\frac{k_r\theta_r}{\sigma_r^2}\left[(-k_r-\Delta_r)T + 2\log\left(1+\frac{(k_r+\Delta_r)\left(e^{\Delta_r T}-1\right)}{2\Delta_r}\right)\right] \\
& - \frac{2\imath\xi\left(1-e^{\Delta_r T}\right)}{2\Delta_r + (-k_r-\Delta_r)\left(1-e^{\Delta_r T}\right)} r_0 \\
& - \frac{\imath\xi\rho\sigma_v - k_v + \sigma_v\rho - \Delta_v}{\sigma_v^2}\frac{1-e^{\Delta_v T}}{1-e^{\Delta_v T}\Lambda} v_0 \\
& - \frac{k_v\theta_v}{\sigma_v^2}\left[\left(\imath\xi\rho\sigma_v - k_v + \sigma_v\rho - \Delta_v\right)T + 2\log\left(\frac{1-\Lambda e^{\Delta_v T}}{1-\Lambda}\right)\right] \\
& - \imath\xi\lambda\mu T + T\lambda(1+\mu)\left[(1+\mu)^{\imath\xi} e^{\frac{\imath\xi}{2}\sigma^2(\imath\xi+1)} - 1\right] - \imath\xi q T\Bigg],
\end{aligned}$$

with

$$\begin{aligned}
\Delta_r &= \sqrt{k_r^2 - 2\sigma_r^2\imath\xi}, \\
\Delta_v &= \sqrt{(\imath\xi\rho\sigma_v - k_v + \sigma_v\rho)^2 + \sigma_v^2(\xi^2 - \imath\xi)}, \\
\Lambda &= \frac{\imath\xi\rho\sigma_v - k_v + \sigma_v\rho - \Delta_v}{\imath\xi\rho\sigma_v - k_v + \sigma_v\rho + \Delta_v}.
\end{aligned}$$

Under measure $\mathbb{Q}^T(\xi)$*, the characteristic function of* s_T*, is given by:*

$$\begin{aligned}
\phi_{s_T}^{\mathbb{Q}^T(\xi)}(\xi) = \exp\Bigg[& -\frac{k_r\theta_r}{\sigma_r^2}\left[\left(-k_r-\Delta_r^T\right)T + 2\log\left(1+\frac{(k_r+\Delta_r^T)\left(e^{\Delta_r^T T}-1\right)}{2\Delta_r^T}\right)\right] \\
& + \frac{2(\imath\xi-1)\left(1-e^{\Delta_r^T T}\right)}{-2\Delta_r^T - (-k_r-\Delta_r^T)\left(1-e^{\Delta_r^T T}\right)} r_0 - \log[P(0,T)] \\
& - \frac{\imath\xi\rho\sigma_v - k_v - \Delta_v^T}{\sigma_v^2}\frac{1-e^{\Delta_v^T T}}{1-e^{\Delta_v^T T}\Lambda^T} v_0 \\
& - \frac{k_v\theta_v}{\sigma_v^2}\left[\left(\imath\xi\rho\sigma_v - k_v - \Delta_v^T\right)T + 2\log\left(\frac{1-\Lambda^T e^{\Delta_v^T T}}{1-\Lambda^T}\right)\right]
\end{aligned}$$

$$-\imath\xi\lambda\mu T + T\lambda\left[(1+\mu)^{\imath\xi}e^{\frac{\imath\xi}{2}\sigma^2(\imath\xi-1)}-1\right]-\imath\xi qT\Bigg],$$

with

$$\begin{aligned}
\Delta_r^T &= \sqrt{k_r^2-2\sigma_r^2(\imath\xi-1)},\\
\Delta_v^T &= \sqrt{(\imath\xi\rho\sigma_v-k_v)^2+\sigma_v^2(\xi^2+\imath\xi)},\\
\Lambda^T &= \frac{\imath\xi\rho\sigma_v-k_v-\Delta_v^T}{\imath\xi\rho\sigma_v-k_v+\Delta_v^T}.
\end{aligned}$$

Remark 4.3. ***The Bakshi, Cao and Chen (1997) model with double exponential Jumps***
From Theorem 4.1, and using equations (3.3.16)*,* (4.2.45) *and* (4.2.23)*, the characteristic function of s_T under measure $\mathbb{Q}^S(\xi)$ is given by:*

$$\begin{aligned}
\phi_{s_T}^{\mathbb{Q}^S(\xi)}(\xi) = \exp\Bigg[&-\frac{k_r\theta_r}{\sigma_r^2}\left[(-k_r-\Delta_r)T+2\log\left(1+\frac{(k_r+\Delta_r)\left(e^{\Delta_r T}-1\right)}{2\Delta_r}\right)\right]\\
&-\frac{2\imath\xi\left(1-e^{\Delta_r T}\right)}{2\Delta_r+(-k_r-\Delta_r)\left(1-e^{\Delta_r T}\right)}r_0\\
&-\frac{\imath\xi\rho\sigma_v-k_v+\sigma_v\rho-\Delta_v}{\sigma_v^2}\frac{1-e^{\Delta_v T}}{1-e^{\Delta_v T}\Lambda}v_0\\
&-\frac{k_v\theta_v}{\sigma_v^2}\left[\left(\imath\xi\rho\sigma_v-k_v+\sigma_v\rho-\Delta_v\right)T+2\log\left(\frac{1-\Lambda e^{\Delta_v T}}{1-\Lambda}\right)\right]\\
&+\imath\xi\lambda\left[1-p\frac{\eta_1}{\eta_1-1}-(1-p)\frac{\eta_2}{\eta_2+1}\right]T\\
&+T\lambda\left(p\frac{\eta_1}{\eta_1-1}+(1-p)\frac{\eta_2}{\eta_2+1}\right)\left[\widetilde{p}\frac{\widetilde{\eta_1}}{\widetilde{\eta_1}-\imath\xi}+\widetilde{q}\frac{\widetilde{\eta_2}}{\widetilde{\eta_2}+\imath\xi}-1\right]-\imath\xi qT\Bigg],
\end{aligned}$$

with

$$\begin{aligned}
\Delta_r &= \sqrt{k_r^2-2\sigma_r^2\imath\xi},\\
\Delta_v &= \sqrt{(\imath\xi\rho\sigma_v-k_v+\sigma_v\rho)^2+\sigma_v^2(\xi^2-\imath\xi)},\\
\Lambda &= \frac{\imath\xi\rho\sigma_v-k_v+\sigma_v\rho-\Delta_v}{\imath\xi\rho\sigma_v-k_v+\sigma_v\rho+\Delta_v},
\end{aligned}$$

$$\widetilde{p} = p\left(p\frac{\eta_1}{\eta_1-1} + (1-p)\frac{\eta_2}{\eta_2+1}\right)^{-1}\frac{\eta_1}{\eta_1-1}, \qquad \widetilde{\eta_1} = \eta_1 - 1,$$
$$\widetilde{q} = (1-p)\left(p\frac{\eta_1}{\eta_1-1} + (1-p)\frac{\eta_2}{\eta_2+1}\right)^{-1}\frac{\eta_2}{\eta_2+1}, \qquad \widetilde{\eta_2} = \eta_2 + 1.$$

Under measure $\mathbb{Q}^T(\xi)$, the characteristic function of s_T, is given by:

$$\begin{aligned}
\phi_{s_T}^{\mathbb{Q}^T(\xi)}(\xi) = \exp\Bigg[& -\frac{k_r\theta_r}{\sigma_r^2}\left[\left(-k_r-\Delta_r^T\right)T + 2\log\left(1+\frac{(k_r+\Delta_r^T)\left(e^{\Delta_r^T T}-1\right)}{2\Delta_r^T}\right)\right] \\
& + \frac{2(\iota\xi-1)\left(1-e^{\Delta_r^T T}\right)}{-2\Delta_r^T-(-k_r-\Delta_r^T)\left(1-e^{\Delta_r^T T}\right)}r_0 - \log[P(0,T)] \\
& - \frac{\iota\xi\rho\sigma_v - k_v - \Delta_v^T}{\sigma_v^2}\frac{1-e^{\Delta_v^T T}}{1-e^{\Delta_v^T T}\Lambda^T}v_0 \\
& - \frac{k_v\theta_v}{\sigma_v^2}\left[\left(\iota\xi\rho\sigma_v - k_v - \Delta_v^T\right)T + 2\log\left(\frac{1-\Lambda^T e^{\Delta_v^T T}}{1-\Lambda^T}\right)\right] \\
& + \iota\xi\lambda\left[1 - p\frac{\eta_1}{\eta_1-1} - (1-p)\frac{\eta_2}{\eta_2+1}\right]T \\
& + T\lambda\left[p\frac{\eta_1}{\eta_1-\iota\xi} + (1-p)\frac{\eta_2}{\eta_2+\iota\xi} - 1\right] - \iota\xi q T\Bigg],
\end{aligned}$$

with

$$\begin{aligned}
\Delta_r^T &= \sqrt{k_r^2 - 2\sigma_r^2(\iota\xi-1)}, \\
\Delta_v^T &= \sqrt{(\iota\xi\rho\sigma_v - k_v)^2 + \sigma_v^2(\xi^2+\iota\xi)}, \\
\Lambda^T &= \frac{\iota\xi\rho\sigma_v - k_v - \Delta_v^T}{\iota\xi\rho\sigma_v - k_v + \Delta_v^T}.
\end{aligned}$$

Chapter 5

Numerical results

In this chapter we will present a numerical implementation for European call and put options under several models, since each model can be seen as a limit case of the Bakshi et al. (1997) model obtained in Remark 4.2 of the model obtained in Remark 4.3. For a detailed verification that in fact each model that we have considered is a limit case of a more general model, we should calculate the limits of the characteristic functions determined in Remarks 4.2 and 4.3. For example, taking the characteristic function of Remark 4.2, the characteristic function of the Black and Scholes (1973) model can be seen as the following limit:

$$\lim_{k_r,\sigma_r,\theta_r,k_v,\sigma_v,\theta_v,\lambda,\mu,\sigma,\rho\to 0} \phi_{s_T} = \phi_{s_T}^{BS}, \tag{5.0.1}$$

which can be computed by setting $k_r = \sigma_r = \theta_r = k_v = \sigma_v = \theta_v = \lambda = \mu = \sigma = \rho = x$ and calculating the following limit:

$$\lim_{x\to 0} \phi_{s_T} = \phi_{s_T}^{BS}. \tag{5.0.2}$$

We will not calculate those limits here, but in our numerical implementation we will make for each model the respective parameters close to zero. The models that we have implemented are:

- Black and Scholes (1973) (bs)
- Heston (1993) (heston)

- Heston (1993) + Cox et al. (1985) (hestoncir), stochastic volatility plus stochastic interest rates
- Bates (1996) (bates)
- Bakshi et al. (1997) (bakshi)
- Kou (2002) (kou)
- Heston (1993) + Kou (2002) (hestonkou), stochastic volatility plus jumps with a double exponential distribution
- Heston (1993) + Cox et al. (1985) + Kou (2002) (bakshikou), stochastic volatility plus jumps with a double exponential distribution plus stochastic interest rates
- Merton (1976) (merton)

The names between the parentheses, are the names that should be used in matlab for each model.
In order to calculate the integral needed for the inversion of the characteristic function, we will use the Gauss-Kronrod quadrature that is already implemented in matlab as ***quadgk*** — for a detailed explanation about this method check, for instance, Laurie (1997) or Gilli et al. (2011). Schmelzle (2010) studied several Gauss quadratures and this one achieved the best overall performance among all.
It is of main importance refer that Δ_r and Δ_v of Remarks 4.2 and 4.3 can be substituted by $-\Delta_r$ and $-\Delta_v$, such alternative setting leads us to the same results and is numerically more stable than the previous one — see Albrecher et al. (2007). Nevertheless we have implemented the first approach without any issues, which can be due to the use of the Gauss-Kronrod quadrature that is a very stable method even for non smooth expressions.

5.1 European call and put options

The value at time zero of a European call option on the asset S, with strike

K and maturity at time $T(>0)$ is given by:

$$C_0 = E^{\mathbb{Q}}\left[e^{-\int_0^T r_s \mathrm{d}s}(S_T - K)^+\right] \tag{5.1.1}$$

$$= E^{\mathbb{Q}}\left[e^{-\int_0^T r_s \mathrm{d}s} S_T \Pi_{S_T \geq K}\right] - KE^{\mathbb{Q}}\left[e^{-\int_0^T r_s \mathrm{d}s} \Pi_{S_T \geq K}\right]. \tag{5.1.2}$$

Using Remarks 1.6 and 1.7 we have:

$$C_0 = S_0 e^{-qT}\left[1 - \mathbb{Q}^S(S_T < K)\right] - KP(0,T)\left[1 - \mathbb{Q}^T(S_T < K)\right]. \tag{5.1.3}$$

In order to calculate the probabilities above we use the Theorem 1.7. Therefore, we have:

$$1 - \mathbb{Q}^S(S_T < K) = 1 - \mathbb{Q}^S\left[\log\left(\frac{S_T}{S_0}\right) < \log\left(\frac{K}{S_0}\right)\right] \tag{5.1.4}$$

$$= 1 - \mathbb{Q}^S\left[s_T < \log\left(\frac{K}{S_0}\right)\right] \tag{5.1.5}$$

$$= 1 - F_{s_T}^{\mathbb{Q}^S}\left[\log\left(\frac{K}{S_0}\right)\right] \tag{5.1.6}$$

$$= \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} Re\left[\frac{e^{-\imath\xi\log\left(\frac{K}{S_0}\right)}\phi_{s_T}^{\mathbb{Q}^S(\xi)}(\xi)}{\imath\xi}\right]\mathrm{d}\xi. \tag{5.1.7}$$

Following the same procedure, we obtain under measure $\mathbb{Q}^T$:

$$1 - \mathbb{Q}^T(S_T < K) = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} Re\left[\frac{e^{-\imath\xi\log\left(\frac{K}{S_0}\right)}\phi_{s_T}^{\mathbb{Q}^T(\xi)}(\xi)}{\imath\xi}\right]\mathrm{d}\xi. \tag{5.1.8}$$

The final expression for the value of a call is given by:

$$C_0 = S_0 e^{-qT}\left[\frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} Re\left[\frac{e^{-\imath\xi\log\left(\frac{K}{S_0}\right)}\phi_{s_T}^{\mathbb{Q}^S(\xi)}(\xi)}{\imath\xi}\right]\mathrm{d}\xi\right] \tag{5.1.9}$$

$$-KP(0,T)\left[\frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} Re\left[\frac{e^{-\imath\xi\log\left(\frac{K}{S_0}\right)}\phi_{s_T}^{\mathbb{Q}^T(\xi)}(\xi)}{\imath\xi}\right]\mathrm{d}\xi\right]. \tag{5.1.10}$$

It is of main importance to deduce the put-call parity in a context where the interest rate is a stochastic process. Therefore, we have:

$$\begin{aligned} C_0 - P_0 &= E^{\mathbb{Q}}\left[e^{-\int_0^T r_s \mathrm{ds}} \left[(S_T - K)^+ - (K - S_T)^+ \right] \right] \\ &= E^{\mathbb{Q}}\left[e^{-\int_0^T r_s \mathrm{ds}} (S_T - K) \right] \\ &= E^{\mathbb{Q}}\left[e^{-\int_0^T r_s \mathrm{ds}} S_T \right] - K E^{\mathbb{Q}}\left[e^{-\int_0^T r_s \mathrm{ds}} \right]. \end{aligned} \tag{5.1.11}$$

Since

$$\frac{S_T}{e^{-qT + \int_0^T r_s \, \mathrm{ds}}}$$

is a martingale under $\mathbb{Q}$, we have:

$$P_0 = C_0 - e^{-qT} S_0 + P(0,T)K, \tag{5.1.12}$$

where P_0 is the value at time zero of a European put option and $P(0,T)$ is the value at time zero of a zero coupon bond.

In Table 5.1 we present the results obtained with an upper limit of 1000 for the numerical integration.

Table 5.1: Numerical results

	BS	Heston	Heston + CIR	Bates	Bakshi	Kou	Heston + Kou	Heston + CIR + Kou	Merton
k_r	$\to 0$	$\to 0$	0.3	$\to 0$	0.3	$\to 0$	$\to 0$	0.3	$\to 0$
σ_r	$\to 0$	$\to 0$	0.05	$\to 0$	0.05	$\to 0$	$\to 0$	0.05	$\to 0$
θ_r	$\to 0$	$\to 0$	0.05	$\to 0$	0.05	$\to 0$	$\to 0$	0.05	$\to 0$
r_0	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04
T	1	1	1	1	1	1	1	1	1
k_v	$\to 0$	2	2	2	2	$\to 0$	2	2	$\to 0$
σ_v	$\to 0$	0.1	0.1	0.1	0.1	$\to 0$	0.1	0.1	$\to 0$
θ_v	$\to 0$	0.01	0.01	0.01	0.01	$\to 0$	0.01	0.01	$\to 0$
v_0	0.1	0.01	0.01	0.01	0.01	0.1	0.01	0.01	0.1
λ	$\to 0$	$\to 0$	$\to 0$	2	2	2	2	2	2
μ	$\to 0$	$\to 0$	$\to 0$	0.2	0.2	$\to 0$	$\to 0$	$\to 0$	0.2
σ	$\to 0$	$\to 0$	$\to 0$	0.1	0.1	$\to 0$	$\to 0$	$\to 0$	0.1
η_1	$\to 0$	$\to 0$	$\to 0$	$\to 0$	$\to 0$	8	8	8	$\to 0$
η_2	$\to 0$	$\to 0$	$\to 0$	$\to 0$	$\to 0$	5	5	5	$\to 0$
p	$\to 0$	$\to 0$	$\to 0$	$\to 0$	$\to 0$	0.4	0.4	0.4	$\to 0$
q	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
ρ	$\to 0$	-0.5	-0.5	-0.5	-0.5	$\to 0$	-0.5	-0.5	$\to 0$
K	100	100	100	100	100	100	100	100	100
S_0	100	100	100	100	100	100	100	100	100
Call	5.5263	5.5470	5.6331	13.8359	13.8923	13.5366	13.5649	13.6379	13.8498
Put	2.6003	2.6209	2.5777	10.9099	10.8370	10.6106	10.6388	10.5826	10.9238
error $\mathbb{Q}^S$	1.1880e-09	1.0239e-09	1.0627e-09	1.0480e-09	1.0588e-09	4.6621e-10	2.2563e-10	2.3320e-10	1.0869e-09
error $\mathbb{Q}^T$	8.1561e-10	9.5437e-10	9.8768e-10	1.0673e-10	1.0929e-10	1.3559e-09	1.3446e-09	1.3396e-09	1.5915e-10

Results for European call and put options; error $\mathbb{Q}^S$ and error $\mathbb{Q}^T$ are the errors of integration on each measure.

Chapter 6

A general dependency structure

For the valuation of some exotic options it is important to have a model that is able to incorporate a general dependency structure, that is, a model that take in to account the correlation between the interest rates, the asset price and its volatility. In general a closed formula for the characteristic function of the asset price is not known, under such conditions the fundamental PDE that the characteristic function satisfies is not affine on the state variables, and the solution for this problem is usually to take an approximation of the terms that are not affine in order to make them affine. For example, in the Bakshi, Cao and Chen (1997) model, if we consider the correlation between the interest rates and the asset price and/or its volatility, the fundamental PDE is no longer affine on the state variables, and to make it affine an approximation can be made — see for example van Haastrecht and Pelsser (2011) and Grzelak and Oosterlee (2011).
Our goal is to reach a model with a general dependency structure and with a fundamental PDE that is exactly affine and quadratic on the state variables.

Following the approach of Section 2.1, and defining T_t as

$$T_t := \int_0^t v^2(s)\,\mathrm{ds}, \tag{6.0.1}$$

whenever the instantaneous activity rate $v(t)$ is not a positive cadlag process, we can have a business time T_t with the same restrictions of the Section 2.1.

The following theorem is a generalization of Theorem 2.1, and it allows us to know the joint characteristic function of N time-changed Lévy processes. Suppose that Y_t is the sum of N time-changed Lévy processes, i.e.,

$$Y_t = \sum_{i=1}^{N} Y_t^i. \tag{6.0.2}$$

Theorem 6.1. *The generalized Fourier transform of the time-changed Lévy process $Y_T = \sum_{i=1}^{N} Y_T^i$ with $Y_T^i \equiv X^i_{T_T{}^i}$ is given by:*

$$\phi_{Y_T}(\xi) \equiv E^{\mathbb{Q}}\left[e^{\imath\xi Y_T}\right] = E^{\mathbb{Q}}\left[e^{\imath\xi \sum_{i=1}^{N} Y_T^i}\right] = E^{\mathbb{Q}}\left[e^{\imath\xi \sum_{i=1}^{N} X^i_{T_T{}^i}}\right] = E^{\mathbb{Q}_{1,\dots,N}(\xi)}\left[e^{-\sum_{i=1}^{N} T_T{}^i \Psi_x^i(\xi)}\right], \tag{6.0.3}$$

where $\mathbb{Q}_{1,\dots,N}(\xi)$ is a complex valued measure that is absolutely continuous with respect to $\mathbb{Q}$, whose Radon-Nikodym derivative is given by

$$\frac{\mathrm{d}\mathbb{Q}_{1,\dots,N}(\xi)}{\mathrm{d}\mathbb{Q}_{1,\dots,N-1}(\xi)} \equiv M_T^N(\xi) = e^{\imath\xi Y_T^N + T_T{}^N \Psi_x^N(\xi)}, \qquad \xi \in \mathbb{C}_\xi, \tag{6.0.4}$$

and

$$\frac{\mathrm{d}\mathbb{Q}_{1,\dots,N-1}(\xi)}{\mathrm{d}\mathbb{Q}_{1,\dots,N-2}(\xi)} \equiv M_T^{N-1}(\xi) = e^{\imath\xi Y_T^{N-1} + T_T{}^{N-1} \Psi_x^{N-1}(\xi)}, \qquad \xi \in \mathbb{C}_\xi, \tag{6.0.5}$$

$$. \tag{6.0.6}$$

$$. \tag{6.0.7}$$

$$. \tag{6.0.8}$$

$$\frac{\mathrm{d}\mathbb{Q}_{1}(\xi)}{\mathrm{d}\mathbb{Q}} \equiv M_T^1(\xi) = e^{\imath\xi Y_T^1 + T_T{}^1 \Psi_x^1(\xi)}, \qquad \xi \in \mathbb{C}_\xi, \tag{6.0.9}$$

where $\mathbb{C}_\xi$ is the subset of $\mathbb{C}$ where $\phi_{Y_T}(\xi)$ is well defined.
In this context N change of measures are needed to obtain the final measure $\mathbb{Q}_{1,\dots,N}(\xi)$.

Proof.

i) Consider the σ-algebra generated by the past values of the processes $(Y_t^i)_{t\in[0,T]}$ and $({T_t}^i)_{t\in[0,T]}$, completed by the null sets, i.e.

$$\mathscr{G} = \sigma(Y_t^1,\dots,Y_t^N,{T_t}^1,\dots,{T_t}^N, t\in[0,T]) \bigvee \mathscr{N},$$

and let $(\Omega,\mathscr{G},\mathbb{P})$ be a probability space with a filtration $(\mathscr{G}_t)_{t\in[0,T]}$.
The proof is recursive, so we will only make the first change of measure; for the other $N-1$ change of measures the scheme is the same. First we need to proof that $M_t^1(\xi)$ is a complex valued $\mathbb{Q}$-martingale with respect to $(\mathscr{G}_t)_{t\in[0,T]}$.
Let us define $M_t^{1,1}(\xi) \equiv e^{\imath\xi X_t^1 + t\Psi_x^1(\xi)}$.

a) $M_t^{1,1}(\xi)$ is adapted to the filtration generated by the process $(X_t^1)_{t\in[0,T]}$ completed by the null sets, $(\mathscr{F}_t)_{t\in[0,T]}$.

b) $E^{\mathbb{Q}}\left[\left|M_t^{1,1}(\xi)\right|\right] \le \left|e^{\imath\xi X_t^1}\right|\left|e^{t\Psi_x^1(\xi)}\right| = \left|e^{\imath(a+\imath b)X_t^1}\right|\left|e^{t\Psi_x^1(\xi)}\right| = \left|e^{\imath a X_t^1 - bX_t^1}\right|\left|e^{t\Psi_x^1(\xi)}\right| \le$

$$\le \left|e^{\imath a X_t^1}\right|\left|e^{-bX_t^1}\right|\left|e^{t\Psi_x^1(\xi)}\right| = \left|e^{-bX_t^1}\right|\left|e^{t\Psi_x^1(\xi)}\right| < \infty, \qquad a,b\in\mathbb{R},$$

since $\Psi_x^1(\xi)$ and X_t^1 are both finite by definition.

c) For $0 \le s < t$ we have:

$$E^{\mathbb{Q}}\left[\frac{M_t^{1,1}(\xi)}{M_s^{1,1}(\xi)}\middle| X_s^1\right] = E^{\mathbb{Q}}\left[e^{\imath\xi(X_t^1-X_s^1)+(t-s)\Psi_x^1(\xi)}\middle| X_s^1\right] = e^{(t-s)\Psi_x^1(\xi)}E^{\mathbb{Q}}\left[e^{\imath\xi(X_{t-s}^1)}\middle| X_s^1\right]$$
$$= e^{(t-s)\Psi_x^1(\xi)}e^{-(t-s)\Psi_x^1(\xi)} = 1.$$

Therefore, $M_t^{1,1}(\xi)$ is a complex valued $\mathbb{Q}$-martingale with respect to $(\mathscr{F}_t)_{t\in[0,T]}$. By abuse of language we denote that filtration by X_s^1. For each fixed $t\in[0,T]$, ${T_t}^1$ is a stopping time that is finite $\mathbb{Q}$-a.s.. Therefore, by Proposition 1.1, $M_t^1(\xi) = M_{{T_t}^1}^{1,1}(\xi)$ is a complex valued $\mathbb{Q}$-martingale with respect to the $(\mathscr{G}_t)_{t\in[0,T]}$.

ii)

$$\phi_{Y_T}(\xi) \equiv E^{\mathbb{Q}}\left[e^{\imath\xi Y_T}\right] = E^{\mathbb{Q}}\left[e^{\imath\xi Y_T{}^1 + T_T{}^1\Psi_x^1(\xi) - T_T{}^1\Psi_x^1(\xi) + \imath\xi\sum\limits_{i=2}^{N} Y_T^i}\right] = E^{\mathbb{Q}}\left[M_T^1(\xi) e^{-T_T{}^1\Psi_x^1(\xi) + \imath\xi\sum\limits_{i=2}^{N} Y_T^i}\right]$$

$$= E^{\mathbb{Q}_1(\xi)}\left[e^{-T_T{}^1\Psi_x^1(\xi) + \imath\xi\sum\limits_{i=2}^{N} Y_T^i}\right].$$

Following the same procedure for the other change of measures we obtain

$$\phi_{Y_T}(\xi) \equiv E^{\mathbb{Q}}\left[e^{\imath\xi Y_T}\right] = E^{\mathbb{Q}_{1,\ldots,N}(\xi)}\left[e^{-\sum\limits_{i=1}^{N} T_T{}^i\Psi_x^i(\xi)}\right]. \tag{6.0.10}$$

■

Hereafter, we will consider $N = 2$ and in order to denote the stochastic process of the interest rate we will consider $X_t^r = t$. Then, $\Psi_x^r(\xi) = -\imath\xi$, and it follows follows that $M_t^r(\xi) = 1$. Thus only one change of measure is needed, and the imposition of $r_t > 0, \forall t$ can be relaxed. Furthermore, in this context we will consider $T_t^r := \int\limits_0^t r(s)\,\mathrm{ds}$.

Remark 6.1. *Consider three correlated Brownian Motions, $\widetilde{W_1}$, $\widetilde{W_2}$ and $\widetilde{W_3}$, with a correlation matrix*

$$C = \begin{pmatrix} 1 & \rho_{v,x} & \rho_{r,x} \\ \rho_{v,x} & 1 & \rho_{v,r} \\ \rho_{r,x} & \rho_{v,r} & 1 \end{pmatrix}, \qquad \rho \in [-1,1],$$

C is real and symmetric. It is also positive definite because its eigenvalues are positive in the case of $\rho_{v,x}, \rho_{r,x}, \rho_{v,r} \in [-1,1]$.
Its Cholesky factorization is, $C = L.L^T$, where

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \rho_{v,x} & \sqrt{1-\rho_{v,x}^2} & 0 \\ \rho_{r,x} & \frac{\rho_{v,r}-\rho_{v,x}\rho_{r,x}}{\sqrt{1-\rho_{v,x}^2}} & \sqrt{1-\rho_{r,x}^2 - \left(\frac{\rho_{v,r}-\rho_{v,x}\rho_{r,x}}{\sqrt{1-\rho_{v,x}^2}}\right)^2} \end{pmatrix}.$$

Consider three independent Brownian motions, W_1, W_2 and W_3. Then:

$$\begin{pmatrix} \widetilde{W_1} \\ \widetilde{W_2} \\ \widetilde{W_3} \end{pmatrix} = L. \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix}.$$

Given two Ito's processes

$$\mathrm{dv_t} = \mu_v(v_t, t)\,\mathrm{dt} + \sigma_v(v_t, t)\,\mathrm{dW_v^{\mathbb{Q}}(t)} \tag{6.0.11}$$

and

$$\mathrm{dr_t} = \mu_r(r_t, t)\,\mathrm{dt} + \sigma_r(r_t, t)\,\mathrm{dW_r^{\mathbb{Q}}(t)}, \tag{6.0.12}$$

consider the following time-changed Lévy process under the risk neutral measure $\mathbb{Q}$ as in Section 4.2:

1. $Y_t = Y_t^v + Y_t^r$,
2. $Y_t^v = X_{T_t{}^v}^v$,
3. $Y_t^r = X_{T_t{}^r}^r$,
4. $X_t^v = W_x^{\mathbb{Q}}(t) + \mu t$,
5. $X_t^r = t$,
6. $\mathrm{dv_t} = \mu_v(v_t, t)\,\mathrm{dt} + \sigma_v(v_t, t)\,\mathrm{dW_v^{\mathbb{Q}}(t)}$,
7. $\mathrm{dr_t} = \mu_r(r_t, t)\,\mathrm{dt} + \sigma_r(r_t, t)\,\mathrm{dW_r^{\mathbb{Q}}(t)}$,
8. $\mathrm{d} < \mathrm{W_x^{\mathbb{Q}}}, \mathrm{W_v^{\mathbb{Q}}} >_{\mathrm{t}} = \rho_{v,x}\,\mathrm{dt}$,
9. $\mathrm{d} < \mathrm{W_x^{\mathbb{Q}}}, \mathrm{W_r^{\mathbb{Q}}} >_{\mathrm{t}} = \rho_{r,x}\,\mathrm{dt}$,
10. $\mathrm{d} < \mathrm{W_v^{\mathbb{Q}}}, \mathrm{W_r^{\mathbb{Q}}} >_{\mathrm{t}} = \rho_{v,r}\,\mathrm{dt}$,

with

$$T_t^v := \int_0^t v^2(s)\,\mathrm{ds}, \tag{6.0.13}$$

$$T_t^r := \int_0^t r(s)\,\mathrm{ds}. \tag{6.0.14}$$

As we have seen in Section 4.2, using Remark 6.1, and denoting

$$a = \sqrt{1-\rho_{v,x}^2},$$
$$b = \frac{\rho_{v,r} - \rho_{v,x}\rho_{r,x}}{\sqrt{1-\rho_{v,x}^2}},$$
$$c = \sqrt{1-\rho_{r,x}^2 - \left(\frac{\rho_{v,r} - \rho_{v,x}\rho_{r,x}}{\sqrt{1-\rho_{v,x}^2}}\right)^2},$$

we have under measure $\mathbb{Q}$:

$$
\begin{aligned}
Y_t &= Y_t^v + Y_t^r \\
Y_t^v &= X_{T_t{}^v}^v \\
Y_t^r &= X_{T_t{}^r}^r \\
X_t^v &= Z_x^{\mathbb{Q}}(t) - \frac{1}{2}t \\
X_t^r &= t \\
\mathrm{dv_t} &= \mu_v(v_t,t)\,\mathrm{dt} + \sigma_v(v_t,t)\left(\rho_{v,x}\,\mathrm{dZ_x^{\mathbb{Q}}(t)} + a\,\mathrm{dZ_v^{\mathbb{Q}}(t)}\right) \\
\mathrm{dr_t} &= \mu_r(r_t,t)\,\mathrm{dt} + \sigma_r(r_t,t)\left(\rho_{r,x}\,\mathrm{dZ_x^{\mathbb{Q}}(t)} + b\,\mathrm{dZ_v^{\mathbb{Q}}(t)} + c\,\mathrm{dZ_r^{\mathbb{Q}}(t)}\right).
\end{aligned}
$$

In order to write the model under the complex valued measure $\mathbb{Q}_v(\xi)$, and using Theorem 6.1, we have:

$$
\begin{aligned}
\frac{\mathrm{d}\mathbb{Q}_v(\xi)}{\mathrm{d}\mathbb{Q}} &= e^{\iota\xi Y_T^v + T_T{}^v \Psi_x^v(\xi)} \\
&= e^{\iota\xi X^v_{\int_0^T v^2(s)\,\mathrm{ds}} + \Psi_x^v(\xi)\int_0^T v^2(s)\,\mathrm{ds}} \\
&= e^{\iota\xi\left(Z_x^{\mathbb{Q}}\left(\int_0^T v^2(s)\,\mathrm{ds}\right) - \frac{1}{2}\int_0^T v^2(s)\,\mathrm{ds}\right) + \left(\frac{1}{2}\iota\xi + \frac{1}{2}\xi^2\right)\int_0^T v^2(s)\,\mathrm{ds}}
\end{aligned}
$$

$$= e^{\imath\xi \int_0^T v_s \, \mathrm{dZ}_{\mathrm{x}}^{\mathbb{Q}}(\mathrm{t}) + \frac{1}{2}\xi^2 \int_0^T v_s^2 \, \mathrm{ds}}.$$

Using the Girsanov's Theorem, we have:

$$\mathrm{dZ}_{\mathrm{x}}^{\mathbb{Q}_{\mathrm{v}}(\xi)}(\mathrm{t}) = \mathrm{dZ}_{\mathrm{x}}^{\mathbb{Q}}(\mathrm{t}) - \imath\xi v_t \mathrm{dt}. \tag{6.0.15}$$

Using the same procedure as used in Section 4.2, we have under measure $\mathbb{Q}^S$:

$$\begin{aligned}
&\mathrm{dZ}_{\mathrm{x}}^{\mathbb{Q}^{\mathrm{S}}}(\mathrm{t}) = \mathrm{dZ}_{\mathrm{x}}^{\mathbb{Q}}(\mathrm{t}) - v_t \, \mathrm{dt}, \\
&\mathrm{dZ}_{\mathrm{v}}^{\mathbb{Q}^{\mathrm{S}}}(\mathrm{t}) = \mathrm{dZ}_{\mathrm{v}}^{\mathbb{Q}}(\mathrm{t}), \\
&\mathrm{dZ}_{\mathrm{r}}^{\mathbb{Q}^{\mathrm{S}}}(\mathrm{t}) = \mathrm{dZ}_{\mathrm{r}}^{\mathbb{Q}}(\mathrm{t}), \\
&\widetilde{\lambda} = \lambda E^{\mathbb{Q}}\left(e^V\right), \\
&\widetilde{f}(\mathrm{V}_i) = \frac{e^{V_i} f(V_i)}{E^{\mathbb{Q}}\left(e^V\right)}.
\end{aligned}$$

Suppose that r_t is cast into an affine term structure model, i.e., the time-t pure discount bond price, with maturity at time $T(\geq t)$, can be written as:

$$P(t,T) = e^{A(t,T) - B(t,T) r_t}.$$

In this case, where $A(.)$ and $B(.)$ are deterministic functions of time, Bjork (1998, Section 17) has shown that $\mu_r(r_t,t)$ and $\sigma_r(r_t,t)$ must be of the form:

$$\begin{aligned}
&\mu_r(r_t,t) = \alpha_t r_t + \beta_t, \\
&\sigma_r(r_t,t) = \sqrt{\gamma_t r_t + \delta_t}, \qquad \alpha_t, \beta_t, \gamma_t, \delta_t \in \mathbb{R}.
\end{aligned}$$

Hence, and using the same procedure used in Section 4.2, we have under measure $\mathbb{Q}^T$

$$\begin{aligned}
&\mathrm{dZ}_{\mathrm{r}}^{\mathbb{Q}^{\mathrm{T}}}(\mathrm{t}) = \mathrm{dZ}_{\mathrm{r}}^{\mathbb{Q}}(\mathrm{t}) - \sigma_r(r_t,t) B(\tau) \, \mathrm{dt}, \\
&\mathrm{dZ}_{\mathrm{x}}^{\mathbb{Q}^{\mathrm{T}}}(\mathrm{t}) = \mathrm{dZ}_{\mathrm{x}}^{\mathbb{Q}}(\mathrm{t}), \\
&\mathrm{dZ}_{\mathrm{v}}^{\mathbb{Q}^{\mathrm{T}}}(\mathrm{t}) = \mathrm{dZ}_{\mathrm{v}}^{\mathbb{Q}}(\mathrm{t}).
\end{aligned}$$

Using equation (6.0.15) and following the same procedure as used in Section 3.2, we have:

$$\mathrm{dZ_x^{\mathbb{Q}_v^S(\xi)}(t)} = \mathrm{dZ_x^{\mathbb{Q}^S}(t)} - \imath\xi v_t \mathrm{dt},$$
$$\mathrm{dZ_x^{\mathbb{Q}_v^T(\xi)}(t)} = \mathrm{dZ_x^{\mathbb{Q}^T}(t)} - \imath\xi v_t \mathrm{dt},$$

which can be rewritten as:

$$\mathrm{dZ_x^{\mathbb{Q}_v^S(\xi)}(t)} = \mathrm{dZ_x^{\mathbb{Q}}(t)} - v_t\,\mathrm{dt} - \imath\xi v_t \mathrm{dt} = \mathrm{dZ_x^{\mathbb{Q}}(t)} - v_t(1+\imath\xi)\,\mathrm{dt},$$
$$\mathrm{dZ_x^{\mathbb{Q}_v^T(\xi)}(t)} = \mathrm{dZ_x^{\mathbb{Q}}(t)} - \imath\xi v_t \mathrm{dt}.$$

We are now ready to write the model under the complex valued measures $\mathbb{Q}_v^S(\xi)$ and $\mathbb{Q}_v^T(\xi)$. Under the measure $\mathbb{Q}_v^S(\xi)$, we have:

$$\phi_{Y_T}(\xi) = E^{\mathbb{Q}_v^S(\xi)}\left[e^{-\left(-\frac{1}{2}\imath\xi+\frac{1}{2}\xi^2\right)\int_0^T v_s^2\,\mathrm{ds}+\imath\xi\int_0^T r_s\,\mathrm{ds}}\right] \tag{6.0.16}$$

$$\begin{aligned}
&Y_t = Y_t^v + Y_t^r\\
&Y_t^v = X_{T_t{}^v}^v\\
&Y_t^r = X_{T_t{}^r}^r\\
&X_t^v = Z_x^{\mathbb{Q}^S}(t) + \frac{1}{2}t\\
&X_t^r = t\\
&\mathrm{dv_t} = \left[\mu_v(v_t,t) + v_t(1+\imath\xi)\sigma_v(v_t,t)\rho_{v,x}\right]\,\mathrm{dt}\\
&+\sigma_v(v_t,t)\left(\rho_{v,x}\,\mathrm{dZ_x^{\mathbb{Q}_v^S(\xi)}(t)} + a\,\mathrm{dZ_v^{\mathbb{Q}_v^S(\xi)}(t)}\right)\\
&\mathrm{dr_t} = \left[\mu_r(r_t,t) + v_t(1+\imath\xi)\sigma_r(r_t,t)\rho_{r,x}\right]\,\mathrm{dt}\\
&+\sigma_r(r_t,t)\left(\rho_{r,x}\,\mathrm{dZ_x^{\mathbb{Q}_v^S(\xi)}(t)} + b\,\mathrm{dZ_v^{\mathbb{Q}_v^S(\xi)}(t)} + c\,\mathrm{dZ_r^{\mathbb{Q}_v^S(\xi)}(t)}\right)
\end{aligned}$$

Under the measure $\mathbb{Q}_v^T(\xi)$, we have:

$$\phi_{Y_T}(\xi) = E^{\mathbb{Q}_v^T(\xi)}\left[e^{-\left(\frac{1}{2}\imath\xi+\frac{1}{2}\xi^2\right)\int_0^T v_s^2\,\mathrm{ds}+\imath\xi\int_0^T r_s\,\mathrm{ds}}\right] \tag{6.0.17}$$

$$
\begin{aligned}
&Y_t = Y_t^v + Y_t^r \\
&Y_t^v = X^v_{T_t{}^v} \\
&Y_t^r = X^r_{T_t{}^r} \\
&X_t^v = Z_x^{\mathbb{Q}^T}(t) - \frac{1}{2}t \\
&X_t^r = t \\
&\mathrm{dv_t} = \left[\mu_v(v_t,t) + v_t \imath\xi\sigma_v(v_t,t)\rho_{v,x}\right]\,\mathrm{dt} \\
&+\sigma_v(v_t,t)\left(\rho_{v,x}\,\mathrm{dZ_x^{\mathbb{Q}_v^T(\xi)}(t)} + a\,\mathrm{dZ_v^{\mathbb{Q}_v^T(\xi)}(t)}\right) \\
&\mathrm{dr_t} = \left[\mu_r(r_t,t) + c\sigma_r^2(r_t,t)B(\tau) + v_t\imath\xi\sigma_r(r_t,t)\rho_{r,x}\right]\,\mathrm{dt} \\
&+\sigma_r(r_t,t)\left(\rho_{r,x}\,\mathrm{dZ_x^{\mathbb{Q}_v^S(\xi)}(t)} + b\,\mathrm{dZ_v^{\mathbb{Q}_v^S(\xi)}(t)} + c\,\mathrm{dZ_r^{\mathbb{Q}_v^S(\xi)}(t)}\right)
\end{aligned}
$$

In order to apply the multi-dimensional version of the Feynman-Kac Theorem we must rewrite the SDE for both r_t and v_t in matrix form. Under measure $\mathbb{Q}_v^S(\xi)$, we have

$$
\begin{pmatrix} \mathrm{dv_t} \\ \mathrm{dr_t} \end{pmatrix} = \begin{pmatrix} \mu_v(v_t,t) + v_t(1+\imath\xi)\sigma_v(v_t,t)\rho_{v,x} \\ \mu_r(r_t,t) + v_t(1+\imath\xi)\sigma_r(r_t,t)\rho_{r,x} \end{pmatrix}\mathrm{dt}
$$

$$
+\begin{pmatrix} \sigma_v(v_t,t)\rho_{v,x} & \sigma_v(v_t,t)a & 0 \\ \sigma_r(r_t,t)\rho_{r,x} & \sigma_r(r_t,t)b & \sigma_r(r_t,t)c \end{pmatrix}\cdot\begin{pmatrix} \mathrm{dZ_x^{\mathbb{Q}_v^S(\xi)}(t)} \\ \mathrm{dZ_v^{\mathbb{Q}_v^S(\xi)}(t)} \\ \mathrm{dZ_r^{\mathbb{Q}_v^S(\xi)}(t)} \end{pmatrix},
$$

and to obtain the generator of the process we only need to compute

$$
\begin{pmatrix} \sigma_v(v_t,t)\rho_{v,x} & \sigma_v(v_t,t)a & 0 \\ \sigma_r(r_t,t)\rho_{r,x} & \sigma_r(r_t,t)b & \sigma_r(r_t,t)c \end{pmatrix}\cdot\begin{pmatrix} \sigma_v(v_t,t)\rho_{v,x} & \sigma_r(r_t,t)\rho_{r,x} \\ \sigma_v(v_t,t)a & \sigma_r(r_t,t)b \\ 0 & \sigma_r(r_t,t)c \end{pmatrix}.
$$

Using the relations

$$
\begin{aligned}
&a^2 = 1-\rho_{v,x}^2 \\
&ab = \rho_{v,r} - \rho_{v,x}\rho_{r,x} \\
&\rho_{r,x}^2 + b^2 + c^2 = 1,
\end{aligned}
$$

we obtain

$$
\begin{pmatrix} \sigma_v^2(v_t,t) & \sigma_r(r_t,t)\sigma_v(v_t,t)\rho_{v,r} \\ \sigma_r(r_t,t)\sigma_v(v_t,t)\rho_{v,r} & \sigma_r^2(r_t,t) \end{pmatrix}.
$$

Therefore, the generator of the process is:

$$
\begin{aligned}
\mathscr{A}^{\mathbb{Q}_v^S(\xi)} &= \left[\mu_v(v_t,t) + v_t(1+\imath\xi)\sigma_v(v_t,t)\rho_{v,x}\right]\frac{\partial}{\partial v_t} \\
&+ \left[\mu_r(r_t,t) + v_t(1+\imath\xi)\sigma_r(r_t,t)\rho_{r,x}\right]\frac{\partial}{\partial r_t} \\
&+ \frac{1}{2}\begin{pmatrix} \sigma_v^2(v_t,t) & \sigma_r(r_t,t)\sigma_v(v_t,t)\rho_{v,r} \\ \sigma_r(r_t,t)\sigma_v(v_t,t)\rho_{v,r} & \sigma_r^2(r_t,t) \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial^2}{\partial v_t^2} & \frac{\partial^2}{\partial v_t \partial r_t} \\ \frac{\partial^2}{\partial r_t \partial v_t} & \frac{\partial^2}{\partial r_t^2} \end{pmatrix},
\end{aligned}
$$

which is the same as

$$
\begin{aligned}
\mathscr{A}^{\mathbb{Q}_v^S(\xi)} &= \left[\mu_v(v_t,t) + v_t(1+\imath\xi)\sigma_v(v_t,t)\rho_{v,x}\right]\frac{\partial}{\partial v_t} \\
&+ \left[\mu_r(r_t,t) + v_t(1+\imath\xi)\sigma_r(r_t,t)\rho_{r,x}\right]\frac{\partial}{\partial r_t} \\
&+ \frac{1}{2}\left[\sigma_v^2(v_t,t)\frac{\partial^2}{\partial v_t^2} + \sigma_r^2(r_t,t)\frac{\partial^2}{\partial r_t^2} + 2\sigma_r(r_t,t)\sigma_v(v_t,t)\rho_{v,r}\frac{\partial^2}{\partial r_t \partial v_t}\right].
\end{aligned}
$$

To obtain the generator of the process under the measure $\mathbb{Q}_v^T(\xi)$, the procedure is the same, and we get:

$$
\begin{aligned}
\mathscr{A}^{\mathbb{Q}_v^T(\xi)} &= \left[\mu_v(v_t,t) + v_t\imath\xi\sigma_v(v_t,t)\rho_{v,x}\right]\frac{\partial}{\partial v_t} \\
&+ \left[\mu_r(r_t,t) + c\sigma_r^2(r_t,t)B(\tau) + v_t\imath\xi\sigma_r(r_t,t)\rho_{r,x}\right]\frac{\partial}{\partial r_t} \\
&+ \frac{1}{2}\left[\sigma_v^2(v_t,t)\frac{\partial^2}{\partial v_t^2} + \sigma_r^2(r_t,t)\frac{\partial^2}{\partial r_t^2} + 2\sigma_r(r_t,t)\sigma_v(v_t,t)\rho_{v,r}\frac{\partial^2}{\partial r_t \partial v_t}\right].
\end{aligned}
$$

The multi-dimensional version of the Feynman-Kac Theorem states that

$$
\frac{\partial V}{\partial t} + \mathscr{A}V(v_t,r_t,t) - r(v_t,r_t,t)V(v_t,r_t,t) = 0, \tag{6.0.18}
$$

with boundary condition $V(v_T,r_T,T)$ has as solution

$$
V(v_t,r_t,t) = E^{\mathbb{Q}}\left[e^{-\int_t^T r(v_t,r_t,t)\,\mathrm{ds}} V(v_T,r_T,T)\middle|\mathscr{F}_t\right]. \tag{6.0.19}
$$

Comparing equations (6.0.16), (6.0.17) and (6.0.19) the process $r(v_t, r_t, t)$ under both measures is:

$$r^{\mathbb{Q}_v^S(\xi)}(v_t, r_t, t) = \left(-\frac{1}{2}\iota\xi + \frac{1}{2}\xi^2\right) v_t^2 - \iota\xi r_t,$$
$$r^{\mathbb{Q}_v^T(\xi)}(v_t, r_t, t) = \left(\frac{1}{2}\iota\xi + \frac{1}{2}\xi^2\right) v_t^2 - \iota\xi r_t,$$

and the boundary condition $V(v_T, r_T, T)$ is equal to 1.

To solve equation (6.0.18) under the measure $\mathbb{Q}_v^S(\xi)$ we will use the trial solution

$$V(v_t, r_t, t) = e^{a(\tau)+b(\tau)r_t+c(\tau)v_t+d(\tau)v_t^2}. \tag{6.0.20}$$

From the form of the generator, and in order for the model to remain affine, we must have $\sigma_v(v_t, t) = \sigma_v$ and $\sigma_r(r_t, t) = \sigma_r$, with σ_v and σ_r real positive constants, and the drift must be of the form $\mu_v(v_t, t) = \alpha_v(t)v_t + \beta_v(t)$ and $\mu_r(r_t, t) = \alpha_r(t)r_t + \beta_r(t)$.[1]
Therefore, we have:

$$\begin{aligned}
\sigma_v(v_t, t) &= \sigma_v \\
\sigma_r(r_t, t) &= \sigma_r \\
\mu_v(v_t, t) &= k_v\theta_v - k_v v_t \\
\mu_r(r_t, t) &= k_r\theta_r - k_r r_t.
\end{aligned}$$

Under these conditions, the PDE (6.0.18) has the following form under measure $\mathbb{Q}_v^S(\xi)$:

$$\begin{aligned}
&\left(-\frac{\partial a}{\partial \tau} - \frac{\partial b}{\partial \tau} r_t - \frac{\partial c}{\partial \tau} v_t - \frac{\partial d}{\partial \tau} v_t^2\right) V \\
&\qquad + \left(k_v\theta_v - k_v v_t + v_t(1+\iota\xi)\sigma_v\rho_{v,x}\right)(c(\tau) + 2d(\tau)v_t)V \\
&\qquad\qquad + \left(k_r\theta_r - k_r r_t + v_t(1+\iota\xi)\sigma_r\rho_{r,x}\right) b(\tau)V \\
&+ \frac{1}{2}\sigma_v^2\left(2d(\tau) + c^2(\tau) + 4c(\tau)d(\tau)v_t + 4d^2(\tau)v_t^2\right)V + \frac{1}{2}\sigma_r^2 b^2(\tau)V \\
&\qquad\qquad + \sigma_r\sigma_v\rho_{v,r}\left(b(\tau)c(\tau) + 2b(\tau)d(\tau)v_t\right)V \\
&\qquad\qquad\qquad + \left(\frac{1}{2}\iota\xi - \frac{1}{2}\xi^2\right) v_t^2 V + \iota\xi r_t V = 0.
\end{aligned} \tag{6.0.21}$$

[1] We will use the form used in Vasicek (1977), but the form used in Hull and White (1990) works too.

Writing equation (6.0.21) in order to r_t, v_t and v_t^2, we obtain the following ODEs:

$$\frac{\partial b}{\partial \tau} = -k_r b(\tau) + \imath\xi, \tag{6.0.22}$$

$$\begin{aligned} -\frac{\partial c}{\partial \tau} + 2k_v\theta_v d(\tau) + (1+\imath\xi)\sigma_v\rho_{v,x}c(\tau) - k_v c(\tau) + (1+\imath\xi)\sigma_r\rho_{r,x}b(\tau) \\ + 2\sigma_v^2 c(\tau)d(\tau) + 2\sigma_r\sigma_v\rho_{v,r}b(\tau)d(\tau) = 0, \end{aligned} \tag{6.0.23}$$

$$\frac{\partial d}{\partial \tau} = 2d(\tau)\big((1+\imath\xi)\sigma_v\rho_{v,x} - k_v\big) + 2\sigma_v^2 d^2(\tau) + \frac{1}{2}\imath\xi - \frac{1}{2}\xi^2, \tag{6.0.24}$$

and

$$\frac{\partial a}{\partial \tau} = k_v\theta_v c(\tau) + k_r\theta_r b(\tau) + \sigma_v^2 d(\tau) + \frac{1}{2}\sigma_v^2 c^2(\tau) + \frac{1}{2}\sigma_r^2 b^2(\tau) + \sigma_r\sigma_v\rho_{v,r}b(\tau)c(\tau), \tag{6.0.25}$$

with $a(0) = b(0) = c(0) = d(0) = 0$.
To obtain the characteristic function under $\mathbb{Q}_v^T(\xi)$ the procedure is the same, but the trial solution is now

$$V(v_t, r_t, t) = e^{a(\tau)+b(\tau)r_t+c(\tau)v_t+d(\tau)v_t^2-\log[P(t,T)]}. \tag{6.0.26}$$

From equation (6.0.22), $b(\tau)$ is given by:

$$b(\tau) = \frac{\imath\xi\left(1-e^{-k_r\tau}\right)}{k_r}. \tag{6.0.27}$$

Using the Riccati′s equation (6.0.24), $d(\tau)$ is given by:

$$d(\tau) = -\frac{2(1+\imath\xi)\sigma_v\rho_{v,x} - 2k_v - \Delta}{4\sigma_v^2}\frac{1-e^{\Delta\tau}}{1-e^{\Delta\tau}\Lambda}, \tag{6.0.28}$$

with

$$\Delta = \sqrt{(2(1+\imath\xi)\sigma_v\rho_{v,x} - 2k_v)^2 - 8\sigma_v^2\left(\frac{1}{2}\imath\xi - \frac{1}{2}\xi^2\right)}, \tag{6.0.29}$$

$$\Lambda = \frac{2(1+\imath\xi)\sigma_v\rho_{v,x} - 2k_v - \Delta}{2(1+\imath\xi)\sigma_v\rho_{v,x} - 2k_v + \Delta}. \tag{6.0.30}$$

Concerning the function $c(\tau)$, equation (6.0.23) can be rewritten as:

$$\begin{aligned} &\frac{\partial c}{\partial \tau} - \left[(1+\imath\xi)\sigma_v\rho_{v,x} - k_v + 2\sigma_v^2 d(\tau)\right]c(\tau) = \\ &\quad 2k_v\theta_v d(\tau) + (1+\imath\xi)\sigma_r\rho_{r,x}b(\tau) + 2\sigma_r\sigma_v\rho_{v,r}b(\tau)d(\tau), \end{aligned}$$

yielding the integrating factor

$$e^{-(1+\imath\xi)\sigma_v\rho_{v,x}\tau+k_v\tau-2\sigma_v^2\int d(\tau)\,\mathrm{d}\tau}.$$

Therefore, $c(\tau)$ is given by:

$$\begin{aligned}
&c(\tau)e^{-(1+\imath\xi)\sigma_v\rho_{v,x}\tau+k_v\tau-2\sigma_v^2\int d(\tau)\,\mathrm{d}\tau}\\
&\qquad=\int e^{-(1+\imath\xi)\sigma_v\rho_{v,x}\tau+k_v\tau-2\sigma_v^2\int d(\tau)\,\mathrm{d}\tau}2k_v\theta_v d(\tau)\,\mathrm{d}\tau\\
&\quad+\int e^{-(1+\imath\xi)\sigma_v\rho_{v,x}\tau+k_v\tau-2\sigma_v^2\int d(\tau)\,\mathrm{d}\tau}(1+\imath\xi)\sigma_r\rho_{r,x}b(\tau)\,\mathrm{d}\tau\\
&\quad+\int e^{-(1+\imath\xi)\sigma_v\rho_{v,x}\tau+k_v\tau-2\sigma_v^2\int d(\tau)\,\mathrm{d}\tau}2\sigma_r\sigma_v\rho_{v,r}b(\tau)d(\tau)\,\mathrm{d}\tau.
\end{aligned} \tag{6.0.31}$$

Since $\int d(\tau)\,\mathrm{d}\tau$ can be computed by Proposition D.1, it follows that:

$$\int d(\tau)\,\mathrm{d}\tau=-\frac{1}{4\sigma_v^2}\left[\left(2(1+\imath\xi)\sigma_v\rho_{v,x}-2k_v-\Delta\right)\tau+2\log\left[\frac{1-\Lambda e^{\Delta\tau}}{1-\Lambda}\right]\right]. \tag{6.0.32}$$

Consequently, the integrating factor can be rewritten as:

$$e^{-\frac{\Delta}{2}\tau+\log\left[\frac{1-\Lambda e^{\Delta\tau}}{1-\Lambda}\right]}=e^{-\frac{\Delta}{2}\tau}\frac{1-\Lambda e^{\Delta\tau}}{1-\Lambda}. \tag{6.0.33}$$

The first integral of equation (6.0.31) is equivalent to:

$$\begin{aligned}
&-2k_v\theta_v\frac{2(1+\imath\xi)\sigma_v\rho_{v,x}-2k_v-\Delta}{4\sigma_v^2}\int e^{-\frac{\Delta}{2}\tau}\frac{1-e^{\Delta\tau}}{1-\Lambda}\,\mathrm{d}\tau\\
&\quad=-2k_v\theta_v\frac{2(1+\imath\xi)\sigma_v\rho_{v,x}-2k_v-\Delta}{4\sigma_v^2}\frac{2e^{-\frac{\Delta}{2}\tau}\left(e^{\Delta\tau}+1\right)}{\Delta(\Lambda-1)}.
\end{aligned} \tag{6.0.34}$$

The second integral of equation (6.0.31) is equivalent to:

$$\begin{aligned}
&(1+\imath\xi)\sigma_r\rho_{r,x}\int e^{-\frac{\Delta}{2}\tau}\frac{1-\Lambda e^{\Delta\tau}}{1-\Lambda}\frac{\imath\xi\left(1-e^{-k_r\tau}\right)}{k_r}\,\mathrm{d}\tau\\
&\quad=(1+\imath\xi)\sigma_r\rho_{r,x}\frac{\imath\xi}{k_r(1-\Lambda)}\int e^{-\frac{\Delta}{2}\tau}\left(1-\Lambda e^{\Delta\tau}\right)\left(1-e^{-k_r\tau}\right)\mathrm{d}\tau\\
&\quad=(1+\imath\xi)\sigma_r\rho_{r,x}\frac{\imath\xi}{k_r(1-\Lambda)}\left[\frac{e^{-\frac{\Delta}{2}\tau}}{-\frac{\Delta}{2}}-\frac{e^{\left(-\frac{\Delta}{2}-k_r\right)\tau}}{-\frac{\Delta}{2}-k_r}-\Lambda\frac{e^{\frac{\Delta}{2}\tau}}{\frac{\Delta}{2}}+\Lambda\frac{e^{\left(\frac{\Delta}{2}-k_r\right)\tau}}{\frac{\Delta}{2}-k_r}\right].
\end{aligned} \tag{6.0.35}$$

The third integral of equation (6.0.31) is equivalent to:

$$\begin{aligned} &2\sigma_r\sigma_v\rho_{v,r}\frac{\imath\xi}{k_r(1-\Lambda)}\int\left(1-e^{\Delta\tau}\right)e^{-\frac{\Delta}{2}\tau}\left(1-e^{-k_r\tau}\right)\mathrm{d}\tau \\ &\quad = 2\sigma_r\sigma_v\rho_{v,r}\frac{\imath\xi}{k_r(1-\Lambda)}\left[\frac{e^{-\frac{\Delta}{2}\tau}}{-\frac{\Delta}{2}}-\frac{e^{\left(-\frac{\Delta}{2}-k_r\right)\tau}}{-\frac{\Delta}{2}-k_r}-\frac{e^{\frac{\Delta}{2}\tau}}{\frac{\Delta}{2}}+\frac{e^{\left(\frac{\Delta}{2}-k_r\right)\tau}}{\frac{\Delta}{2}-k_r}\right]. \end{aligned} \tag{6.0.36}$$

Then, $c(\tau)$ is given by:

$$\begin{aligned} c(\tau) &= e^{\frac{\Delta}{2}\tau}\frac{1-\Lambda}{1-\Lambda e^{\Delta\tau}}\left[-2k_v\theta_v\frac{2(1+\imath\xi)\sigma_v\rho_{v,x}-2k_v-\Delta}{4\sigma_v^2}\frac{2e^{-\frac{\Delta}{2}\tau}\left(e^{\Delta\tau}+1\right)}{\Delta(\Lambda-1)}\right] \\ &+ e^{\frac{\Delta}{2}\tau}\frac{1-\Lambda}{1-\Lambda e^{\Delta\tau}}\left[(1+\imath\xi)\sigma_r\rho_{r,x}\frac{\imath\xi}{k_r(1-\Lambda)}\left[\frac{e^{-\frac{\Delta}{2}\tau}}{-\frac{\Delta}{2}}-\frac{e^{\left(-\frac{\Delta}{2}-k_r\right)\tau}}{-\frac{\Delta}{2}-k_r}-\Lambda\frac{e^{\frac{\Delta}{2}\tau}}{\frac{\Delta}{2}}+\Lambda\frac{e^{\left(\frac{\Delta}{2}-k_r\right)\tau}}{\frac{\Delta}{2}-k_r}\right]\right] \\ &\quad + e^{\frac{\Delta}{2}\tau}\frac{1-\Lambda}{1-\Lambda e^{\Delta\tau}}\left[2\sigma_r\sigma_v\rho_{v,r}\frac{\imath\xi}{k_r(1-\Lambda)}\left[\frac{e^{-\frac{\Delta}{2}\tau}}{-\frac{\Delta}{2}}-\frac{e^{\left(-\frac{\Delta}{2}-k_r\right)\tau}}{-\frac{\Delta}{2}-k_r}-\frac{e^{\frac{\Delta}{2}\tau}}{\frac{\Delta}{2}}+\frac{e^{\left(\frac{\Delta}{2}-k_r\right)\tau}}{\frac{\Delta}{2}-k_r}\right]\right], \end{aligned} \tag{6.0.37}$$

which is the same as:

$$\begin{aligned} c(\tau) &= k_v\theta_v\frac{2(1+\imath\xi)\sigma_v\rho_{v,x}-2k_v-\Delta}{\sigma_v^2\Delta}\frac{e^{\Delta\tau}+1}{1-\Lambda e^{\Delta\tau}} \\ &+ (1+\imath\xi)\sigma_r\rho_{r,x}\frac{\imath\xi}{k_r}\left[-\frac{2}{\Delta}\frac{1}{1-\Lambda e^{\Delta\tau}}+\frac{1}{\frac{\Delta}{2}+k_r}\frac{e^{-k_r\tau}}{1-\Lambda e^{\Delta\tau}}-\frac{2\Lambda}{\Delta}\frac{e^{\Delta\tau}}{1-\Lambda e^{\Delta\tau}}+\frac{\Lambda}{\frac{\Delta}{2}-k_r}\frac{e^{(\Delta-k_r)\tau}}{1-\Lambda e^{\Delta\tau}}\right] \\ &\quad + 2\sigma_r\sigma_v\rho_{v,r}\frac{\imath\xi}{k_r}\left[-\frac{2}{\Delta}\frac{1}{1-\Lambda e^{\Delta\tau}}+\frac{1}{\frac{\Delta}{2}+k_r}\frac{e^{-k_r\tau}}{1-\Lambda e^{\Delta\tau}}-\frac{2}{\Delta}\frac{e^{\Delta\tau}}{1-\Lambda e^{\Delta\tau}}+\frac{1}{\frac{\Delta}{2}-k_r}\frac{e^{(\Delta-k_r)\tau}}{1-\Lambda e^{\Delta\tau}}\right]. \end{aligned} \tag{6.0.38}$$

Finally, function $a(\tau)$ follows by simply integrating equation (6.0.25) and using the explicit solutions (6.0.27), (6.0.28) and (6.0.38).

Conclusion

In this work, the characteristic function of the Bakshi et al. (1997) model was deduced by using the time-changed Lévy process technique. Therefore, we were able to deduce the pricing formulas for European-style standard call and put options, and a Gauss-Kronrod quadrature was used for the inversion of the characteristic function. In the last chapter we have proposed an exact model with stochastic volatility, stochastic interest rates, jumps and a full correlation scheme. We were able to obtain the exact characteristic function of the asset price under this model, which is affine in the state variables and in the square of the volatility. A different approach for the stochastic interest rate can be made by defining the business time as $T_t := \int_0^t r^2(s)\,\mathrm{ds}$, but further investigation is needed in order to understand the consequences of this new business time for the characteristic function, in particular if it remains affine on the state variables, on the square of the volatility, and possibly, on the square of the interest rate. A closer look on the last chapter suggests that the characteristic function under such scenario needs to have an additional term in $r_t v_t$. To understand the consequences of our full correlation model, a deeper investigation is needed, and could be the subject of future works.

Appendices

Appendix A

Matlab Code function 1

```
function [call,put,errbndQS,errbndQT] = bakshi(kr,sigmar,thetar,
    rzero,T,kv,sigmav,thetav,vzero,intensity,mu,sigma,q,rho,K,S0,
    limitesuperior)

%%% Zero coupon bond

delta = ( kr.^2 + 2.*sigmar.^2 ).^0.5;

B = ( 2.*(1 - exp(delta.*T)) )./( 2.*delta + (kr + delta).*(exp(
    delta.*T) - 1));

A = - ((kr.*thetar)./(sigmar.^2)).*( (-delta - kr).*T + 2.*log(
    1 + (kr + delta).*(exp(delta.*T) - 1)./(2.*delta)) );

P = exp( A + B.*rzero);

%%% Measure QS

deltarQS = @(x)(kr.^2 - 2.*(sigmar.^2).*1i.*x).^0.5;

deltavQS = @(x)((1i.*x.*rho.*sigmav - kv + sigmav.*rho).^2 + (
    sigmav.^2).*(x.^2 - 1i.*x)).^0.5;

lambdaQS = @(x)(1i.*x.*rho.*sigmav - kv + sigmav.*rho - deltavQS
    (x))./(1i.*x.*rho.*sigmav - kv + sigmav.*rho + deltavQS(x));

CFQS = @(x)exp( - ((kr.*thetar)./(sigmar.^2)).*( (-kr - deltarQS
    (x)).*T + 2.*log(1 + (kr + deltarQS(x)).*(exp(deltarQS(x).*T)
```

```
 - 1)./(2.*deltarQS(x)))) - (2.*1i.*x.*(1 - exp(deltarQS(x).*
T))))./(2.*deltarQS(x) + (-kr - deltarQS(x)).*(1 - exp(
deltarQS(x).*T))).*rzero - ((1i.*x.*rho.*sigmav - kv + sigmav
.*rho - deltavQS(x))./(sigmav.^2)).*((1 - exp(deltavQS(x).*T)
)./(1 - exp(deltavQS(x).*T).*lambdaQS(x))).*vzero - (kv.*
thetav)./(sigmav.^2).*((1i.*x.*rho.*sigmav - kv + sigmav.*rho
 - deltavQS(x)).*T + 2.*log((1 - exp(deltavQS(x).*T).*
lambdaQS(x))./(1 - lambdaQS(x)))) - 1i.*x.*intensity.*mu.*T +
 T.*intensity.*(1 + mu).*( ((1 + mu).^(1i.*x)).*exp( ((1i.*x)
./2).*(sigma.^2).*(1i.*x + 1) ) - 1) - 1i.*x.*q.*T);

%%% Measure QT

deltarQT = @(x)(kr.^2 - 2.*(sigmar.^2).*(1i.*x - 1)).^0.5;

deltavQT = @(x)((1i.*x.*rho.*sigmav - kv).^2 + (sigmav.^2).*(x
    .^2 + 1i.*x)).^0.5;

lambdaQT = @(x)(1i.*x.*rho.*sigmav - kv - deltavQT(x))./(1i.*x.*
    rho.*sigmav - kv + deltavQT(x));

CFQT = @(x)exp( - ((kr.*thetar)./(sigmar.^2)).*( (-kr - deltarQT
    (x)).*T + 2.*log(1 + (kr + deltarQT(x)).*(exp(deltarQT(x).*T)
     - 1)./(2.*deltarQT(x)))) - (2.*(1i.*x - 1).*(1 - exp(
    deltarQT(x).*T)))./(2.*deltarQT(x) + (-kr - deltarQT(x)).*(1
    - exp(deltarQT(x).*T))).*rzero - log(P) - ((1i.*x.*rho.*
    sigmav - kv - deltavQT(x))./(sigmav.^2)).*((1 - exp(deltavQT(
    x).*T))./(1 - exp(deltavQT(x).*T).*lambdaQT(x))).*vzero - (kv
    .*thetav)./(sigmav.^2).*((1i.*x.*rho.*sigmav - kv - deltavQT(
    x)).*T + 2.*log((1 - exp(deltavQT(x).*T).*lambdaQT(x))./(1 -
    lambdaQT(x)))) - 1i.*x.*intensity.*mu.*T + T.*intensity.*(
    ((1 + mu).^(1i.*x)).*exp( ((1i.*x)./2).*(sigma.^2).*(1i.*x -
    1) ) - 1) - 1i.*x.*q.*T);

%%% price of a call option

FICFQS = @(x)real( exp(-1i.*x.*log(K./S0)).*CFQS(x)./(1i.*x) );

FICFQT = @(x)real( exp(-1i.*x.*log(K./S0)).*CFQT(x)./(1i.*x) );

[QS,errbndQS] = quadgk(FICFQS,0,limitesuperior,'AbsTol',1e-12,'
    RelTol',1e-8);

[QT,errbndQT] = quadgk(FICFQT,0,limitesuperior,'AbsTol',1e-12,'
    RelTol',1e-8);
```

```
call = S0.*exp(-q.*T).*( 1./2 + 1./pi.*QS ) - K.*P.*( 1./2 + 1./
    pi.*QT );

put = call - exp(- q.*T).*S0 + P.*K;
```

Appendix B

Matlab Code function 2

```
function [call,put,errbndQS,errbndQT] = bakshikou(kr,sigmar,
    thetar,rzero,T,kv,sigmav,thetav,vzero,intensity,eta1,eta2,p,q
    ,rho,K,S0,limitesuperior)

%%% Zero coupon bond

delta = ( kr.^2 + 2.*sigmar.^2 ).^0.5;

B = ( 2.*(1 - exp(delta.*T)) )./( 2.*delta + (kr + delta).*(exp(
    delta.*T) - 1));

A = - ((kr.*thetar)./(sigmar.^2)).*( (-delta - kr).*T + 2.*log(
    1 + (kr + delta).*(exp(delta.*T) - 1)./(2.*delta)) );

P = exp( A + B.*rzero);

%%% Measure QS

deltarQS = @(x)(kr.^2 - 2.*(sigmar.^2).*1i.*x).^0.5;

deltavQS = @(x)((1i.*x.*rho.*sigmav - kv + sigmav.*rho).^2 + (
    sigmav.^2).*(x.^2 - 1i.*x)).^0.5;

lambdaQS = @(x)(1i.*x.*rho.*sigmav - kv + sigmav.*rho - deltavQS
    (x))./(1i.*x.*rho.*sigmav - kv + sigmav.*rho + deltavQS(x));

ptil = p.*( p.*(eta1./(eta1 - 1)) + (1 - p).*(eta2./(eta2 + 1)))
    .^(-1).*(eta1./(eta1 - 1));
```

```
qtil = (1 - p).*( p.*(eta1./(eta1 - 1)) + (1 - p).*(eta2./(eta2
    + 1))).^(-1).*(eta2./(eta2 + 1));

eta1til = eta1 - 1;

eta2til = eta2 + 1;

CFQS = @(x)exp( - ((kr.*thetar)./(sigmar.^2)).*( (-kr - deltarQS
    (x)).*T + 2.*log(1 + (kr + deltarQS(x)).*(exp(deltarQS(x).*T)
     - 1)./(2.*deltarQS(x)))) - (2.*1i.*x.*(1 - exp(deltarQS(x).*
    T)))./(2.*deltarQS(x) + (-kr - deltarQS(x)).*(1 - exp(
    deltarQS(x).*T))).*rzero - ((1i.*x.*rho.*sigmav - kv + sigmav
    .*rho - deltavQS(x))./(sigmav.^2)).*((1 - exp(deltavQS(x).*T)
    )./(1 - exp(deltavQS(x).*T).*lambdaQS(x))).*vzero - (kv.*
    thetav)./(sigmav.^2).*((1i.*x.*rho.*sigmav - kv + sigmav.*rho
     - deltavQS(x)).*T + 2.*log((1 - exp(deltavQS(x).*T).*
    lambdaQS(x))./(1 - lambdaQS(x)))) + 1i.*x.*intensity.*(1 - p
    .*(eta1./(eta1 - 1)) - (1 - p).*(eta2./(eta2 + 1))).*T + T.*
    intensity.*(p.*(eta1./(eta1 - 1)) + (1 - p).*(eta2./(eta2 +
    1))).*(ptil.*(eta1til./(eta1til - 1i.*x)) + qtil.*(eta2til./(
    eta2til + 1i.*x)) - 1) - 1i.*x.*q.*T);

%%% Measure QT

deltarQT = @(x)(kr.^2 - 2.*(sigmar.^2).*(1i.*x - 1)).^0.5;

deltavQT = @(x)((1i.*x.*rho.*sigmav - kv).^2 + (sigmav.^2).*(x
    .^2 + 1i.*x)).^0.5;

lambdaQT = @(x)(1i.*x.*rho.*sigmav - kv - deltavQT(x))./(1i.*x.*
    rho.*sigmav - kv + deltavQT(x));

CFQT = @(x)exp( - ((kr.*thetar)./(sigmar.^2)).*( (-kr - deltarQT
    (x)).*T + 2.*log(1 + (kr + deltarQT(x)).*(exp(deltarQT(x).*T)
     - 1)./(2.*deltarQT(x)))) - (2.*(1i.*x - 1).*(1 - exp(
    deltarQT(x).*T)))./(2.*deltarQT(x) + (-kr - deltarQT(x)).*(1
    - exp(deltarQT(x).*T))).*rzero - log(P) - ((1i.*x.*rho.*
    sigmav - kv - deltavQT(x))./(sigmav.^2)).*((1 - exp(deltavQT(
    x).*T))./(1 - exp(deltavQT(x).*T).*lambdaQT(x))).*vzero - (kv
    .*thetav)./(sigmav.^2).*((1i.*x.*rho.*sigmav - kv - deltavQT(
    x)).*T + 2.*log((1 - exp(deltavQT(x).*T).*lambdaQT(x))./(1 -
    lambdaQT(x)))) + 1i.*x.*intensity.*(1 - p.*(eta1./(eta1 - 1))
     - (1 - p).*(eta2./(eta2 + 1))).*T + T.*intensity.*(p.*(eta1
    ./(eta1 - 1i.*x)) + (1 - p).*(eta2./(eta2 + 1i.*x)) - 1) - 1i
```

```
    .*x.*q.*T);

%%% price of a call option

FICFQS = @(x)real( exp(-1i.*x.*log(K./S0)).*CFQS(x)./(1i.*x) );

FICFQT = @(x)real( exp(-1i.*x.*log(K./S0)).*CFQT(x)./(1i.*x) );

[QS,errbndQS] = quadgk(FICFQS,0,limitesuperior,'AbsTol',1e-12,'
    RelTol',1e-8);

[QT,errbndQT] = quadgk(FICFQT,0,limitesuperior,'AbsTol',1e-12,'
    RelTol',1e-8);


call = S0.*exp(-q.*T).*( 1./2 + 1./pi.*QS ) - K.*P.*( 1./2 + 1./
    pi.*QT );

put = call - exp(- q.*T).*S0 + P.*K;
```

Appendix C

Matlab Code function 3

```
function [call,put,errbndQS,errbndQT] = model

%%% model: bs for Black Scholes ; heston for heston model ;
    hestoncir
%%% for heston model with stochastic interest rates following a
    CIR
%%% process; bates for bates model ; bakshi for bakshi, cao and
    chen model;
%%% hestonkou for the heston model with double exponential
    distributed jumps;
%%% bakshikou for the bakshi model with double exponential
    distributed jumps;
%%% kou for the kou model;
%%% merton for the jump-diffusion Merton Model.

prompt = 'Write bs for Black Scholes ; heston for heston model
    ;\nhestoncir for heston model with stochastic interest rates
    following a CIR process ;\nbates for bates model ; bakshi for
     bakshi, cao and chen model ;\nkou for the kou model ;\
    nhestonkou for the heston model with double exponential
    distributed jumps ;\nbakshikou for the bakshi model with
    double exponential distributed jumps;\nmerton for the jump-
    diffusion Merton Model.\n';

str = input(prompt,'s');

format long
```

```
switch(str)

    case 'bs'
        prompt = 'S0\n';
        S0 = input(prompt);
        prompt = 'K\n';
        K = input(prompt);
        prompt = 'r0\n';
        rzero = input(prompt);
        prompt = 'Time to maturity\n';
        T = input(prompt);
        prompt = 'Volatility\n';
        vzero = input(prompt).^2;
        prompt = 'Dividend yield\n';
        q = input(prompt);
        prompt = 'Upper limit\n';
        limitesuperior = input(prompt);

        [call,put,errbndQS,errbndQT] = bakshi(1e-6,1e-6,1e-6,
            rzero,T,1e-6,1e-6,1e-6,vzero,0,0,0,q,0,K,S0,
            limitesuperior);

    case 'heston'
        prompt = 'S0\n';
        S0 = input(prompt);
        prompt = 'K\n';
        K = input(prompt);
        prompt = 'r0\n';
        rzero = input(prompt);
        prompt = 'Time to maturity\n';
        T = input(prompt);
        prompt = 'v0\n';
        vzero = input(prompt);
        prompt = 'Dividend yield\n';
        q = input(prompt);
        prompt = 'kv\n';
        kv = input(prompt);
        prompt = 'thetav\n';
        thetav = input(prompt);
        prompt = 'sigmav\n';
        sigmav = input(prompt);
        prompt = 'rho\n';
        rho = input(prompt);
        prompt = 'Upper limit\n';
        limitesuperior = input(prompt);
```

```
    [call,put,errbndQS,errbndQT] = bakshi(1e-6,1e-6,1e-6,
        rzero,T,kv,sigmav,thetav,vzero,0,0,0,q,rho,K,S0,
        limitesuperior);

case 'hestoncir'
    prompt = 'S0\n';
    S0 = input(prompt);
    prompt = 'K\n';
    K = input(prompt);
    prompt = 'r0\n';
    rzero = input(prompt);
    prompt = 'Time to maturity\n';
    T = input(prompt);
    prompt = 'v0\n';
    vzero = input(prompt);
    prompt = 'Dividend yield\n';
    q = input(prompt);
    prompt = 'kv\n';
    kv = input(prompt);
    prompt = 'thetav\n';
    thetav = input(prompt);
    prompt = 'sigmav\n';
    sigmav = input(prompt);
    prompt = 'kr\n';
    kr = input(prompt);
    prompt = 'thetar\n';
    thetar = input(prompt);
    prompt = 'sigmar\n';
    sigmar = input(prompt);
    prompt = 'rho\n';
    rho = input(prompt);
    prompt = 'Upper limit\n';
    limitesuperior = input(prompt);

    [call,put,errbndQS,errbndQT] = bakshi(kr,sigmar,thetar,
        rzero,T,kv,sigmav,thetav,vzero,0,0,0,q,rho,K,S0,
        limitesuperior);

case 'bates'
    prompt = 'S0\n';
    S0 = input(prompt);
    prompt = 'K\n';
    K = input(prompt);
    prompt = 'r0\n';
```

```
    rzero = input(prompt);
    prompt = 'Time to maturity\n';
    T = input(prompt);
    prompt = 'v0\n';
    vzero = input(prompt);
    prompt = 'Dividend yield\n';
    q = input(prompt);
    prompt = 'kv\n';
    kv = input(prompt);
    prompt = 'thetav\n';
    thetav = input(prompt);
    prompt = 'sigmav\n';
    sigmav = input(prompt);
    prompt = 'intensity\n';
    intensity = input(prompt);
    prompt = 'mu\n';
    mu = input(prompt);
    prompt = 'sigma\n';
    sigma = input(prompt);
    prompt = 'rho\n';
    rho = input(prompt);
    prompt = 'Upper limit\n';
    limitesuperior = input(prompt);

    [call,put,errbndQS,errbndQT] = bakshi(1e-6,1e-6,1e-6,
       rzero,T,kv,sigmav,thetav,vzero,intensity,mu,sigma,q,
       rho,K,S0,limitesuperior);

case 'bakshi'
    prompt = 'S0\n';
    S0 = input(prompt);
    prompt = 'K\n';
    K = input(prompt);
    prompt = 'r0\n';
    rzero = input(prompt);
    prompt = 'Time to maturity\n';
    T = input(prompt);
    prompt = 'v0\n';
    vzero = input(prompt);
    prompt = 'Dividend yield\n';
    q = input(prompt);
    prompt = 'kv\n';
    kv = input(prompt);
    prompt = 'thetav\n';
    thetav = input(prompt);
```

```
        prompt = 'sigmav\n';
        sigmav = input(prompt);
        prompt = 'kr\n';
        kr = input(prompt);
        prompt = 'thetar\n';
        thetar = input(prompt);
        prompt = 'sigmar\n';
        sigmar = input(prompt);
        prompt = 'intensity\n';
        intensity = input(prompt);
        prompt = 'mu\n';
        mu = input(prompt);
        prompt = 'sigma\n';
        sigma = input(prompt);
        prompt = 'rho\n';
        rho = input(prompt);
        prompt = 'Upper limit\n';
        limitesuperior = input(prompt);

        [call,put,errbndQS,errbndQT] = bakshi(kr,sigmar,thetar,
           rzero,T,kv,sigmav,thetav,vzero,intensity,mu,sigma,q,
           rho,K,S0,limitesuperior);

    case 'kou'
        prompt = 'S0\n';
        S0 = input(prompt);
        prompt = 'K\n';
        K = input(prompt);
        prompt = 'r0\n';
        rzero = input(prompt);
        prompt = 'Time to maturity\n';
        T = input(prompt);
        prompt = 'Volatility\n';
        vzero = input(prompt).^2;
        prompt = 'Dividend yield\n';
        q = input(prompt);
        prompt = 'Intensity\n';
        intensity = input(prompt);
        prompt = 'eta1\n';
        eta1 = input(prompt);
        prompt = 'eta2\n';
        eta2 = input(prompt);
        prompt = 'p\n';
        p = input(prompt);
        prompt = 'Upper limit\n';
```

```
    limitesuperior = input(prompt);

    [call,put,errbndQS,errbndQT] = bakshikou(1e-6,1e-6,1e-6,
       rzero,T,1e-6,1e-6,1e-6,vzero,intensity,eta1,eta2,p,q
       ,1e-6,K,S0,limitesuperior);

case 'merton'
    prompt = 'S0\n';
    S0 = input(prompt);
    prompt = 'K\n';
    K = input(prompt);
    prompt = 'r0\n';
    rzero = input(prompt);
    prompt = 'Time to maturity\n';
    T = input(prompt);
    prompt = 'Volatility\n';
    vzero = input(prompt).^2;
    prompt = 'Dividend yield\n';
    q = input(prompt);
    prompt = 'intensity\n';
    intensity = input(prompt);
    prompt = 'mu\n';
    mu = input(prompt);
    prompt = 'sigma\n';
    sigma = input(prompt);
    prompt = 'Upper limit\n';
    limitesuperior = input(prompt);

    [call,put,errbndQS,errbndQT] = bakshi(1e-6,1e-6,1e-6,
       rzero,T,1e-6,1e-6,1e-6,vzero,intensity,mu,sigma,q,0,K
       ,S0,limitesuperior);

case 'hestonkou'
    prompt = 'S0\n';
    S0 = input(prompt);
    prompt = 'K\n';
    K = input(prompt);
    prompt = 'r0\n';
    rzero = input(prompt);
    prompt = 'Time to maturity\n';
    T = input(prompt);
    prompt = 'v0\n';
    vzero = input(prompt);
    prompt = 'Dividend yield\n';
    q = input(prompt);
```

```
        prompt = 'kv\n';
        kv = input(prompt);
        prompt = 'thetav\n';
        thetav = input(prompt);
        prompt = 'sigmav\n';
        sigmav = input(prompt);
        prompt = 'Intensity\n';
        intensity = input(prompt);
        prompt = 'eta1\n';
        eta1 = input(prompt);
        prompt = 'eta2\n';
        eta2 = input(prompt);
        prompt = 'p\n';
        p = input(prompt);
        prompt = 'rho\n';
        rho = input(prompt);
        prompt = 'Upper limit\n';
        limitesuperior = input(prompt);

        [call,put,errbndQS,errbndQT] = bakshikou(1e-6,1e-6,1e-6,
            rzero,T,kv,sigmav,thetav,vzero,intensity,eta1,eta2,p,
            q,rho,K,S0,limitesuperior);

    case 'bakshikou'
        prompt = 'S0\n';
        S0 = input(prompt);
        prompt = 'K\n';
        K = input(prompt);
        prompt = 'r0\n';
        rzero = input(prompt);
        prompt = 'Time to maturity\n';
        T = input(prompt);
        prompt = 'v0\n';
        vzero = input(prompt);
        prompt = 'Dividend yield\n';
        q = input(prompt);
        prompt = 'kv\n';
        kv = input(prompt);
        prompt = 'thetav\n';
        thetav = input(prompt);
        prompt = 'sigmav\n';
        sigmav = input(prompt);
        prompt = 'kr\n';
        kr = input(prompt);
        prompt = 'thetar\n';
```

```
        thetar = input(prompt);
        prompt = 'sigmar\n';
        sigmar = input(prompt);
        prompt = 'Intensity\n';
        intensity = input(prompt);
        prompt = 'eta1\n';
        eta1 = input(prompt);
        prompt = 'eta2\n';
        eta2 = input(prompt);
        prompt = 'p\n';
        p = input(prompt);
        prompt = 'rho\n';
        rho = input(prompt);
        prompt = 'Upper limit\n';
        limitesuperior = input(prompt);

        [call,put,errbndQS,errbndQT] = bakshikou(kr,sigmar,
           thetar,rzero,T,kv,sigmav,thetav,vzero,intensity,eta1,
           eta2,p,q,rho,K,S0,limitesuperior);

        otherwise
        error('Unexpected model, try again.');

end
```

Appendix D

Riccati′s equation

Proposition D.1. ***Riccati′s equation***

Consider the following Riccati′s equation with constant parameters:

$$b'(t) = a + cb(t) + db(t)^2, \tag{D.0.1}$$

$$b(0) = 0. \tag{D.0.2}$$

The solution of such problem can be written as:

$$b(t) = -\frac{c-\Delta}{2d}\frac{1-e^{\Delta t}}{1-e^{\Delta t}\Lambda}, \tag{D.0.3}$$

$$\Delta = \sqrt{c^2 - 4da}, \tag{D.0.4}$$

$$\Lambda = \frac{c-\Delta}{c+\Delta}. \tag{D.0.5}$$

Moreover, if we consider $c'(t) = k_1 b(t)$ with $c(0) = 0$, $c(t)$ as the form:

$$c(t) = -\frac{k_1}{2d}\left[(c-\Delta)t + 2\log\left(\frac{1-\Lambda e^{\Delta t}}{1-\Lambda}\right)\right]. \tag{D.0.6}$$

Proof.

First we take equation (D.0.1), with initial condition (D.0.2). We can rewrite the right hand side of equation (D.0.1) as:

$$b'(t) = d\left(b(t) + \frac{c+\Delta}{2d}\right)\left(b(t) + \frac{c-\Delta}{2d}\right), \tag{D.0.7}$$

because,

$$b(t) = \frac{-c \pm \sqrt{c^2 - 4ad}}{2d} = \frac{-c \pm \Delta}{2d}, \qquad \Delta = \sqrt{c^2 - 4ad}, \tag{D.0.8}$$

solves

$$a + cb(t) + db(t)^2 = 0. \tag{D.0.9}$$

Equation (D.0.7) can be written as:

$$\frac{b'(t)}{d\left(b(t) + \frac{c+\Delta}{2d}\right)\left(b(t) + \frac{c-\Delta}{2d}\right)} = 1. \tag{D.0.10}$$

It is easy to check that the above equation as the following similar form

$$\frac{1}{\Delta}\left(-\frac{b'(t)}{b(t) + \frac{c+\Delta}{2d}} + \frac{b'(t)}{b(t) + \frac{c-\Delta}{2d}}\right) = 1, \tag{D.0.11}$$

which is the same as:

$$-\frac{b'(t)}{b(t) + \frac{c+\Delta}{2d}} + \frac{b'(t)}{b(t) + \frac{c-\Delta}{2d}} = \Delta. \tag{D.0.12}$$

Integrating in both sides, we have:

$$-\log\left(b(t) + \frac{c+\Delta}{2d}\right) + \log\left(b(t) + \frac{c-\Delta}{2d}\right) = \Delta t + H, \qquad H \in \mathbb{R}. \tag{D.0.13}$$

Thereby, we can found $b(t)$ from equation (D.0.13), and it turns to be as:

$$b(t) = \frac{(c+\Delta)e^{\Delta t + H} - (c - \Delta)}{2d\left(1 - e^{\Delta t + H}\right)}. \tag{D.0.14}$$

Using equations (D.0.14) and (D.0.2), we have that:

$$e^H = \frac{c - \Delta}{c + \Delta} = \Lambda. \tag{D.0.15}$$

Using equation (D.0.15), equation (D.0.14) can be written as:

$$b(t) = -\frac{c - \Delta}{2d}\frac{1 - e^{\Delta t}}{1 - e^{\Delta t}\Lambda}. \tag{D.0.16}$$

Which yields with equation (D.0.3).
For equation (D.0.6), we know that $c'(t) = k_1 b(t)$ with $c(0) = 0$, therefore, using equation (D.0.16) we have:

$$c'(t) = -k_1 \frac{c-\Delta}{2d} \frac{1-e^{\Delta t}}{1-e^{\Delta t}\Lambda}, \tag{D.0.17}$$

integrating in both sides, we have:

$$c(t) = -k_1 \frac{c-\Delta}{2d} \int \frac{1-e^{\Delta t}}{1-e^{\Delta t}\Lambda} \, \mathrm{dt} + G, \qquad G \in \mathbb{R}. \tag{D.0.18}$$

It is easy to check that:

$$\frac{1-e^{\Delta t}}{1-e^{\Delta t}\Lambda} = 1 + \frac{(\Lambda-1)e^{\Delta t}}{1-e^{\Delta t}\Lambda}, \tag{D.0.19}$$

using equation (D.0.19), equation (D.0.18) becomes:

$$c(t) = -k_1 \frac{c-\Delta}{2d} \left[t + \frac{\Lambda-1}{-\Lambda\Delta} \int \frac{-\Lambda\Delta e^{\Delta t}}{1-\Lambda e^{\Delta t}} \, \mathrm{dt} \right] + G \tag{D.0.20}$$

$$= -k_1 \frac{c-\Delta}{2d} \left[t + \frac{\Lambda-1}{-\Lambda\Delta} \log\left(1-\Lambda e^{\Delta t}\right) \right] + G. \tag{D.0.21}$$

Since $c(0) = 0$, we can obtain G from equation (D.0.21),

$$G = k_1 \frac{c-\Delta}{2d} \left[\frac{\Lambda-1}{-\Lambda\Delta} \log(1-\Lambda) \right]. \tag{D.0.22}$$

Using equation (D.0.22), equation (D.0.21) becomes:

$$c(t) = -k_1 \frac{c-\Delta}{2d} \left[t + \frac{1-\Lambda}{\Lambda\Delta} \log\left(\frac{1-\Lambda e^{\Delta t}}{1-\Lambda} \right) \right]. \tag{D.0.23}$$

Because

$$\frac{1-\Lambda}{\Lambda\Delta} = \frac{2}{c-\Delta}, \tag{D.0.24}$$

equation (D.0.6) follows:

$$c(t) = -\frac{k_1}{2d} \left[(c-\Delta)t + 2\log\left(\frac{1-\Lambda e^{\Delta t}}{1-\Lambda} \right) \right]. \tag{D.0.25}$$

■

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