FIXED RATE MORTGAGE VALUATION USING A CONTINGENT CLAIMS APPROACH

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ABSTRACT

In this thesis a methodology based on the Cox, Ingersoll and Ross (1985a) equilibrium model is developed in order to evaluate British fixed rate mortgage contracts. The spot interest rate and the house price are used as state variables and it is assumed that the borrowers' decision making process is driven by a purely economic rationale. In other words, the framework can be classified as a frictionless model. Both, repayment and ("without profits") endowment mortgages, with specifications based on the common provisions inherent in the British mortgage market, are evaluated. The model also provides values for mortgage indemnity guarantees, the corresponding coinsurance, the embedded options to terminate the loan through prepayment or default and the bond-type annuity corresponding to the mortgage monthly payments.

As the partial differential equation that gives the values of the different features of the mortgage contracts does not have a closed-form solution, an explicit finite difference method was used to solve the problem. Numerical results for the value of the different components of the mortgage contracts were determined under different economic scenarios. The relationship between the evolution of the parameters used to characterise the economic environment and the value of the different mortgage-related assets is in line with the underlying economic intuition.

The comparison between repayment and endowment mortgages leads to the conclusion that the value of the prepayment option tends to be higher in the latter case.

This implies that, ceteris paribus, the borrower needs to pay more in order to reach an equilibrium combination in "without profits" endowment mortgage contracts.

Changes in the contractual features inherent to the specification of the mortgage products lead to different equilibrium coupon rates and different values for the mortgage-related assets. Consequently, studies that do not take into account fundamental features embedded in mortgage contracts, like the prepayment and the default options or the early redemption penalty, tend to produce biases that might lead to misleading conclusions.

DECLARATION

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

DEDICATION

To my beloved wife Luzia, our sons José Henrique and António Gonçalo, and my parents António and Luciana.

To the future.

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THE AUTHOR

The author obtained a *"litentia gradum"* in Management [1984] from the Higher Institute of Economics (ISE) of the Technical University of Lisbon (UTL). He then joined the Azores University (1984-1985) as a Lecturer in Accounting and Finance. During this period he also worked in consultancy for the Azores Regional Government in the evaluation of investment projects. In the following academic year, while serving in the Portuguese army, he joined the Economics Faculty of the New University of Lisbon (UNL), where he taught Cost Accounting and Finance (1985-1987). He then joined the Higher Institute of Economics and Management (ISEG) of the Technical University of Lisbon (UTL), where he has taught Financial Accounting (1987-1990), Corporate Finance (1990-1993) and Investments (1991-1993). In 1992, he obtained an MBA from the Technical University of Lisbon, winning the prize for the best student of the course. After joining ISEG he has done consulting and training work with Partex, SA, Norma, SA, Associação Portuguesa de Bancos, Espaço Atlântico, SA and other Portuguese companies. In 1989 he became a Director of Tradingpor, SA. In January 1994, he joined the Doctoral Programme of the Manchester Business School, starting his doctoral research.

Valeu a pena ? Tudo vale a pena
Se a alma não é pequena.
Quem quer passar o Bojador
tem de passar além da dor.
Deus ao mar o perigo e o abismo deu,
Mas nele é que espelhou o céu.

Fernando Pessoa (in "Mensagem")

Was it worth while? It is worth while, all,
If the soul is not small.
Whoever means to sail behind the Cape
Must double sorrow - no escape.
Peril and abyss has God to the sea given
And yet made it the mirror of heaven.

Pessoa, Fernando, "Portuguese Ocean", in "Message"

Translated by, Griffin, Jonathan, "Fernando Pessoa. Selected Poems",

1982, 2nd Edition, Penguin Books, London.

CHAPTER 1

Introduction

1.1. A General Overview of the Real Estate and Mortgage Markets

The importance of the mortgage and real estate markets is widely recognized in well developed economies. Despite this, only a few would attribute to these markets the weight they deserve in terms of the global allocation of wealth. For instance in 1990, residential real estate and land accounted for more than half the wealth in the US[1].

In the UK, there are no publicly available statistics conveying a reasonably accurate notion of the situation at this level. However, it is still possible to find a measure of the relative importance of mortgage lending in the UK. Figure 1.1 shows the value of mortgage loans in the UK plotted against the value of the Public Sector debt. It can be easily observed that mortgage lending always exceeds, and in some years is nearly double, the size of the Public Sector debt.

[1] According to the "Balance Sheets for the US Economy 1949-1990", published by the Board of Governors of the Federal Reserve System (1990) the value of residential structures and land was \$ 8.35 trillion, and the USA's net worth amounted to \$ 16.24 trillion.

According to Breeden (1993), in the early 1990s, mortgage lending in the US was also much greater than the value of the Public Sector debt. This gives an immediate hint about the economic significance of the mortgage industry.

In the UK, despite the strong depression that affected the housing sector during the late 1980s and beginning of the 1990s, the size of the mortgage market continued to grow systematically during the whole period, as can be observed in Figure 1.2. Nowadays, housing is assumed to be one of the most basic needs in a developed society. With low savings levels in the UK as in other developed economies, and what now seems to be a natural tendency for the UK to keep a high and possibly increasing level of homeownership (see Chinloy and Megbolugbe, 1994), it may be concluded that there is a reasonable probabililty that the mortgage market will remain a major pool of wealth in this country.

Figure 1.3 portrays the mortgage advances, by main type of mortgage product, in the UK during a period of more than twenty five years. Endowment mortgages and repayment mortgages constitute more than 70% of the market. Until a few years ago, variable rate mortgages had a complete dominance of mortgage lending in the UK. However, as it can be seen in Figure 1.4, in recent years a significant percentage of the new mortgage advances have had fixed rate components. In other words, despite the prevalence of variable rate instruments, fixed rates are also present in the market with a weight that is not insignificant.

Clearly, it is important to understand the risks and returns of mortgage securities, and to have frameworks capable of evaluating them.

1.2. Motivation for the Research

The main aim of this thesis consists in developing a contingent claims setting whose general characteristics will provide a representation of fixed rate mortgages and mortgage related products that is closer to the British reality than those provided by the existing models of a similar type. The final result is a theoretical framework prepared for the valuation of "stylized" fixed rate mortgages and mortgage related products, in a continuous-time setting, where the state-variables are assumed to evolve stochastically.

The main reason for focusing on fixed rate mortgages is an operational one - with the computing power available it would be very difficult, if not impossible, to tackle the valuation of variable rate mortgages. Nevertheless, as will be explained later in more detail, the approach proposed here is readily extendible to variable rate mortgages once adequate computing power is available.

Besides that, the relative importance of these type of financial products, where the burden of the management of interest rate exposure is left in the hands of a specialized institution, will tend naturally to grow, since the ability of a financial institution to perform well in this role will also tend to be higher than that of the common borrower.

It must be emphasized that the financial assets modeled are only "stylized" versions of products traded in the British mortgage market. There are two main reasons for this. In the first place, as Breeden (1993) suggests "mortgages are ... too complicated to value precisely, even with the Black and Scholes model and the many improvements developed in the ... subsequent years". Therefore, at this stage in the development of theory, the natural aim will be to improve the adherence of the models to reality, not to give a perfect representation of it.

In addition, the precise specification of the mortgage products diverges significantly, not only between contracts, but also, for the same type of contract, between lenders. The guidelines that were adopted during the execution of this work. were:

i) whenever possible, model the assets under study in a general and flexible way. The use of this approach will eventually allow for the framework to be used as the basis of future exercises in which the precise specifications will be closer to those that correspond to the different products available in reality. In order to achieve this result, will be necessary to exclude some features and to add others, but the main framework can be kept;

ii) keep the modeling exercise inside the bounds of what it is clearly possible to solve mathematically with recourse to the tools employed. In some cases, this represents the reality in a significantly simplified way.

The US mortgage market is a natural source of contrast and reference in the field. Almost all the contingent claims literature on the valuation of this type of product has its origin in the US and obviously takes into consideration the specific design of the products available there. The present work is no exception and, whenever necessary, contrasts will be made with the North American mortgage market.

The motivation for undertaking this research includes several factors. In the first place, as suggested by Kau and Keenan (1995), mortgages are one of the most complex assets available in the financial marketplace. It is important that attempts to model mortgages in contexts increasingly closer to reality were made in order for us to have an increasingly better notion about the peculiarities of the contract and its reaction to the several factors that might influence its value.

In the US the application of the continuous-time stock and bond option pricing methodology (Black and Scholes, 1973; Merton, 1973; Cox, Ingersoll and Ross, 1985a,b, etc.) to mortgage valuation has been one of the focal points in the field since the beginning of the 1980s, when Dunn and McConnell (1981a,b) published their seminal papers in the area. However, the use of such techniques has not been so common in the UK, where one of the earlier examples is Ward (1987). This fact is especially relevant, not only because the markets have different structures, but also because the products available in each one have different characteristics (see, Douetil, 1994 and Lea, 1994).

The structural features of both markets show some relevant differences (see, Chinloy and Megbolugbe, 1994) at various levels. In a short summary, it can be said that the British market is dominated by a mortgage instrument that seems not to have a perfect counterpart in the US - the endowment mortgage; the American mortgage market is heavily influenced by the Government through the provision of default risk insurance, whilst the British market relies on privately issued insurance; and, finally, the histories of both markets do not match perfectly, with special relevance to the recent and long depression of house prices in the UK that was not experienced to the same extent in the US (see, Lea, 1994). In terms of characteristics of the mortgage products available, the dissimilarities include different maturities, different prepayment penalties and arrangement fees, and different insurance coverages. As a result, the cash-flows inherent to apparently similar mortgages are, in fact, different implying that the corresponding market values also differ. This is equivalent to saying that, from a financial modeling point of view, despite having similar names, the different structure of cash-flows makes UK and US mortgages different products. Imediately follows that models already available in the US finance literature do not seem able to provide reasonable valuations for the UK. Therefore, it is necessary to develop research modeling specifically British mortgage products.

Additionally, the dominant mortgage instrument in the UK, the endowment mortgage (see, Figure 1.3), is apparently non existent in the US market and its valuation under a contingent claims framework that considers both the options held by the borrower seems never to have been attempted[2].

Another factor that is important to mention is the recent introduction into the UK market of capped mortgage insurance products.

The present work addresses these particularities providing valuation frameworks for repayment and endowment mortgages, and the corresponding capped mortgage insurance products, that take into consideration the specific features of the British market.

[2] The only attempt, known by the author, to model an endowment mortgage using a contingent claims context is given by Chinloy (1995), whose framework is completely different from that proposed in this work since it does not take into account the prepayment option held by the borrower.

1.3. Research Content and Methodology

The general valuation framework used in this work is based on the Cox, Ingersoll and Ross (1985a) equilibrium model. The corresponding valuation equation provides the value of the assets under study, given the terminal and boundary conditions of the problem. In this case a two state variable version of the model was used in order to cope with the two natural sources of uncertainty associated with mortgage products: term structure risk and default risk. All the changes in the term structure of interest rates are assumed to be driven by the evolution of the spot interest rate. The other state variable is the house price, the source of default risk.

The first steps that are taken in the calculations consist of the determination of the terminal and boundary conditions for each asset studied. Given these terminal and boundary conditions, it is possible to use the model to price the assets for all the other moments in time (in other words, when the terminal conditions do not apply). The intrinsic complexity of mortgage contracts, with embedded options to prepay or default, leads to a situation where a closed form solution for the model is probably impossible to obtain. Consequently, it was necessary to use a numerical solution technique. The nature of the problem led naturally to the choice of an explicit finite difference method. Using this methodology, a series of simulation results were generated which identified the sensitivity of the different assets to changes in the various parameters of the model.

The contribution inherent to the thesis can be summarized by saying that the work just adds to the literature on contingent claims pricing frameworks for the valuation of "without profits" endowment mortgages, British repayment mortgages, Mortgage

Indemnity Guarantees and the corresponding coinsurance. These models consider both the default and the prepayment embedded options.

1.4. Organization of the Thesis

The organization of this thesis is as follows:

Chapter 1 provides an introduction to the whole work. It gives an intuitive notion of the importance of the mortgage and real estate markets in the UK, introduces the main purposes of the research, the methodological approach used, the content of the thesis and its structure and organization.

Chapter 2, reviews the academic literature on the contingent claims valuation of mortgages and "mortgage related" products and gives a brief notion about the organisation and institutional features of the British mortgage market.

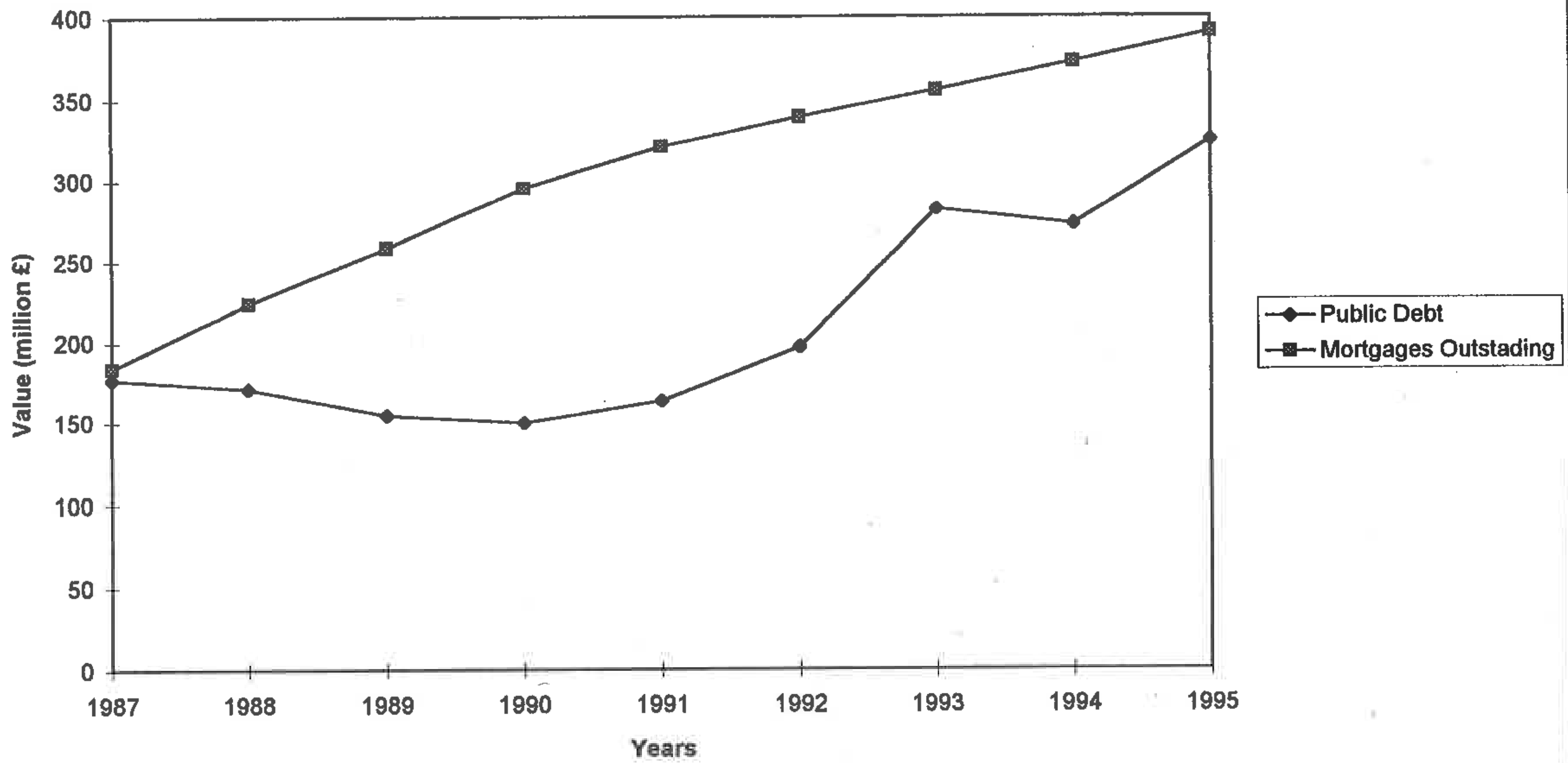
In Chapter 3, the general valuation model that is used to price the assets under study is developed. As the valuation model does not have a closed-form solution it is necessary to use numerical methods to solve it.

Chapter 4 presents the transformations and the overall procedure that were required in order to reach the numerical solution of the problem.

In Chapter 5, several scenarios for the value of the key parameters of the model are studied and conclusions are extracted about the value of the different assets under scrutiny.

Finally, Chapter 6 concludes the thesis identifying the most important contributions and suggesting areas for further research in the field.

Figure 1.1. Mortgages Outstanding/ Public Sector Debt in the UK (1987/95)



Source: Financial Statistics (Central Statistical Office)

Figure 1.2. Evolution of the Mortgage Market in the UK (1980/95)

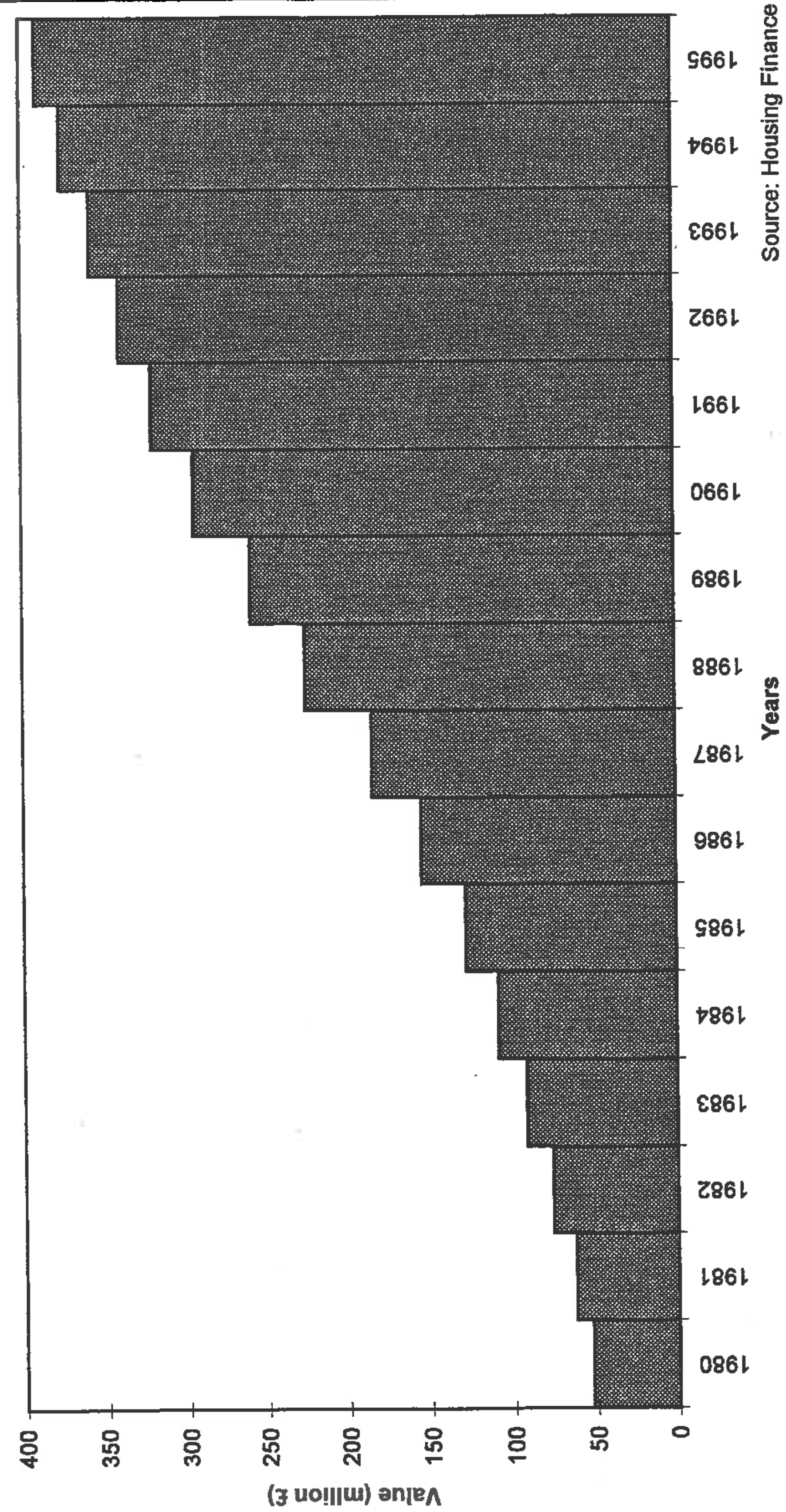
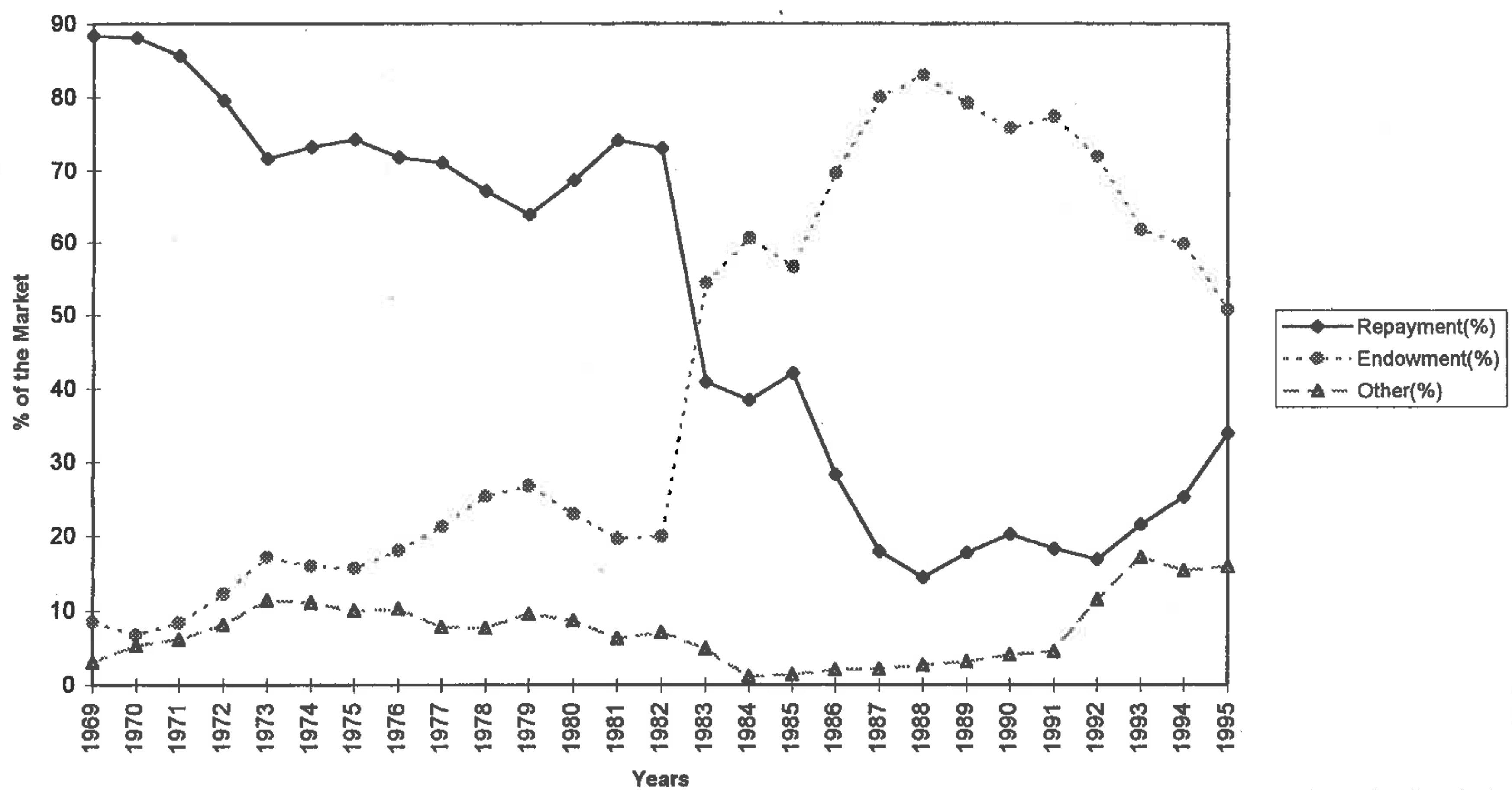
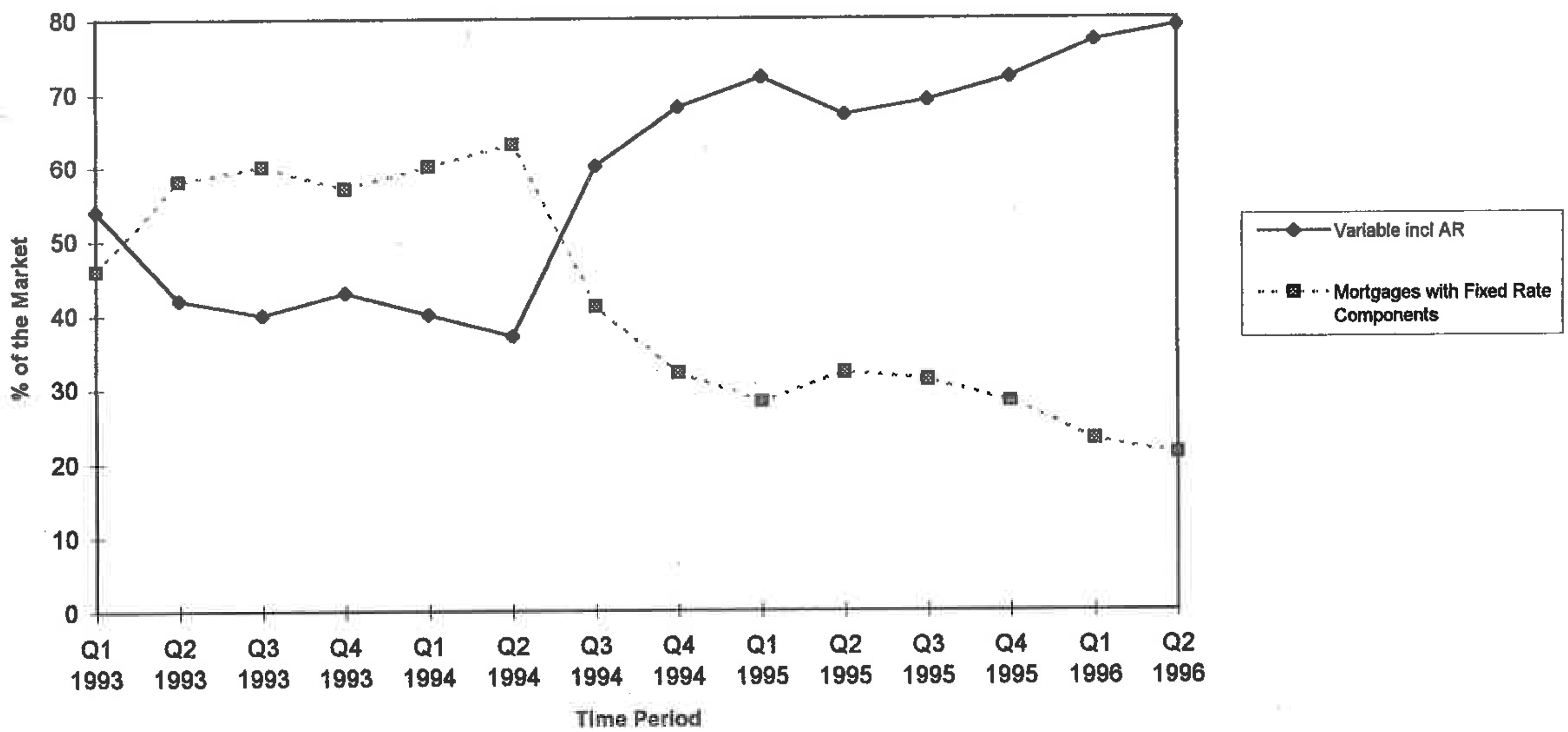


Figure 1.3. The Main Products on the British Mortgage Market (1969/95)



Source: CML Information Services Fact Sheet

Figure 1.4. "Pure" Variable Rate Mortgages/Mortgages with Fixed Rate Components (1993/1996)



Source: Housing Finance, several numbers

CHAPTER 2

An Overview of the Contingent Claims Approach to Mortgage and Mortgage Insurance Valuation

2.1. Introduction

Option valuation has attracted the curiosity of the academic community since at least the beginning of the 20th century. The most remarkable example of this interest is probably the work of Louis Bachelier (1900), whose attempts to model the behavior of stock market prices, according to Dimand (1993), Mandelbrot (1989) and Merton (1992, 1996), anticipated by several years the work of Einstein on Brownian motion. Despite this continued attention, the great breakthrough was registered only in the beginning of the 1970s with the publication of papers by Black and Scholes (1973) and by Merton (1973). Under some strong assumptions about interest rate and volatility behavior, both articles provide closed-form solutions to the valuation of simple options on stocks. Following these seminal contributions, a whole new approach to finance has been under development, extending and applying the conceptual setting of these authors to the valuation of all types of explicit and embedded options. In brief, any right or set of rights that gives its holder the possibility of making future decisions capable of affecting the value or the timing of the cash-flows from some venture has a value. The contingent claims approach to the valuation of financial assets takes into account this basic principle, and the notion of arbitrage-free markets, in order to produce a theoretical framework under which it is possible to value assets with implicit or explicit option characteristics.

The buyer of an ordinary mortgage receives a "financial package" consisting of a loan and two options: *i)* an American call option to prepay the loan at any time, or, in other terms, an option to acquire the full set of rights over the house by paying-off the remaining debt; and *ii)* a European compound put option (a succession of European put options) to default on the loan at any payment date. Of course, it only makes sense for the borrower to exercise the right to exchange the house for the remaining payments at a payment date since earlier exercise confers some financial benefit (in the time value of the money) to the lender.

The major insight inherent in the models developed under this approach comes from the fact that either the underlying partial equilibrium (arbitrage) or the alternative general equilibrium framework allow for quantitative predictions to be made about mortgage and mortgage related products prices, instead of simple qualitative conjectures.

The present chapter provides an overview of this growing body of knowledge. It attempts to identify areas of future research and rationalises the choice of the approach to the valuation of mortgage and mortgage-related products which is used in the following chapters of the thesis.

The chapter has eight sections. The first introduces the topic. The second describes the various mortgage instruments available in the market. The third presents the basic ideas that lie behind the use of contingent claims frameworks in the valuation of mortgage and mortgage related products. The discussion of the alternative types of valuation models available is introduced in the fourth section, alongside the presentation of matters related to the approaches that can be used to tackle the two alternative ways of terminating the mortgage prior to the maturity: default and prepayment. The fifth section analyses and discusses the valuation of mortgage insurance products. A summary of the main empirical research on mortgage valuation is presented in section six. The seventh section summarizes the primary research areas that present opportunities for future investigation. Finally, section eight concludes the chapter and justifies the main research option taken by the author.

2.2. Mortgages: an Overview of the Contracts and the Markets

The residential mortgage is one of the most common and oldest financial contracts. Despite being a popular product, sold to non-specialists for many decades, it presents some features that are relatively unusual in the financial markets and that make it especially difficult to evaluate. Before starting to describe its main characteristics, from a financial modeling perspective, it may be helpful to note the most relevant elements in the valuation of a financial contract.

The valuation of a financial asset depends on two main factors: *i)* its cash-flow structure and *ii)* the economic environment in which it is traded.

In a contingent claims valuation framework, the elements that are used to characterize the economic environment are the most relevant sources of risk inherent in the asset under study. This theme will be addressed with more detail, later on, in the fourth section.

The cash-flow structure of a mortgage is determined by the contract rate and the other contractual features of the loan.

Some important contractual features are characteristic of almost all types of mortgages. Especially relevant is the fact that the term to maturity is particularly long, reaching usually twenty five years in the UK and thirty years in the USA. Other important features generally encompassing all residential mortgage contracts are the possible sources of early termination: the borrower is given the option to terminate the loan prior to the maturity date, either through prepayment or through default.

Several other contractual features vary with the specific characteristics of the underlying contract, requiring individual specification.

There are several types of residential mortgages, the most important of which are the endowment mortgage and the repayment mortgage.

Types of Residential Mortgages:

i) Endowment Mortgages

During the last decade the prevailing type of mortgage available in the UK market has been the endowment mortgage. It combines a bullet loan payable at termination, with an endowment insurance policy. Under this contract the borrower is required to make a combined payment that includes interest only, on the amount borrowed but no repayment of the capital sum (i.e. a non amortizing loan) and an endowment policy premium. The lender is the beneficiary of this policy. If the borrower dies, the issuer of the insurance policy pays off his debt (life insurance component of the endowment policy). If this does not happen during the term of the loan, and the borrower also does not default, the insurance policy accumulates the necessary value to pay off the loan at maturity leaving the borrower owning an unmortgaged house (investment component of the endowment policy)[3].

The endowment policy can differ normally from mortgage to mortgage according to the preferences of the borrower. He may take a policy whose future value is bigger than ("with-profits" endowment mortgage), equal to ("without profits" endowment mortgage) or less than the value of the debt (see, Chinloy and Megbolugbe, 1994). The most popular option and also the one that is proposed in the publicity brochures of most banks and building societies is the "with-profits" endowment mortgage.

ii) Repayment Mortgages

The contractual nature and the cash-flow structure of a repayment mortgage are much more straightforward than those of an endowment mortgage. The borrower still assumes the obligation of making monthly payments until the term of the loan. The difference is that the payments are now aimed both at paying interest on the unpaid balance and at repaying this unpaid balance over the period of the loan. During the early years of the contract almost all of the amount paid consists of interest, but gradually, as the unpaid balance decreases, the proportion of the payment consisting of interest decreases until, at the final payment, no more is owned.

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3 This aim is not automatically guaranteed for all types of endowment mortgages. Some types of endowment mortgage involve a degree of risk in that, at maturity, the value of the investment component of the endowment may be insufficient to repay the full amount of the debt.

iii) Other Residential Mortgages

There are several other less common residential mortgage contracts. In the UK, probably the most important are Pension Mortgages and Personal Equity Plan Mortgages (PEPs).

Pension Mortgages are contracts available only to self-employed people, where the borrowers pay interest only on their debt and at the same time make contributions to a personal pension plan. At the maturity of the loan the proceeds of this pension plan are used to pay the loan.

PEPs are mortgages in which the borrower is asked to pay interest only on the debt and simultaneously to contribute to a series of personal equity plans. The principal is repaid when these personal equity plans are cashed in.

In the US, the residential mortgage spectrum also includes some contracts whose relative weight is clearly minor. Two examples of these type of contracts are the Price Level Adjusted Mortgages (PLAMs), in which the monthly payments are periodically adjusted in order to take into consideration the evolution of the inflation rate, and Graduated Payment Mortgages, in which the amount of the monthly payment rises at a pre-defined rate during the life of the loan.

B) Contract Rate Specification

The other main element that exerts a significant influence over the cash-flow structure is the type of contract rate inherent to the mortgage.

There are two major groups of providers on the supply side of the British residential mortgage market: building societies and banks. Historically, building societies have dominated this market, but with the liberalization of the financial services regulation that took place in the 1980s and the transformation of some of the biggest building societies into banks, the general picture has changed significantly.

Building societies tend to set their mortgage rates relative to the evolution of the prevailing money market rates (see, Chinloy and Megbolugbe, 1994) and banks determine their mortgage rates as a function of their respective base lending rates[4], which are indirectly related to the prevailing money market rates. Thus, regardless of the type of lender, the evolution of the money market seems to exert an important influence in the evolution of the mortgage rates.

There are two main types of contractual rate specification: variable rates and fixed rates.

i) Variable Rate Mortgages

As the name suggests, under this type of contract rate, the monthly payments made by the borrower evolve in accordance with the lender's mortgage rate. In the unlikely event that the mortgage rate will not change during the life of the loan, the amount of the monthly payment will remain unchanged.

In the UK, the most common rate adjustment regime allows for the contract rate to be changed whenever the lender's mortgage rate changes. Nonetheless, contracts that limit the number of rate and payment adjustments to one per year are becoming increasingly common (see Chinloy and Megbolugbe, 1994).

In the US a similar type of mortgage is traded under the name of adjustable rate mortgage (ARM). However, despite being a variable rate contract, an ARM is not

[4] Usually:

Mortgage Rate = Base Lending Rate + Margin exactly the same as a variable rate mortgage (VRM). An ARM is always a repayment mortgage, but in the UK both repayment and endowment mortgages can have variable rates. Even between repayment mortgages the differences are significant. For instance, most ARMs have lifetime caps. This feature is uncommon in the UK, but, as mentioned before, it has some similarity with the capped rate mortgages available here. Other characteristics that seem to be unavailable in the British mortgage market are the yearly caps and floors on the contract rate, the periodicity of the contract rate adjustments, and the insurance coverage provided by Government agencies.

Despite the apparent similarities, these products generate cash-flow patterns sufficiently different that the models developed to evaluate one are not ideally suited to evaluate the other.

ii) Fixed Rate Mortgages

With fixed rate mortgages (FRMs), monthly payments are independent of the evolution of the interest rates for a pre-determined period of time that can last until the maturity of the loan or alternatively only for a fraction of its expected time length. During this period, mortgage payments depend only upon the rate that was agreed between both parties in the contract. After that period expires, if the mortgage loan has not reached maturity, the contract rate will usually either change to the prevailing variable rate or to a new fixed rate.

In the US, FRMs constitute the dominant contractual feature in the mortgage market. In the UK, as we already know, the situation is completely different. However, it is important to note that fixed rates are now available for periods between six months and 25 years, and the weight of these type of contracts is not negligible.

iii) Capped Rate Mortgages

Under this type of loan, during a pre-determined period, the mortgage rate has a ceiling above which it cannot move. In other words, the borrower's monthly payments are capped.

Whenever the evolution of a mortgage contract rate becomes contractually constrained, the lender automatically assumes interest rate risk, usually in exchange for a higher level of the coupon. Thus, if the borrower is allowed to terminate the loan prematurely, this higher rate will not be obtained. When faced with a decrease in the market rates, the borrower will tend naturally to repay his loan and to refinance his house investment under better interest rate conditions. In order to reduce the problem, the UK market lenders tend to impose early redemption penalties.

Lending institutions are usually specialized in the management of interest rate exposures. However, at least in the UK market, interest rate risk is not usually perceived as the most significant source of uncertainty embedded in a mortgage contract. The credit risk inherent in the default option held by the borrower has created serious trouble for the British building societies and insurance industry, and deserves close attention. The coverage of these risks is one of the reasons why mortgage and insurance industries are so closely tied in the UK (see Douetil, 1994). In fact, there are several insurance products which are sold in parallel with mortgages. Probably the most important and best-known of them is the mortgage indemnity guarantee.

C) Insurance-Related Features

i) Mortgage Indemnity Guarantees

In the British mortgage market, the risk of default embedded in high loan-to-value (LTV) ratio mortgages is partially covered (for the lender) by an insurance contract called a Mortgage Indemnity Guarantee (MIG). The details of the policy are not perfectly standardized, but the common industry standard classifies a mortgage as a high LTV loan if the corresponding ratio exceeds 0,75.

The coverage given by the policy includes only what Chinloy and Megbolugbe (1994) refer as "top-slice" loss. This policy was originally aimed at covering the difference between the value of the debt and the value of the property, in case of default. However, since the beginning of the 1990s, when the major insurers realized that they were facing losses estimated to be from £ 3 billion to £ 4 billion in this product alone (see, Douetil, 1994), the contracts were changed in order to limit the amount of the exposure. Nowadays, despite having the same aim, the policies available in the market are substantially different from those that were traded before 1992. The most relevant differences are related to the capping of the exposure and the requirement imposed on lenders to always share some part of the risk independently of the value of the claim. The specific details vary from case to case, but the rule seems to be that, at worse, the insurer's exposure will be capped at the difference between the original value of the debt and the value that corresponds to the "limit LTV ratio"[5]. The coinsurance clause seems also not to be perfectly standardized but, under normal circumstances, any lender might expect to be asked by the insurer to cover at least 20% of the loss.

[5] Usually 75% of the value of the house.

The premiums also diverge from lender to lender. Normally premiums are paid as a lump sum upon completion of the deal[6] and reach 3.5% to 7% of the value covered (see, Chinloy and Megbolugbe, 1994). However, some lenders are now offering the possibility of spreading the value of the MIG premium over a few years and adding it to the borrower's monthly payment.

Several other insurance products are in some way connected with mortgages, i.e. building and contents insurance, fire insurance and unemployment insurance. However, only those which can make a difference in the value of different mortgage contracts will be considered here. Life insurance products are embedded in some mortgage contracts and only sold as a complement in others. They constitute an obvious source of difference in terms of final valuation of the real products and consequently deserve some special attention.

ii) Life Assurance

In contrast with endowments, repayment mortgages do not have any built-in life cover. Thus, it is relatively common for the holders of these type of loans to take a separate insurance policy that covers against death.

[6] In some cases this fee is added to the value of the mortgage.

iii) Endowment Insurance Policy

Endowment mortgages have a built-in life assurance component. As mentioned before, the aim is not only the accumulation of an amount that will enable the borrower to pay the loan at termination, but also to provide life coverage that will enable the mortgage to be paid by the insurer if the borrower dies during the life of the loan.

From a financial modeling point of view, the most relevant point about this component of endowment loans is the fact that, independently of the type of mortgage rate that is inherent to the contract, the interest rate on the insurance policy is fixed. The final amount of the corresponding premium is determined by this interest rate, the age and sex composition of the household[7], the maturity of the loan, and the "profit clause" of the contract.

As suggested in the previous chapter, almost all the research done on the contingent claims valuation of mortgage contracts originates in the US. Consequently, models aimed at the valuation of repayment mortgages are relatively common, but this is not so for endowment mortgages.

The classic valuation of endowment insurance products is based on traditional certainty models that consider discrete time frameworks such as those proposed by Grant and Kingsnorth (1968), Fletcher (1960), Morris (1980) and Stoodley (1936, 1938). Examples of stochastic valuation frameworks for UK endowments were published as early as 1979 (Ford and Masters, 1979). Unfortunately, these cases consider a stochastic term structure but ignore the value of the options embedded in the contracts and normally do not take into account the evolution of any other source

[7] This will determine the probabilities of survival of the household during the life of the loan.

of risk. The only case of a endowment mortgage contingent claims model known to the author is Chinloy (1995), whose valuation framework considers only the default option.

At the repayment mortgage level, the situation is completely different. In spite of the fact that the available models usually consider simplified versions of mortgage contracts available in the USA, it is already possible to have a reasonable insight on how this type of mortgage should be valued and what are the main factors that affect its valuation.

Research analyzing FRMs using a contingent claims approach includes Buser and Hendershott (1984), Epperson et al. (1985), Giliberto and Ling (1992), Hall (1985), Hilliard et al. (1994), Leung and Sirmans (1990), Kau et al. (1992, 1995) and Maris and White (1989).

On the other hand, work dedicated to the valuation of ARMs and related products include Berk and Roll (1988), Buser, Hendershott and Sanders (1985), Cox, Ingersoll and Ross (1980), Findlay and Capozza (1977), Houston et al. (1991), Kau et al. (1985, 1990, 1993a), McConnell and Singh (1994b), Schwartz and Torous (1991), Sobti and Sykes (1993), and Stanton and Wallace (1995a, 1995b).

The need to differentiate their products from those of the competition leads mortgage lenders to create new types of mortgages and sometimes to modify the specifications for the same type of contract. This diversity makes it very difficult to design models automatically able to reproduce real life contracts.

In order to keep the valuation problem inside the boundaries of what is mathematically possible to solve with the tools currently available, it is desirable to simplify the features of the contracts studied. This means that the normal trend in the literature is to consider the main features of the mortgage products but to simplify as much as possible. Therefore, the specifications of the products modeled usually correspond to mere stylized versions of the original contracts[8].

The work developed in this thesis is no exception. It considers the fixed rate versions of both repayment and endowment-type mortgages. It also takes into account both the options held by the borrower to terminate the contract. However, it does not consider the life insurance clauses embedded in some contracts and it ignores the impact of the right to recourse held by the lenders and (or) the providers of MIGs[9]. Finally, it considers only one type of endowment mortgage: the "without profits" contract.

2.3. The Basics of Mortgage Valuation Using Contingent Claims Analysis

The contingent claims valuation of mortgage contracts is only a subset of the more general field that researches the valuation of financial assets under uncertainty. Whenever the environment is assumed to be uncertain, the valuation function needs to

[8] An explicit admission of this phenomenon for American mortgages and the way in which they are modeled can be found in the Presidential Address delivered in 1989 to the American Real Estate and Urban Economics Association by James Follain (Follain, 1990).

[9] In the UK, as in 80% of the states in USA (see, Vandell, 1993), mortgage lenders and (or) mortgage insurers have the right to foreclose on the assets of the borrowing household. However, all the available mortgage pricing models ignore this feature. The rationale for this is related to two factors: the need to keep the valuation frameworks inside the limits of what is mathematically tractable, and what seems to be a notion generally accepted in the industry according to which attempts to seize the assets of households that already lost one of their most basic assets, are normally doomed to produce very low or even negative net financial rewards (given the costs of the process).

take into consideration, in a probabilistic way, the possible states of nature that can take place at any moment in the future. The classic textbook net present value calculation that values the asset under study as if there were no risk is inadequate. Neither the cash-flows nor the rates that will be necessary to use in order to discount those cash-flows are known with certainty.

A stochastic economic environment needs to be taken into account. In this case, the future value of a mortgage is not known and it is also impossible to anticipate if the conditions under which termination takes place will ever occur.

Under normal circumstances, in a world of uncertainty in which economic agents react to risk in different ways, it would be necessary to take into account investors' preferences. Fortunately, one of the main achievements of the contingent claims approach to value financial assets consists in establishing that the role of preferences in the valuation of derivative assets is relatively marginal.

As its name suggests, a derivative asset is one whose cash-flows are completely dependent on the evolution of the corresponding underlying asset. It is a superfluous asset whose behaviour, unlike that of its underlying asset(s), is redundant in the fundamental characterization of the economy.

If the economy is populated with decision-makers who behave in a rational way, preferring systematically more to less, identical assets must always have the same price. Knowing the prices of a set of (underlying) traded assets and making some simple risk adjustments, it is possible to evaluate a derivative asset as if the world were risk neutral (see, Duffie, 1996; Jarrow, 1988; and Huang and Litzenberger, 1988). In other words, a derivative asset can be evaluated as the expected (present) value of its future cash-flows.

Here, mortgages are assumed to be derivative assets, whose payoffs depend on the evolution of some variables that are necessary to characterize the real economy. However, mortgages are not, necessary to characterize that underlying economy.

2.4. Mortgage Valuation Models

The valuation of any financial asset is always an attempt to adequately relate its future cash-flows to the corresponding discounting factors. The economic environment exerts the decisive influence over the evolution of these elements.

In this case the evolution of both the future payments and the discount factors is uncertain. We are faced with two sources of risk. Therefore, it is necessary to consider the evolution of at least two variables to characterize those sources of risk.

The major challenge inherent in financial modeling consists in finding mathematical representations of these sources of risk and taking it adequately into consideration in the valuation process.

2.4.1. Term Structure Modeling

The second section of the present chapter described in some detail the uncommonly long term to maturity of mortgage contracts. This factor makes it particularly inappropriate to use a flat term structure of interest rates in mortgage modeling. Therefore, the literature on the subject that will be reviewed here considers only stochastic term structures.

Interest rates are natural state variables in any mortgage valuation model. In fact, the value of any financial asset whose cash-flows do not change continuously in direct relationship with the evolution of the interest rates is necessarily affected by that evolution, as a result of the process of continuous discounting that takes place whenever a present value is calculated using a continuous-time stochastic valuation framework.

Under normal circumstances in the UK, a mortgage is expected to be outstanding for up to twenty five years. Therefore, potentially, all interest rates up to this maturity can assume the role of state variables in a mortgage valuation model. Unfortunately, with the tools that are available at the moment, it is only possible to reach a solution for a valuation problem of this kind if a small number of interest rates is assumed to drive the behavior of all the others. The common solution consists in assuming that all the interest rates are driven by one or two exogenous rates - the state variables of the model, in the interest dimension.

Different interest rate processes have been used in mortgage valuation models. The most common specification is probably that originally proposed by Cox, Ingersoll and Ross (CIR, 1985b). It is a mean reverting process with local variance proportional to the level of the spot rate[10], which always prevents interest rates from reaching negative values. It has been used in numerous articles including Dunn and McConnell (1981a, 1981b), Kau et al. (1987, 1990, 1992, 1993a, 1993b, 1995), Schwartz and Torous (1992), Titman and Torous (1989), Stanton (1995), Stanton and Wallace (1995a, 1995b) and Yang and Maris (1996). This specification exhibits an intrinsic theoretical appeal, since it is originally derived from a general equilibrium framework. Besides

[10] The local standard deviation is proportional to the square root of the interest rate.

that, with only four parameters, it is also a parsimonious specification as compared with some of the alternatives available, such as those no-arbitrage models proposed by Black, Derman and Toy (1990) or Black and Karasinski (1991). These are probably the reasons why the CIR model continues to be one of the most common, or even the dominant, specification of term structure risk in mortgage modeling.

In spite of the existence of some studies (e.g. Chan et al., 1992 and Chen and Yang, 1995), based on time series estimates of the parameters, whose results favour the performance of no-arbitrage models in the representation of the term structure, the use of this type of specifications in mortgage valuation models is, up to now, marginal or even non-existent.

In mortgage modeling, the parsimony of the specification is a characteristic whose relevance should be emphasized. The complexity of the numerical algorithms that are used to solve the valuation problem imposes the need to keep all the modeled features as simple as possible, in order to reduce the intricacy of the calculations. This helps explain why two-state term structure models are seldom used in the field[11]. Despite presenting a clear advantage in terms of the number of degrees of freedom used to describe the actual term structure, this type of model is normally used only in contexts in which the term structure is considered the only source of uncertainty. A classic example of this is given by Schwartz and Torous (1992). After the publication of several articles in which a two-state variable term structure is used (e.g. Schwartz and Torous 1989a, 1989b, 1991) to tackle valuation problems that do not consider the default option, those authors revert to a single factor term structure specification in an article in which both prepayment and default option are considered. In fact, when the

[11]Examples of articles that use this approach to term structure modeling include Brennan and Schwartz (1985) and Schwartz and Torous (1989a, 1989b, 1991)

put option to default is ignored, the consideration of an additional state variable to describe the term structure of interest rates does not create a major computational problem. However, if both options are considered, the use of two state variables to describe the term structure and another one to account for credit risk will impose the solution of a partial differential equation in three state variables. With the intrinsic complexity of the mortgage products, this becomes a very difficult task to perform, and I am not aware of any published attempt to do it.

Another factor that might help to explain the dominance of this field by single factor specifications of term structure risk is the research done on the subject by Babbs (1991), Buser, Hendershott and Sanders (1990) and Steeley (1989a,b). They conclude that the bulk of yield curve variation is accounted for by a single factor. Therefore, it is not indispensable to use two interest rates as state variables because it is always possible to define a single state variable process whose results, in terms of pricing, closely approximate those that would be obtainable with a given two state variable specification (see, Hendershott and Van Order, 1987 and Kau and Keenan, 1995).

In summary, whenever both early termination options are to be considered, as is the case here, single factor term structure specifications are the rule.

The CIR model, for all the reasons mentioned, has become a benchmark and seems to continue to be the most common specification for the term structure risk in mortgage valuation models. Its overall performance in mortgage valuation frameworks was recently analyzed using a curve fitting approach (Archer and Ling, 1995), with good comparative results, even in relation to the Black, Derman and Toy (1991) no-arbitrage model. Therefore, this will be the specification used to represent the term structure risk in this thesis.

2.4.2. The Modeling of Early Termination

Besides the term structure of interest rates, the other main factor that it is necessary to consider, in order to characterize the economic environment, is the stochastic process followed by the asset that underlies the mortgage contract.

2.4.2.1. Default

Academic discussion about the motives that drive mortgage default is not new. Jackson and Kaserman (1980), presented a study in which the two main competing theories of mortgage default were empirically tested. The first of these alternatives asserts that borrowers observe a strictly optimizing behavior, maximizing the financial gain or minimizing the financial loss that results from the rational comparison between the financial costs and benefits associated with the continuation or termination of the periodic payments, that are inherent to the underlying mortgage contract. This view is referred as the "equity" theory of default.

An alternative hypothesis, normally referred to as the "ability to pay" theory, holds that mortgage borrowers tend to avoid defaulting their loans if their current income stream is enough to fulfill the periodic payment without disproportionate financial burden.

The conclusion of the Jackson and Kaserman (1980) study supported the claim that the equity effects were clearly dominant. These results were later confirmed in a series of other similar studies by Foster and Van Order (1984, 1985) and Waller (1988). In studies aimed at examining the impact of microeconomic characteristics on the probability of default Sa-Aadu (1988) and Cunningham and Capone (1990) both concluded that equity elements are the most relevant in the explanation of the probability of default. More recently, Quigley and Van Order (1995), in a similar work, examined residential mortgage default using micro data on household choice and individual house prices, and once again concluded that there is a solid connection between mortgagor equity, at the individual level, and mortgagor default probabilities[12].

Despite being a good indicator of the ability of frictionless models to evaluate mortgage contracts, the empirical evidence mentioned above does not completely rule out the existence of significant transaction costs (see, Quigley and Van Order, 1995). Most option-theoretic approaches to mortgage valuation define transaction costs as the costs of exercising the underlying options (e.g. Kau et al. 1992, 1993a,b). This includes selling and purchasing fees, taxes and more important, limitations on future access to credit. It is obvious that, if such costs are not negligible, they will tend to reduce default rates. There are even some authors who believe that this is the fundamental factor to take into consideration if one wants to understand and explain mortgage default (see, Quercia and Stegman, 1992 and Vandell, 1993). Early empirical research done in the field (e.g. Campbell and Dietrich, 1983; Vandell and Thibodeau, 1985) suggests that the observed levels of default are much lower than the frictionless models would predict, and higher levels of transaction costs would be necessary to provide some rational explanation for the data. However, these studies did not take into

[12] Lambrecht, Perraudin and Satchell (1996) approach the same problem with slightly different results. However, the way in which the proxies for each hypothesis are specified and specially the absence of consideration of the value of the options to bring the loan to an early termination in their treatment of equity, weaken their conclusions.

consideration the value of the options to terminate the loan in the future. In fact, whenever an option to terminate the loan is exercised, the borrower automatically gives away the possibility of exercising both early termination options in the future. As a consequence, borrowers tend to default not immediately after the value of the house falls below the present value of the future payments, but only after the value of the house reaches levels that are lower than the value of the mortgage, which includes both options to terminate the loan. Therefore, the conclusions of those studies need to be taken with some reserve.

More recent empirical work done on the subject reiterates the existence of this kind of cost, but asserts that its magnitude is considerably smaller than what is assumed by those who believe that transaction costs are the key element to determine mortgage default (see, Kau, Keenan and Kim, 1994 and Quigley and Van Order, 1995).

Kau and Kim (1994), in a theoretical paper, show that observed deferral in default, generally credited to transactions costs, can be explained as the result of entirely rational choices, in an economic environment that completely ignores the existence of that kind of cost. Titman and Torous (1989) using a sample of non-prepayable, fixed rate commercial mortgage loans were also able to explain the evolution of default premia despite using a "ruthless" zero-transaction cost mortgage valuation model.

In summary, the evidence available so far does not support the claim that transaction costs should be explicitly taken into consideration in mortgage valuation exercises.

All the previously mentioned studies refer to the US mortgage market. However, there is no conceptual reason to support the idea that borrowers' behavior is different in the UK from what it is in the US. Thus, the determinants of mortgage default in the

UK should not be different from those in the US. In a perfectly competitive market where information is freely and simultaneously available to all the economic agents, borrowers will exercise their options whenever this action will increase their wealth. Considering a world where transaction costs and reputation costs are irrelevant, mortgage borrowers might increase their wealth by defaulting on their loans when the market value of their mortgages surpasses the value of their houses. In order to take into account such a situation in a contingent claims valuation framework it is necessary to use the value of the underlying real estate asset as one of the state variables in the model. This is usually achieved through the application of a standard log-normal process (geometric Brownian motion) to represent the evolution of the property value (see, Cunningham and Hendershott, 1984; Epperson et al., 1985; Kau et al. 1990, 1992, 1993a, 1993b, 1995; Schwartz and Torous, 1992 and Titman and Torous, 1989). The value of the asset is assumed to evolve at a constant rate that is continually disturbed by a stochastic factor.

The present work follows this trend and also employs a log-normal process to take into account the evolution of the house price.

2.4.2.2. Prepayment

When the borrower of a mortgage contract chooses to terminate the mortgage, exercising his call option to prepay the loan, because he thinks that it is possible to refinance at a more favorable rate, we are faced with an optimal prepayment. In this case, the decision is motivated only by the attempt of the borrower to minimize the cost of the loan. In the absence of credit risk, the only factor that influences his decision is the evolution of the term structure of interest rates.

This is the type of prepayment decision that is commonly modeled in frictionless mortgage valuation models. The normal procedure consists of supposing that a prepayment will take place whenever the market value of the mortgage surpasses the market value of a hypothetical refinancing loan.

If this optimal call policy were a good representation of the real world, a mortgage should never be repaid when its market value is less than its call price (call option is out-of-the-money). Furthermore, a mortgage loan should always be called when its value exceeds its repayment price (call option is in-the-money). However, some borrowers do not prepay their loans when the contract rate surpasses the rate at which it would be possible for them to refinance the loan in the market. Similarly, other borrowers prepay their loans in adverse situations in which the refinancing rate is superior to the contract rate (see, Foster and Van Order, 1985, and Green and Shoven, 1986). This type of prepayment is normally called suboptimal prepayment[13] because, despite having some appeal to the borrower, it does not minimize the market cost of the mortgage, as given by the frictionless mortgage valuation models. The reasons that underlie such decisions are related to micro-level personal circumstances that cannot be captured by the currently available contingent claims valuation models.

In order to cope with the problem, some researchers have attached prepayment functions to their contingent claims models in order to allow for suboptimal prepayments. Exogenous payments are usually modeled as Poisson processes whose mean rate of arrival is determined by the external prepayment function (see, Dunn and

[13] In this case suboptimal does not mean irrational.

McConnell, 1981b; Schwartz and Torous, 1989a). This type of feature is especially important in the valuation of secondary mortgage market products, like CMOs, whose cash-flows are allocated serially to different groups of investors and whose values are very sensitive to the prepayment record of the underlying pool of mortgages.

There is a whole stream of research studies that completely ignore the default option, focusing only on prepayments (e.g. Buser, Hendershott and Sanders, 1985, 1990; Dunn and McConnell, 1981a,b; McConnell and Singh, 1993, 1994a; Schwartz and Torous, 1989a,b). The products studied are normally fully insured mortgages or the corresponding mortgage-backed securities. The rationale for dropping the investigation on the put option to default is related to two different factors. In the first place, apparently default does not influence the cash-flows of a fully insured mortgage, since in case of default, the insurer covers the loss. If this is so, default would be equivalent to prepayment, for it generates similar cash-flows. This is true, if we consider only the point of view of the lender. However, from the point of view of the economic agent who holds the default decision, this is not exactly correct. When default takes place the borrower exchanges the house for the value of the mortgage. The insurance pays off the difference between the outstanding debt and the present value of the future mortgage payments, up to the limit of the coverage, but only in a deferred moment. Therefore, there is a significant difference between value and timing of the cash-flows for each one of the economic agents in case of default, and it is not correct to assume that a fully insured mortgage is equivalent to one which is default-free. Prepayment and default are different features motivated by different circumstances and should be considered separately.

The second factor that contributes to the development of mortgage valuation models that ignore the default option is related to the need to reserve computing power, in order to cope with exogenous prepayment[14]. As mentioned above, this is especially relevant when the products under study are CMOs, IOs or POs for whose pricing the endogenously determined prepayment behavior seems particularly ineffective.

The main part of this thesis consists in studying mortgages, not mortgage-backed securities. Consequently, the potential advantages of using an exogenously determined prepayment function would be relatively minor or even non-existent, and always at the expense of dropping one important state variable, with the corresponding consequences in terms of the study of default. It would achieve little in exchange for something that potentially seems much more important. Furthermore, recently the field has witnessed a renewed emphasis in the attempts to develop and use rational prepayment methods in mortgage valuation models (e.g. Giliberto and Ling, 1992; Harding, 1994; Kau et al. 1993a, 1995; McConnell and Singh, 1994a; Stanton, 1995). The main reason for this is related to the intrinsic nature of all the empirical models and their natural weaknesses. Changes in the environment tend to undermine the ability of models based on historical fits to continue to produce good predictions, because their parameters tend to show some instability over time (lack of robustness). Consequently, with the increase in the volume of mortgage-based products traded in the secondary markets, there is a renewed pressure towards the development of models based on sound economic theory instead of past correlations (see, Harding, 1994).

[14] An additional reason for this is the attempt to improve the term structure modeling characteristics of the models, through the use of an additional state variable to describe the term structure. The relevance of this exercise was discussed in the previous subsection.

Given all these reasons, the present study does not consider suboptimal prepayment.

2.5. *Valuation of Mortgage Default Insurance*

Finance and insurance are research fields whose subject matters present some degree of overlap. Both are mainly concerned with the evaluation and the allocation of risk in the economy (Brennan, 1993). In spite of this proximity, the basic methodological approaches used by these disciplines continue to exhibit a fundamental divide: the key paradigm of actuarial science is of a statistic nature, whereas, in the financial economics world the central role is assumed by the rational behavior of the economic agent and, consequently, by concepts like equilibrium and arbitrage.

According to Boyle (1990), the application of continuous-time stochastic modeling in the actuarial world started with the publication of Lundberg's (1909) seminal work. However, despite the publication of stochastic frameworks of great sophistication such as those presented by Ford and Masters (1979), Forfar et al. (1987), and Wilkie (1976, 1995), the main financial paradigms seem to continue to exert a relatively minor impact in the actuarial world. Nonetheless, the gap appears to be narrowing. Since the beginning of the 1960s, some tools of economic analysis, including the expected utility hypothesis and game theory, have been introduced to the insurance research field due to the works of Karl Borch, whose research opened a series of bridges between both fields[15] (for a review, see Boyle, 1990).

For obvious reasons, the relationship between contingent claims analysis and insurance theory was established only some years later (Black and Scholes breakthrough was only published in 1973). However, the power of the option-theoretic

[15] Part of this work was later on published in Borch (1974).

approach to provide valuation of contingent liabilities was used almost immediately after the publication of the contingent claims analysis seminal papers (Black and Scholes, 1973 and Merton, 1973). Some of the initial work, in the valuation of insurance contracts, was done originally by Merton (1977) and Smith (1979), and later on developed, by Albizzati and Geman (1994), Brys and De Varenne (1994), Brennan (1993), Cummins (1988), Lai and Gendron (1994), Merton (1989), Shimko (1992) and others.

"Real estate related" insurance topics have also been researched in Cunningham and Hendershott (1984), Clauretie and Jameson (1990), Doherty and Garven (1986), Kau et al. (1993b) , Sharp (1989), and Shiller and Weiss (1994).

The valuation of private mortgage insurance contracts is a relatively minor subject in the US because, as previously mentioned, mortgage insurance is normally provided by Government agencies. As a consequence, besides the work of Kau et al. (1993b), examples of contingent claims research in the area are scarce. One example is Cunningham and Hendershott (1984), where an attempt is made to find the market value of the default insurance guarantees provided by the Federal Housing Administration (FHA).

The valuation of MIGs, despite the structural similarities with the work of Kau et al. (1993b), poses also new problems since the underlying insurance contracts are significantly different.

2.6. Empirical Research on Mortgage Pricing

According to Hendershott and Van Order (1987), in the USA, the main motivation for early mortgage research was the need to explain changes in yield differentials between Treasury bonds and GNMA[16] securities - a secondary mortgage market security guaranteed by a government agency. A possible explanation could come from the evolution of the value of the call option embedded in mortgages. Hendershott, Shilling and Villani (1984) studied this phenomenon using statistical regression techniques. Their conclusions support that claim as reasonable.

Examples of this type of "contingent claims related" empirical research are given also by Hendershott and Van Order (1990), and O'Keefe and Van Order (1990). Both studies regress prices of mortgage-backed securities on variables that, according to the option- theoretic approach to mortgage pricing, should have an impact in the value of these securities. In both cases the conclusions support the idea that the principal properties of the contingent claims model hold.

Quigley and Van Order (1995) is another example of this type of approach in a slightly different context. The paper studies the relationship between homeowner equity at the individual level and homeowner default probabilities, concluding that "in the money options" are exercised and that the probability of default approaches one at moderate values of negative equity.

The formal mortgage pricing literature does not follow this trend. Attempts to develop models able to produce outputs capable of being directly tested against market prices or yields are almost non-existent. Three exceptions known to the author are Hall

[16] GNMA stands for Government National Mortgage Association (often also called "Ginnie Mae"). It is a US government agency that guarantees the timely payment of principal and interest on mortgage-backed securities collateralized by mortgages guaranteed or insured by agencies like the FHA, the VA or the FmHA.

(1985), Titman and Torous (1989) and Giliberto and Ling (1992). Hall (1985) used a single state variable to value the call option embedded in GNMA mortgage-backed securities, having concluded that it was priced adequately by the market. Unfortunately, the study does not take into consideration the default option held by the borrower. Titman and Torous (1989) study the default option on commercial mortgages using a two-factor model. The results confirm that the option is priced properly. Finally, Giliberto and Ling (1992) use also a two-factor model for residential mortgages. According to the results, the contingent claims model produces an unbiased prediction of the changes in real mortgage rates. Unfortunately, the study does not provide a role for default.

Apart from the above mentioned pieces of research, the interaction between the predominantly atheoretical empirical work, normally aimed at the explanation of mortgage behavior, and the contingent claims mortgage literature is nearly non-existent if one excludes the use of empirical estimates to parametrize the models.

In spite of this mainly theoretical orientation there is wide evidence of the importance of the option-theoretic approach in mortgage pricing. Investment banks including Salomon Brothers, Goldman Sachs & Co., Citicorp and others (see Hendershott and Van Order, 1987) use this type of model in their day to day activities. Sadly, those models are proprietary material whose precise specifications are unknown.

The present thesis is no exception. It is a theoretical piece of research on mortgage pricing.

2.7. Probable Future Research Directions

The quality of the mortgage valuation models can be improved by introducing features capable of describing the contracts and the economic environment in a way that makes the final framework closer to reality. This can be achieved through two different routes: *i)* studies that will conform with the dominant theoretical frictionless framework without any major concessions to micro-level factors that this type of model seems incapable of capturing, and *ii)* research aimed at determining and incorporating those micro-factors in models that will almost necessarily incorporate some option components, but will probably have a different overall structure.

In terms of improvements to the option-theoretic framework, perhaps the most likely developments are the prospects of improving the way in which the term structure and the house prices are modeled. Despite what was previously mentioned about the ability of single factor models to model the term structure of interest rates, the introduction of additional state variables could improve, even if slightly, the global quality of the outcome. The ability of the log-normal process to model house prices needs to be examined carefully, as does the treatment of houses as pure financial assets. In fact, as Kau and Keenan (1995) state, they are consumption goods, whose payoffs are service flows, not dividends.

The research questions and strategies are obviously broader on the latter investigation route. The main problem with the "classical" approach is that it does not consider what Vandell (1995) refers as the mortgagor "non-ruthless" behavior. Not only does it seem that common mortgages do not default as soon as the "ruthless" models tend to suggest, but also their prepayment behavior seems not to be adequately captured by the theoretical frameworks available at the moment. One of the difficulties derives from the fact that these models are not capable of considering properly the idiosyncratic problems of each household, like unemployment, change of workplace and divorce (Vandell, 1995).

In an area so unexplored as this one, the questions for future research are necessarily numerous. A non-exhaustive sample includes:

i) What is the role of household idiosyncratic problems like unemployment, death and divorce in the default and prepayment decisions ? *ii)* What is the influence exerted by changes in the workplace of household members in the decision to prepay the mortgage ? *iii)* What is the relation between household solvency and the decisions to prepay (solvency excess - moving to a better house) and default (solvency problems - inability to cope with mortgage payments) ? *iv)* How do the borrowers perceive the hidden costs associated with default ? *v)* What are the transaction costs linked with the decisions of prepayment and default ? What are the relative shares of lenders and borrowers in these costs ? *vi)* What is the relevance of the lender's right to access the household assets in the default behaviour ? *vii)* In case the relevance of these additional sources of risk are empirically validated, how might they be incorporated in a general valuation model without leaving it practically intractable ?

In some cases the design of models capable of incorporating more sources of risk and consequently more state-variables is not a major difficulty in itself. Unfortunately if the actual framework is used without any major transformation, the models will become almost impossible to solve. Much more challenging is the task of designing or identifying mathematical tools capable of providing solutions for these frameworks of increased complexity. Therefore, this is also a major theme for future research.

2.8. Conclusion

This chapter presents a short review of the option-theoretic mortgage valuation models and also a general survey of the British mortgage contracts and markets.

In spite of the complexity of the financial product under study and the commonly recognized shortcomings inherent in the frictionless mortgage valuation models, option theory has evolved as an appropriate approach to mortgage evaluation (Kau and Keenan, 1995). Notwithstanding, it is obvious that there is more to mortgage valuation than contingent claims analysis.

It is possible that the next generation of mortgage valuation models will be able to incorporate micro-level characteristics capable of explaining the idiosyncrasies of the borrower's behaviour, that are impossible to capture with the current modeling exercises. If this stage is reached, it is probable that a new type of model will outperform the frictionless ones that currently dominate the field. However, before reaching this level, it is desirable to develop and apply the frictionless models as much as possible in order to create a broad and sound layer for future research. To achieve that goal, it is advisable to take into account as many real-life features as possible. As Kau et al. (1993a) have shown, the interaction of the different financial assets embedded in the mortgage leads to results that diverge significantly from those suggested by models that consider simplified versions of the contract, in which some features are ignored.

Faced with this situation it was necessary for the author to decide between two distinct research orientations: *i)* to attempt to develop a model incorporating contingent claims and some micro-level characteristics of mortgage behavior, or *ii)* to try to extend the applicability of the frictionless contingent claims mortgage valuation model to new products and (or) a new economic environment.

The former alternative is naturally more appealing and exciting. However, without losing interest in eventually pursuing that kind of research in the future, it looked more natural at this stage to go for something that is firmly established and that still needs improvement. In other words to decide in favour of the latter option.

CHAPTER 3

Mortgage and Mortgage Indemnity Valuation

3.1. Introduction

This chapter presents a contingent claims valuation framework applicable to some debt and insurance instruments used in the British mortgage market.

Two types of mortgages will be considered: the repayment mortgage and the "without profits" endowment mortgage. In the case of repayment mortgages, the monthly payment covers the interest and a portion of the capital borrowed. Initially, the amount of capital paid is low, as almost all the payment is interest. Over the years the outstanding balance reduces and interest becomes an increasingly smaller part of each monthly payment. Endowment mortgage mechanics are different. During the life of the loan, the borrower (mortgagor) pays interest on all of the principal and makes contributions to an endowment assurance policy whose main purpose is to build up capital from which the loan will be repaid at maturity[17].

[17] The endowment insurance policy has also another purpose - to pay for a life insurance policy to assure the lender that the loan will be paid even if the borrower dies. With repayment mortgages, the borrower usually faces a cost for this term assurance. This feature will be ignored here.

The risk characteristics of the endowment mortgage make it more difficult to model than the repayment mortgage. This is probably the reason why apparently there is no literature evaluating such products under a contingent claims framework in which both options held by the borrower are simultaneously taken into account.

"Without profits" endowment mortgages are contracts in which the borrower is automatically assured that the value of the endowment assurance policy will grow with certainty to the exact level of capital necessary to pay the principal at maturity.

In parallel with mortgages, this chapter focuses also on mortgage indemnity guarantees (MIGs). A MIG is an insurance policy whose value depends on the expected behaviour of the borrower. The borrower's behaviour is influenced by the value of his property and the value of the options that he holds as a natural result of being a borrower in a mortgage contract. The value of these options is affected not only by the value of the house, but also by the evolution of the term structure of interest rates that ultimately will determine the value of the loan. Consequently, it can said that the value of the mortgage is linked to the value of the underlying property, the term structure of interest rates and time. Simultaneously, as insurance is directly related to default and default depends on the value of the whole mortgage, it can also be said that, in order to value MIGs in this framework, it is necessary to have previously determined the values of the mortgage and its embedded options.

As mentioned in Chapter 2, current UK MIGs, with their capped guarantee, are different from its American counterparts. Having been severely hurt by non-capped mortgage indemnity policies during the last house price recession, British insurers developed new products in which the value of their liabilities is not a mere function of the amount of the loss suffered by the lender, but they have a upper limit. Also, instead of covering all the costs that result from a default, the current MIGs automatically involve the lender in the process of loss coverage, making him responsible for one part of these costs - the coinsurance. This work provides also a model to calculate the value of this potential cost for the lender.

The main goals of the chapter can be summarized in the following terms:

1) To identify a framework under which "stylized" fixed rate repayment mortgage contracts, based on British examples with specific idiosyncrasies and differences in relation to USA counterparts, can be valued;
2) To develop a setting in which the contingent claims valuation of "stylized" fixed rate endowment mortgages can take place;
3) To develop a valuation framework applicable to MIGs and capable of valuing also coinsurance - the fraction of the default risk that is assumed by the lender under the terms of the most recent MIG contracts.

The chapter is organized as follows. The next section, presents the foundations for the valuation models used in this work. This includes the definition of the two-state variable valuation equation, the identification of the mortgage contract components and the definition of the value of the mortgage payments to be made by the borrowers. The third section defines the conceptual framework for the valuation of repayment mortgages and its insurance related products - mortgage indemnity guarantees (MIG) and the corresponding coinsurance. Special emphasis is given to the terminal conditions which must be imposed on the variables. The fourth section presents a similar framework for the valuation of endowment mortgages, and its insurance-related products. Section 5 concludes the chapter.

3.2. Contingent Claims Model

3.2.1 Valuation Framework

This section presents a contingent claims model for pricing residential mortgages.

The model assumes that the factors affecting the returns from a mortgage can be summarized by two state variables: the spot interest rate, *r(t)*, that follows a mean-reverting square root diffusion process (Cox, Ingersoll and Ross, 1985b) and the value of the underlying house, *H(t)*, that is assumed to follow a lognormal diffusion process (for a discussion, see Merton, 1973).

These stochastic processes are depicted in equations (3.1) and (3.2):

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dz_r \qquad (3.1)$$

where

$\kappa \equiv$ speed of adjustment in the mean reverting process,

$\theta \equiv$ central location or long term mean of the short-term interest rate *r(t)* (steady state spot rate),

$\sigma \equiv$ instantaneous standard deviation of the (interest rate) disturbance,

$z_r \equiv$ standardized Wiener process,

and

$$\frac{\partial H}{H} = (\mu - \delta)dt + v dz_H \qquad (3.2)$$

where:

$\mu \equiv$ instantaneous average rate of house price appreciation,

$\delta \equiv$ "dividend-type" per unit service flow provided by the house,

$v \equiv$ instantaneous standard deviation of the house price,

$z_H \equiv$ standardized Wiener process.

The unpredictable changes in both variables are assumed to be related through the following expression:

$$dz_r(t)\, dz_H(t) = \rho dt \qquad (3.3)$$

where ρ denotes the instantaneous correlation coefficient between both Wiener processes.

Mortgages are conceived here as derivative assets. Their prices are assumed to depend on the evolution of the global economy, via the term structure of interest rates and house prices, but it is also assumed that mortgages are redundant in the characterization of this global economy. In other terms, in spite of the need to consider consumer preferences, technology and other constraints to determine the evolution of the state variables, once the house price and the term structure are determined, the value of the mortgage is set through a process of arbitrage inference. All other factors that might exert some influence will only be taken into consideration through the market price of risk associated with each state variable. However, if the state variable is a traded asset the risk adjustment does not exist (no external factor will need to be taken into account). This is the case with the house price. Consequently, the only external parameter to consider in our case would be the market price of risk associated with the variable representing the term structure of interest rates, - the spot interest rate, *r(t)*.

Cox, Ingersoll and Ross (CIR, 1985a) derived a general methodology for the valuation of contingent claims in an equilibrium framework. This framework will be used in this work under certain conditions. It will be assumed that either *i)* the market price of risk is null, which implies that the instantaneous expected rate of return on interest rate derivative products is independent of their maturity or settlement date[18]; or *ii)* that any risk premium is embedded in the parameters κ and θ that characterize the term structure (for a seminal approach to this idea see Cox, Ingersoll and Ross, 1979).

Under this framework, with the stochastic processes represented in (3.1) and (3.2), it is known from standard arguments in finance (see, for instance Cox, Ingersoll and Ross, 1985a,b; Epperson et al., 1985, and Kau et al., 1992, 1993a) that the partial differential equation (PDE) for the valuation of any asset *F(r,H,t)*, whose value is a function only of the two mentioned state-variables and time, takes the form:

[18] This is equivalent to assuming that the Cox, Ingersoll and Ross (1981) Local Expectations Hypothesis (LEH) holds.

$$\frac{1}{2}H^2 v^2 \frac{\partial^2 F}{\partial H^2} + \rho H\sqrt{r}v\sigma \frac{\partial^2 F}{\partial H \partial r} + \frac{1}{2} r\sigma^2 \frac{\partial^2 F}{\partial r^2}$$

$$+\kappa(\theta - r)\frac{\partial F}{\partial r} + (r-\delta)H\frac{\partial F}{\partial H} + \frac{\partial F}{\partial t} - rF = 0 \quad (3.4)$$

The valuation procedure developed in this work considers two forms of endogenous termination prior to maturity, namely, prepayment and default. Therefore, it is not possible to utilize forward valuation procedures (like the Monte Carlo method). Such numerical procedures do not pose any problems to the valuation of promised payments when no form of termination is allowed, but cannot hold for any form of endogenous termination (see Kau et al., 1990). As a consequence, it is necessary to use a backward valuation procedure, using the fact that the assets under analysis can be valued at expiration, given the specific nature of the underlying contracts and the current economic conditions. In other words, as the value of the mortgage is affected by the options to prepay and default in the future, it is necessary to work backwards, in an iterative manner, with the value of the assets in later moments in time feeding into the value of the same assets in previous periods. Once the value of the different financial assets embedded in a mortgage is known at termination, it is possible to use equation (3.4), above, to solve for the value of these assets in previous moments in time. Appropriately small sequential time steps backwards were used to solve the equation and give values of the mortgage and all the mortgage related assets immediately after the previous mortgage payment date. These values, in conjunction with the value of that mortgage payment provide a new set of terminal conditions which can be used as the basis for the solution of the PDE again, working backwards in time to a still earlier payment.

The process can be repeated iteratively until the initial moment in the life of the loan is reached and the value of these assets at origination is finally determined. The use of a backward solution framework of this type leads to the (recursive) solution of the valuation equation, relative to each one of the assets under study, for as many times as the months in the life of the mortgage under study.

3.2.2. The Mortgage Contract

The study considers fixed rate repayment mortgages and fixed rate endowment mortgages. In order to embody both products, the formulation that will be used is as general as possible.

A) Notation:

Some of the notation that will be used is as follows:

n = the life of the mortgage in months;

$\eta(i)$ = the time of the i^{th} month (i^{th} payment date);

$TD(t)$ = borrower's total debt in case of early termination (includes outstanding principal, any accrued interest and early termination penalty);

L = amount of the loan;

c = the fixed coupon rate (repayment mortgages);

d = the fixed coupon rate (endowment mortgages)[19];

f = the assurance policy and zero coupon bond component of the endowment coupon;

g = the interest rate component of the endowment coupon ;

Obviously, $d = g + f$.

B) Components of the Mortgage Value:

In order to calculate the value of the mortgage for a borrower it is necessary to take into account not only the present value of future payments promised to the lender, but also the value of the options implicit in the contract. The following notation is used:

$V_B(H,r,t)$ = Value at time t of the mortgage for borrower (mortgagor);

$A(r,t)$ = Value at time t of the remaining mortgage payments;

$C(H,r,t)$ = Value at time t of the call option to prepay the mortgage;

$D(H,r,t)$ = Value at time t of the option to default on the mortgage;

$J(H,r,t)$ = Value at time t of the joint option to terminate the mortgage.

At any point in time the value of the joint option to terminate the mortgage prior to maturity will be given by:

$$J(H,r,t) = C(H,r,t) + D(H,r,t) \tag{3.5}$$

[19] In this framework d is not exactly an interest rate. Its value is driven by the interest rate but it incorporates simultaneously the interest and the necessary amount paid to the mortgage assurance policy in order to make it grow to the value of the principal by the termination of the loan.

Consequently, the value of the mortgage to the borrower will be given by:

$$V_B\,(H,r,t) = A(r,t) - C(H,r,t) - D(H,r,t) = A(r,t) - J(H,r,t) \tag{3.6}$$

Given the number of elements involved in the definition of each variable, when there is no risk of major confusion, arguments will be dropped.

The value of a mortgage that has a MIG associated to it is not the same for the lender as it is for the borrower. Although it depends on the contract, the MIG is not part of the contract. Besides that, after the deal is signed, only the lender can benefit from the guarantee provided by this insurance policy. There is no reason why the borrower should take the MIG into consideration in his rational decision making process in order to minimize the mortgage costs (minimize V_B)[20]. If the decision is to default on the contract, then the MIG will produce payoffs from which only the lender will benefit.

The following notation regarding the value of the MIG, and the coinsurance is used:

$I(H,r,t)$ = the value of the MIG at time t;

$CI(H,r,t)$ = the value of the coinsurance (share of the default coverage that is assumed by the lender) at time t.

The value of the contract for the lender will be given by:

[20] Exception made for the initial moment in the life of the loan.

$$V_L(H,r,t) = V_B(H,r,t) + I(H,r,t) \qquad (3.7)$$

The value of the mortgage to the lender, V_L, is only obtainable once V_B is known. In fact, the borrower's behaviour plays a major role in the definition of the mortgage value, and this work is structured to derive V_B in the first place. Consequently, from now onwards, unless otherwise stated, all references to "mortgage value" should be understood as "mortgage value to the borrower".

At payment dates, a distinction will be made between the value of an asset immediately before and immediately after each payment. The notation used will be:

$F^-(H,r,t)$ = Value of the asset F immediately before a payment is made;

$F^+(H,r,t)$ = Value of the asset F immediately after a payment is made.

C) Payments and Balance:

Repayment Mortgages

The value of each payment, MP, is determined in order to allow the principal to be paid in full by the end of the contract:

$$MP = \frac{\left(\frac{c}{12}\right)\left[1+\left(\frac{c}{12}\right)\right]^n O(0)}{\left[1+\left(\frac{c}{12}\right)\right]^n - 1} \qquad (3.8)$$

where $O(0)$ represents the amount of the debt at the origination of the loan.

The outstanding balance after each payment date, $\eta(i)$, is given by the following expression:

$$O(i) = \frac{\left[\left(1+\frac{c}{12}\right)^{n} - \left(1+\frac{c}{12}\right)^{i}\right]}{\left(1+\frac{c}{12}\right)^{n} - 1} O(0) \qquad (3.9)$$

Endowment Mortgages

In the endowment mortgage case, the value of each monthly payment is a function of the need to pay in full the interest on the principal borrowed and of simultaneously making an installment for a fund which generates a return to provide an amount for exactly repaying this principal at the termination of the loan. Its formulation is the following:

$$MP = O(0)\frac{d}{12} \qquad (3.10)$$

The present value of the outstanding balance immediately after any payment is given by $O(i)=O(0)=L$.

3.3. The Valuation of Repayment Mortgages

3.3.1. The Mortgage Payments

In general terms, the valuation of the monthly payments is easier than the valuation of the mortgage, its embedded options or the mortgage indemnity. In the former case, we are faced with a bond-type financial asset that does not incorporate complex features. Given the cash-flows, its value is only a function of the term structure of interest rates. In the latter case, the value of the financial assets is not only a function of the term structure but also a function of the house price. Besides that, in this latter case, it is necessary to deal with an American option, with its inherent free-boundary (the prepayment option) and also a compound European option (the default option).

The valuation procedure differs between the maturity of the loan and the other payment dates.

A) Maturity of the Loan

By the end of the mortgage loan (at termination), the value of the payment due, A^-, is by definition equal to *MP*, the value of the last monthly payment, which corresponds to the unpaid balance at that moment. So, the terminal condition for the loan is:

$$A^-(r,t) = MP \qquad \text{for } t = \eta(n) \qquad (3.11)$$

B) Other Payment Dates:

With each monthly payment the present value of the borrower's debt, A, is reduced by the amount paid, MP. Consequently, for all other payment dates the valuation equation, (3.4), may be solved using the following terminal condition:

$$A^{-}(r,t) = A^{+}(r,t) + MP \qquad \text{for } t = \eta(1), \ldots, \eta(n-1) \qquad (3.12)$$

Adding the natural boundary conditions to the terminal condition just stated, it is possible to obtain the value of A at the origination of the loan, $A[r(0),0]$.

3.3.2 Mortgage Valuation

As suggested in equation (3.6), the mortgage value, V_B, is a function not only of the value of the amortizing payments, A, but also of the value of the joint option to terminate the mortgage, J. Both the default and prepayment options require considering the house price, H, as a state variable.

If the house price exceeds the value of the remaining payments, for a rational borrower, it may be preferable to sell the house than to default. Clearly, the house price has a direct impact on the value of the default option. The same does not happen with prepayment. The decision to pay the loan in advance might be affected by the evolution of the term structure of interest rates or by some personal considerations of the borrower[21], but not necessarily by the value of the underlying asset. Thus, in the first instance, the house price will not be considered in the valuation of the prepayment

[21] Not considered in this study.

option. However, the exercise of the default option terminates the loan, which implies automatically that the prepayment option expires valueless. So, indirectly, the house price also affects prepayment. As a consequence of this interaction between both options to terminate the loan, they cannot be considered separately. The borrower, in his rational attempt to minimize the cost of the mortgage, will consider its value as a whole *[$V_B = A - J$]*. Therefore, ultimately, the borrower's net position on the loan will be given by $H - V_B$. The conditions on default and prepayment might be tacitly understood from its expected behaviour at this level.

At termination, the borrower has two alternatives: to pay or to default and lose the house as a consequence. Thus, the value of the mortgage at termination is given by:

$$V_B^-(H,r,t) = \min\{MP, H\} \qquad \text{for } t = \eta(n) \qquad (3.13)$$

Using the same kind of reasoning, the terminal condition for the other payment dates is:

$$V_B^-(H,r,t) = \min\{[V_B^+(H,r,t) + MP], H\}$$

$$\text{for } t = \eta(1), \ldots, \eta(n-1) \qquad (3.14)$$

3.3.3. Boundary Conditions

Besides the natural boundary conditions applied when the state-variables take on extreme values, it is necessary to enforce a time-dependent boundary condition in order to take into account the prepayment option held by the borrower[22]. This is a free-boundary condition, which can be operationalized using either a boundary tracking method (for a synthesis see Crank, 1984) or through the use of a transformation capable of reducing the problem to a fixed boundary one, from which the free-boundary can be inferred afterwards (for a review see Wilmott, Dewynne and Howison, 1993).

The same kind of free-boundary does not exist for default because, by definition, default occurs at a payment date when the borrower does not comply with his obligation to pay. Consequently, the terminal conditions for default at each payment date automatically cover all of the possible situations.

3.3.4. The Value of the Borrower's Debt in Case of Early Termination of the Mortgage

If the borrower decides to prepay the mortgage, the total amount of his debt will be a function of the sum of the principal owed with the accrued interest corresponding to the period between the last payment and the prepayment date. Besides this amount, in the UK, it is also necessary to consider the early termination penalty contractually defined. The way in which this penalty is calculated is not standardized. Herein it is modeled as a percentage of the amount of the outstanding balance plus the accrued interest at the moment of the termination. Under these circumstances, the amount of the total debt will be given by:

[22] The details related to the specification of the boundary conditions will be presented in the next chapter, in conjunction with all of the other features of the numerical analysis approach used to solve the problem.

$$TD(t) = \{(1+\pi)\{1+c[t-\eta(i)]\}O(i) \qquad \text{for} \quad \eta(i) \leq t \leq \eta(i+1) \tag{3.15}$$

Where π represents the early termination penalty imposed to the borrower.

3.3.5. Default

As reported in Chapter 2, mortgage default has received widespread attention in academic literature during the last few years. Special emphasis has been given to the contingent claims approach to the default decision.

The approach used in this work[23] is in line with the type of reasoning presented for instance by McDonald and Siegel (1986), in a completely different field and using a completely different approach. Their article focuses on the valuation of the option to choose the timing of an investment decision. The main argument centres around the fact that a decision to invest in a non-repeatable project will annihilate the possibility of making the same investment in the future and consequently will mean the loss of the value of the corresponding option. Here, a positive value is attributed to the possibility of terminating the loan in the future. The default decision is assumed not to be simply triggered ("ruthlessly") by any situation in which the value of the remaining payments exceeds the value of the house. Instead, default is assumed to occur when the value of the mortgage to the borrower, V_B, exceeds the house value, H. As V_B is the difference between the value of the remaining payments, A, and the value of the joint option to terminate the mortgage, J, a triggering situation happens when:

[23] Which draws on Kau et al. (1993a).

$$A(r,t) > H + [C(H,r,t)+D(H,r,t)] \tag{3.16}$$

The payoff to the borrower is given by $A - H$.

By defaulting, the borrower not only gives away the house but also loses the option to terminate the loan either through default or through prepayment. In defaulting, both options are lost because the loan terminates and (further) default and prepayment options expire valueless. On a payment date $\eta(i)$, if the borrower pays, the value of his option to default is automatically reduced as a consequence of the reduction in the time to maturity of his option.

Maturity of the Loan:

At the maturity of the loan, the default option will be valueless if the value of the house surpasses the value of the monthly payment. Otherwise, it will have a value equal to the difference between the latter and the former:

$$D^{-}(H,r,t) = \max\{0,[MP - H]\} \qquad \text{for } t = \eta(n) \tag{3.17}$$

Other Payment Dates:

The terminal condition for default on the other payment dates is the following:

$D^{-}(H,r,t) =$

- $D^{+}(H,r,t)$ *if* $V_B^{-}(H,r,t) = V_B^{+}(H,r,t) + MP$ (no default)

- $A^{-}(r,t) - H$ $\quad$ *if* $V_B^{-}(H,r,t) = H$ $\quad$ (default)

for $t = \eta(1), \ldots, \eta(n-1)$ $\quad$ (3.18)

3.3.6. Prepayment

Prepayment is now the only component of the equation (3.6) whose value is not yet known.

Maturity of the Loan:

At the maturity of the loan, prepayment cannot be of any benefit to the borrower. At this time the option is valueless. Consequently, the terminal condition at maturity is the following:

$$C^{-}(H,r,t) = 0 \qquad \text{for } t = \eta(n) \qquad (3.19)$$

Other Payment Dates:

At any other payment dates, prepayment will only be of any value if default does not take place (a defaulted loan cannot be prepaid). If the contract is not terminated, the value of prepayment becomes equal to its value in the future:

$C^{-}(H,r,t) =$

- $C^{+}(H,r,t)$ $\quad$ *if* $V_B^{-}(H,r,t) = V$

- 0 if $V_B^-(H,r,t) = H$ (default)

for $t = \eta(1), \ldots, \eta(n-1)$ (3.20)

An important point to make is related to the interrelationships between the valuation of the different components of the mortgage contract and the possibility of determining the value of the prepayment option in an alternative way. Table 3.1 illustrates the terminal conditions for the valuation of those different components and also for the valuation of the mortgage itself. It can be seen that, in contrast to the approach just presented, C can be calculated by difference $(C = A - V_B - D)$, once the values of A, V_B and D are determined.

Table 3.1. Terminal Conditions for the Value of the Components of the Mortgage Contract

Components of the Mortgage Contract	Value in Case of Continuation (no default)	Value in Case of Default
A^-	$A^+ + MP$	$A^+ + MP$
C^-	C^+	0
D^-	D^+	$A^- - H$
V_B^-	$V_B^+ + MP$	H
$C^- = A^- - V_B^- - D^-$	$(A^+ + MP) - (V_B^+ + MP) - D^+ = A^+ - V_B^+ - D^+ = C^+$	$(A^+ + MP) - (A^- - H) - H = 0$

3.3.7. Mortgage Indemnity Valuation

The mortgage indemnity policy contract is an insurance asset not included in the "mortgage package" of embedded assets and options but its value is dependent on the mortgage's expected performance.

The main aim of the present sub-section is to develop a framework capable of valuing British Mortgage Indemnity Guarantees and the corresponding coinsurance assumed by the lender in each contract.

3.3.7.1. Mortgage Indemnity Guarantees

The MIG is a contract according to which an insurer agrees to pay a fraction of the total loss suffered by a mortgage lender on each loan included in a specific pool of mortgages. The precise characteristics of British MIG contracts vary from case to case in accordance with the agents involved, the nature of the insured mortgage pools and the economic environment during which the contract is signed. The features of the contract that will be considered here are based on common characteristics of recent MIGs. The insurer agrees to pay a fraction, γ, of the total loss suffered by the lender if this loss stays inside some interval. If the loss exceeds the top limit of the interval, a maximum (cap) indemnity, Γ, is paid.

The loss will be considered to be the difference between the value of the borrower's total debt and the value of the house, $\{TD(t) - H\}$.

In insurance industry jargon, the coverage is called the guarantee. If the average loan-to-value ratio (LTV) of the mortgages in the pool is represented by x[24] and the

[24] $x \leq 1$.

"normal" LTV is represented by y[25] (where $x > y$), the top limit of the interval just mentioned is given by $\frac{1}{\gamma}\{[xH(0)]-[yH(0)]\}$[26]. So, the insurer's maximum exposure has a value of $\{[xH(0)]-[yH(0)]\}=(x-y)H(0)$ and the guarantee, $G[\eta(i)]$, is given by:

$G[\eta(i)] =$

- $\gamma\,\{TD^{-}[\eta(i)] - H\}$ if $\{\{TD[\eta(i)]\}-H\}\leq\frac{1}{\gamma}\{[xH(0)]-[yH(0)]\}$
- $\{[xH(0)]-[yH(0)]\}=\Gamma$

  if $\{\{TD[\eta(i)]\}-H\}>\frac{1}{\gamma}\{[xH(0)]-[yH(0)]\}$

for all $\eta(i)$ (3.21)

If at any payment date, $\eta(i)$, the option to default is not exercised, the value of the MIG will be reduced to the value of the coverage against future default.

Maturity of the Loan:

The combination of all these features gives a general terminal condition of the following form for the MIG at termination:

$I^{-}(H,r,t)$ =

- 0 if $V_B^-(H,r,t)=MP$ (no default)

There are many different levels of LTV ratios. Information on the subject is scarce. However, contacts with professionals working for several of the main players in the British mortgage market lead to author to believe that the most common LTV ratio for new mortgages in the UK is 0.95.

[25] As mentioned in the previous chapter, commonly the "normal" LTV ratio is assumed to be 0.75.

[26] $\gamma\left\{\frac{1}{\gamma}\{[xH(0)]-[yH(0)]\}\right\}=\{[xH(0)]-[yH(0)]\}=\Gamma$

The normal level of the parameter γ in MIG contracts is 0.8. This is also the level assumed for the variable in the present thesis.

- $min\{[\gamma(MP - H)], \Gamma\}$

if $V_B^-(H,r,t) = H$ (default)

for $t = \eta(n)$ (3.22)

Other Payment Dates:

$I^-(H,r,t) =$

- $I^+(H,r,t)$ if $V_B^-(H,r,t) = V_B^+(H,r,t) + MP$ (no default)

- $min\{\{\gamma[TD^-(t) - H]\}, \Gamma\}$

if $V_B^-(H,r,t) = H$ (default)

for $t = \eta(1), \ldots, \eta(n-1)$ (3.23)

3.3.7.2. Coinsurance

The coinsurance is not an independent asset or contract. Coinsurance is the technical word given to the potential loss not covered by the MIG - that includes any loss above the cap. Its valuation can be relevant not only to the insurer, but also for the lender and for third party insurers eventually interested in selling coverage for this type of risk.

As happens with the value of the MIG, if at any payment date, $\eta(i)$, the option to default is not exercised, the value of the coinsurance will be reduced to the value of this type of risk in case of future default.

Letting $CI(H,r,t)$ represent the value of coinsurance, the corresponding terminal conditions will be given by the following expressions:

Maturity of the Loan:

$CI^-(H,r,t) =$

- 0 if $V_B^-(H,r,t) = MP$ (no default)

- $\max\{[(1-\gamma)(MP - H)], [(MP - H) - \Gamma]\}$

  if $V_B^-(H,r,t) = H$ (default)

  for $t = \eta(n)$ (3.24)

Other Payment Dates:

$CI^-(H,r,t) =$

- $CI^+(H,r,t)$ if $V_B^-(H,r,t) = V_B^+(H,r,t) + MP$ (no default)

- $\max\{\{(1-\gamma)\{TD^-[\eta(i)] - H\}\}, \{\{TD^-[\eta(i)] - H\} - \Gamma\}\}$

  if $V_B^-(H,r,t) = H$ (default)

  for $t = \eta(1), \ldots, \eta(n-1)$ (3.25)

By definition, at any payment date the value of the coinsurance is equal to the difference between the value of the potential loss and the value of the insurance coverage. So, in aggregate

$CI^-(H,r,t) + I^-(H,r,t) =$

- $\{CI^{+}(H,r,t) + I^{+}(H,r,t)\}$ (no default)

- $\{TD^{-}(t) - H\}$ (default)

3.3.8. The Equilibrium Condition

In an equilibrium framework the terms of any contract cannot be set arbitrarily. The need to avoid arbitrage imposes the condition that the value of the assets exchanged must be equal, in order for both parties to be able to accept the deal. The value of the contract to the lender must be equal to the amount lent.

A very important factor to ponder here is the non-reimbursable amount (arrangement fee) paid by the borrower at origination, whose main aim consists in acting as a deterrent to early termination. The other important factor to take into account is the MIG, which benefits only the lender.

Considering what was just mentioned, the equilibrium condition for the mortgage is the following:

$$V_B[H(0),r(0),0]+I[H(0),r(0),0]=(1-\xi)L \qquad (3.26)$$

Where, ξ is any arrangement fee imposed at the beginning of the deal, expressed in terms of percentage of the amount lent.

In order to reach an equilibrium situation it is necessary to find a contract rate, c, capable of balancing equation (3.26). In this work, this was achieved through the use of a secant iteration technique, in line with those proposed by Gerald and Wheatley (1994) and Press et al. (1992).

It is important to note that, in spite of not being a part of the contract, the MIG, affects the equilibrium contract rate, *c*, and consequently the values of all the assets under study.

3.4. The Valuation of "Without Profits" Endowment Mortgages

3.4.1. General Remarks

In general terms the treatment that must be given to a "without profits" fixed rate endowment mortgage in order to find its value under the contingent claims framework proposed in this thesis is very similar to that used to value fixed rate repayment mortgages.

As mentioned in Chapter 2, the amount and the structure of the cash-flows inherent to both types of mortgages are completely different. However, the assets embedded in both products and the insurance products that are related to them (MIG and coinsurance) are the same.

In conceptual terms, the terminal conditions that constitute one of basic factors for the solution of the valuation equation do not change. In other words, in spite of the cash-flow differences that will necessarily generate discrepancies in the final value of both products, there are no structural differences in terms of the financial assets underlying both types of mortgages or the corresponding interrelationships (both are mortgages). The only factors that are necessary to take into consideration are the differences in what is paid by the borrower, either during the normal life of the loan or for early termination.

3.4.2. The Determination of the Value of Different Components of the Monthly Payments

A borrower who contracts a "without profits" endowment mortgage has a package that includes *i)* the loan, *ii)* the present value of a zero coupon bond whose value at maturity will be equal to the amount borrowed and *iii)* an assurance policy which will cover the risk of default in case of his death during the life of the mortgage (life insurance component of the term assurance policy).

The second and third components can be understood as "tranches" of an implicit loan that will need to be redeemed by the borrower. The price of each one of these "tranches" is available in the market and so it is possible to find the total market value of the implicit loan. As a consequence, and thinking in terms of the "global package", the borrower needs to pay not only monthly interest over the effective amount of the loan, but also interest and principal on a second implicit loan whose value is the sum of the present values of the life insurance component of the term assurance policy and the zero coupon bond.

From equation (3.10) the functional form for the monthly payments in the case of an endowment mortgage is already known ($MP = O(0)\frac{d}{12}$). Therefore, the problem now consists of the determination of the values of the two components, *f*, the assurance policy and zero coupon bond component of the endowment coupon, and g, the interest rate component of the endowment coupon, which are included in the global rate d. Assuming that the average level of risk does not differ between borrowers and mortgages of both types, the interest component of both products must be similar ($g = c$) and so the problem becomes restricted to a single variable, f.

The approach used here to model the "without profits" endowment mortgage consists in finding a way of expressing the payments that the borrower will need to make in order to pay for the second and third elements of the package, in terms of the amount borrowed (L). In other words, the idea is to find a fixed differential that will be added to the interest rate. The end result will be a global rate which, when applied to the amount of the loan, will give the value of the global payment made by the borrower at each payment date.

In the case of a repayment mortgage, the life assurance policy is not included in the contract but normally constitutes one of the requirements imposed by the lender in granting the loan. It is a commonly traded product whose price is relatively easy to get from any insurance company that sells this type of policy.

On the other hand, working under a CIR framework, the value at time t of a zero coupon bond that pays off £1 at time T, $TBnd(t)$, will be given by the known closed form solution (Cox, Ingersoll and Ross, 1985b):

$$TBnd(r,t,T) = \left[\mathrm{A}(t,T)e^{-\mathrm{B}(t,T)r}\right] \tag{3.27}$$

Where:

$$A(t,T)=\left\{\frac{\phi_1 e^{\phi_2\tau}}{\phi_2\left[e^{\phi_1\tau}-1\right]+\phi_1}\right\}^{\phi_3} \quad (3.28)$$

$$B(t,T)=\frac{e^{\phi_1\tau}-1}{\phi_2\left[e^{\phi_1\tau}-1\right]+\phi_1} \quad (3.29)$$

$$\phi_1=\sqrt{(\kappa+\lambda)^2+2\sigma^2} \quad (3.30)$$

$$\phi_2=\frac{\kappa+\lambda+\phi_1}{2} \quad (3.31)$$

$$\phi_3=\frac{2\kappa\theta}{\sigma^2} \quad (3.32)$$

$$\tau=T-t \quad (3.33)$$

κ, θ, σ and r will keep the same meanings attributed to them in section 3.2, and λ designates the market price of risk, assumed to be zero.

Under these modeling constraints, considering *[TBnd(t)L]* as the value at time t of a portfolio of zero coupon bonds whose value will grow to the amount of *O(0)=L* at time *T*, and letting *AP* represent the value at time *0* of the life assurance component of the endowment insurance policy, the value of the (constant) monthly additional payment, SP, that the borrower will need to pay in order to redeem the so-called implicit loan will be given by[27]:

$$SP = \{[TBnd(0)L] + AP\} \frac{\left(\frac{g}{12}\right)\left[1+\left(\frac{g}{12}\right)\right]^n}{\left[1+\left(\frac{g}{12}\right)\right]^n - 1} \qquad (3.34)$$

And the rate differential will be given by:

$$f = 12\left[\frac{SP}{O(0)}\right] \qquad (3.35)$$

As mentioned above, in spite of not being a normal contractual feature in repayment mortgage contracts, the life assurance policy is commonly required by the mortgage lenders from any borrower. Therefore, the distinction between repayment and endowment mortgages at this level is more formal than substantial. In order to maintain some consistency between the treatments given to both types of mortgage contracts, AP will consequently be ignored. However, given what was presented above, it is important to stress that the explicit consideration of this feature under this framework would not bring any sort of problem.

[27] As mentioned above, it is assumed that $g = c$.

3.4.3. The Value of the Borrower's Debt in Case of Early Termination of the Mortgage

Once again in this respect, endowment mortgages differ from repayment ones. As the debt outstanding is a constant, and the full payment of this amount will only take place if the loan reaches maturity, any early termination will imply that the total amount of the debt will be given by the sum of *i)* the principal, *ii)* the unpaid part of the amount necessary to buy the zero coupon bond and the life assurance policy at the origination of the contract and *iii)* the interest accrued on both, deducted from the value assumed by the zero coupon bond portfolio at the time net of the early repayment penalty:

$$TD(t) = \{[O(0) + U(i)]\{1 + d[t - \eta(i)]\}\} - \{(1-\pi)TBnd(t)L\}$$

$$\text{for} \quad \eta(i) \leq t \leq \eta(i+1) \qquad (3.36)$$

Where π represents the early repayment penalty contractually imposed on the borrower, and $U(i)$ stands for the value, at the moment of the termination, of the unpaid part of TP, the amount of the implicit loan that includes the zero coupon bond portfolio and the life insurance component of the endowment policy. This latter component is defined in the following terms:

$$U(i) = \frac{\left[\left(1+\frac{g}{12}\right)^n - \left(1+\frac{g}{12}\right)^i\right]}{\left(1+\frac{g}{12}\right)^n - 1} TP \qquad (3.37)$$

where:

$$TP = TBnd(0)L + AP^{28} \tag{3.38}$$

3.4.4. The Equilibrium Condition

The formulation of the equilibrium condition does not change compared with the repayment mortgage modeling process. It is given by equation (3.26).

3.5. Conclusion

This chapter presents a theoretical framework for valuing UK mortgages and MIG.

At the repayment mortgage valuation level, the terminal conditions imposed take into account the specific nature of the early repayment penalties included in most UK fixed rate mortgages. Besides that, a framework to approach the valuation of a "without profits" endowment mortgage is also developed.

At the insurance valuation level, the terminal conditions account for the features included in some recent UK policies like the cap and the definition of the guarantee as a proportion of the loss. Simultaneously, the framework allows for the valuation of coinsurance, the potential loss not covered by the MIG but covered by the lender.

The framework presented can be easily extended to cope with the variable rate counterparts of both products analyzed. In these cases it is necessary to cope with additional problems related to the need to overcome the path-dependency inherent in

[28] In this case, as AP is ignored, we have:

$$TP = TBnd(0)L$$

the evolution of variable rates. However, the methodologies proposed by Kishimoto (1989), Kau et al. (1993a) and Hilliard et al. (1995) permit those difficulties to be overcome. The only constraint in these cases is related to the computing power necessary to tackle the problem. The need to add an additional auxiliary state variable to track the evolution of the contract rate adds an extra spatial dimension to the problem. This requires the use of a powerful computing system in order reach a solution in a reasonable period of time.

No apparent closed-form solutions are available for the resulting PDE (3.4). Closed-form solutions become more implausible, if not impossible, in presence of the special problems created by the free-boundary imposed by the American option to prepay the loan, and also by the need to cope with the idiosyncrasies dictated by the compound European option to default. As a consequence a decision was made to solve the problem numerically. The next chapter presents a numerical procedure to solve (3.4) using a finite-difference method.

Given the rate of improvement in computing power, it might be helpful to a reader of this thesis intending to perform such calculations a few years from now, to know that the "fixed rate calculations" carried out for this thesis required run times of a substantial part of a day on a 90 Mhz Pentium PC (once programs were developed several PCs were used simultaneously).

Appendix 1

Formulas for the Values of the Monthly Payments and the Outstanding Balance

1. Repayment Mortgages

Formulas (3.8) and (3.9) do not present a perfectly "common" way to define the value of the monthly payments and the outstanding balance in actuarial terms. This derives basically from the fact that the cash-flows are referred to the terminal period of the contract. Given this unusual condition it is desirable to demonstrate the validity of the formulas and to explain their rationale.

A) Value of the Monthly Payments:

As mentioned, the PDE (3.4) must be solved using a backward procedure. In order to follow this procedure it is necessary to start the valuation process from the terminal moment in the life of the financial derivative under consideration. This means that it is necessary to refer all the cash-flows to this final moment instead of using the more common actuarial procedure of referring all the cash-flows to the initial moment in the life of the loan. Obviously, the ultimate formulation will change as a consequence.

The first step in defining the value of each monthly payment is to recognise that the future (capitalised) value of the outstanding debt in the terminal period of the contract must be equal to the future (capitalised) value of all the payments when this value is also referred to the terminal moment of the contract[29]:

$$O(0)\left(1+\frac{c}{12}\right)^{n} = MP\left(1+\frac{c}{12}\right)^{n}\left[\frac{1-\left(1+\frac{c}{12}\right)^{-n}}{\frac{c}{12}}\right]$$

Developing the right-hand side in the first place and isolating *MP* afterwards it is possibly to get successively:

$$O(0)\left(1+\frac{c}{12}\right)^{n} = MP\left[\frac{\left(1+\frac{c}{12}\right)^{n}-1}{\frac{c}{12}}\right]$$

$$MP = \frac{O(0)\left(1+\frac{c}{12}\right)^{n}\left(\frac{c}{12}\right)}{\left(1+\frac{c}{12}\right)^{n}-1}$$

B) Value of the Outstanding Balance Immediately After a Payment Date $\eta(i)$:

At any payment date, $\eta(i)$, the outstanding debt can be expressed in the following terms:

[29] The symbols used in this Appendix are defined in the main text.

$$O(i)=\left[O(0)-MP\frac{1-\left(1+\frac{c}{12}\right)^{-i}}{\frac{c}{12}}\right]\left(1+\frac{c}{12}\right)^{i}$$

Making the substitution for *MP* and solving successively, it is possible to get:

$$O(i)=\left\{O(0)-\left[\frac{O(0)\left(1+\frac{c}{12}\right)^{n}\frac{c}{12}}{\left(1+\frac{c}{12}\right)^{n}-1}\right]\left[\frac{1-\left(1+\frac{c}{12}\right)^{-i}}{\frac{c}{12}}\right]\right\}\left(1+\frac{c}{12}\right)^{i}$$

$$=\left\{\frac{O(0)\left(1+\frac{c}{12}\right)^{n}-O(0)\left(1+\frac{c}{12}\right)^{n}-O(0)+O(0)\left[\left(1+\frac{c}{12}\right)^{n}\left(1+\frac{O(i)}{12}\right)^{-i}\right]}{\left(1+\frac{c}{12}\right)^{n}-1}\right\}\left(1+\frac{c}{12}\right)^{i}$$

$$=\frac{O(0)\left[\left(1+\frac{c}{12}\right)^{n}-\left(1+\frac{c}{12}\right)^{i}\right]}{\left[1+\left(\frac{c}{12}\right)\right]^{n}-1}$$

2. Endowment Mortgages

In the endowment mortgage case, by definition, the amount of debt outstanding after each payment is equal to the amount of the loan. Consequently, the only unknown that it is necessary to find is the value of the monthly payments that is equal to:

$$MP = O(0)\frac{d}{12}$$

CHAPTER 4

Numerical Solution of a Two-State Variable Contingent Claims Mortgage Valuation Model Using the Explicit Finite Difference Method

4.1. Introduction

The contingent claims analysis leads to the modeling of many derivative assets as partial differential equations (PDEs). A few of these models - usually the ones that use the simplest and strongest assumptions - allow for analytic or closed-form solutions. But this is the exception, not the rule. The natural tendency to model increasingly sophisticated assets, and to relax the strongest assumptions in order to reduce the distance between models and reality, leads naturally to the development of valuation frameworks of enlarged complexity for which no closed-form solutions are available.

In a few cases, it is possible to find mathematical transformations capable of providing the necessary conditions for the problem to be solved in closed-form.

However, this path has also its drawbacks. Besides the fact that closed-form solutions are rare, in some situations where it is mathematically possible to follow this route, the transformed version of the problem becomes so far away from the original that it is difficult to relate the transformed variables to the original ones in order to get the desired information. Thus, even in some of these uncommon cases where the original problem allows itself to be transformed in such a way that the derivation of a closed-form solution is possible, there are situations where it might be better to attempt an accurate numerical solution of the correct model than a closed-form solution of the wrong model (see, Wilmott, Dewynne and Howison, 1993).

This chapter presents a numerical procedure for the solution of a contingent claims valuation model aimed at valuing "mortgage related" products. The theoretical cornerstone underlying this framework is the Cox, Ingersoll and Ross (1985a,b) model. It can be said that the CIR (1985b) interest rate model in some way represents an attempt to overcome the strongest limitations of the classical term-structure models (e.g. Vasicek, 1977, or Brennan and Schwartz, 1982). Under these classical models the functional form of the market price of risk and the stochastic processes governing the interest rate(s) are provided exogenously. Consequently, there is no guarantee that the corresponding results are consistent with a general financial markets equilibrium. CIR (1985b), in contrast, is based on the comprehensive context of an intertemporal capital asset pricing model (CIR, 1985a). Departing from a set of strong assumptions about consumption preferences of the economic agents, the authors were able to derive endogenously the stochastic process followed by the instantaneous risk-free interest rate. The resulting general valuation framework allows for the consideration of an arbitrary, but finite, number of state-variables, each one representing a different source of risk[30].

Despite being able to overcome the weaknesses of the older term structure models, from the point of view of complexity of solution, the CIR model contains many of the same kind of problematic features as the classic models. The valuation equation is of identical type: a second-order PDE whose terminal and boundary conditions describe the cash-flow structure of the asset under valuation. In the specific problem tackled here, there is a common characteristic in mortgage design that will add to the natural complexity of the model: the "embedded" features that allow for early termination of the loan by the borrower.

The numerical techniques used in the valuation of derivative assets may be classified as forward and backward methods. Both are widely used and have various advantages and disadvantages. Forward methods, like Monte Carlo simulation, tackle the valuation problem on the basis of information about present and past events. This makes them perfectly able to cope with the "path-dependent" behaviour, that make the valuation of variable rate mortgages so problematic, but incapable of dealing with early exercise of any options ("American-type" features). In contrast, backward techniques like the binomial and trinomial lattices or the finite difference methods provide the solution of the valuation equation on the basis of present and future information. Consequently, considering the current state of the art in numerical techniques for the solution of PDEs, the valuation of mortgage products or the corresponding mortgage indemnity guarantees are best done using backward procedures.

[30] In order to keep the model tractable it is necessary to reduce the number of state-variables to only a few (normally one or two).

The two major types of backward procedures are lattices and finite difference methods. They are used to solve similar types of derivative valuation problems (Hull, 1997) and each one has its advantages (see, Geske and Shastri, 1985). Lattices are more intuitive and easy to implement than finite differences. Despite that, according to Oakes (1992a,b), finite difference methods seem to be most widely used. Finite difference methods are more general than lattice methods, the knowledge of the joint probability distribution of the sources of risk is not required (Courtadon, 1990), they are able to compute simultaneously several starting contingent claim prices for different values of the risk parameters, or if we want to put it in other words, they can be more powerful when several option values are to be calculated simultaneously (Trigeorgis, 1996). On these grounds, and giving special consideration to the fact that a mortgage valuation problem similar to the one that is under study in the present work was already successfully solved using a finite difference algorithm (Kau et al., 1992, 1993a,b, 1995), the option to work with a finite difference method seems to be the most appropriate in this case. Unfortunately, despite their advantages, there are also some difficulties and shortcomings associated with these methods. The algorithms were generally developed to tackle problems of a similar nature in the Natural Sciences or Engineering. Despite some structural similarity, the problems are different in detail and, consequently, the use of the algorithms is not straightforward. Normally, only the canonical forms of the problems were studied in detail. The contingent claim valuation equations tend to be more complex than these canonical forms, so the behavior of the solution and the accuracy of the algorithms can be completely different. This is a source of potential trouble and requires the development of specially designed adaptations.

In addition to the previous obstacle there is limited availability of "pre-packaged" software capable of undertaking the solution of the models. Consequently, it is necessary to design a specific version of the algorithm and the corresponding computer code to put it into operation.

This type of problem gains special relevance in the case tackled in this work: the application of a finite difference method to solve a two-state dimensional real estate finance model that presents several potentially problematic characteristics:

i) the existence of a mixed derivative term;

ii) the existence of a "free boundary";

iii) the presence of variable coefficients.

The chapter is organized as follows: Section 2 provides a general presentation of finite difference methods as a tool to solve PDEs, discusses the alternatives available in this case, justifies the decision taken, and presents the generic characteristics of the Explicit Finite Differences method. Section 3 develops a framework in which the technique is applied in the solution of a contingent claims model meant for the valuation of mortgage related derivative assets. Section 4 discusses the methodology used to tackle the problems posed by the existence of a free boundary, and Section 5 concludes the chapter.

4.2. The Finite Difference Methodology

Finite difference methods are techniques wherein the domain of interest of the problem under study is discretized using a set of nodes and the information flowing between these nodes is generated through the use of Taylor series expansions. The domain of the solution of the PDE is covered by a rectilinear grid or net with a finite number of mesh points. At each node all the derivative terms in the equation are replaced by finite difference approximations that approximate its value. The differential equation is replaced by a finite difference equivalent expression that takes into consideration the approximate value of the function at some neighboring points. The result is a set of algebraic equations representing the PDE at each one of the mesh points. A numerical algorithm is then used in order to solve directly or iteratively these equations.

This section presents the basics of the methodology, drawing on classic works by Lapidus and Pinder (1982) and Ames (1992) .

4.2.1 Finite Difference Approximations

The representation of the equations will be exhibited in the first instance for a simple one space variable system, and afterwards extended to the more complex two-space variable system.

4.2.1.1 Notation

Considering that $u(x)$ represents a continuous function of the single independent variable x, the x domain is discretized into a rectilinear grid of points whose sides are parallel to the x and t axis. The grid spacings in the x and t directions are given by h and k. The grid points are given by:

$$u(x_j) \equiv u(jh) \equiv u_j \qquad \text{(with } j = 0, 1, 2, \ldots\text{)}$$

The representation of the difference and differential equations at the points $x = jh$ and $t = mk$ will be represented by U_j^m and u_j^m respectively.

The substitution of the location x_j by jh indicates that the coordinates of each node are specified as the product of the integer j and the grid spacing h, that here is assumed to be constant. The integer j indicates the position of the point along the x coordinate in relation to a certain moment in time. Similarly, this moment is given by the product of the integer m and the grid spacing for the time dimension, k, that here is also assumed to be constant.

When the problem is two-space-dimensional, the function can be specified at any one of its nodes as:

$$u(x_j, y_i) \equiv u(jh, il) \equiv u_{j,i} \qquad \text{(with } j = 0, 1, 2, \ldots \text{ and } i = 0, 1, 2, \ldots\text{)}$$

In this case, the spacing in the x direction is given by h and in the y direction by l. The location of u along the x and y coordinates is given by j and i. The representation of the difference and differential equations at the point $x = jh$, $y = il$ and $t = mk$ will be denoted by $U^m_{j,i}$ and $u^m_{j,i}$, respectively.

4.2.1.2 Taylor Series Expansions

Finite difference methods are based on Taylor series expansions of the functions under study and their derivatives. If $u(x)$ and its derivatives are single-valued, finite and continuous functions of x, the Taylor series expansion for $u(x)$ can be written at the point x_r either as:

$$u(x_j + h) = u(x_j) + hu_{x|j} + \frac{h^2}{2!}u_{xx|j} + \frac{h^3}{3!}u_{xxx|j} + \ldots \tag{4.1}$$

Or as:

$$u(x_j - h) = u(x_j) - hu_{x|j} + \frac{h^2}{2!}u_{xx|j} - \frac{h^3}{3!}u_{xxx|j} + \ldots \tag{4.2}$$

From (4.1) neglecting second and higher powers of h it is possible to obtain:

$$u_{x|j} \approx \frac{u(x_j + h) - u(x_j)}{h} \equiv \frac{u_{j+1} - u_j}{h} \approx \frac{U_{j+1} - U_j}{h} \tag{4.3}$$

This is the so-called forward difference approximation to a first derivative, since the differencing is in the forward t direction.

Applying a similar procedure to (4.2) it is possible to obtain:

$$u_{x|j} \approx \frac{u(x_j)-u(x_j-h)}{h} \equiv \frac{u_j-u_{j-1}}{h} \approx \frac{U_j-U_{j-1}}{h} \quad (4.4)$$

This is the backward finite difference approximation since the differencing is in the backward direction.

Because in both cases the series have been subject to discretional truncation, the approximations contain errors. The dimension of these errors is a function of the size of the largest term of the truncated series. In other words, the order of the error is given by the first element of the truncated series. In equations (4.3) and (4.4) the error is of order *h*, *O(h)*.

Similarly, subtracting equations (4.1) and (4.2) and solving for $u_{x|j}$ gives:

$$u_{x|j} \approx \frac{U_{j+1}-U_{j-1}}{2h} \quad (4.5)$$

with a truncation error of order $O(h^2)$.

Applying the same sort of rationale, this is reasonably called a central difference approximation to a first derivative term.

The accuracy of the approximations is obviously directly proportional to the order of the error[31]. As a consequence, whenever possible, it is common to use central differences in finite difference algorithms.

Using similar procedures, it is also possible to determine approximations for higher order derivatives. For example, adding (4.1) and (4.2), neglecting $O(h^4)$ and higher

[31] Or, if preferred, directly proportional to the highest order terms included in the expansion.

order terms, and solving for the second spacial derivative, the following central difference approximation will be found:

$$u_{xx|j} = \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} \tag{4.6}$$

whose truncation error is $O(h^2)$.

These results may be extended in a straightforward manner to derive many finite difference approximations to derivatives in higher dimensions. The most important in the context of this chapter, given the nature of the problem to be solved, are the following:

$$u_{x|j,i} = \frac{U_{j+1,i} - U_{j,i}}{h} \tag{4.7}$$

$$u_{y|j,i} = \frac{U_{j,i+1} - U_{j,i}}{l} \tag{4.8}$$

Equations (4.7) and (4.8) give the forward difference approximations to first order partial derivatives of an equation in two state variables, with truncation errors of $O(h)$ and $O(l)$, respectively.

Similarly, equations (4.9) and (4.10) represent the backward approximations to the same derivatives:

$$u_{x|j,i} = \frac{U_{j,i} - U_{j-1,i}}{h} \tag{4.9}$$

$$u_{y|j,i} = \frac{U_{j,i} - U_{j,i-1}}{l} \qquad (4.10)$$

The truncation errors continue to be of order *O(h)* and *O(l)*, respectively.

Finally, equations (4.11) and (4.12) give central difference approximations to both first order partial derivatives of an equation in two state variables, with truncation errors of *O(h)* and *O(l)*, respectively.

$$u_{x|j,i} = \frac{U_{j+1,i} - U_{j-1,i}}{2h} \qquad (4.11)$$

$$u_{y|j,i} \approx \frac{U_{j,i+1} - U_{j,i-1}}{2l} \qquad (4.12)$$

The two-dimensional equivalents of (4.6) are obtained using the same type of procedure:

$$u_{xx|j,i} = \frac{U_{j+1,i} - 2U_{j,i} + U_{j-1,i}}{h^2} \qquad (4.13)$$

$$u_{yy|j,i} = \frac{U_{j,i+1} - 2U_{j,i} + U_{j,i-1}}{l^2} \qquad (4.14)$$

Equations (4.13) and (4.14) have truncating errors of $O(h^2)$ and $O(l^2)$.

The last important extension required is the one needed to handle mixed derivative terms. Using Taylor series again to expand successively the first derivative gives, after some rearrangement, the following final result:

$$u_{xy}|_{j,i} = \frac{U_{j+1,i+1} - U_{j+1,i-1} - U_{j-1,i+1} + U_{j-1,i-1}}{4hl} \qquad (4.15)$$

The truncation error is here of dimension $O(h^2)+O(l^2)$.

When $h = l$, (4.15) becomes:

$$u_{xy}|_{j,i} = \frac{U_{j+1,i+1} - U_{j+1,i-1} - U_{j-1,i+1} + U_{j-1,i-1}}{4h^2} \qquad (4.16)$$

4.2.2 Alternative Options

There are two main approaches to implementing finite difference methods in the solution of PDEs: explicit and implicit finite difference schemes. Explicit and implicit finite difference algorithms can be obtained through many different representations of the time and space differentials (see, Lapidus and Pinder, 1982, pages 155-162). According to the formulation adopted in Wilmott et al. (1993), both schemes represent the space differentials by symmetric central differences. The major distinction between the two alternatives lies in the way in which the time differentials are represented.

If a forward difference is used to represent the time differential, the resulting difference equation involves only one grid point at the advanced time level[32]. This method is called the explicit finite difference approximation (Smith, 1985). At any moment in time, the value of the function at the next time step is represented in terms of its own value in several space points at the current time step.

Alternatively, if a backward difference is used to represent the time differential, the resulting difference involves more than one grid point at the advanced time level. This method is called the implicit finite difference method. In this case, in order to move to the next time step, the algorithm imposes the need to solve a set of simultaneous equations whose dimension is a function of the number of space steps to be considered (for a detailed review of the methodology see, for instance, Morton and Mayers, 1994).

Obviously, the structure of solution of the implicit finite difference method is more complex than that of the explicit method but this does not mean that it should be avoided in the solution of PDEs. The global performance of any method is assessed not only on this basis but also in terms of its accuracy, consistency, stability and convergence[33].

[32] It is necessary to emphasize that "advanced time level" does not necessarily mean a future moment in time. This will happen when the underlying equation is of the forward parabolic type, but the inverse will happen if the equation is of the backward parabolic type. In other words, when the underlying PDE is a backward parabolic one, the use of symmetric central differences at the current time level to represent the space differentials in conjunction with the use of forward differences to represent the time differential will lead to a situation in which the value of the variable under study at a certain point in space and time is expressed in terms of the value of the same variable in several space points at a posterior moment in time. Therefore, "advanced time level" means the moment in time that is treated subsequently to the current one by the algorithm.

[33] A full discussion of such concepts, which is out of the scope of the present work, can be found in Richtmyer and Morton (1967).

Accuracy is a measure of the deviation between the numerical solutions given by the algorithmic solution of the difference equations (discretized version of the problem) and the exact solution of the differential equation.

The stability of a finite difference method is a characteristic that assures that the deviation between the solutions of the differential and difference equations at a fixed point in time is always bounded. In other words, an algorithm is said to be stable if the rounding errors that arise in numerical solutions as a consequence of the use of finite precision arithmetic are not magnified at each iteration (Wilmott et al., 1995).

Consistency is the characteristic that assures that the difference equation converges to the correct differential equation as the grid spacing tends to zero (Mitchell and Griffiths, 1980). In other words, the truncation errors tend to zero, as the finite difference mesh is refined.

Finally, a numerical method is considered to be convergent if the difference between the theoretical solutions of the differential and difference equations at a fixed point in time tend to zero uniformly as the dimension of the space steps in the grid tends also to zero (Mitchell and Griffiths, 1980).

In order to assure convergence it is necessary for all the perturbations in the computed solution to be bound (stability) and also that truncation errors go to zero with the refinement of the mesh (consistency).

As Mitchell and Griffiths (1980) state, "there is no best method for approximating difference formulae". There are positive and negative characteristics associated with each, and only in the face of each situation it is possible to decide what method is the best suited for its solution. In general, implicit methods have better stability characteristics than explicit methods but, on the other hand, explicit algorithms are easier to solve (Lapidus and Pinder, 1982). The best method in each case is the one which on average performs best in terms of the factors mentioned above, under the specific conditions in which it is to be applied.

As noted by Rebonato (1996), the use of finite difference methods has been the major line of attack to solve complex contingent claims problems arising in modern finance. The introduction of the method in the field is normally credited to Schwartz (1975, 1977) and Brennan and Schwartz (1978, 1979), and since the publication of the latter work its use seems to have been growing steadily. Several algorithms have been used.

Courtadon (1990), has reviewed the use of simple one-dimensional methods. Besides the explicit and the implicit methods, the article describes in detail the implementation of the author's version of the Crank-Nicolson method, first published in Courtadon (1982).

Clewlow (1990), provides a deeper survey which includes two dimensional methods. The paper describes and analyses the implementation of ADI algorithms similar to those used by Gibson and Schwartz (1990) or Schwartz and Torous (1992) and also linear Hopscotch algorithms like those used by Bailey (1987, 1989) and Titman and Torous (1989). Explicit methods in two and more dimensions have also been used for instance by Kau et al. (1993a,b) and Dempster and Hutton (1995, 1996).

The performance of implicit methods is sometimes presented as superior to the performance of explicit methods in finance applications. However, this is not always the case. The expansion of the power and speed of the computers in the last few years has made the computational difficulties related to the increased number of steps, that is inherent in the explicit finite difference method[34], progressively smaller. Additionally, in explicit finite difference methods, each time step tends to be "computationally cheaper" since, in contrast to what happens with implicit methods, it does not involve the solution of a system simultaneous equations.

Another important factor to be considered is that complex problems in finance involving several space dimensions sometimes become so intricate and difficult to program that it is preferable to use explicit methods. Dempster and Hutton (1995, 1996), produced two studies where one of the main conclusions was precisely that, if a sufficiently fast computer is available, the solution of cross-currency valuation functions by the explicit finite difference method is probably the best choice available.

The present work has several complex features, namely, a free boundary corresponding to an American option, several interconnected valuation functions and a stream of hundreds of European options (European compound option). Under the framework that is proposed, all the solutions are given by the same algorithm. Thus, it is advisable to find a method that is as straightforward to program as possible.

Given all the previous reasons and also the fact that the only published work in real estate finance that dealt with a framework of the same kind (Kau et al., 1992, 1993a, 1993b, 1995) used an explicit finite difference method, a decision was made to use this algorithm.

[34] As a consequence of the need to overcome the stability constraint.

4.3. A Framework for the Solution of the Model Using an Explicit Finite Difference Method

4.3.1. The Structure of the Solution

At any moment in time, the value of all assets under study in the present work is a function of the future states of nature. Therefore, it is necessary to solve the problem starting from termination and moving towards the initial moment in the life of the loan. Figure 4.1 illustrates the general procedure.

Once the moment of the last monthly payment of a mortgage is reached, the global value of the contract (V_B), its embedded assets (A, D, C) and the insurance products related to it (I, CI) are easily determined. These values constitute the final terminal conditions for the solution of the valuation equation (3.4). The introduction of the corresponding set of boundary conditions, specifying the value of the different assets in the situations where the state variables reach extreme values, and the consideration of the free boundary portraying the behavior of the assets in prepayment circumstances, will allow for the solution of the PDE.

The recursive nature of the definition of the terminal conditions presented in Chapter 3, with the values of the assets in later moments exerting their influence over the valuation process in earlier moments through the terminal conditions, allows for the procedure to be systematically repeated until the values at origination are determined. Therefore, equation (3.4) is solved in relation to each one of the variables as many times as the number of months that correspond to the life of the loan. In the present study, the life of the loan is considered to be 25 years. Consequently, PDE (3.4) is solved 300 times in relation to each asset.

4.3.2. Transformed Version of the Original PDE

The numerical solution of equation (3.4) imposes the need to overcome some minor difficulties. The first one is related to its infinite domain. Arbitrage arguments require that the time value of money cannot be negative. As a consequence, in the interest rate dimension, r, the natural domain of the equation is the space $(0, \infty)$. Similarly, in a house market without any major inefficiencies and also without transaction costs, the same kind of arguments hold. Houses provide the basis for the fulfillment of several needs in our society, and the number of houses is not infinite. Thus, house prices should also be, at least, non-negative. In other words in the house price dimension, H, the natural domain of the equation is similar to the interest rate dimension, $(0, \infty)$.

Unfortunately, infinite boundary conditions are very difficult to handle by numerical methods. The situation calls for the use of a transformed version of the original PDE capable of minimizing the problems that arise as a consequence of this specific feature. Instead of working directly with r and H, two transformed variables were defined and used.

In the r dimension the following transformation was chosen:

$$y = \frac{1}{1 + \psi r} \tag{4.17}$$

for some constant $\psi > 0$.

Under this transformation the variable y will assume a value of 0 for "$r = \infty$" and a value of 1 for $r = 0$. The inverse transformation is:

$$r = \frac{1-y}{\psi y} \qquad (4.18)$$

In the house price dimension, H, the transformation used was of the same kind:

$$x = \frac{1}{1+\omega H} \qquad (4.19)$$

for some constant $\omega > 0$.

Similarly to what happened in the interest rate dimension, the variable x will assume a value of 0 for "$H = \infty$" and a value of 1 for $H = 0$. The inverse transformation is:

$$H = \frac{1-x}{\omega x} \qquad (4.20)$$

Under these transformations, the infinite area $(0, \infty) \times (0, \infty)$ is mapped onto the unit square, $(0, 1) \times (0,1)$.

The number of points on a given y grid that correspond to small values of r are inversely proportional to the value assumed by ψ. In contrast, the smaller the value assumed by ψ, the more points on a given y grid will correspond to values of r greater than, lets say, 3%. In this case, the area of major interest is the one that provides values of r that more plausibly occur in the market during the time horizon considered. Thus, values ranging from 2%-3% to 20%-25% are the most important for this kind of study. Considering this fact, it was necessary to choose a value of ψ capable of providing a reasonable compromise between those conflicting tendencies. In this case, a value of $\psi = 10$ was chosen. The middle point in the y grid, then corresponds to $r = 10\%$[35].

All assets under study are expressed in relation to par. The homogeneity of the contract in H allows for the use of a discretionary initial house value. Consequently, a value of 1 is used. The values of the different assets under study can be found afterwards through a mere scale transformation - multiplication of the value expressed in relation to par by the actual value of the house.

The fact that a house price value of 1 is going to be used makes the problem easier in the transformed house price dimension. The value $\omega = 1$ was chosen, which is equivalent to using no scale factor in the H transformation.

The time variable was also transformed. As Wilmott et al. (1993) mention, parabolic PDEs should preferably be solved in the forward dimension. In order to achieve this, the following transformation was used[36]:

$$\tau = T - t \tag{4.21}$$

The inverse transformation is $t = T - \tau$.

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[35] As we will see in Chaper 5, there are a few exceptions to this rule. In order to analyse the relationship between the spot and the long term average of the interest rate (term structure effects) ψ's of 8.33 ($r = 12\%$) and 12.5 ($r = 8\%$) are also used.

[36] It is important to note that this transformation is not imperative. Some authors have solved parabolic PDEs in the contingent claims field using finite difference algorithms that work in the backward time dimension (see Hull, 1997 and Courtadon, 1990).

After all of these transformations, the PDE (3.4) changes significantly. In order to rewrite it, it is necessary to make several substitutions[37], that are mentioned below.

The main aim is to express the original function in terms of the new variables, so:

$$W(x,y,\tau) = F(r(y),H(x),t(\tau)) \tag{4.22}$$

The derivatives of the original function need also to be expressed in terms of the new variables. For the first and second derivatives in r, respectively, the results can be expressed in the following way:

$$\frac{\partial F}{\partial r} = \frac{\partial W}{\partial y}\frac{\partial y}{\partial r} \tag{4.23}$$

$$\frac{\partial^2 F}{\partial r^2} = \frac{\partial^2 W}{\partial y^2}\left(\frac{\partial y}{\partial r}\right)^2 + \frac{\partial^2 y}{\partial r^2}\left(\frac{\partial W}{\partial y}\right) \tag{4.24}$$

The first and second derivatives in H provide obviously similar results:

$$\frac{\partial F}{\partial H} = \frac{\partial W}{\partial x}\frac{\partial x}{\partial H} \tag{4.25}$$

$$\frac{\partial^2 F}{\partial H^2} = \frac{\partial^2 W}{\partial x^2}\left(\frac{\partial x}{\partial H}\right)^2 + \frac{\partial^2 x}{\partial H^2}\left(\frac{\partial W}{\partial x}\right) \tag{4.26}$$

Equation (3.4) also contains a mixed derivative. Its expression in terms of the new variables is the following:

$$\frac{\partial^2 F}{\partial H \partial r} = \frac{\partial^2 W}{\partial x \partial y}\frac{\partial y}{\partial r}\left(\frac{\partial x}{\partial H}\right) \tag{4.27}$$

[37] For a more detailed explanation of all the substitutions used see Appendix 2.

All the previous proposed substitutions contain elements that are functions of the derivative of a new state variable with respect to the original one. As a consequence, it is necessary to determine their values. In the transformed r dimension the results can be expressed in the following way:

$$\frac{\partial y}{\partial r} = -\psi y^2 \tag{4.28}$$

$$\frac{\partial^2 y}{\partial r^2} = 2\psi^2 y^3 \tag{4.29}$$

In the H dimension the results are the following:

$$\frac{\partial x}{\partial H} = -\omega x^2 \tag{4.30}$$

$$\frac{\partial^2 x}{\partial H^2} = 2\omega^2 x^3 \tag{4.31}$$

Finally, the second derivatives of the original functions in relation to the original state variables incorporate the squares of the first derivatives of the new variable with respect to the original one. The corresponding results can be expressed in the following way:

$$\left(\frac{\partial y}{\partial r}\right)^2 = \psi^2 y^4 \tag{4.32}$$

$$\left(\frac{\partial x}{\partial H}\right)^2 = \omega^2 x^4 \tag{4.33}$$

Considering all these substitutions, the formulation for the transformed version of PDE (3.4) is the following:

$$\frac{1}{2}H(x)^2 v^2 \omega^2 x^4 \frac{\partial^2 W}{\partial x^2} + \rho H(x)\sqrt{r(y)}\, v\sigma\psi\omega x^2 y^2 \frac{\partial^2 W}{\partial x \partial y}$$

$$+\frac{1}{2}r(y)\sigma^2\psi^2 y^4 \frac{\partial^2 W}{\partial y^2} + \left\{H(x)^2 v^2 \omega^2 x^3 - \left[\left(r(y)-\delta\right)H(x)\omega x^2\right]\right\}\frac{\partial W}{\partial x}$$

$$+\left\{r(y)\sigma^2\psi^2 y^3 - \left[\kappa\left(\theta - r(y)\right)\psi y^2\right]\right\}\frac{\partial W}{\partial y} - \frac{\partial W}{\partial \tau(t)} - r(y)W = 0 \tag{4.34}$$

4.3.3. Finite Difference Representation of the PDE

This sub-section presents the finite difference representation of the transformed PDE at the interior nodes and also at all boundaries when the free boundary influences the results. In order to make the exposition clearer, the section will be subdivided into components in which the interior nodes, the lower boundaries, the upper boundaries and the corners of the grid will be treated separately.

4.3.3.1. Interior Nodes

The state space will be spanned into discrete intervals whose complete set will be called the lattice. The transformed spot interest rate dimension will be represented in the interval $[0, 1]$, subdivided into I intervals such that $Il = y_I = 1$ and $il = y_i$. Similarly, the house price will be represented in the interval $[0, 1]$, subdivided into J intervals such that $Jl = x_J = 1$ and $jl = x_j$. Finally, the time to maturity, τ, is defined in the interval $[0, T]$, subdivided into N intervals such that, $Ns= \tau_N =1$ and $ns = \tau_n$. The value of the asset, $F(r,H,t)$, will be approximated by $U_{i,j}^{n}$.

The solution will be based on a grid where 50 steps are used to discretize each space dimension. So, $h = l = 0.02$.

In spite of the common use of this type of numerical algorithm in the finance and real estate literature in recent years (see, for instance Brennan and Schwartz, 1978 and Hull and White, 1990, for applications to one space dimension valuation equations and Kau et al. 1987, 1990, 1992, 1993a, 1995 for applications to two space dimensions valuation equations), a formal error theory for this problem does not exist (Kau et al., 1995). However, from the Lax Equivalence Theorem, it is known that the stability and the convergence of a finite difference scheme is necessary and sufficient to assure convergence (see, Lapidus and Pinder, 1982, page 163).

Given the spatial lattice defined above, a grid of 66 time steps a month ($k \approx 0.00126$) was used in order to assure numerical stability and consistency[36], which is in line with what Kau et al. (1993a) used when they solved the same equation in a three-dimensional framework.

Using the notation defined above, all interior nodes of the lattice - nodes corresponding to points where $0 < y < 1$ and $0 < x < 1$ - can be expressed in terms of difference equations where the most accurate central difference approximations are used to represent the space derivatives. The time derivative, where such a procedure is not possible (see Wilmott at al., 1993, page 270), is represented by a forward difference approximation. This leads the PDE (4.34) to be approximated by the following difference equation:

$$\frac{1}{2}H(x)^2 v^2 \omega^2 x^4 \frac{U^n_{i,j+1} - 2U^n_{i,j} + U^n_{i,j-1}}{h^2}$$

$$+\rho H(x)\sqrt{r(y)}v\sigma\psi\omega x^2 y^2 \frac{U^n_{i+1,j+1} - U^n_{i+1,j-1} - U^n_{i-1,j+1} + U^n_{i-1,j-1}}{4lh}$$

$$+\frac{1}{2}r(y)\sigma^2\psi^2 y^4 \frac{U^n_{i+1,j} - 2U^n_{i,j} + U^n_{i-1,j}}{l^2}$$

$$+\left\{H(x)^2 v^2 \omega^2 x^3 - \left[(r(y)-\delta)H(x)\omega x^2\right]\right\}\frac{U^n_{i,j+1} + U^n_{i,j-1}}{2h}$$

$$+\left\{r(y)\sigma^2\psi^2 y^3 - \left[\kappa(\theta - r(y))\psi y^2\right]\right\}\frac{U^n_{i+1,j} + U^n_{i-1,j}}{2l}$$

36 Appendix 3 presents a more detailed account of the approach used to tackle the issues related to the stability, consistency and convergence of the numerical procedure employed here.

$$-\frac{U_{i,j}^{n+1}+U_{i,j}^{n}}{s}-r(y)U_{i,j}^{n}=0 \tag{4.35}$$

After some rearrangement, it is possible to find a relationship that gives the value of the asset under valuation at a certain time-step as a function of its own value at the previous time-step:

$$\begin{aligned}
U_{i,j}^{n+1} = &\left\{1-\left[H(x)^2 v^2 \omega^2 x^4\left(\frac{s}{h^2}\right)\right]-\left[r(y)\sigma^2\psi^2 y^4\left(\frac{s}{l^2}\right)\right]-r(y)s\right\}U_{i,j}^{n} \\
&+\left\{\frac{1}{2}H(x)^2 v^2\omega^2 x^4\left(\frac{s}{h^2}\right)\right\}\left(U_{i,j+1}^{n}+U_{i,j-1}^{n}\right) \\
&+\left\{\left\{H(x)^2 v^2\omega^2 x^3-\left[(r(y)-\delta)H(x)\omega x^2\right]\right\}\left(\frac{s}{2h}\right)\right\}\left(U_{i,j+1}^{n}-U_{i,j-1}^{n}\right) \\
&+\left\{\frac{1}{2}r(y)\sigma^2\psi^2 y^4\left(\frac{s}{l^2}\right)\right\}\left(U_{i+1,j}^{n}+U_{i-1,j}^{n}\right) \\
&+\left\{\left\{r(y)\sigma^2\psi^2 y^3-\left[\kappa(\theta-r(y))\psi y^2\right]\right\}\left(\frac{s}{2l}\right)\right\}\left(U_{i+1,j}^{n}-U_{i-1,j}^{n}\right) \\
&+\rho H(x)\sqrt{r(y)}\,v\sigma\psi\omega y^2 x^2\left(\frac{s}{4lh}\right)\left(U_{i+1,j+1}^{n}-U_{i+1,j-1}^{n}-U_{i-1,j+1}^{n}+U_{i-1,j-1}^{n}\right)
\end{aligned} \tag{4.36}$$

Alternatively, if there is a preference for a representation of the finite difference scheme where the coefficients of each approximation are perfectly isolated:

$$U_{i,j}^{n+1} = \left\{1-\left[H(x)^2 v^2 \omega^2 x^4\left(\frac{s}{h^2}\right)\right]-\left[r(y)\sigma^2\psi^2 y^4\left(\frac{s}{l^2}\right)\right]-r(y)s\right\}U_{i,j}^{n}$$

$$+\left\{\left[\frac{1}{2}H(x)^2 v^2\omega^2 x^4\left(\frac{s}{h^2}\right)\right]+\left\{\left\{H(x)^2 v^2\omega^2 x^3-\left[\left(r(y)-\delta\right)H(x)\omega x^2\right]\right\}\left(\frac{s}{2h}\right)\right\}\right\}U^n_{i,j+1}$$

$$+\left\{\left[\frac{1}{2}H(x)^2 v^2\omega^2 x^4\left(\frac{s}{h^2}\right)\right]-\left\{\left\{H(x)^2 v^2\omega^2 x^3-\left[\left(r(y)-\delta\right)H(x)\omega x^2\right]\right\}\left(\frac{s}{2h}\right)\right\}\right\}U^n_{i,j-1}$$

$$+\left\{\left[\frac{1}{2}r(y)\sigma^2\psi^2 y^4\left(\frac{s}{l^2}\right)\right]+\left\{\left\{r(y)\sigma^2\psi^2 y^3-\left[\kappa\left(\theta-r(y)\right)\psi y^2\right]\right\}\left(\frac{s}{2l}\right)\right\}\right\}U^n_{i+1,j}$$

$$+\left\{\left[\frac{1}{2}r(y)\sigma^2\psi^2 y^4\left(\frac{s}{l^2}\right)\right]-\left\{\left\{r(y)\sigma^2\psi^2 y^3-\left[\kappa\left(\theta-r(y)\right)\psi y^2\right]\right\}\left(\frac{s}{2l}\right)\right\}\right\}U^n_{i-1,j}$$

$$+\rho H(x)\sqrt{r(y)}v\sigma\psi\omega y^2 x^2\left(\frac{s}{4lh}\right)\left(U^n_{i+1,j+1}-U^n_{i+1,j-1}-U^n_{i-1,j+1}+U^n_{i-1,j-1}\right) \quad (4.37)$$

The coefficients are a function not only of the model parameters, but also of the values assumed by the transformed space variables; so they are variable.

Unfortunately this type of finite difference scheme is prone to generate stability problems.

In order to keep the errors associated with the finite difference representation of the differential equation inside acceptable bounds, it is necessary to guarantee that all the U^n coefficients in equation (4.37) are positive (Merton and Mayers, 1994)[37]. This feature will assure the respect of the so-called maximum principle, helping to secure the stability of the scheme. In order to achieve this result there are two main possibilities. One consists in the reduction of the spacial step, which substantially reduces the operational viability of explicit finite difference algorithms. The other calls for changes in the finite difference representation of the first derivative.

The coefficients of the second derivative terms in equation (4.34) are always positive, but the same cannot be said about the coefficients of the first derivative terms.

[37] As it will be seen in Chapter 5, in the present study, the last element on the right hand side is normally null because a $\rho = 0$ is considered.

Consequently, it is necessary to devise a strategy capable of circumventing the problem and avoiding the instability problems that are associated with it. According to Morton and Mayers (1994), page 47, the solution consists in using forward or backward differences for the first derivative terms instead of using central differences.

In our transformed valuation equation, as the coefficients are variable, the problem is even more acute, since, as depicted in Figure 4.2, the sign of the coefficients of the first derivative terms of equation (4.34) changes across the grid. Consequently, it is necessary to use what Morton and Mayers (1994) refer to as "upwind differencing". In other words, to use a forward difference approach when the coefficient of the first derivative term is positive and a backward difference approximation when the coefficient of the first derivative term is negative[38].

As equation (4.34) is a two dimensional PDE, there are four potential combinations of first derivative signs. Under normal circumstances, as can be seen in Fig. 4.2, only two of those will tend to occur. However, the computer code that was written to solve the problem (see Appendix 4) allows for the choice of the approximation to be dictated automatically as a function of the sign of the first derivative in each dimension. Consequently, all alternatives are considered.

Equation (4.38) provides an example of the application of this type of procedure in a case where backward approximation was used to approximate the first derivatives in both space dimensions:

$$U_{i,j}^{n+1} = \Bigg\{1 - \left[H(x)^2 v^2 \omega^2 x^4 \left(\frac{s}{h^2}\right)\right] - \left[r(y)\sigma^2 \psi^2 y^4 \left(\frac{s}{l^2}\right)\right] - r(y)s$$

[38] I would like to express my gratitude to Prof. Joan Walsh from the Department of Mathematics of the University of Manchester for offering me some of her insight about the subject and giving me helpful advice on how to implement this scheme in the present study.

$$+\left\{\left\{H(x)^2 v^2\omega^2x^3-\left[(r(y)-\delta)H(x)\alpha x^2\right]\right\}\left(\frac{s}{h}\right)\right\}$$

$$+\left\{\left\{r(y)\sigma^2\psi^2y^3-\left[\kappa(\theta-r(y))\psi y^2\right]\right\}\left(\frac{s}{l}\right)\right\}U_{i,j}^n$$

$$+\left\{\left[\frac{1}{2}H(x)^2 v^2\omega^2x^4\left(\frac{s}{h^2}\right)\right]\right\}U_{i,j+1}^n$$

$$+\left\{\left[\frac{1}{2}r(y)\sigma^2\psi^2y^4\left(\frac{s}{l^2}\right)\right]\right\}U_{i+1,j}^n$$

$$+\left\{\left[\frac{1}{2}H(x)^2 v^2\omega^2x^4\left(\frac{s}{h^2}\right)\right]-\left\{H(x)^2v^2\omega^2x^3-\left[(r(y)-\delta)H(x)\alpha x^2\right]\right\}\left(\frac{s}{h}\right)\right\}U_{i,j-1}^n$$

$$+\left\{\left[\frac{1}{2}r(y)\sigma^2\psi^2y^4\left(\frac{s}{l^2}\right)\right]-\left\{\left\{r(y)\sigma^2\psi^2y^3-\left[\kappa(\theta-r(y))\psi y^2\right]\right\}\left(\frac{s}{l}\right)\right\}\right\}U_{i-1,j}^n$$

$$+\rho H(x)\sqrt{r(y)}v\sigma\psi\omega y^2x^2\left(\frac{s}{4lh}\right)\left(U_{i+1,j+1}^n-U_{i+1,j-1}^n-U_{i-1,j+1}^n+U_{i-1,j-1}^n\right) \quad (4.38)$$

4.3.3.2. Upper Boundary Conditions in the Transformed State Variables

The use of transformed variables is only a way of simplifying the solution of the model. The relevant state variables continue to be the original ones. Thus, the discussion of the problem's boundary conditions ought to be done in terms of those original state variables.

The upper boundary conditions in the transformed variables ($y = x = 1$) correspond to the lower boundary conditions in the original state variables ($r = H = 0$). Therefore, this sub-section deals with what are in fact lower boundary conditions in the original state variables.

All the formulation related to the boundary conditions of the problem, that needs to consider degenerate versions of equation (3.4), will only be presented in terms of differential equations. The conversion of these into difference equations is done using the same principles that were used to create (4.38). In general terms, whenever possible, central differences are used. When this is not convenient in terms of stability, an upwinding difference approximation scheme is used.

4.3.3.2.1. The House Price Dimension

When $H = 0$ the value of the mortgage is certainly higher than the value of the house. Consequently, the borrower's rational behavior will lead to certain default. In these circumstances, prepayment is of no value and so:

$$C(0,r) = 0 \tag{4.39}$$

Simultaneously, in case of default, the value of the mortgage becomes equal to the value of the house. Therefore:

$$V_B(0,r) = H = 0 \tag{4.40}$$

$$D(0,r) = A(r) \tag{4.41}$$

In terms of "insurance related" products the situation is not so straightforward. It is necessary to consider the degenerated form assumed by equation (3.4) in this circumstance. When the variable under analysis is the MIG, the corresponding PDE, uniquely in r, is the following:

$$\frac{1}{2} r\sigma^2 \frac{\partial^2 I}{\partial r^2} + \kappa(\theta - r)\frac{\partial I}{\partial r} + \frac{\partial I}{\partial t} - rI = 0 \qquad (4.42)$$

Similarly, when the variable under scrutiny is the coinsurance, the resulting degenerate version of the valuation equation is the following:

$$\frac{1}{2} r\sigma^2 \frac{\partial^2 CI}{\partial r^2} + \kappa(\theta - r)\frac{\partial CI}{\partial r} + \frac{\partial CI}{\partial t} - rCI = 0 \qquad (4.43)$$

4.3.3.2.2. The Spot Interest Rate Dimension

In a situation where the interest rate, r, is null there is no discounting. Remembering that the dimension of the time step used in the numerical procedure is represented by s, and taking into consideration that the value of the interest rate in the next moment in time is certain to be $\kappa\theta s$, the boundary condition for the value of the future payments, A, at any moment in time will be given by:

$$A(0,t) = A(\kappa\theta s,\ t+s) \qquad (4.44)$$

The value of all of the other assets, $F(0,H,t)$, will be given by the solution of the degenerate form of the PDE (3.4) when r is considered to be null.

It is important to note that a null value for the state variable does not imply a null derivative in relation to this state variable. Consequently, in contrast with what is mentioned by Kau et al. (1995), the degenerate version of the main PDE that corresponds to this boundary still contains a first derivative term in r, because this element is not multiplied by the variable, and therefore does not vanish:

$$\frac{1}{2}H^2 v^2 \frac{\partial^2 F}{\partial H^2} - \delta H \frac{\partial F}{\partial H} + \kappa\theta \frac{\partial F}{\partial r} + \frac{\partial F}{\partial t} = 0 \tag{4.45}$$

4.3.3.3. Lower Boundary Conditions in the Transformed State Variables

In accordance with what was stated in the previous sub-section, the lower boundary conditions in the transformed state variables correspond to the situations in which the original PDE reaches its upper boundaries. In other words, the situations to be analysed correspond to cases in which $r \to \infty$ and $H \to \infty$.

4.3.3.3.1. The House Price Dimension

When $H \to \infty$, default cannot have any value, but the same cannot be said about prepayment. The value of this variable is a function of the relationship between the amount of the debt and the value of the mortgage. The degenerate form of the equation that will give its value is the following:

$$\frac{1}{2} r\sigma^2 \frac{\partial^2 C}{\partial r^2} + \kappa(\theta - r)\frac{\partial C}{\partial r} + \frac{\partial C}{\partial t} - rC = 0 \tag{4.46}$$

In agreement with the previous statement, according to which under this condition the default option is valueless:

$$\lim_{H\to\infty} D(H,r) = 0 \quad (4.47)$$

Being so, the value of the mortgage will be given by:

$$\lim_{H\to\infty} V_B(H,r) = A(r) - \lim_{H\to\infty} C(H,r) \quad (4.48)$$

In a situation in which default is not possible the insurance coverage is valueless. Thus:

$$\lim_{H\to\infty} I(H,r) = 0 \quad (4.49)$$

$$\lim_{H\to\infty} CI(H,r) = 0 \quad (4.50)$$

4.3.3.3.2. The Spot Interest Rate Dimension

When $r \to \infty$ any future payment is valueless. Consequently, all assets that involve future payments are also valueless:

$$\lim_{r\to\infty} A(r) = 0 \quad (4.51)$$

$$\lim_{r\to\infty} C(H,r) = 0 \quad (4.52)$$

$$\lim_{r\to\infty} D(H,r) = 0 \quad (4.53)$$

$$\lim_{r\to\infty} V_B(H,r) = 0 \quad (4.54)$$

$$\lim_{r\to\infty} I(H,r) = 0 \quad (4.55)$$

$$\lim_{r\to\infty} CI(H,r) = 0 \quad (4.56)$$

4.3.3.4. Corners of the Grid

The so-called "corners" of the grid are points at which boundaries in different dimensions of the grid are observed simultaneously. Corners exist where both the state variables reach extreme values. In other words, $r = 0$ or "$r = \infty$" and simultaneously $H = 0$ or "$H = \infty$".

4.3.3.4.1. Corners in the Upper Boundary of the Transformed Interest Rate Dimension of the Grid

Obviously this situation corresponds to the boundary where $r = 0$. The value of the function at these points will depend not only on the value of r, but also on the value of H.

Since the value of the interest rate in the next moment in time is certain to be $\kappa\theta s$, whenever $H = 0$, the value of any asset $F(0,0,t)$ will be given by:

$$F(0,0,t) = F(0,\kappa\theta s,\ t+s) \tag{4.57}$$

Similarly, whenever $H \to \infty$, the value of the assets will be given by:

$$\lim_{H\to\infty} F(H,0,t) = \lim_{H\to\infty} F(H,\kappa\theta s,t+s) \tag{4.58}$$

4.3.3.4.2. Corners in the Lower Boundary of the Transformed Interest Rate Dimension of the Grid

In contrast with the previous sub-section, the points that are considered here correspond to situations in which $r \to \infty$.

In this case, as happens along all the other points of the upper boundary in the r dimension, the assets involving payments in the future are of no consequence. Consequently, the equations that give the value of the different assets there continue to hold. Thus:

$$\lim_{r \to \infty} A(r) = 0 \tag{4.59}$$

$$\lim_{r \to \infty} C(H,r) = 0 \tag{4.60}$$

$$\lim_{r \to \infty} D(H,r) = 0 \tag{4.61}$$

$$\lim_{r \to \infty} V_B(H,r) = 0 \tag{4.62}$$

$$\lim_{r \to \infty} I(H,r) = 0 \tag{4.63}$$

$$\lim_{r \to \infty} CI(H,r) = 0 \tag{4.64}$$

4.4. The Free Boundary

This section presents a discussion of the implications of the free boundary imbedded in the problem and the way in which it was treated in the numerical solution of the model.

4.4.1. Prepayment Region

Prepayment can take place at any time. Therefore, it is necessary to identify the free boundary given by the combinations (H,r) that correspond to the points in which prepayment first takes place.

Given the assumption that was made about the rational behavior of the borrowers, at every moment in time, the value of the total debt must be at least as high as the value of the mortgage:

$$V_B \leq TD \tag{4.65}$$

Consequently, the prepayment boundary is given by the region where a "value-matching" condition is observed:

$$V_B = TD \tag{4.66}$$

Besides this, it is also necessary to observe a "high-order contact" or "smooth-pasting" condition requiring that both functions meet tangentially at the boundary (see

Merton, 1973). Putting this in a different way, it is required that not only the values of the functions V_B and TD, but also their slopes, should match at the boundary.

In the repayment mortgage case the derivatives of TD with respect to the space variables are easy to calculate:

$$\frac{\partial TD}{\partial H} = 0 \tag{4.67}$$

$$\frac{\partial TD}{\partial r} = 0 \tag{4.68}$$

Therefore, it is necessary to specify the following conditions:

$$\frac{\partial V_B}{\partial H} = \frac{\partial TD}{\partial H} = 0 \tag{4.69}$$

and

$$\frac{\partial V_B}{\partial r} = \frac{\partial TD}{\partial r} = 0 \tag{4.70}$$

when $V_B = TD$.

In the endowment mortgage case, the diverse nature of the outstanding balance, which includes a component that is an interest rate product, leads to significant differences at the interest rate derivative level:

$$\frac{\partial V_B}{\partial H} = \frac{\partial TD}{\partial H} = 0 \tag{4.71}$$

and

$$\frac{\partial V_B}{\partial r} = \frac{\partial TD}{\partial r} = (1-\pi)LABe^{-Br} \tag{4.72}$$

where , as mentioned in Chapter 3, A and B are parameters in the CIR closed-form solution of the default free bond valuation problem.

If prepayment takes place, the default option becomes immediately valueless, either in the repayment mortgage case or in the endowment mortgage case. The same happens with the insurance products, MIG and coinsurance, because after the repayment there is no possibility of default. In summary, default, insurance and coinsurance only make sense outside the prepayment region.

The need to observe smoothness in the solutions for these variables makes it easier to identify the conditions that they need to observe along the prepayment boundary. Therefore, in the prepayment boundary it will be necessary to observe:

$$\frac{\partial D}{\partial H} = \frac{\partial D}{\partial r} = \frac{\partial I}{\partial H} = \frac{\partial I}{\partial r} = \frac{\partial CI}{\partial H} = \frac{\partial CI}{\partial r} = 0 \tag{4.73}$$

An important aspect that it is necessary to mention is related to the interaction between the "normal" boundary conditions and the free boundary. Obviously, inside the prepayment region the valuation function obeys a different regime. Consequently, it is necessary to expand this regime to the boundary in order to assure the smoothness of the solution near the boundaries that "touch" the prepayment region. Figures 4.3.A and 4.3.B give a graphical representation of the process that was followed in order to extend the valuation regime inherent in the prepayment region to the adjoining boundaries.

4.4.2. Numerical Treatment of the Free Boundary

Free boundaries are difficult to treat numerically. As mentioned in the previous chapter, there are two approaches to deal with such features: boundary tracking methods or the use of transformations capable of reducing the original problem to a fixed boundary one, from which the free-boundary can be inferred afterwards. The solution adopted here is one of the latter type. Drawing on Berger, Cement and Rogers (1975) the problem is converted into a non-linear PDE with a fixed boundary.

The valuation equation originally assumed the form:

- $\frac{\partial V_B}{\partial t} + \mathcal{L}V_B = 0$ if $V_B < TD$
- $V_B = TD$ otherwise (4.74)

where $\mathcal{L}$ is the second-order linear operator in (3.4).

In the repayment mortgage case, noting that when $V_B = TD$:

$$\frac{\partial V_B}{\partial t} + \mathcal{L}V_B = \frac{\partial TD}{\partial t} + \mathcal{L}TD = (1+\pi)cO - rTD \qquad (4.75)$$

the valuation equation can be rewritten in the following form:

$\frac{\partial V_B}{\partial t} + \mathcal{L}V =$

- 0 if $V_B < TD$
- $(1+\pi)cO - rTD$ if $V_B = TD$ (4.76)

Where the problem is now defined for the entire (H,r) space.

In the endowment mortgage case, the different formulation of the value of the outstanding balance at any moment in time, leads again to a more intricate solution:

$$\frac{\partial V_B}{\partial t} + \pounds V_B = \frac{\partial TD}{\partial t} + \pounds TD = \frac{1}{2} r\sigma^2 \frac{\partial^2 TD}{\partial r^2} + [\kappa(\theta - r)]\frac{\partial TD}{\partial r} + \frac{\partial TD}{\partial t} - rTD \qquad (4.77)$$

where:

$$\frac{\partial TD}{\partial r} = (1-\pi)LABe^{-Br} \qquad (4.78)$$

$$\frac{\partial^2 TD}{\partial r^2} = -(1-\pi)LAB^2 e^{-Br} \qquad (4.79)$$

$$\frac{\partial TD}{\partial t} = -(1-\pi)L\left\{ rAe^{-Br}\left\{ \frac{\{(\phi_1 e^{\phi_1 \tau})\{\phi_2(e^{\phi_1 \tau}-1)+\phi_1\}\} - \{\phi_2(\phi_1 e^{\phi_1 \tau})(e^{\phi_1 \tau}-1)\}}{\{\phi_2[e^{\phi_1 \tau}-1]+\phi_1\}^2} \right\}\right\}$$

$$+(1-\pi)L\left\{ e^{-Br}\left\{ \phi_3 \left\{ \frac{\phi_1 e^{\phi_2 \tau}}{\phi_2[e^{\phi_1 \tau}-1]+\phi_1} \right\}^{\phi_3 - 1} \left\{ \frac{\{(\phi_1\phi_2 e^{\phi_2 \tau})\{\phi_2(e^{\phi_1 \tau}-1)+\phi_1\}\} - \{(\phi_1\phi_2 e^{\phi_1 \tau})(\phi_1 e^{\phi_2 \tau})\}}{\{\phi_2[e^{\phi_1 \tau}-1]+\phi_1\}^2} \right\}\right\}\right\}$$

$$+Od \qquad (4.80)$$

ϕ_1, ϕ_2 and ϕ_3 continue to assume the same meaning that was attributed to them in Chapter 3: parameters in the CIR closed-form solution of the default-free bond valuation problem.

In accordance, the valuation equation can be rewritten in the following form:

$$\frac{\partial V_B}{\partial t} + \pounds V =$$

- *0*   *if* $V_B < TD$

$$\bullet \quad \frac{1}{2} r\sigma^2 \frac{\partial^2 TD}{\partial r^2} + \kappa(\theta - r)\frac{\partial TD}{\partial r} + \frac{\partial TD}{\partial t} - rTD \qquad \text{if } V_B = TD \tag{4.81}$$

4.4.3. The Default Region

The first element to note here is that default only makes sense outside the prepayment region, since a prepaid loan cannot be defaulted. In the second place it is also necessary to note that, as mentioned in the previous chapter, rational default only makes sense on payment dates. Thus the terminal conditions for default[39] are sufficient to provide the full specification of the default boundary. From the inspection of these terminal conditions it can be easily seen that A is continuous over all of the default boundary because its value is not in any way affected by D. The same happens with V_B, but it is not applicable to the other variables. Their values are always in some way related to the value of default, but this is not immediately reflected in the terminal condition of their valuation equations. Fortunately, these discontinuities in the terminal conditions do not impede the existence of solutions for the valuation PDEs.

4.5. Conclusion

This chapter presents the framework that is necessary to implement the numerical solution of the valuation models developed in chapter 3. In order for this to become possible, choices were made in relation to the type of numerical solution and the

[39] Given by equations 3.17 and 3.18.

specific algorithm to use. A decision was made to use an explicit finite difference method.

In order to implement this algorithm successfully for the solution of the models in question, it was necessary to transform the original valuation equation, whose domain is bi-infinite, into another one, whose domain is the unit square. The next step includes the identification of the common boundary conditions of the problem. The prepayment free-boundary was also identified. Finally, difference equations capable of providing the background for the implementation of the model, at all points of the grid, were developed and presented.

Figure 4.1. General Structure of the Solution of the Mortgage Valuation Problem

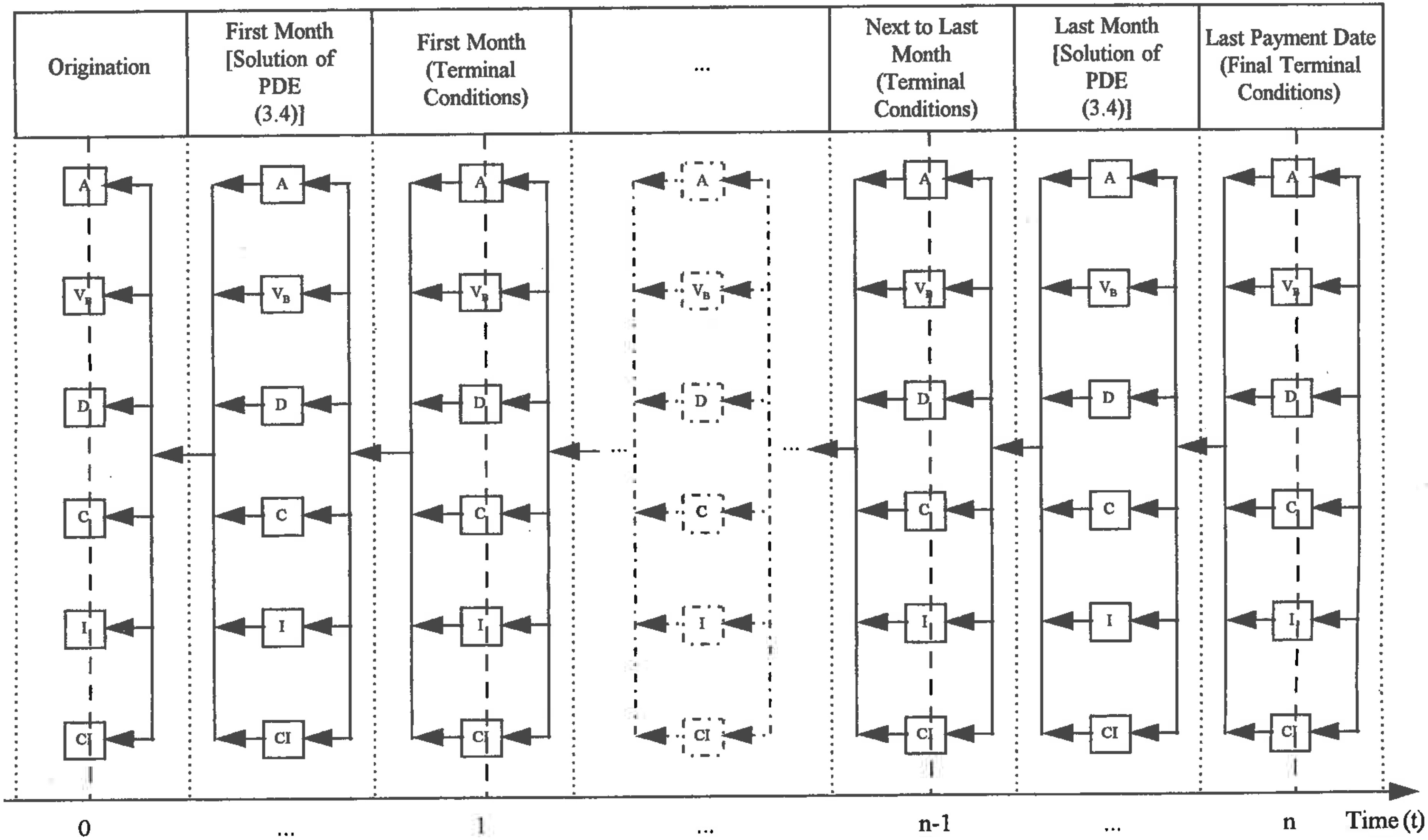
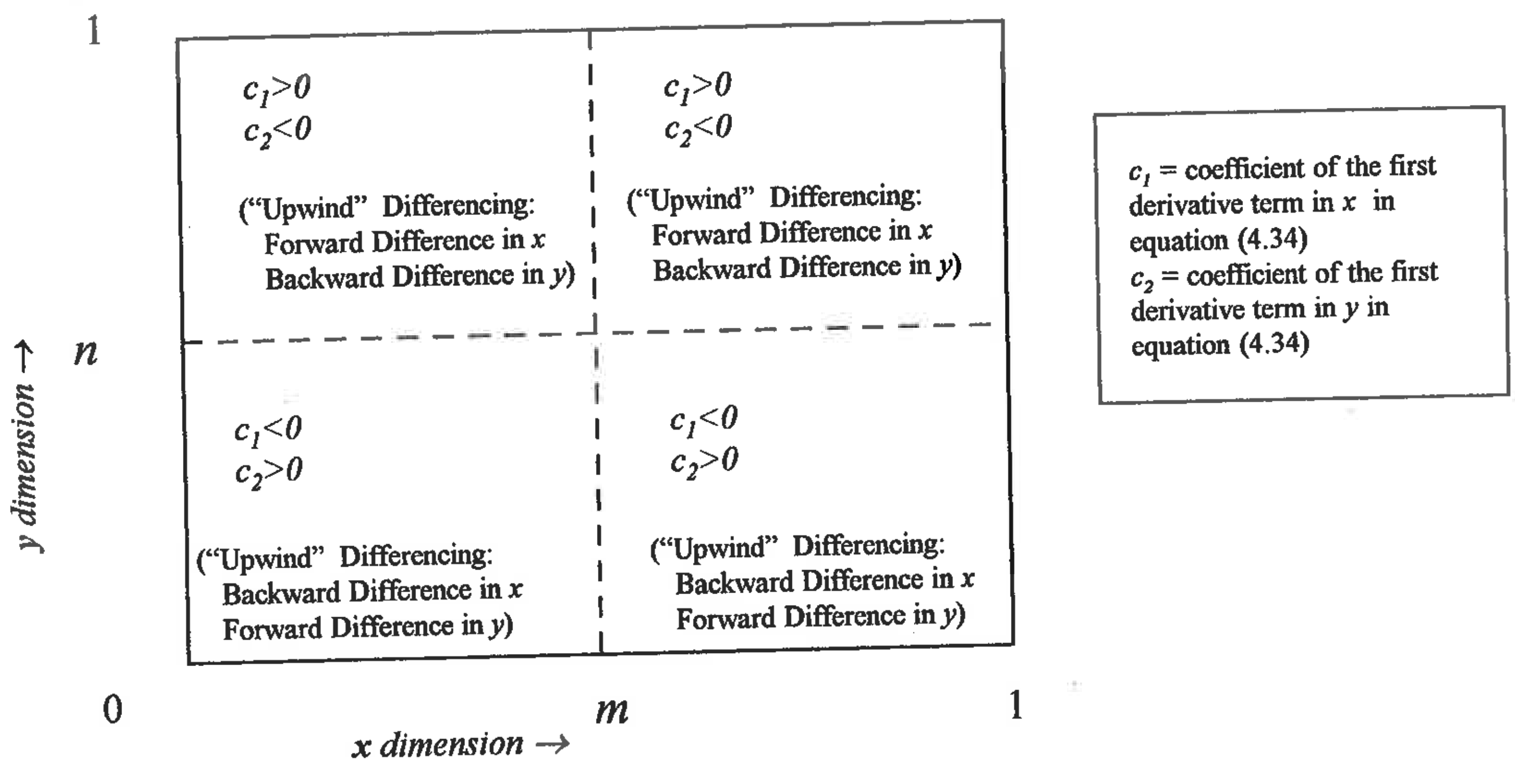


Figure 4.2. Coefficients of the First Derivative Terms Across The Grid



Note: This figure gives a general idea about the signs of the coefficients of the first derivative terms in equation (4.34). The exact position of m and n will change with the parameters that are included in those coefficients.

Figure 4.3.A. “Normal” Boundaries / Free Boundary and Prepayment Region

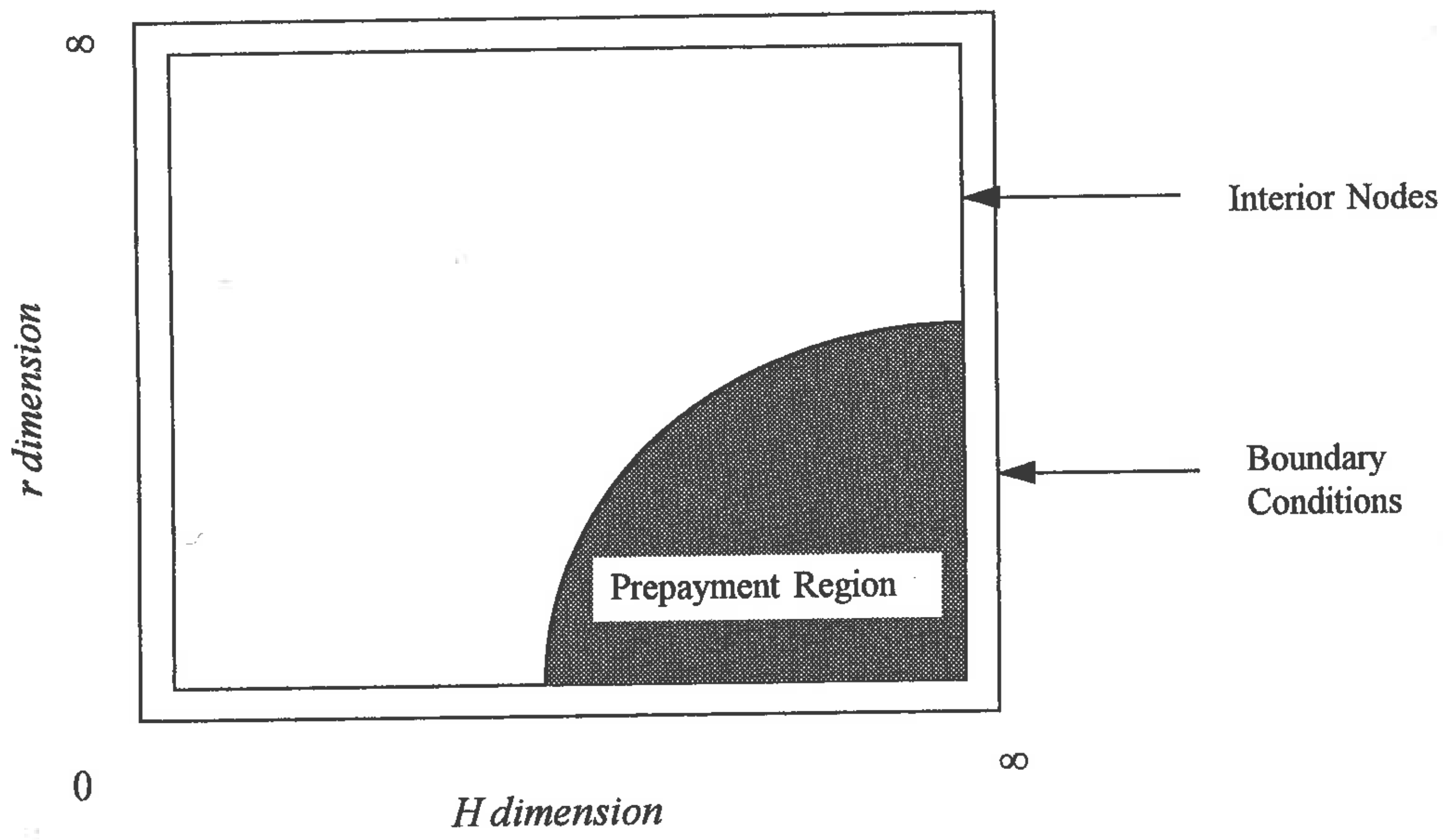
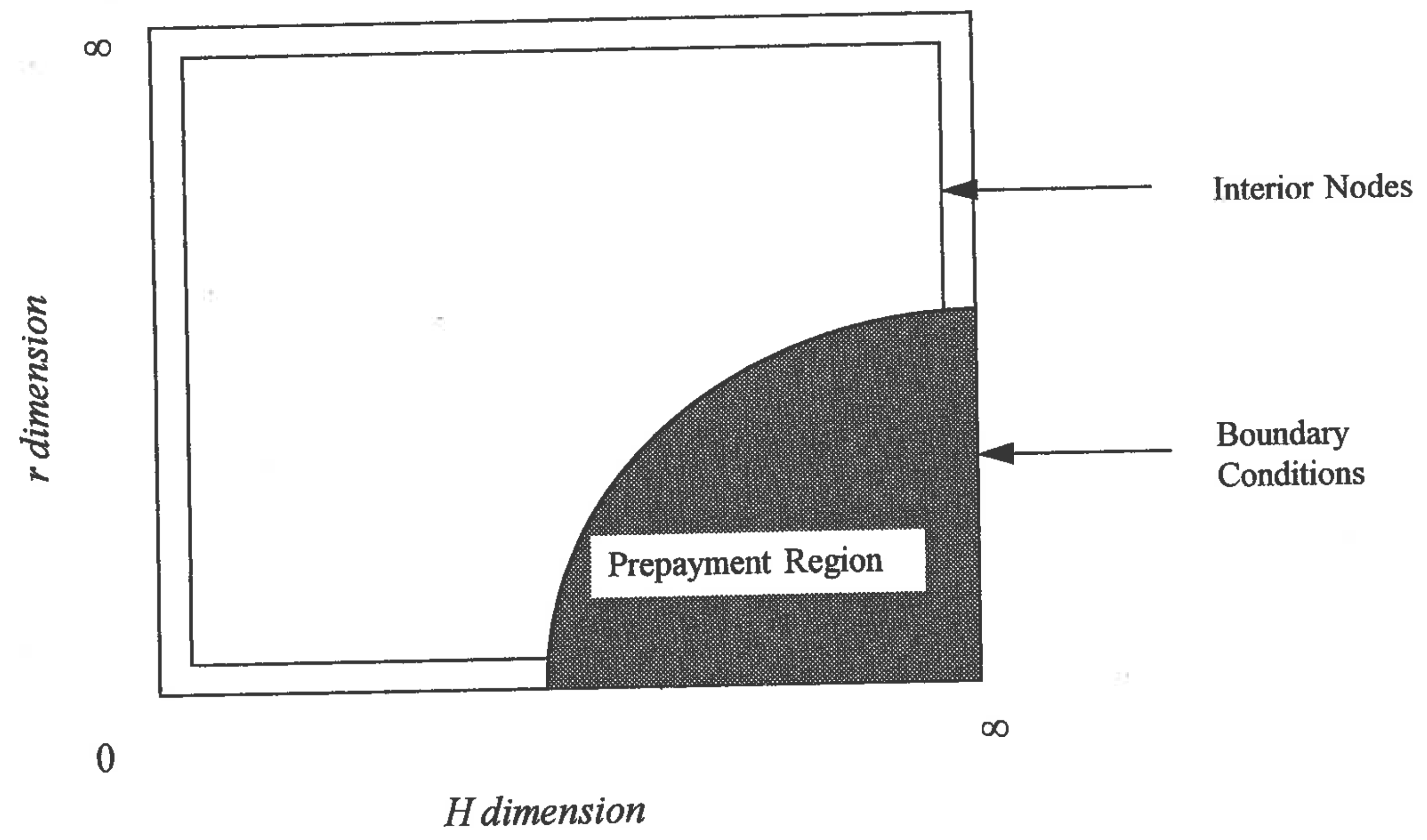


Figure 4.3.B. Influence Exerted By the Free Boundary and the Prepayment Region on the "Normal" Boundary Conditions



Appendix 2

Substitutions Used in the Transformation of the Original PDE

The transformation of the original parabolic PDE (3.4), whose domain is infinite, into a forward parabolic PDE, whose domain is the unit square, requires the substitution of the original state variables and also the substitution of the time variable.

The derivatives of the original function with respect to the original state variables are easy to obtain using the chain rule.

The first derivative in r is straightforward:

$$\frac{\partial F}{\partial r}=\frac{\partial W}{\partial y}\frac{\partial y}{\partial r}$$

The second derivative is expressed in the following terms:

$$\frac{\partial^2 F}{\partial r^2}=\frac{\partial}{\partial r}\left(\frac{\partial W}{\partial y}\frac{\partial y}{\partial r}\right)$$

The final result after the application of the chain rule is:

$$\frac{\partial^2 F}{\partial r^2}=\frac{\partial^2 W}{\partial y^2}\left(\frac{\partial y}{\partial r}\right)^2+\frac{\partial^2 y}{\partial r^2}\left(\frac{\partial W}{\partial y}\right)$$

Similarly, in the H dimension the first derivative is given by:

$$\frac{\partial F}{\partial H}=\frac{\partial W}{\partial x}\frac{\partial x}{\partial H}$$

The second derivative is given by:

$$\frac{\partial^2 F}{\partial H^2}=\frac{\partial^2 W}{\partial x^2}\left(\frac{\partial x}{\partial H}\right)^2+\frac{\partial^2 y}{\partial H^2}\left(\frac{\partial W}{\partial x}\right)$$

The treatment of the time derivative provides the following results:

$$\frac{\partial F}{\partial t}=\frac{\partial W}{\partial \tau}\frac{\partial \tau}{\partial t}$$

$$\frac{\partial F}{\partial t}=-\frac{\partial W}{\partial \tau}$$

The application of the chain rule to the mixed derivative case leads to an apparently more complex expression:

$$\frac{\partial^2 F}{\partial H \partial r}=\frac{\partial}{\partial r}\left(\frac{\partial W}{\partial x}\frac{\partial x}{\partial H}\right)$$

$$\frac{\partial^2 F}{\partial H \partial r}=\frac{\partial^2 W}{\partial x \partial y}\frac{\partial y}{\partial r}\left(\frac{\partial x}{\partial H}\right)+\frac{\partial^2 x}{\partial H \partial r}\left(\frac{\partial W}{\partial x}\right)$$

The state variables are independent from each other. Neither $r = r(H)$ nor $H = H(r)$. Consequently, $\frac{\partial H}{\partial r}=\frac{\partial r}{\partial H}=0$ and the previous expression is simplified to:

$$\frac{\partial^2 F}{\partial H \partial r}=\frac{\partial^2 W}{\partial x \partial y}\frac{\partial y}{\partial r}\left(\frac{\partial x}{\partial H}\right)$$

All the previous expressions for the derivatives of the original function with respect to the state variables involve derivatives of the new state variables with respect to the original ones. Thus, it is necessary to find the corresponding expressions.

In the r dimension, the expression for the first derivative is:

$$\frac{\partial y}{\partial r}=\frac{\partial}{\partial r}\left(\frac{1}{1+\psi r}\right)$$

After some rearrangement it is possible to obtain:

$$\frac{\partial y}{\partial r} = -\psi y^2$$

The second derivative is given by:

$$\frac{\partial^2 y}{\partial r^2} = \frac{\partial}{\partial r}\left[\frac{-\psi}{(1+\psi r)^2}\right]$$

$$\frac{\partial^2 y}{\partial r^2} = 2\psi^2 y^3$$

Similarly, in the *H* dimension the first derivative is:

$$\frac{\partial x}{\partial H} = \frac{\partial}{\partial H}\left(\frac{1}{1+\omega H}\right)$$

After some rearrangement:

$$\frac{\partial x}{\partial H} = -\omega x^2$$

The expression for the second derivative is:

$$\frac{\partial^2 x}{\partial H^2} = \frac{\partial}{\partial H}\left[\frac{-\omega}{(1+\omega H)^2}\right]$$

$$\frac{\partial^2 x}{\partial H^2} = 2\omega^2 x^3$$

Finally, it is also necessary to calculate the values of the squares of the first derivatives of the new state variables with respect to the former ones. The corresponding expressions are:

$$\left(\frac{\partial y}{\partial r}\right)^2 = \psi^2 y^4$$

$$\left(\frac{\partial x}{\partial H}\right)^2 = \omega^2 x^4$$

Appendix 3

Consistency, Stability and Convergence of the Numerical Procedure

A detailed study of the errors associated with most of the PDEs directly associated with the contingent claims valuation of financial assets is still to be done. The present problem is no exception (see, Kau et al., 1995, page 35). As a result, the studies that employ this type of finite difference technique tend not to emphasise the subject (see, Kau et al. 1990, 1993a, 1995). However, one can employ the model parabolic PDE in two dimensions (two dimensional heat flow equation) as an approximation capable of providing a reasonable insight into the conditions which must be fulfilled in order to reach convergence. This is the more commonly observed approach used, for instance, in Hull and White (1990), page 92, Brennan and Schwartz (1977), page 1707 or Brennan and Schwartz (1978), page 464. Following this approach, from Morton and Mayers (1994), pages 60-61, it can be said that, in order for the numerical solution of a PDE of the type under study to converge and be stable as $h, l, k \to 0$, the following condition must be satisfied:

$$\Lambda \frac{k}{h^2} + \Theta \frac{k}{l^2} \leq \frac{1}{2}$$

where:

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$$\Lambda \frac{k}{h^2} + \Theta \frac{k}{l^2} \leq \frac{1}{2}$$

where:

Λ = coefficient of the second derivative term in x in equation (4.34);

Θ = coefficient of the second derivative term in y in equation (4.34);

h = grid spacing in the x dimension;

l = grid spacing in the y dimension;

k = grid spacing in the time dimension.

Developing this expression for the problem under study leads to:

$$\frac{1}{2}H(x)^2 v^2\omega^2x^4\left(\frac{k}{h^2}\right)+\frac{1}{2}r(y)\sigma^2\psi^2y^4\left(\frac{k}{l^2}\right)\leq\frac{1}{2}$$

As $h = l$:

$$\frac{\left\{\left[H(x)^2 v^2\omega^2x^4\right]+\left[r(y)\sigma^2\psi^2y^4\right]\right\}k}{h^2}\leq 1$$

The value of the left-hand side changes with the parameters that are included in the coefficients of the second derivative terms of (4.34). These changes will be reflected within the same lattice when the variations affect x, $H(x)$, y, $r(y)$, ψ and ω. Otherwise, when the variations are related to the parameters that characterize the economic environment (ν and σ), the changes will be felt only between different sets of economic environment parameters. As a consequence of this phenomenon, it is necessary to define a step size for the time variable capable of providing a guarantee that the condition will be respected for each point of the grid and each set of economic environment parameters. In order to achieve this result with a reasonable margin of security, it was decided to use 66 time steps a month corresponding to a $k \approx 0.00126$. This grid spacing assures the satisfaction of the stability and consistency condition in all circumstances considered in the present work. Table 4.1. illustrates this fact for a few points of the grid - the interior corners of the grid and the central point of the lattice. The parameters that underlie the illustration were chosen from those used in the numerical results presented in Chapter 5, in order to give the highest possible values for Λ and Θ, and therefore portray the situations that potentially could cause the most problems.

Table 4.1. Stability and Consistency Conditions at Some Relevant Points of the Grid

Position in the grid				Scale Factors		Economic Environment		Λ	Θ	$\Lambda\frac{k}{h^2}+\Theta\frac{k}{l^2}$
x	$H(x)$	y	$r(y)$	ω	ψ	ν	σ	$\frac{1}{2}H(x)^2\nu^2\omega^2x^4$	$\frac{1}{2}r(y)^2\sigma^2\psi^2y^4$	
0.98	0.02	0.02	3.92	1	12.5	0.10	0.10	1.9E-06	4.9E-06	7.6E-06
0.98	0.02	0.98	0.002	1	12.5	0.10	0.10	1.9E-06	0.0012	0.0037
0.02	49	0.02	3.92	1	12.5	0.10	0.10	1.9E-06	4.9E-06	7.6E-06
0.02	49	0.98	0.002	1	12.5	0.10	0.10	1.9E-06	0.0012	0.0037
0.50	1.0	0.50	0.08	1	12.5	0.10	0.10	0.00031	0.0039	0.0133

Note: The space step is $h = l = 0.02$, and the time step is $k \approx 0.001262$.

Appendix 4

FORTRAN Program to Solve a Repayment Mortgage Valuation Problem Without Early Repayment Penalty

```
C
C   ######################################################################
C   #                                                                    #
C   #                                                                    #
C   #  FORTRAN PROGRAM TO DETERMINE THE VALUE OF A FIXED-RATE            #
C   #           REPAYMENT MORTGAGE, ITS EMBEDDED OPTIONS                 #
C   #                 AND INSURANCE-RELATED PRODUCTS                     #
C   #                                                                    #
C   #                                                                    #
C   ######################################################################
C
      DIMENSION OLDA(52),R(52),OB(302),X(52),Y(52),H(52),YJY2(52),YJY3(5
     12),YJY4(52),DPN0(52),DPN3(52),DPN10(52),DPN2(52),DPN7(52)
      DIMENSION OLDC(52,52),OLDD(52,52),OLDV(52,52),OLDI(52,52),OLDCOI(5
     12,52),XJX2(52),XJX3(52),XJX4(52),HJX2(52)
      REAL KOVL,KOVL2,KAPLAM(52),KAPPA,LAMBDA,NEWA(52),NIU,NIU2,KOVH2
      REAL MAXCO,NEWC(52,52),NEWD(52,52),NEWV(52,52),NEWI(52,52),NEWCOI(
     152,52),M,KOVH,KOV4LH,NLTV
      CHARACTER*8 edate@
      CHARACTER*8 time@
      CHARACTER*10 FNAME
      CHARACTER*8 atime
      CHARACTER*8 adate
C   ----------------------------------------------------------
C   Fortran (non essential) specific to the Salford compiler follows:
C   ----------------------------------------------------------
      CALL GET_MEMORY_INFO@(NP1,NP2,NP3,NP4,NP5,NP6,NP7)
      WRITE(6,200)NP2,NP4,NP7
  200 FORMAT(1X,'EXTENDED MEMORY PAGES ',I6/1X,'REMAINING EXTENDED
PAGES
     1',I6,1X,'NUMBER OF PAGE TURNS SO FAR ',I6)
      CALL SET_TRAP_ON_PAGE_TURN@
      call cissue('cls',k)
      if(k.ne.0)call cou@('DOS command to clear screen failed!')
C   ----------------------------------------------------------
C   GET INFORMATION FROM THE USER, THEN START THE CLOCK:
C   ----------------------------------------------------------
      WRITE(6,201)
  201 FORMAT(1X,'Do you want to print to a file, Jose?  (Type 0 for no)'
     1)
```

```
      READ(6,101)JREPLY
  101 FORMAT(I4)
      if(JREPLY.NE.0) THEN
      WRITE(6,202)
  202 FORMAT(1X,'PLEASE TYPE OUTPUT FILENAME (MAXIMUM 6 CHARACTERS) ')
      READ(6,102) FNAME
  102 FORMAT(A6)
      OPEN(1,FILE=FNAME)
      ENDIF
C     ----------------------------------------------------------

C     Clock@ is specific to the Salford compiler.  It measures cpu time.
C     Be aware that if this program were to be run in a multi-tasking
C     environment (perhaps under Windows 95) clock@ would record cpu
C     time for ALL activities, not just for this program!
C     ----------------------------------------------------------
      call clock@(start)
C     ****************************************************************
C     EXTERNAL PARAMETERS
C     ****************************************************************
      NISST = 66
      KAPPA = 0.25
      LAMBDA = 0.00
      SIGMA = 0.05
      PSI = 10.0
      THETA = 0.10
      OMEGA = 1.0
      NIU = 0.05
      RHO = 0.0
      RLTV = 0.95
      NLTV = 0.75
      ZIND = 0.8
      DELTA = 0.075
      MAXIT = 30
      TOLVAL = 0.0001
      PVPN = 0.01
C
      SIG2 = SIGMA*SIGMA
      PSI2 = PSI*PSI
      PSI3 = PSI*PSI2
```

```
      S2PS2 = SIG2*PSI2
      S2PS3 = SIG2*PSI3
      OMEG2 = OMEGA*OMEGA
    . NIU2 = NIU*NIU
C     ******************************************************************
C     The following statements (and others which will be obvious, later
C     in the program) are commented out for initial runs. Although in a
C     later version of the program a more automated procedure could,
C     clearly, be programmed, here it is necessary (but not especially
C     inconvenient) to discover, from the initial runs, two contract
C     rates giving mortgage mortgage values either side of its
C     equilibrium value. The code below will then implement a secant
C     iteration technique that will determine the equilibrium rate.
C     In other words, the rate at which the mortgage contract has equal
C     value to lender and borrower.
C     ==================================================================
C     ROOT0 =
C     VALUE0 =
C     ROOT1 =
C     VALUE1 =
C     IF(ABS(VALUE0).LT.ABS(VALUE1)) THEN
C     SWAP1 = ROOT0
C     ROOT0 = ROOT1
C     ROOT1 = SWAP1
C     SWAP2 = VALUE0
C     VALUE0 = VALUE1
C     VALUE1 = SWAP2
C     ELSE
C     ROOT0 = ROOT0
C     ROOT1 = ROOT1
C     VALUE0 = VALUE0
C     VALUE1 = VALUE1
C     ENDIF
C     ==================================================================
      NHPST = 50
      NIRST = 50
      CRATE = 0.11
C     ******************************************************************
C     INTERNAL PARAMETERS
C     ******************************************************************
```

```
C     All the variables that start with I are in some way related to
C     time. IY is the counter for years (number of years to maturity),
C     IM is the counter for months (number of months to the final of the
C     year) and IS is the counter for steps
C     ------------------------------------------------------------------
      IMMIN = 1
      IMMAX = 300
      IMINC = 1
      DISINC = 1.0/(12.0*FLOAT(NISST))
      ISMIN = 1
      ISMAX = ISMIN+NISST
      ISINC = 1
C     ------------------------------------------------------------------
C     The counters for the other variables (interest rate - dimension y;
C     house price - dimension X)  all start with a J.
C     JY is the counter in the interest rate dimension. The structure of
C     the variables is similar to the previous ones, so it will not be
C     necessary to give detailed discriptions of the contents
C     ------------------------------------------------------------------
      JYMIN = 1
      JYMAX = JYMIN + NIRST
      JYINC = 1
C     ------------------------------------------------------------------
C     Y is the transformed variable in the interest rate dimension. So,
C     the grid will be linear in Y but not in R (the interest rate)
C     ------------------------------------------------------------------
      YMIN = 0.0
      YMAX = 1.0
      YINC = (YMAX - YMIN)/FLOAT(NIRST)
C     ------------------------------------------------------------------
C     JX is the counter for the house price dimension.
C     ------------------------------------------------------------------
      JXMIN = 1
      JXMAX = JXMIN + NHPST
      JXINC = 1
C     ------------------------------------------------------------------
C     X is the transformed variable in the house price dimension.
C     Again, the grid is linear in X but not in H (the house prices)
C     ------------------------------------------------------------------
      XMIN = 0.0
```

```
      XMAX = 1.0
      XINC = (XMAX - XMIN)/FLOAT(NHPST)
C
      KOVL = DISINC/YINC
      KOVL2 = DISINC/(YINC*YINC)
      KOVH = DISINC/XINC
      KOVH2 = DISINC/(XINC*XINC)
      KOV4LH = DISINC/(4.0*XINC*YINC)
C     ------------------------------------------------
C     End of parameters section of the program
C     ------------------------------------------------
C
C     Print headings and use time and date functions
C     ------------------------------------------------
      atime = time@()
      adate = edate@()
      write(1,203)atime,adate
  203 format(1x,'Time: ',a8,2x,'On ',a8,' (Day/Month/Year)')
      write(1,204)
  204 format(1x)
      write(1,205)
  205 format(12x,'MORTGAGE CALCULATIONS'/11X,'**** REP VERSION *** ')
      write(1,204)
      write(1,206)
  206 format(3x,'Jose Pereira'/)
C     ------------------------------------------------
C     End of print headings section of the program
C     ------------------------------------------------
      JYHALF=JYMIN+NINT(DFLOAT(NIRST)/2.0)
      JXHALF=JXMIN+NINT(DFLOAT(NHPST)/2.0)
C     ==============================================================
C     DO 10 K=1,MAXIT,1
C     ROOT2=ROOT0-(VALUE0*((ROOT0-ROOT1)/(VALUE0-VALUE1)))
C     CRATE=ROOT2
C     ==============================================================
C     SET ARRAYS TO ZERO
C     ------------------------------------------------
      DO 1001 IM = IMMIN,IMMAX,IMINC
      OB(IM) = 0.0
 1001 CONTINUE
```

```
      DO 1002 JY = JYMIN,JYMAX,JYINC
      Y(JY) =     0.0
      R(JY) =     0.0
      OLDA(JY) =  0.0
      NEWA(JY) =  0.0
      YJY2(JY) =  0.0
      YJY3(JY) =  0.0
      YJY4(JY) =  0.0
      DPN0(JY) =  0.0
      DPN2(JY) =  0.0
      DPN3(JY) =  0.0
      DPN7(JY) =  0.0
      DPN10(JY) = 0.0
      KAPLAM(JY) = 0.0
 1002 CONTINUE
      DO 1003 JX = JXMIN,JXMAX,JXINC
      X(JX) =    0.0
      H(JX) =    0.0
      XJX2(JX) = 0.0
      XJX3(JX) = 0.0
      XJX4(JX) = 0.0
      HJX2(JX) = 0.0
 1003 CONTINUE
      DO 1004 JY=JYMIN,JYMAX,JYINC
      DO 1005 JX=JXMIN,JXMAX,JXINC
      OLDC(JY,JX) =   0.0
      OLDD(JY,JX) =   0.0
      OLDCOI(JY,JX) = 0.0
      OLDI(JY,JX) =   0.0
      OLDV(JY,JX) =   0.0
      NEWC(JY,JX) =   0.0
      NEWD(JY,JX) =   0.0
      NEWCOI(JY,JX) = 0.0
      NEWI(JY,JX) =   0.0
 1005 CONTINUE
 1004 CONTINUE
C     ----------------------------------------------------------------
C     End of section of the program setting arrays to zero
C     ----------------------------------------------------------------
C     PRELIMINARY CALCULATIONS:
```

```
C     Loops which follow will give the value of both state
C     variables and its transformed versions.
C     ----------------------------------------------------------
      Y(JYMAX) = YMAX
      YJY2(JYMAX) = Y(JYMAX)*Y(JYMAX)
      YJY3(JYMAX) = YJY2(JYMAX)*Y(JYMAX)
      YJY4(JYMAX) = YJY3(JYMAX)*Y(JYMAX)
      Y(JYMIN) = YMIN
      YJY2(JYMIN) = 0.0
      YJY3(JYMIN) = 0.0
      YJY4(JYMIN) = 0.0
      X(JXMAX) = XMAX
      XJX2(JXMAX) = X(JXMAX)*X(JXMAX)
      XJX3(JXMAX) = XJX2(JXMAX)*X(JXMAX)
      XJX4(JXMAX) = XJX3(JXMAX)*X(JXMAX)
      X(JXMIN) = XMIN
      XJX2(JXMIN) = 0.0
      XJX3(JXMIN) = 0.0
      XJX4(JXMIN) = 0.0
      DO 1010 JY = JYMIN+1,JYMAX-1,JYINC
      Y(JY) = YMIN + ((DFLOAT(JY)-DFLOAT(JYMIN))*YINC)
      YJY2(JY) = Y(JY)*Y(JY)
      YJY3(JY) = YJY2(JY)*Y(JY)
      YJY4(JY) = YJY3(JY)*Y(JY)
1010  CONTINUE
      DO 1011 JX = JXMIN+1,JXMAX-1,JXINC
      X(JX) = XMIN + ((DFLOAT(JX)-DFLOAT(JXMIN))*XINC)
      XJX2(JX) = X(JX)*X(JX)
      XJX3(JX) = XJX2(JX)*X(JX)
      XJX4(JX) = XJX3(JX)*X(JX)
1011  CONTINUE
      R(JYMAX) = 0.0
      KAPLAM(JYMAX) = KAPPA*THETA
      DO 1012 JY = JYMIN+1, JYMAX-1, JYINC
      R(JY) = (1.0-Y(JY))/(PSI*Y(JY))
      KAPLAM(JY) = KAPPA*(THETA-R(JY)) - LAMBDA*R(JY)
1012  CONTINUE
      H(JXMAX) = 0.0
      HJX2(JXMAX) = 0.0
      DO 1013 JX = JXMIN+1,JXMAX-1,JXINC
```

```
      H(JX) = (1.0 - X(JX))/(OMEGA*X(JX))
      HJX2(JX) = H(JX)*H(JX)
 1013 CONTINUE
C***********************************************************************
C     THE SOLUTION OF THE PROBLEM STARTS HERE
C***********************************************************************
      M=(((CRATE/12.0)*(1+(CRATE/12.0))**(DFLOAT(IMMAX)))/(((1.0+(CRATE/
     112.0))**(DFLOAT(IMMAX)))-1.0))*RLTV
      A1 = (1.0+(CRATE/12.0))**(DFLOAT(IMMAX))
      A2 = (1.0+(CRATE/12.0))**(DFLOAT(IMMAX-1))
      A3 = ((DFLOAT(1)+(CRATE/12.0))**(DFLOAT(IMMAX)))-DFLOAT(1)
      OB(IMMIN) = ((A1-A2)/A3)*RLTV
      TD = M
      MAXCO = (RLTV - NLTV)*1.0
C     ******************************************************************
C     FINAL TERMINAL CONDITIONS
C     ******************************************************************
      DO 2001 JY = JYMIN,JYMAX,JYINC
      NEWA(JY) = M
      OLDA(JY) = NEWA(JY)
 2001 CONTINUE
      DO 2002 JY=JYMIN,JYMAX,JYINC
      DO 2003 JX=JXMIN+1,JXMAX,JXINC
      OLDV(JY,JX) = MIN(H(JX),M)
      OLDC(JY,JX) = 0.0
      OLDD(JY,JX) = MAX(0.0,(M-H(JX)))
      OLDI(JY,JX) = MAX(0.0,MIN((ZIND*(M-H(JX))),MAXCO))
      OLDCOI(JY,JX) = MAX(((1-ZIND)*(M-H(JX))),((M-H(JX))-MAXCO),0.0)
 2003 CONTINUE
 2002 CONTINUE
      OLDC(JY,JXMIN) = 0.0
      OLDD(JY,JXMIN) = 0.0
      OLDV(JY,JXMIN) = MIN(TD,(OLDA(JY)-OLDD(JY,JX)-OLDC(JY,JX)))
      OLDI(JY,JXMIN) = 0.0
      OLDCOI(JY,JXMIN) = 0.0
C     ******************************************************************
C     THE NEXT STEP WILL CONSIST IN THE SOLUTION OF THE PDE IN ORDER
C     TO DETERMINE THE VALUES OF THE DIFFERENT VARIABLES IN THE
C      BEGINNING OF THE LAST MONTH
C     ******************************************************************
```

```
      IM=IMMIN
      DO 3000 IS=ISMIN,ISMAX,ISINC
      A1 = (1.0+(CRATE/12.0))**(DFLOAT(IMMAX))
      A2 = (1.0+(CRATE/12.0))**(DFLOAT(IMMAX-1))
      A3 = ((DFLOAT(1)+(CRATE/12.0))**(DFLOAT(IMMAX))-DFLOAT(1)
      OB(IM) = ((A1-A2)/A3)*RLTV
      TD = (DFLOAT(1) + (CRATE*((FLOAT(ISMAX)-FLOAT(IS))/(DFLOAT(NISST)*
     1DFLOAT(12)))))*OB(IM)
      DO 3001 JY = JYMIN+1,JYMAX-1,JYINC
      DPN0(JY) = R(JY)*S2PS2*YJY4(JY)*KOVL2
      DPN3(JY) = -PSI*YJY2(JY)*KAPLAM(JY)
      DPN10(JY) = (R(JY)*S2PS2*YJY3(JY))
      DPN4 = DPN3(JY)+DPN10(JY)
      IF(DPN4.LT.0) THEN
      DPN2(JY)=(0.5*DPN0(JY)*OLDA(JY+1))+((DPN3(JY)+DPN10(JY))*KOVL*OLDA
     1(JY))
      DPN7(JY)=(0.5*DPN0(JY)*OLDA(JY-1))-((DPN3(JY)+DPN10(JY))*KOVL*OLDA
     1(JY-1))
      ELSE
      DPN2(JY)=(0.5*DPN0(JY)*OLDA(JY+1))+((DPN3(JY)+DPN10(JY))*KOVL*OLDA
     1(JY+1))
      DPN7(JY)=(0.5*DPN0(JY)*OLDA(JY-1))-((DPN3(JY)+DPN10(JY))*KOVL*OLDA
     1(JY))
      ENDIF
      DPN1 = ((1.0-DPN0(JY)-R(JY)*DISINC)*OLDA(JY))
      NEWA(JY) = DPN1 + DPN2(JY) + DPN7(JY)
 3001 CONTINUE
C    ********************************************************************
C     BOUNDARY CONDITIONS
C    ********************************************************************
      NEWA(JYMIN) = 0.0
      NEWA(JYMAX) = OLDA(JYMAX-1)
C    ---------------------------------------------------------
C     (i) CORNERS OF THE GRID
C    ---------------------------------------------------------
C     CORNER OF THE GRID IN WHICH H=0.0 AND R=0.0
C    ---------------------------------------------------------
      NEWA(JYMAX) = OLDA(JYMAX-1)
      NEWV(JYMAX,JXMAX) = OLDV(JYMAX-1,JXMAX)
      NEWC(JYMAX,JXMAX) = OLDC(JYMAX-1,JXMAX)
```

```
      NEWD(JYMAX,JXMAX) = OLDD(JYMAX-1,JXMAX)
      NEWI(JYMAX,JXMAX) = OLDI(JYMAX-1,JXMAX)
      NEWCOI(JYMAX,JXMAX) = OLDCOI(JYMAX-1,JXMAX)
      OLDV(JYMAX,JXMAX) = NEWV(JYMAX,JXMAX)
      OLDC(JYMAX,JXMAX) = NEWC(JYMAX,JXMAX)
      OLDD(JYMAX,JXMAX) = NEWD(JYMAX,JXMAX)
      OLDI(JYMAX,JXMAX) = NEWI(JYMAX,JXMAX)
      OLDCOI(JYMAX,JXMAX) = NEWCOI(JYMAX,JXMAX)
C     ----------------------------------------------------
C      CORNER OF THE GRID IN WHICH H IS INFINITE AND R=0.0.
C     ----------------------------------------------------
      NEWA(JYMAX) = OLDA(JYMAX-1)
      NEWV(JYMAX,JXMIN) = OLDV(JYMAX-1,JXMIN)
      NEWC(JYMAX,JXMIN) = OLDC(JYMAX-1,JXMIN)
      NEWD(JYMAX,JXMIN) = OLDD(JYMAX-1,JXMIN)
      NEWI(JYMAX,JXMIN) = OLDI(JYMAX-1,JXMIN)
      NEWCOI(JYMAX,JXMIN) = OLDCOI(JYMAX-1,JXMIN)
      OLDV(JYMAX,JXMIN) = NEWV(JYMAX,JXMIN)
      OLDC(JYMAX,JXMIN) = NEWC(JYMAX,JXMIN)
      OLDD(JYMAX,JXMIN) = NEWD(JYMAX,JXMIN)
      OLDI(JYMAX,JXMIN) = NEWI(JYMAX,JXMIN)
      OLDCOI(JYMAX,JXMIN) = NEWCOI(JYMAX,JXMIN)
C     ----------------------------------------------------
C      CORNER OF THE GRID IN WHICH BOTH H AND R ARE INFINITE
C     ----------------------------------------------------
      NEWA(JYMIN) = 0.0
      NEWV(JYMIN,JXMIN) = 0.0
      NEWC(JYMIN,JXMIN) = 0.0
      NEWD(JYMIN,JXMIN) = 0.0
      NEWI(JYMIN,JXMIN) = 0.0
      NEWCOI(JYMIN,JXMIN) = 0.0
      OLDV(JYMIN,JXMIN) = NEWV(JYMIN,JXMIN)
      OLDC(JYMIN,JXMIN) = NEWC(JYMIN,JXMIN)
      OLDD(JYMIN,JXMIN) = NEWD(JYMIN,JXMIN)
      OLDI(JYMIN,JXMIN) = NEWI(JYMIN,JXMIN)
      OLDCOI(JYMIN,JXMIN) = NEWCOI(JYMIN,JXMIN)
C     ----------------------------------------------------
C      CORNER OF THE GRID IN WHICH H=0.0 AND R IS INFINITE
C     ----------------------------------------------------
      NEWA(JYMIN) = 0.0
```

```
      NEWV(JYMIN,JXMAX) = 0.0
      NEWC(JYMIN,JXMAX) = 0.0
      NEWD(JYMIN,JXMAX) = 0.0
      NEWI(JYMIN,JXMAX) = 0.0
      NEWCOI(JYMIN,JXMAX) = 0.0
      OLDV(JYMIN,JXMAX) = NEWV(JYMIN,JXMAX)
      OLDC(JYMIN,JXMAX) = NEWC(JYMIN,JXMAX)
      OLDD(JYMIN,JXMAX) = NEWD(JYMIN,JXMAX)
      OLDI(JYMIN,JXMAX) = NEWI(JYMIN,JXMAX)
      OLDCOI(JYMIN,JXMAX) = NEWCOI(JYMIN,JXMAX)
C     -----------------------------------------
C     (ii) EDGES OF THE GRID
C     -----------------------------------------
C     H(JX) = 0, R(JY) VARIES
C     -----------------------------------------
      DO 3002 JY=JYMIN+1,JYMAX-1,JYINC
      OLDC(JY,JXMAX) = 0.0
      OLDD(JY,JXMAX) = NEWA(JY)
      OLDV(JY,JXMAX) = 0.0
      CTRL=((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPPA*(THETA-R(JY))-(LAMBDA*R(J
     1Y)))*PSI*YJY2(JY)))
      IF(CTRL.LT.0) THEN
      NEWI(JY,JXMAX)=((1-(R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC)
     1)*OLDI(JY,JXMAX))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*(OLDI(JY+1
     2,JXMAX)+OLDI(JY-1,JXMAX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPPA*(T
     3HETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDI(JY,JXMAX)-OL
     4DI(JY-1,JXMAX)))
       OLDI(JY,JXMAX) = NEWI(JY,JXMAX)
       NEWCOI(JY,JXMAX)=((1-(R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)-(R(JY)*DISIN
     1C))*OLDCOI(JY,JXMAX))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*(OLDCO
     2I(JY+1,JXMAX)+OLDCOI(JY-1,JXMAX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((
     3KAPPA*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDCOI(JY
     4,JXMAX)-OLDCOI(JY-1,JXMAX)))
       OLDCOI(JY,JXMAX) = NEWCOI(JY,JXMAX)
       ELSE
       NEWI(JY,JXMAX)=((1-(R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC)
     1)*OLDI(JY,JXMAX))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*(OLDI(JY+1
     2,JXMAX)+OLDI(JY-1,JXMAX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPPA*(T
     3HETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDI(JY+1,JXMAX)-
     4OLDI(JY,JXMAX)))
```

```
      OLDI(JY,JXMAX) = NEWI(JY,JXMAX)
      NEWCOI(JY,JXMAX)=((1-(R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)-(R(JY)*DISIN
     1C))*OLDCOI(JY,JXMAX))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*(OLDCO
     2I(JY+1,JXMAX)+OLDCOI(JY-1,JXMAX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((
     3KAPPA*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDCOI(JY
     4+1,JXMAX)-OLDCOI(JY,JXMAX)))
      OLDCOI(JY,JXMAX) = NEWCOI(JY,JXMAX)
      ENDIF
 3002 CONTINUE
C     ----------------------------------------------
C     R = 0.0, H VARIES
C     ----------------------------------------------
      DO 3003 JX=JXMIN+1,JXMAX-1,JXINC
      NEWA(JYMAX) = OLDA(JYMAX-1)
      NEWV(JYMAX,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2))*OLDV(JYMA
     1X,JX))+((.5*HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)*(OLDV(JYMAX,JX+1)+
     2OLDV(JYMAX,JX-1)))+(((HJX2(JX)*NIU2*OMEG2*XJX3(JX))+(DELTA*H(JX)*O
     3MEGA*XJX2(JX)))*KOVH*(OLDV(JYMAX,JX+1)-OLDV(JYMAX,JX)))-((KAPPA*TH
     4ETA*PSI*YJY2(JY)*KOVL)*(OLDV(JYMAX,JX)-OLDV(JYMAX-1,JX)))
      OLDV(JYMAX,JX) = NEWV(JYMAX,JX)
      NEWD(JYMAX,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2))*OLDD(JYMA
     1X,JX))+((.5*HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)*(OLDD(JYMAX,JX+1)+
     2OLDD(JYMAX,JX-1)))+(((HJX2(JX)*NIU2*OMEG2*XJX3(JX))+(DELTA*H(JX)*O
     3MEGA*XJX2(JX)))*KOVH*(OLDD(JYMAX,JX+1)-OLDD(JYMAX,JX)))-((KAPPA*TH
     4ETA*PSI*YJY2(JY)*KOVL)*(OLDV(JYMAX,JX)-OLDV(JYMAX-1,JX)))
      OLDD(JYMAX,JX) = NEWD(JYMAX,JX)
      NEWC(JYMAX,JX) = MAX(0.0,(NEWA(JYMAX)-OLDV(JYMAX,JX)-OLDD(JYMAX,JX
     1)))
      OLDC(JYMAX,JX) = NEWC(JYMAX,JX)
      NEWI(JYMAX,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2))*OLDI(JYMA
     1X,JX))+((.5*HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)*(OLDI(JYMAX,JX+1)+
     2OLDI(JYMAX,JX-1)))+(((HJX2(JX)*NIU2*OMEG2*XJX3(JX))+(DELTA*H(JX)*O
     3MEGA*XJX2(JX)))*KOVH*(OLDI(JYMAX,JX+1)-OLDI(JYMAX,JX)))-((KAPPA*TH
     4ETA*PSI*YJY2(JY)*KOVL)*(OLDV(JYMAX,JX)-OLDV(JYMAX-1,JX)))
      OLDI(JYMAX,JX) = NEWI(JYMAX,JX)
      NEWCOI(JYMAX,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2))*OLDCOI(
     1JYMAX,JX))+((.5*HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)*(OLDCOI(JYMAX,
     2JX+1)+OLDCOI(JYMAX,JX-1)))+(((HJX2(JX)*NIU2*OMEG2*XJX3(JX))+(DELTA
     3*H(JX)*OMEGA*XJX2(JX)))*KOVH*(OLDCOI(JYMAX,JX+1)-OLDCOI(JYMAX,JX))
     4)-((KAPPA*THETA*PSI*YJY2(JY)*KOVL)*(OLDV(JYMAX,JX)-OLDV(JYMAX-1,JX
```

```
      5)))
      OLDCOI(JYMAX,JX) = NEWCOI(JYMAX,JX)
 3003 CONTINUE
C     ----------------------------------------------------------------
C     THE BOUNDARY NEXT IS WHEN H IS INFINITE AND R VARIES
C     ----------------------------------------------------------------
      DO 3004 JY=JYMIN+1,JYMAX-1,JYINC
      CTRL=((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPPA*(THETA-R(JY))-(LAMBDA*R(J
     1Y)))*PSI*YJY2(JY)))
      IF(CTRL.LT.0) THEN
      NEWC(JY,JXMIN)=((1-(R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC)
     1)*OLDC(JY,JXMIN))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*(OLDC(JY+1
     2,JXMIN)+OLDC(JY-1,JXMIN)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPPA*(T
     3HETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDC(JY,JXMIN)-OL
     4DC(JY-1,JXMIN)))
      ELSE
      NEWC(JY,JXMIN)=((1-(R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC)
     1)*OLDC(JY,JXMIN))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*(OLDC(JY+1
     2,JXMIN)+OLDC(JY-1,JXMIN)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPPA*(T
     3HETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDC(JY+1,JXMIN)-
     4OLDC(JY,JXMIN)))
      ENDIF
      OLDC(JY,JXMIN) = NEWC(JY,JXMIN)
      NEWV(JY,JXMIN) = NEWA(JY) - OLDC(JY,JXMIN)
      NEWD(JY,JXMIN) = 0.0
      NEWI(JY,JXMIN) = 0.0
      NEWCOI(JY,JXMIN) = 0.0
      OLDV(JY,JXMIN) = NEWV(JY,JXMIN)
      OLDD(JY,JXMIN) = NEWD(JY,JXMIN)
      OLDI(JY,JXMIN) = NEWI(JY,JXMIN)
      OLDCOI(JY,JXMIN) = NEWCOI(JY,JXMIN)
 3004 CONTINUE
C     ----------------------------------------------------------------
C     R IS INFINITE AND H VARIES
C     ----------------------------------------------------------------
      DO 3005 JX=JXMIN+1,JXMAX-1,JXINC
      NEWA(JYMIN) = 0.0
      NEWC(JYMIN,JX) = 0.0
      NEWV(JYMIN,JX) = 0.0
      NEWD(JYMIN,JX) = 0.0
```

```
      NEWI(JYMIN,JX) = 0.0
      NEWCOI(JYMIN,JX) = 0.0
      OLDC(JYMIN,JX) = NEWC(JYMIN,JX)
      OLDV(JYMIN,JX) = NEWV(JYMIN,JX)
      OLDD(JYMIN,JX) = NEWD(JYMIN,JX)
      OLDI(JYMIN,JX) = NEWI(JYMIN,JX)
      OLDCOI(JYMIN,JX) = NEWCOI(JYMIN,JX)
 3005 CONTINUE
C     *****************************************************************
C     INTRODUCTION OF THE ALGORITHM TO SOLVE THE PDE
C     *****************************************************************
      IJXCT = JXMIN
      IJYCT = JYMAX
      DO 3500 JX = JXMIN+1,JXMAX-1,JXINC
      ISTATE = 1
      DO 3501 JY = JYMIN+1,JYMAX-1,JYINC
      FDX=((HJX2(JX)*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX
     12(JX)))
      FDY=((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPPA*(THETA-R(JY))-(LAMBDA*R(J
     1Y)))*PSI*YJY2(JY)))
      IF(FDX.LT.0) THEN
      IXDS=1
      ELSE
      IXDS=2
      ENDIF
      IF(FDY.LT.0) THEN
      IYDS=1
      ELSE
      IYDS=2
      ENDIF
      IF((IXDS.EQ.2).AND.(IYDS.EQ.2)) THEN
      NEWV(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
     1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDV(JY,JX))+((.5*HJX2(JX)*NIU
     22*OMEG2*XJX4(JX)*KOVH2)*(OLDV(JY,JX+1)+OLDV(JY,JX-1)))+(((HJX2(JX)
     3*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX)))*KOVH*(
     4OLDV(JY,JX+1)-OLDV(JY,JX)))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*
     5(OLDV(JY+1,JX)+OLDV(JY-1,JX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPP
     6A*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDV(JY+1,JX)
     7-OLDV(JY,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA
     8*XJX2(JX)*KOV4LH*(OLDV(JY+1,JX+1)-OLDV(JY+1,JX-1)-OLDV(JY-1,JX+1)+
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     9OLDV(JY-1,JX-1)))
      ELSEIF((IXDS.EQ.1).AND.(IYDS.EQ.2)) THEN
      NEWV(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
     1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDV(JY,JX))+((.5*HJX2(JX)*NIU
     22*OMEG2*XJX4(JX)*KOVH2)*(OLDV(JY,JX+1)+OLDV(JY,JX-1)))+(((HJX2(JX)
     3*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX)))*KOVH*(
     4OLDV(JY,JX)-OLDV(JY,JX-1)))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*
     5(OLDV(JY+1,JX)+OLDV(JY-1,JX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPP
     6A*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDV(JY+1,JX)
     7-OLDV(JY,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA
     8*XJX2(JX)*KOV4LH*(OLDV(JY+1,JX+1)-OLDV(JY+1,JX-1)-OLDV(JY-1,JX+1)+
     9OLDV(JY-1,JX-1)))
      ELSEIF((IXDS.EQ.2).AND.(IYDS.EQ.1)) THEN
      NEWV(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
     1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDV(JY,JX))+((.5*HJX2(JX)*NIU
     22*OMEG2*XJX4(JX)*KOVH2)*(OLDV(JY,JX+1)+OLDV(JY,JX-1)))+(((HJX2(JX)
     3*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX)))*KOVH*(
     4OLDV(JY,JX+1)-OLDV(JY,JX)))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*
     5(OLDV(JY+1,JX)+OLDV(JY-1,JX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPP
     6A*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDV(JY,JX)-O
     7LDV(JY-1,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA
     8*XJX2(JX)*KOV4LH*(OLDV(JY+1,JX+1)-OLDV(JY+1,JX-1)-OLDV(JY-1,JX+1)+
     9OLDV(JY-1,JX-1)))
      ELSEIF((IXDS.EQ.1).AND.(IYDS.EQ.1)) THEN
      NEWV(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
     1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDV(JY,JX))+((.5*HJX2(JX)*NIU
     22*OMEG2*XJX4(JX)*KOVH2)*(OLDV(JY,JX+1)+OLDV(JY,JX-1)))+(((HJX2(JX)
     3*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX)))*KOVH*(
     4OLDV(JY,JX)-OLDV(JY,JX-1)))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*
     5(OLDV(JY+1,JX)+OLDV(JY-1,JX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPP
     6A*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDV(JY,JX)-O
     7LDV(JY-1,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA
     8*XJX2(JX)*KOV4LH*(OLDV(JY+1,JX+1)-OLDV(JY+1,JX-1)-OLDV(JY-1,JX+1)+
     9OLDV(JY-1,JX-1)))
      ENDIF
C     --------------------------------------------------------
C     IF THE NEXT CONDITION IS RESPECTED THE MORTGAGOR WILL BE IN THE
C     CONTINUATION REGION, IN TERMS OF PREPAYMENT.
C     --------------------------------------------------------
      IF(NEWV(JY,JX).LT.TD) THEN
```

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OLDV(JY,JX) = NEWV(JY,JX)
IF((IXDS.EQ.2).AND.(IYDS.EQ.2)) THEN
NEWD(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDD(JY,JX))+((.5*HJX2(JX)*NIU
22*OMEG2*XJX4(JX)*KOVH2)*(OLDD(JY,JX+1)+OLDD(JY,JX-1)))+(((HJX2(JX)
3*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX)))*KOVH*(
4OLDD(JY,JX+1)-OLDD(JY,JX)))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*
5(OLDD(JY+1,JX)+OLDD(JY-1,JX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPP
6A*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDD(JY+1,JX)
7-OLDD(JY,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA
8*XJX2(JX)*KOV4LH*(OLDD(JY+1,JX+1)-OLDD(JY+1,JX-1)-OLDD(JY-1,JX+1)+
9OLDD(JY-1,JX-1)))
NEWC(JY,JX)=MAX(0.0,(NEWA(JY)-OLDV(JY,JX)-OLDD(JY,JX)))
NEWI(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDI(JY,JX))+((.5*HJX2(JX)*NIU
22*OMEG2*XJX4(JX)*KOVH2)*(OLDI(JY,JX+1)+OLDI(JY,JX-1)))+(((HJX2(JX)
3*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX)))*KOVH*(
4OLDI(JY,JX+1)-OLDI(JY,JX)))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*
5(OLDI(JY+1,JX)+OLDI(JY-1,JX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPP
6A*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDI(JY+1,JX)
7-OLDI(JY,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA
8*XJX2(JX)*KOV4LH*(OLDI(JY+1,JX+1)-OLDI(JY+1,JX-1)-OLDI(JY-1,JX+1)+
9OLDI(JY-1,JX-1)))
NEWCOI(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2
1*PSI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDCOI(JY,JX))+((.5*HJX2(JX)
2*NIU2*OMEG2*XJX4(JX)*KOVH2)*(OLDCOI(JY,JX+1)+OLDCOI(JY,JX-1)))+(((
3HJX2(JX)*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX))
4)*KOVH*(OLDCOI(JY,JX+1)-OLDCOI(JY,JX)))+((0.5*R(JY)*SIG2*PSI2*YJY4
5(JY)*KOVL2)*(OLDCOI(JY+1,JX)+OLDCOI(JY-1,JX)))+(((R(JY)*SIG2*PSI2*
6YJY3(JY))-((KAPPA*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOV
7L*(OLDCOI(JY+1,JX)-OLDCOI(JY,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGM
8A*PSI*YJY2(JY)*OMEGA*XJX2(JX)*KOV4LH*(OLDCOI(JY+1,JX+1)-OLDCOI(JY+
91,JX-1)-OLDCOI(JY-1,JX+1)+OLDCOI(JY-1,JX-1)))
OLDD(JY,JX) = NEWD(JY,JX)
OLDC(JY,JX) = NEWC(JY,JX)
OLDI(JY,JX) = NEWI(JY,JX)
OLDCOI(JY,JX) = NEWCOI(JY,JX)
ELSEIF((IXDS.EQ.1).AND.(IYDS.EQ.2)) THEN
NEWD(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDD(JY,JX))+((.5*HJX2(JX)*NIU
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22*OMEG2*XJX4(JX)*KOVH2)*(OLDD(JY,JX+1)+OLDD(JY,JX-1)))+(((HJX2(JX)
3*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX)))*KOVH*(
4OLDD(JY,JX)-OLDD(JY,JX-1)))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*
5(OLDD(JY+1,JX)+OLDD(JY-1,JX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPP
6A*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDD(JY+1,JX)
7-OLDD(JY,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA
8*XJX2(JX)*KOV4LH*(OLDD(JY+1,JX+1)-OLDD(JY+1,JX-1)-OLDD(JY-1,JX+1)+
9OLDD(JY-1,JX-1)))
 NEWC(JY,JX)=MAX(0.0,(NEWA(JY)-OLDV(JY,JX)-OLDD(JY,JX)))
 NEWI(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDI(JY,JX))+((.5*HJX2(JX)*NIU
22*OMEG2*XJX4(JX)*KOVH2)*(OLDI(JY,JX+1)+OLDI(JY,JX-1)))+(((HJX2(JX)
3*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX)))*KOVH*(
4OLDI(JY,JX)-OLDI(JY,JX-1)))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*
5(OLDI(JY+1,JX)+OLDI(JY-1,JX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPP
6A*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDI(JY+1,JX)
7-OLDI(JY,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA
8*XJX2(JX)*KOV4LH*(OLDI(JY+1,JX+1)-OLDI(JY+1,JX-1)-OLDI(JY-1,JX+1)+
9OLDI(JY-1,JX-1)))
 NEWCOI(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2
1*PSI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDCOI(JY,JX))+((.5*HJX2(JX)
2*NIU2*OMEG2*XJX4(JX)*KOVH2)*(OLDCOI(JY,JX+1)+OLDCOI(JY,JX-1)))+(((
3HJX2(JX)*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX))
4)*KOVH*(OLDCOI(JY,JX)-OLDCOI(JY,JX-1)))+((0.5*R(JY)*SIG2*PSI2*YJY4
5(JY)*KOVL2)*(OLDCOI(JY+1,JX)+OLDCOI(JY-1,JX)))+(((R(JY)*SIG2*PSI2*
6YJY3(JY))-((KAPPA*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOV
7L*(OLDCOI(JY+1,JX)-OLDCOI(JY,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGM
8A*PSI*YJY2(JY)*OMEGA*XJX2(JX)*KOV4LH*(OLDCOI(JY+1,JX+1)-OLDCOI(JY+
91,JX-1)-OLDCOI(JY-1,JX+1)+OLDCOI(JY-1,JX-1)))
 OLDD(JY,JX) = NEWD(JY,JX)
 OLDC(JY,JX) = NEWC(JY,JX)
 OLDI(JY,JX) = NEWI(JY,JX)
 OLDCOI(JY,JX) = NEWCOI(JY,JX)
 ELSEIF((IXDS.EQ.2).AND.(IYDS.EQ.1)) THEN
 NEWD(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDD(JY,JX))+((.5*HJX2(JX)*NIU
22*OMEG2*XJX4(JX)*KOVH2)*(OLDD(JY,JX+1)+OLDD(JY,JX-1)))+(((HJX2(JX)
3*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX)))*KOVH*(
4OLDD(JY,JX+1)-OLDD(JY,JX)))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*
5(OLDD(JY+1,JX)+OLDD(JY-1,JX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPP
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6A*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDD(JY,JX)-O
7LDD(JY-1,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA
8*XJX2(JX)*KOV4LH*(OLDD(JY+1,JX+1)-OLDD(JY+1,JX-1)-OLDD(JY-1,JX+1)+
9OLDD(JY-1,JX-1)))
 NEWC(JY,JX)=MAX(0.0,(NEWA(JY)-OLDV(JY,JX)-OLDD(JY,JX)))
 NEWI(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDI(JY,JX))+((.5*HJX2(JX)*NIU
22*OMEG2*XJX4(JX)*KOVH2)*(OLDI(JY,JX+1)+OLDI(JY,JX-1)))+(((HJX2(JX)
3*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX)))*KOVH*(
4OLDI(JY,JX+1)-OLDI(JY,JX)))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*
5(OLDI(JY+1,JX)+OLDI(JY-1,JX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPP
6A*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDI(JY,JX)-O
7LDI(JY-1,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA
8*XJX2(JX)*KOV4LH*(OLDI(JY+1,JX+1)-OLDI(JY+1,JX-1)-OLDI(JY-1,JX+1)+
9OLDI(JY-1,JX-1)))
 NEWCOI(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2
1*PSI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDCOI(JY,JX))+((.5*HJX2(JX)
2*NIU2*OMEG2*XJX4(JX)*KOVH2)*(OLDCOI(JY,JX+1)+OLDCOI(JY,JX-1)))+(((
3HJX2(JX)*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX))
4)*KOVH*(OLDCOI(JY,JX+1)-OLDCOI(JY,JX)))+((0.5*R(JY)*SIG2*PSI2*YJY4
5(JY)*KOVL2)*(OLDCOI(JY+1,JX)+OLDCOI(JY-1,JX)))+(((R(JY)*SIG2*PSI2*
6YJY3(JY))-((KAPPA*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOV
7L*(OLDCOI(JY,JX)-OLDCOI(JY-1,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGM
8A*PSI*YJY2(JY)*OMEGA*XJX2(JX)*KOV4LH*(OLDCOI(JY+1,JX+1)-OLDCOI(JY+
91,JX-1)-OLDCOI(JY-1,JX+1)+OLDCOI(JY-1,JX-1)))
 OLDD(JY,JX) = NEWD(JY,JX)
 OLDC(JY,JX) = NEWC(JY,JX)
 OLDI(JY,JX) = NEWI(JY,JX)
 OLDCOI(JY,JX) = NEWCOI(JY,JX)
 ELSEIF((IXDS.EQ.1).AND.(IYDS.EQ.1)) THEN
 NEWD(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDD(JY,JX))+((.5*HJX2(JX)*NIU
22*OMEG2*XJX4(JX)*KOVH2)*(OLDD(JY,JX+1)+OLDD(JY,JX-1)))+(((HJX2(JX)
3*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX)))*KOVH*(
4OLDD(JY,JX)-OLDD(JY,JX-1)))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*
5(OLDD(JY+1,JX)+OLDD(JY-1,JX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPP
6A*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDD(JY,JX)-O
7LDD(JY-1,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA
8*XJX2(JX)*KOV4LH*(OLDD(JY+1,JX+1)-OLDD(JY+1,JX-1)-OLDD(JY-1,JX+1)+
9OLDD(JY-1,JX-1)))
```

```
      NEWC(JY,JX)=MAX(0.0,(NEWA(JY)-OLDV(JY,JX)-OLDD(JY,JX)))
      NEWI(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
     1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDI(JY,JX))+((.5*HJX2(JX)*NIU
     •22*OMEG2*XJX4(JX)*KOVH2)*(OLDI(JY,JX+1)+OLDI(JY,JX-1)))+(((HJX2(JX)
     3*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX)))*KOVH*(
     4OLDI(JY,JX)-OLDI(JY,JX-1)))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*
     5(OLDI(JY+1,JX)+OLDI(JY-1,JX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPP
     6A*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDI(JY,JX)-O
     7LDI(JY-1,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA
     8*XJX2(JX)*KOV4LH*(OLDI(JY+1,JX+1)-OLDI(JY+1,JX-1)-OLDI(JY-1,JX+1)+
     9OLDI(JY-1,JX-1)))
      NEWCOI(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2
     1*PSI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDCOI(JY,JX))+((.5*HJX2(JX)
     2*NIU2*OMEG2*XJX4(JX)*KOVH2)*(OLDCOI(JY,JX+1)+OLDCOI(JY,JX-1)))+(((
     3HJX2(JX)*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX))
     4)*KOVH*(OLDCOI(JY,JX)-OLDCOI(JY,JX-1)))+((0.5*R(JY)*SIG2*PSI2*YJY4
     5(JY)*KOVL2)*(OLDCOI(JY+1,JX)+OLDCOI(JY-1,JX)))+(((R(JY)*SIG2*PSI2*
     6YJY3(JY))-((KAPPA*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOV
     7L*(OLDCOI(JY,JX)-OLDCOI(JY-1,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGM
     8A*PSI*YJY2(JY)*OMEGA*XJX2(JX)*KOV4LH*(OLDCOI(JY+1,JX+1)-OLDCOI(JY+
     91,JX-1)-OLDCOI(JY-1,JX+1)+OLDCOI(JY-1,JX-1)))
      OLDD(JY,JX) = NEWD(JY,JX)
      OLDC(JY,JX) = NEWC(JY,JX)
      OLDI(JY,JX) = NEWI(JY,JX)
      OLDCOI(JY,JX) = NEWCOI(JY,JX)
      ENDIF
C     ---------------------------------------------------------
C     IF THE PROGRAM REACHES THE NEXT STAGE THIS MEANS THAT THE
C     MORTGAGOR WILL BE IN THE PREPAYMENT REGION
C     ---------------------------------------------------------
      ELSE IF(ISTATE.EQ.1)THEN
      NEWV(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
     1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDV(JY,JX))+((.5*HJX2(JX)*NIU
     22*OMEG2*XJX4(JX)*KOVH2)*(OLDV(JY,JX+1)+OLDV(JY,JX-1)))+(HJX2(JX)*N
     3IU2*OMEG2*XJX3(JX)*KOVH*(OLDV(JY,JX+1)-OLDV(JY,JX)))+((0.5*R(JY)
     4*SIG2*PSI2*YJY4(JY)*KOVL2)*(OLDV(JY+1,JX)+OLDV(JY-1,JX)))+(R(JY)*S
     5IG2*PSI2*YJY3(JY)*KOVL*(OLDV(JY,JX)-OLDV(JY-1,JX)))+(RHO*H(JX)*S
     6QRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA*XJX2(JX)*KOV4LH*(OLDV(JY+1
     7,JX+1)-OLDV(JY+1,JX-1)-OLDV(JY-1,JX+1)+OLDV(JY-1,JX-1)))-(CRATE*OB
     8(IM)*DISINC)+(R(JY)*TD*DISINC)
```

```
 OLDV(JY,JX)=NEWV(JY,JX)
 NEWD(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDD(JY,JX))+((.5*HJX2(JX)*NIU
22*OMEG2*XJX4(JX)*KOVH2)*(OLDD(JY,JX+1)+OLDD(JY,JX-1)))+(HJX2(JX)*N
3IU2*OMEG2*XJX3(JX)*KOVH*(OLDD(JY,JX+1)-OLDD(JY,JX)))+((0.5*R(JY)
4*SIG2*PSI2*YJY4(JY)*KOVL2)*(OLDD(JY+1,JX)+OLDD(JY-1,JX)))+(R(JY)*S
5IG2*PSI2*YJY3(JY)*KOVL*(OLDD(JY,JX)-OLDD(JY-1,JX)))+(RHO*H(JX)*S
6QRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA*XJX2(JX)*KOV4LH*(OLDD(JY+1
7,JX+1)-OLDD(JY+1,JX-1)-OLDD(JY-1,JX+1)+OLDD(JY-1,JX-1)))
 OLDD(JY,JX)=NEWD(JY,JX)
 NEWC(JY,JX)=MAX(0.0,(NEWA(JY)-OLDV(JY,JX)-OLDD(JY,JX)))
 OLDC(JY,JX)=NEWC(JY,JX)
 NEWI(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDI(JY,JX))+((.5*HJX2(JX)*NIU
22*OMEG2*XJX4(JX)*KOVH2)*(OLDI(JY,JX+1)+OLDI(JY,JX-1)))+(HJX2(JX)*N
3IU2*OMEG2*XJX3(JX)*KOVH*(OLDI(JY,JX+1)-OLDI(JY,JX)))+((0.5*R(JY)
4*SIG2*PSI2*YJY4(JY)*KOVL2)*(OLDI(JY+1,JX)+OLDI(JY-1,JX)))+(R(JY)*S
5IG2*PSI2*YJY3(JY)*KOVL*(OLDI(JY,JX)-OLDI(JY-1,JX)))+(RHO*H(JX)*S
6QRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA*XJX2(JX)*KOV4LH*(OLDI(JY+1
7,JX+1)-OLDI(JY+1,JX-1)-OLDI(JY-1,JX+1)+OLDI(JY-1,JX-1)))
 OLDI(JY,JX)=NEWI(JY,JX)
 NEWCOI(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2
1*PSI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDCOI(JY,JX))+((.5*HJX2(JX)
2*NIU2*OMEG2*XJX4(JX)*KOVH2)*(OLDCOI(JY,JX+1)+OLDCOI(JY,JX-1)))+(HJ
3X2(JX)*NIU2*OMEG2*XJX3(JX)*KOVH*(OLDCOI(JY,JX+1)-OLDCOI(JY,JX)))
4+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*(OLDCOI(JY+1,JX)+OLDCOI(JY-
51,JX)))+(R(JY)*SIG2*PSI2*YJY3(JY)*KOVL*(OLDCOI(JY,JX)-OLDCOI(JY-
61,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA*XJX2(J
7X)*KOV4LH*(OLDCOI(JY+1,JX+1)-OLDCOI(JY+1,JX-1)-OLDCOI(JY-1,JX+1)+O
8LDCOI(JY-1,JX-1)))
 OLDCOI(JY,JX)=NEWCOI(JY,JX)
 IJXCT = MAX(JX,IJXCT)
 IJYCT = MIN(JY,IJYCT)
 ISTATE = 2
 ELSE
 NEWV(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDV(JY,JX))+((.5*HJX2(JX)*NIU
22*OMEG2*XJX4(JX)*KOVH2)*(OLDV(JY,JX+1)+OLDV(JY,JX-1)))+(HJX2(JX)*N
3IU2*OMEG2*XJX3(JX)*KOVH*(OLDV(JY,JX+1)-OLDV(JY,JX)))+((0.5*R(JY)
4*SIG2*PSI2*YJY4(JY)*KOVL2)*(OLDV(JY+1,JX)+OLDV(JY-1,JX)))+(R(JY)*S
```

```
  5IG2*PSI2*YJY3(JY)*KOVL*(OLDV(JY,JX)-OLDV(JY-1,JX)))+(RHO*H(JX)*S
  6QRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA*XJX2(JX)*KOV4LH*(OLDV(JY+1
  7,JX+1)-OLDV(JY+1,JX-1)-OLDV(JY-1,JX+1)+OLDV(JY-1,JX-1)))-(CRATE*OB
  8(IM)*DISINC)+(R(JY)*TD*DISINC)
   OLDV(JY,JX) = NEWV(JY,JX)
   NEWD(JY,JX) = 0.0
   OLDD(JY,JX) = NEWD(JY,JX)
   NEWC(JY,JX) = MAX(0.0,(NEWA(JY)-OLDV(JY,JX)-OLDD(JY,JX)))
   OLDC(JY,JX) = NEWC(JY,JX)
   NEWI(JY,JX) = 0.0
   OLDI(JY,JX) = NEWI(JY,JX)
   NEWCOI(JY,JX) = 0.0
   OLDCOI(JY,JX) = NEWCOI(JY,JX)
   ISTATE = 2
   ENDIF
   OLDA(JY) = NEWA(JY)
3501 CONTINUE
3500 CONTINUE
   IF (IJYCT.LE.JYMAX) THEN
   DO 3700 JY=IJYCT,JYMAX-1
   OLDV(JY,JXMIN)=TD
   OLDC(JY,JXMIN)=MAX(0.0,(NEWA(JY)-OLDV(JY,JXMIN)-OLDD(JY,JXMIN)))
3700 CONTINUE
   END IF
   IF (IJXCT.GT.JXMIN) THEN
   DO 3701 JX=JXMIN,IJXCT
   OLDV(JYMAX,JX)=TD
   OLDD(JYMAX,JX)=0.0
   OLDI(JYMAX,JX)=0.0
   OLDCOI(JYMAX,JX)=0.0
   OLDC(JYMAX,JX)=MAX(0.0,(NEWA(JYMAX)-OLDV(JYMAX,JX)-OLDD(JYMAX,JX))
  1)
3701 CONTINUE
   END IF
C
   OLDA(JYMIN) = NEWA(JYMIN)
   OLDA(JYMAX) = NEWA(JYMAX)
C
3000 CONTINUE
C    ***********************************************************************
```

```
C    OTHER 299 MONTHS OF THE CONTRACT
C    *******************************************************************
     DO 4000 IM = IMMIN+1,IMMAX,IMINC
     A1 = ((1.0+(CRATE/12.0))**(DFLOAT(IMMAX)))
     A2 =  (1.0+(CRATE/12.0))**(DFLOAT(IMMAX-IM))
     A3 = ((DFLOAT(1)+(CRATE/12.0))**(DFLOAT(IMMAX)))-DFLOAT(1)
     OB(IM) = ((A1-A2)/A3)*RLTV
     TD = OB(IM-1)+M
C    *******************************************************************
C    TERMINAL CONDITIONS
C    *******************************************************************
     DO 4001 JY=JYMIN,JYMAX,JYINC
     atime = time@()
     adate = edate@()
     NEWA(JY) = OLDA(JY)+M
     OLDA(JY) = NEWA(JY)
4001 CONTINUE
     DO 4002 JY = JYMIN,JYMAX,JYINC
     DO 4003 JX = JXMIN+1,JXMAX,JXINC
     OLAPLM = OLDV(JY,JX)+M
     NEWV(JY,JX) = MIN(OLAPLM,H(JX))
     OLDV(JY,JX) = NEWV(JY,JX)
C    ---------------------------------------------------------------
C    IF THE NEXT CONDITION HOLDS THE BORROWER WILL BE IN THE
C    CONTINUATION AREA IN TERMS OF DEFAULT (NO DEFAULT AREA)
C    ---------------------------------------------------------------
     IF (OLDV(JY,JX).EQ.OLAPLM) THEN
     NEWC(JY,JX) = OLDC(JY,JX)
     NEWD(JY,JX) = OLDD(JY,JX)
     NEWI(JY,JX) = OLDI(JY,JX)
     NEWCOI(JY,JX) = OLDCOI(JY,JX)
     OLDC(JY,JX) = NEWC(JY,JX)
     OLDD(JY,JX) = NEWD(JY,JX)
     OLDI(JY,JX) = NEWI(JY,JX)
     OLDCOI(JY,JX) = NEWCOI(JY,JX)
C    ---------------------------------------------------------------
C    IF THE PROGAM REACHES THE NEXT BRANCH THIS WILL MEAN THAT THE
C    BORROWER IS IN THE DEFAULT REGION
C    ---------------------------------------------------------------
     ELSE
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      NEWC(JY,JX) = 0.0
      NEWD(JY,JX) = OLDA(JY)-H(JX)
      NEWI(JY,JX) = MAX(0.0,MIN((ZIND*(TD-H(JX))),MAXCO))
      NEWCOI(JY,JX) = MAX(0.0,MAX(((1.0-ZIND)*(TD-H(JX))),((TD-H(JX))-MA
     1XCO)))
      OLDC(JY,JX) = NEWC(JY,JX)
      OLDD(JY,JX) = NEWD(JY,JX)
      OLDI(JY,JX) = NEWI(JY,JX)
      OLDCOI(JY,JX) = NEWCOI(JY,JX)
      ENDIF
 4003 CONTINUE
C    -----------------------------------------------------
C    H IS INFINITE (JX= JXMIN)
C    -----------------------------------------------------
      NEWV(JY,JXMIN) = OLDV(JY,JXMIN)+M
      NEWD(JY,JXMIN) = 0.0
      NEWC(JY,JXMIN) = OLDA(JY)-NEWV(JY,JXMIN)-NEWD(JY,JXMIN)
      NEWI(JY,JXMIN) = 0.0
      NEWCOI(JY,JXMIN) = 0.0
      OLDV(JY,JXMIN) = NEWV(JY,JXMIN)
      OLDC(JY,JXMIN) = NEWC(JY,JXMIN)
      OLDD(JY,JXMIN) = NEWD(JY,JXMIN)
      OLDI(JY,JXMIN) = NEWI(JY,JXMIN)
      OLDCOI(JY,JXMIN) = NEWCOI(JY,JXMIN)
 4002 CONTINUE
C    ***************************************************************
C    SOLUTION OF THE PDE
C    ***************************************************************
C
      DO 5000 IS = ISMIN,ISMAX,ISINC
      A1 = ((1.0+(CRATE/12.0))**(DFLOAT(IMMAX)))
      A2 =  (1.0+(CRATE/12.0))**(DFLOAT(IMMAX-IM))
      A3 = ((DFLOAT(1)+(CRATE/12.0))**(DFLOAT(IMMAX)))-DFLOAT(1)
      OB(IM) = ((A1-A2)/A3)*RLTV
      TD = (DFLOAT(1) + (CRATE*((DFLOAT(ISMAX)-DFLOAT(IS))/(DFLOAT(NISST
     1)*DFLOAT(12)))))*OB(IM)
C    -----------------------------------------------------
C    VALUE OF A
C    -----------------------------------------------------
      DO 5001 JY = JYMIN+1,JYMAX-1,JYINC
```

```
      DPN0(JY) = R(JY)*S2PS2*YJY4(JY)*KOVL2
      DPN3(JY) = -PSI*YJY2(JY)*KAPLAM(JY)
      DPN10(JY) = (R(JY)*S2PS2*YJY3(JY))
      DPN4 = DPN3(JY)+DPN10(JY)
      IF(DPN4.LT.0) THEN
      DPN2(JY)=(0.5*DPN0(JY)*OLDA(JY+1))+((DPN3(JY)+DPN10(JY))*KOVL*OLDA
     1(JY))
      DPN7(JY)=(0.5*DPN0(JY)*OLDA(JY-1))-((DPN3(JY)+DPN10(JY))*KOVL*OLDA
     1(JY-1))
      ELSE
      DPN2(JY)=(0.5*DPN0(JY)*OLDA(JY+1))+((DPN3(JY)+DPN10(JY))*KOVL*OLDA
     1(JY+1))
      DPN7(JY)=(0.5*DPN0(JY)*OLDA(JY-1))-((DPN3(JY)+DPN10(JY))*KOVL*OLDA
     1(JY))
      ENDIF
      DPN1 = ((1.0-DPN0(JY)-R(JY)*DISINC)*OLDA(JY))
      NEWA(JY) = DPN1 + DPN2(JY) + DPN7(JY)
 5001 CONTINUE
      NEWA(JYMIN) = 0.0
      NEWA(JYMAX) = OLDA(JYMAX-1)
C     ----------------------------------------------------
C     OTHER VARIABLES
C     ****************************************************************
C     BOUNDARIES OVER ALL THE STATE SPACE
C     ****************************************************************
C      (i) CORNERS OF THE GRID
C     ----------------------------------------------------
C     H=0.0 AND R=0.0
C     ----------------------------------------------------
      NEWA(JYMAX) = OLDA(JYMAX-1)
      NEWV(JYMAX,JXMAX) = OLDV(JYMAX-1,JXMAX)
      NEWC(JYMAX,JXMAX) = OLDC(JYMAX-1,JXMAX)
      NEWD(JYMAX,JXMAX) = OLDD(JYMAX-1,JXMAX)
      NEWI(JYMAX,JXMAX) = OLDI(JYMAX-1,JXMAX)
      NEWCOI(JYMAX,JXMAX) = OLDCOI(JYMAX-1,JXMAX)
      OLDV(JYMAX,JXMAX) = NEWV(JYMAX,JXMAX)
      OLDC(JYMAX,JXMAX) = NEWC(JYMAX,JXMAX)
      OLDD(JYMAX,JXMAX) = NEWD(JYMAX,JXMAX)
      OLDI(JYMAX,JXMAX) = NEWI(JYMAX,JXMAX)
      OLDCOI(JYMAX,JXMAX) = NEWCOI(JYMAX,JXMAX)
```

```
C     ----------------------------------------------------------------
C     H IS INFINITE AND R=0.0
C     ----------------------------------------------------------------
      NEWA(JYMAX) = OLDA(JYMAX-1)
      NEWV(JYMAX,JXMIN) = OLDV(JYMAX-1,JXMIN)
      NEWC(JYMAX,JXMIN) = OLDC(JYMAX-1,JXMIN)
      NEWD(JYMAX,JXMIN) = OLDD(JYMAX-1,JXMIN)
      NEWI(JYMAX,JXMIN) = OLDI(JYMAX-1,JXMIN)
      NEWCOI(JYMAX,JXMIN) = OLDCOI(JYMAX-1,JXMIN)
      OLDV(JYMAX,JXMIN) = NEWV(JYMAX,JXMIN)
      OLDC(JYMAX,JXMIN) = NEWC(JYMAX,JXMIN)
      OLDD(JYMAX,JXMIN) = NEWD(JYMAX,JXMIN)
      OLDI(JYMAX,JXMIN) = NEWI(JYMAX,JXMIN)
      OLDCOI(JYMAX,JXMIN) = NEWCOI(JYMAX,JXMIN)
C     ----------------------------------------------------------------
C     H AND R ARE INFINITE
C     ----------------------------------------------------------------
      NEWA(JYMIN) = 0.0
      NEWV(JYMIN,JXMIN) = 0.0
      NEWC(JYMIN,JXMIN) = 0.0
      NEWD(JYMIN,JXMIN) = 0.0
      NEWI(JYMIN,JXMIN) = 0.0
      NEWCOI(JYMIN,JXMIN) = 0.0
      OLDV(JYMIN,JXMIN) = NEWV(JYMIN,JXMIN)
      OLDC(JYMIN,JXMIN) = NEWC(JYMIN,JXMIN)
      OLDD(JYMIN,JXMIN) = NEWD(JYMIN,JXMIN)
      OLDI(JYMIN,JXMIN) = NEWI(JYMIN,JXMIN)
      OLDCOI(JYMIN,JXMIN) = NEWCOI(JYMIN,JXMIN)
C     ----------------------------------------------------------------
C     H = 0.0 AND R IS INFINITE
C     ----------------------------------------------------------------
      NEWA(JYMIN) = 0.0
      NEWV(JYMIN,JXMAX) = 0.0
      NEWC(JYMIN,JXMAX) = 0.0
      NEWD(JYMIN,JXMAX) = 0.0
      NEWI(JYMIN,JXMAX) = 0.0
      NEWCOI(JYMIN,JXMAX) = 0.0
      OLDV(JYMIN,JXMAX) = NEWV(JYMIN,JXMAX)
      OLDC(JYMIN,JXMAX) = NEWC(JYMIN,JXMAX)
      OLDD(JYMIN,JXMAX) = NEWD(JYMIN,JXMAX)
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```
      OLDI(JYMIN,JXMAX) = NEWI(JYMIN,JXMAX)
      OLDCOI(JYMIN,JXMAX) = NEWCOI(JYMIN,JXMAX)
C     ------------------------------------------------
C     (ii) EDGES OF THE GRID
C     ------------------------------------------------
C     H(JX) = 0.0, R(JY) VARIES
C     ------------------------------------------------
      DO 5002 JY=JYMIN+1,JYMAX-1,JYINC
      OLDC(JY,JXMAX) = 0.0
      OLDD(JY,JXMAX) = NEWA(JY)
      OLDV(JY,JXMAX) = 0.0
      CTRL=((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPPA*(THETA-R(JY))-(LAMBDA*R(J
     1Y)))*PSI*YJY2(JY)))
      IF(CTRL.LT.0) THEN
      NEWI(JY,JXMAX)=((1-(R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC)
     1)*OLDI(JY,JXMAX))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*(OLDI(JY+1
     2,JXMAX)+OLDI(JY-1,JXMAX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPPA*(T
     3HETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDI(JY,JXMAX)-OL
     4DI(JY-1,JXMAX)))
      OLDI(JY,JXMAX) = NEWI(JY,JXMAX)
      NEWCOI(JY,JXMAX)=((1-(R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)-(R(JY)*DISIN
     1C))*OLDCOI(JY,JXMAX))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*(OLDCO
     2I(JY+1,JXMAX)+OLDCOI(JY-1,JXMAX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((
     3KAPPA*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDCOI(JY
     4,JXMAX)-OLDCOI(JY-1,JXMAX)))
      OLDCOI(JY,JXMAX) = NEWCOI(JY,JXMAX)
      ELSE
      NEWI(JY,JXMAX)=((1-(R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC)
     1)*OLDI(JY,JXMAX))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*(OLDI(JY+1
     2,JXMAX)+OLDI(JY-1,JXMAX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPPA*(T
     3HETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDI(JY+1,JXMAX)-
     4OLDI(JY,JXMAX)))
      OLDI(JY,JXMAX) = NEWI(JY,JXMAX)
      NEWCOI(JY,JXMAX)=((1-(R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)-(R(JY)*DISIN
     1C))*OLDCOI(JY,JXMAX))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*(OLDCO
     2I(JY+1,JXMAX)+OLDCOI(JY-1,JXMAX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((
     3KAPPA*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDCOI(JY
     4+1,JXMAX)-OLDCOI(JY,JXMAX)))
      OLDCOI(JY,JXMAX) = NEWCOI(JY,JXMAX)
      ENDIF
```

```
 5002 CONTINUE
C    -------------------------------------------------
C    R IS 0.0 AND H VARIES
C    -------------------------------------------------
      DO 5003 JX=JXMIN+1,JXMAX-1,JXINC
      NEWA(JYMAX) = OLDA(JYMAX-1)
      NEWV(JYMAX,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2))*OLDV(JYMA
     1X,JX))+((.5*HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)*(OLDV(JYMAX,JX+1)+
     2OLDV(JYMAX,JX-1)))+(((HJX2(JX)*NIU2*OMEG2*XJX3(JX))+(DELTA*H(JX)*O
     3MEGA*XJX2(JX)))*KOVH*(OLDV(JYMAX,JX+1)-OLDV(JYMAX,JX)))-((KAPPA*TH
     4ETA*PSI*YJY2(JY)*KOVL)*(OLDV(JYMAX,JX)-OLDV(JYMAX-1,JX)))
      OLDV(JYMAX,JX) = NEWV(JYMAX,JX)
      NEWD(JYMAX,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2))*OLDD(JYMA
     1X,JX))+((.5*HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)*(OLDD(JYMAX,JX+1)+
     2OLDD(JYMAX,JX-1)))+(((HJX2(JX)*NIU2*OMEG2*XJX3(JX))+(DELTA*H(JX)*O
     3MEGA*XJX2(JX)))*KOVH*(OLDD(JYMAX,JX+1)-OLDD(JYMAX,JX)))-((KAPPA*TH
     4ETA*PSI*YJY2(JY)*KOVL)*(OLDV(JYMAX,JX)-OLDV(JYMAX-1,JX)))
      OLDD(JYMAX,JX) = NEWD(JYMAX,JX)
      NEWC(JYMAX,JX) = MAX(0.0,(NEWA(JYMAX)-OLDV(JYMAX,JX)-OLDD(JYMAX,JX
     1)))
      OLDC(JYMAX,JX) = NEWC(JYMAX,JX)
      NEWI(JYMAX,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2))*OLDI(JYMA
     1X,JX))+((.5*HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)*(OLDI(JYMAX,JX+1)+
     2OLDI(JYMAX,JX-1)))+(((HJX2(JX)*NIU2*OMEG2*XJX3(JX))+(DELTA*H(JX)*O
     3MEGA*XJX2(JX)))*KOVH*(OLDI(JYMAX,JX+1)-OLDI(JYMAX,JX)))-((KAPPA*TH
     4ETA*PSI*YJY2(JY)*KOVL)*(OLDV(JYMAX,JX)-OLDV(JYMAX-1,JX)))
      OLDI(JYMAX,JX) = NEWI(JYMAX,JX)
      NEWCOI(JYMAX,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2))*OLDCOI(
     1JYMAX,JX))+((.5*HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)*(OLDCOI(JYMAX,
     2JX+1)+OLDCOI(JYMAX,JX-1)))+(((HJX2(JX)*NIU2*OMEG2*XJX3(JX))+(DELTA
     3*H(JX)*OMEGA*XJX2(JX)))*KOVH*(OLDCOI(JYMAX,JX+1)-OLDCOI(JYMAX,JX))
     4)-((KAPPA*THETA*PSI*YJY2(JY)*KOVL)*(OLDV(JYMAX,JX)-OLDV(JYMAX-1,JX
     5)))
      OLDCOI(JYMAX,JX) = NEWCOI(JYMAX,JX)
 5003 CONTINUE
C    -------------------------------------------------
C    H IS INFINITE AND R VARIES
C    -------------------------------------------------
      DO 5004 JY=JYMIN+1,JYMAX-1,JYINC
      CTRL=((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPPA*(THETA-R(JY))-(LAMBDA*R(J
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1Y)))*PSI*YJY2(JY)))
 IF(CTRL.LT.0) THEN
 NEWC(JY,JXMIN)=((1-(R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC)
1)*OLDC(JY,JXMIN))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*(OLDC(JY+1
2,JXMIN)+OLDC(JY-1,JXMIN)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPPA*(T
3HETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDC(JY,JXMIN)-OL
4DC(JY-1,JXMIN)))
 ELSE
 NEWC(JY,JXMIN)=((1-(R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC)
1)*OLDC(JY,JXMIN))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*(OLDC(JY+1
2,JXMIN)+OLDC(JY-1,JXMIN)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPPA*(T
3HETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDC(JY+1,JXMIN)-
4OLDC(JY,JXMIN)))
 ENDIF
 OLDC(JY,JXMIN) = NEWC(JY,JXMIN)
 NEWV(JY,JXMIN) = NEWA(JY) - OLDC(JY,JXMIN)
 NEWD(JY,JXMIN) = 0.0
 NEWI(JY,JXMIN) = 0.0
 NEWCOI(JY,JXMIN) = 0.0
 OLDV(JY,JXMIN) = NEWV(JY,JXMIN)
 OLDD(JY,JXMIN) = NEWD(JY,JXMIN)
 OLDI(JY,JXMIN) = NEWI(JY,JXMIN)
 OLDCOI(JY,JXMIN) = NEWCOI(JY,JXMIN)
5004 CONTINUE
C ------------------------------------------------------------
C R IS INFINITE AND H VARIES
C ------------------------------------------------------------
 DO 5005 JX=JXMIN+1,JXMAX-1,JXINC
 NEWA(JYMIN) = 0.0
 NEWC(JYMIN,JX) = 0.0
 NEWV(JYMIN,JX) = 0.0
 NEWD(JYMIN,JX) = 0.0
 NEWI(JYMIN,JX) = 0.0
 NEWCOI(JYMIN,JX) = 0.0
 OLDC(JYMIN,JX) = NEWC(JYMIN,JX)
 OLDV(JYMIN,JX) = NEWV(JYMIN,JX)
 OLDD(JYMIN,JX) = NEWD(JYMIN,JX)
 OLDI(JYMIN,JX) = NEWI(JYMIN,JX)
 OLDCOI(JYMIN,JX) = NEWCOI(JYMIN,JX)
5005 CONTINUE
```

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C     ******************************************************************
C     ALGORITHM TO SOLVE THE PDE
C     ******************************************************************
      IJXCT = JXMIN
      IJYCT = JYMAX
      DO 5500 JX = JXMIN+1,JXMAX-1,JXINC
      ISTATE = 1
      DO 5501 JY = JYMIN+1,JYMAX-1,JYINC
      FDX=((HJX2(JX)*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX
     12(JX)))
      FDY=((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPPA*(THETA-R(JY))-(LAMBDA*R(J
     1Y)))*PSI*YJY2(JY)))
      IF(FDX.LT.0) THEN
      IXDS=1
      ELSE
      IXDS=2
      ENDIF
      IF(FDY.LT.0) THEN
      IYDS=1
      ELSE
      IYDS=2
      ENDIF
      IF((IXDS.EQ.2).AND.(IYDS.EQ.2)) THEN
      NEWV(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
     1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDV(JY,JX))+((.5*HJX2(JX)*NIU
     22*OMEG2*XJX4(JX)*KOVH2)*(OLDV(JY,JX+1)+OLDV(JY,JX-1)))+(((HJX2(JX)
     3*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX)))*KOVH*(
     4OLDV(JY,JX+1)-OLDV(JY,JX)))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*
     5(OLDV(JY+1,JX)+OLDV(JY-1,JX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPP
     6A*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDV(JY+1,JX)
     7-OLDV(JY,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA
     8*XJX2(JX)*KOV4LH*(OLDV(JY+1,JX+1)-OLDV(JY+1,JX-1)-OLDV(JY-1,JX+1)+
     9OLDV(JY-1,JX-1)))
      ELSEIF((IXDS.EQ.1).AND.(IYDS.EQ.2)) THEN
      NEWV(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
     1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDV(JY,JX))+((.5*HJX2(JX)*NIU
     22*OMEG2*XJX4(JX)*KOVH2)*(OLDV(JY,JX+1)+OLDV(JY,JX-1)))+(((HJX2(JX)
     3*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX)))*KOVH*(
     4OLDV(JY,JX)-OLDV(JY,JX-1)))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*
     5(OLDV(JY+1,JX)+OLDV(JY-1,JX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPP
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     6A*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDV(JY+1,JX)
     7-OLDV(JY,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA
     8*XJX2(JX)*KOV4LH*(OLDV(JY+1,JX+1)-OLDV(JY+1,JX-1)-OLDV(JY-1,JX+1)+
     9OLDV(JY-1,JX-1)))
      ELSEIF((IXDS.EQ.2).AND.(IYDS.EQ.1)) THEN
      NEWV(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
     1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDV(JY,JX))+((.5*HJX2(JX)*NIU
     22*OMEG2*XJX4(JX)*KOVH2)*(OLDV(JY,JX+1)+OLDV(JY,JX-1)))+(((HJX2(JX)
     3*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX)))*KOVH*(
     4OLDV(JY,JX+1)-OLDV(JY,JX)))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*
     5(OLDV(JY+1,JX)+OLDV(JY-1,JX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPP
     6A*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDV(JY,JX)-O
     7LDV(JY-1,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA
     8*XJX2(JX)*KOV4LH*(OLDV(JY+1,JX+1)-OLDV(JY+1,JX-1)-OLDV(JY-1,JX+1)+
     9OLDV(JY-1,JX-1)))
      ELSEIF((IXDS.EQ.1).AND.(IYDS.EQ.1)) THEN
      NEWV(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
     1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDV(JY,JX))+((.5*HJX2(JX)*NIU
     22*OMEG2*XJX4(JX)*KOVH2)*(OLDV(JY,JX+1)+OLDV(JY,JX-1)))+(((HJX2(JX)
     3*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX)))*KOVH*(
     4OLDV(JY,JX)-OLDV(JY,JX-1)))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*
     5(OLDV(JY+1,JX)+OLDV(JY-1,JX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPP
     6A*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDV(JY,JX)-O
     7LDV(JY-1,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA
     8*XJX2(JX)*KOV4LH*(OLDV(JY+1,JX+1)-OLDV(JY+1,JX-1)-OLDV(JY-1,JX+1)+
     9OLDV(JY-1,JX-1)))
      ENDIF
C     ------------------------------------------------
C     IF THE NEXT CONDITION IS RESPECTED THE MORTGAGOR WILL BE IN THE
C     CONTINUATION REGION
C     ------------------------------------------------
      IF(NEWV(JY,JX).LT.TD) THEN
      OLDV(JY,JX) = NEWV(JY,JX)
      IF((IXDS.EQ.2).AND.(IYDS.EQ.2)) THEN
      NEWD(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
     1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDD(JY,JX))+((.5*HJX2(JX)*NIU
     22*OMEG2*XJX4(JX)*KOVH2)*(OLDD(JY,JX+1)+OLDD(JY,JX-1)))+(((HJX2(JX)
     3*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX)))*KOVH*(
     4OLDD(JY,JX+1)-OLDD(JY,JX)))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*
     5(OLDD(JY+1,JX)+OLDD(JY-1,JX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPP
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     6A*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDD(JY+1,JX)
     7-OLDD(JY,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA
     8*XJX2(JX)*KOV4LH*(OLDD(JY+1,JX+1)-OLDD(JY+1,JX-1)-OLDD(JY-1,JX+1)+
     •9OLDD(JY-1,JX-1)))
      NEWC(JY,JX)=MAX(0.0,(NEWA(JY)-OLDV(JY,JX)-OLDD(JY,JX)))
      NEWI(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
     1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDI(JY,JX))+((.5*HJX2(JX)*NIU
     22*OMEG2*XJX4(JX)*KOVH2)*(OLDI(JY,JX+1)+OLDI(JY,JX-1)))+(((HJX2(JX)
     3*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX)))*KOVH*(
     4OLDI(JY,JX+1)-OLDI(JY,JX)))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*
     5(OLDI(JY+1,JX)+OLDI(JY-1,JX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPP
     6A*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDI(JY+1,JX)
     7-OLDI(JY,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA
     8*XJX2(JX)*KOV4LH*(OLDI(JY+1,JX+1)-OLDI(JY+1,JX-1)-OLDI(JY-1,JX+1)+
     9OLDI(JY-1,JX-1)))
      NEWCOI(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2
     1*PSI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDCOI(JY,JX))+((.5*HJX2(JX)
     2*NIU2*OMEG2*XJX4(JX)*KOVH2)*(OLDCOI(JY,JX+1)+OLDCOI(JY,JX-1)))+(((
     3HJX2(JX)*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX))
     4)*KOVH*(OLDCOI(JY,JX+1)-OLDCOI(JY,JX)))+((0.5*R(JY)*SIG2*PSI2*YJY4
     5(JY)*KOVL2)*(OLDCOI(JY+1,JX)+OLDCOI(JY-1,JX)))+(((R(JY)*SIG2*PSI2*
     6YJY3(JY))-((KAPPA*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOV
     7L*(OLDCOI(JY+1,JX)-OLDCOI(JY,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGM
     8A*PSI*YJY2(JY)*OMEGA*XJX2(JX)*KOV4LH*(OLDCOI(JY+1,JX+1)-OLDCOI(JY+
     91,JX-1)-OLDCOI(JY-1,JX+1)+OLDCOI(JY-1,JX-1)))
      OLDD(JY,JX) = NEWD(JY,JX)
      OLDC(JY,JX) = NEWC(JY,JX)
      OLDI(JY,JX) = NEWI(JY,JX)
      OLDCOI(JY,JX) = NEWCOI(JY,JX)
      ELSEIF((IXDS.EQ.1).AND.(IYDS.EQ.2)) THEN
      NEWD(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
     1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDD(JY,JX))+((.5*HJX2(JX)*NIU
     22*OMEG2*XJX4(JX)*KOVH2)*(OLDD(JY,JX+1)+OLDD(JY,JX-1)))+(((HJX2(JX)
     3*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX)))*KOVH*(
     4OLDD(JY,JX)-OLDD(JY,JX-1)))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*
     5(OLDD(JY+1,JX)+OLDD(JY-1,JX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPP
     6A*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDD(JY+1,JX)
     7-OLDD(JY,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA
     8*XJX2(JX)*KOV4LH*(OLDD(JY+1,JX+1)-OLDD(JY+1,JX-1)-OLDD(JY-1,JX+1)+
     9OLDD(JY-1,JX-1)))
```

```
      NEWC(JY,JX)=MAX(0.0,(NEWA(JY)-OLDV(JY,JX)-OLDD(JY,JX)))
      NEWI(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
     1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDI(JY,JX))+((.5*HJX2(JX)*NIU
     22*OMEG2*XJX4(JX)*KOVH2)*(OLDI(JY,JX+1)+OLDI(JY,JX-1)))+(((HJX2(JX)
     3*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX)))*KOVH*(
     4OLDI(JY,JX)-OLDI(JY,JX-1)))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*
     5(OLDI(JY+1,JX)+OLDI(JY-1,JX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPP
     6A*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDI(JY+1,JX)
     7-OLDI(JY,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA
     8*XJX2(JX)*KOV4LH*(OLDI(JY+1,JX+1)-OLDI(JY+1,JX-1)-OLDI(JY-1,JX+1)+
     9OLDI(JY-1,JX-1)))
      NEWCOI(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2
     1*PSI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDCOI(JY,JX))+((.5*HJX2(JX)
     2*NIU2*OMEG2*XJX4(JX)*KOVH2)*(OLDCOI(JY,JX+1)+OLDCOI(JY,JX-1)))+(((
     3HJX2(JX)*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX))
     4)*KOVH*(OLDCOI(JY,JX)-OLDCOI(JY,JX-1)))+((0.5*R(JY)*SIG2*PSI2*YJY4
     5(JY)*KOVL2)*(OLDCOI(JY+1,JX)+OLDCOI(JY-1,JX)))+(((R(JY)*SIG2*PSI2*
     6YJY3(JY))-((KAPPA*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOV
     7L*(OLDCOI(JY+1,JX)-OLDCOI(JY,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGM
     8A*PSI*YJY2(JY)*OMEGA*XJX2(JX)*KOV4LH*(OLDCOI(JY+1,JX+1)-OLDCOI(JY+
     91,JX-1)-OLDCOI(JY-1,JX+1)+OLDCOI(JY-1,JX-1)))
      OLDD(JY,JX) = NEWD(JY,JX)
      OLDC(JY,JX) = NEWC(JY,JX)
      OLDI(JY,JX) = NEWI(JY,JX)
      OLDCOI(JY,JX) = NEWCOI(JY,JX)
      ELSEIF((IXDS.EQ.2).AND.(IYDS.EQ.1)) THEN
      NEWD(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
     1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDD(JY,JX))+((.5*HJX2(JX)*NIU
     22*OMEG2*XJX4(JX)*KOVH2)*(OLDD(JY,JX+1)+OLDD(JY,JX-1)))+(((HJX2(JX)
     3*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX)))*KOVH*(
     4OLDD(JY,JX+1)-OLDD(JY,JX)))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*
     5(OLDD(JY+1,JX)+OLDD(JY-1,JX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPP
     6A*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDD(JY,JX)-O
     7LDD(JY-1,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA
     8*XJX2(JX)*KOV4LH*(OLDD(JY+1,JX+1)-OLDD(JY+1,JX-1)-OLDD(JY-1,JX+1)+
     9OLDD(JY-1,JX-1)))
      NEWC(JY,JX)=MAX(0.0,(NEWA(JY)-OLDV(JY,JX)-OLDD(JY,JX)))
      NEWI(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
     1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDI(JY,JX))+((.5*HJX2(JX)*NIU
     22*OMEG2*XJX4(JX)*KOVH2)*(OLDI(JY,JX+1)+OLDI(JY,JX-1)))+(((HJX2(JX)
```

```
3*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX)))*KOVH*(
4OLDI(JY,JX+1)-OLDI(JY,JX)))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*
5(OLDI(JY+1,JX)+OLDI(JY-1,JX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPP
6A*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDI(JY,JX)-O
7LDI(JY-1,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA
8*XJX2(JX)*KOV4LH*(OLDI(JY+1,JX+1)-OLDI(JY+1,JX-1)-OLDI(JY-1,JX+1)+
9OLDI(JY-1,JX-1)))
 NEWCOI(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2
1*PSI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDCOI(JY,JX))+((.5*HJX2(JX)
2*NIU2*OMEG2*XJX4(JX)*KOVH2)*(OLDCOI(JY,JX+1)+OLDCOI(JY,JX-1)))+(((
3HJX2(JX)*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX))
4)*KOVH*(OLDCOI(JY,JX+1)-OLDCOI(JY,JX)))+((0.5*R(JY)*SIG2*PSI2*YJY4
5(JY)*KOVL2)*(OLDCOI(JY+1,JX)+OLDCOI(JY-1,JX)))+(((R(JY)*SIG2*PSI2*
6YJY3(JY))-((KAPPA*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOV
7L*(OLDCOI(JY,JX)-OLDCOI(JY-1,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGM
8A*PSI*YJY2(JY)*OMEGA*XJX2(JX)*KOV4LH*(OLDCOI(JY+1,JX+1)-OLDCOI(JY+
91,JX-1)-OLDCOI(JY-1,JX+1)+OLDCOI(JY-1,JX-1)))
 OLDD(JY,JX) = NEWD(JY,JX)
 OLDC(JY,JX) = NEWC(JY,JX)
 OLDI(JY,JX) = NEWI(JY,JX)
 OLDCOI(JY,JX) = NEWCOI(JY,JX)
 ELSEIF((IXDS.EQ.1).AND.(IYDS.EQ.1)) THEN
 NEWD(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDD(JY,JX))+((.5*HJX2(JX)*NIU
22*OMEG2*XJX4(JX)*KOVH2)*(OLDD(JY,JX+1)+OLDD(JY,JX-1)))+(((HJX2(JX)
3*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX)))*KOVH*(
4OLDD(JY,JX)-OLDD(JY,JX-1)))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*
5(OLDD(JY+1,JX)+OLDD(JY-1,JX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPP
6A*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDD(JY,JX)-O
7LDD(JY-1,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA
8*XJX2(JX)*KOV4LH*(OLDD(JY+1,JX+1)-OLDD(JY+1,JX-1)-OLDD(JY-1,JX+1)+
9OLDD(JY-1,JX-1)))
 NEWC(JY,JX)=MAX(0.0,(NEWA(JY)-OLDV(JY,JX)-OLDD(JY,JX)))
 NEWI(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDI(JY,JX))+((.5*HJX2(JX)*NIU
22*OMEG2*XJX4(JX)*KOVH2)*(OLDI(JY,JX+1)+OLDI(JY,JX-1)))+(((HJX2(JX)
3*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX)))*KOVH*(
4OLDI(JY,JX)-OLDI(JY,JX-1)))+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*
5(OLDI(JY+1,JX)+OLDI(JY-1,JX)))+(((R(JY)*SIG2*PSI2*YJY3(JY))-((KAPP
6A*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOVL*(OLDI(JY,JX)-O
```

```
     7LDI(JY-1,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA
     8*XJX2(JX)*KOV4LH*(OLDI(JY+1,JX+1)-OLDI(JY+1,JX-1)-OLDI(JY-1,JX+1)+
     9OLDI(JY-1,JX-1)))
      NEWCOI(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2
     1*PSI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDCOI(JY,JX))+((.5*HJX2(JX)
     2*NIU2*OMEG2*XJX4(JX)*KOVH2)*(OLDCOI(JY,JX+1)+OLDCOI(JY,JX-1)))+(((
     3HJX2(JX)*NIU2*OMEG2*XJX3(JX))-((R(JY)-DELTA)*H(JX)*OMEGA*XJX2(JX))
     4)*KOVH*(OLDCOI(JY,JX)-OLDCOI(JY,JX-1)))+((0.5*R(JY)*SIG2*PSI2*YJY4
     5(JY)*KOVL2)*(OLDCOI(JY+1,JX)+OLDCOI(JY-1,JX)))+(((R(JY)*SIG2*PSI2*
     6YJY3(JY))-((KAPPA*(THETA-R(JY))-(LAMBDA*R(JY)))*PSI*YJY2(JY)))*KOV
     7L*(OLDCOI(JY,JX)-OLDCOI(JY-1,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGM
     8A*PSI*YJY2(JY)*OMEGA*XJX2(JX)*KOV4LH*(OLDCOI(JY+1,JX+1)-OLDCOI(JY+
     91,JX-1)-OLDCOI(JY-1,JX+1)+OLDCOI(JY-1,JX-1)))
      OLDD(JY,JX) = NEWD(JY,JX)
      OLDC(JY,JX) = NEWC(JY,JX)
      OLDI(JY,JX) = NEWI(JY,JX)
      OLDCOI(JY,JX) = NEWCOI(JY,JX)
      ENDIF
C     ------------------------------------------------------------
C     IF THE PROGRAM REACHES THE NEXT STAGE THIS MEANS THAT THE
C     MORTGAGOR WILL BE IN THE PREPAYMENT REGION
C     ------------------------------------------------------------
      ELSE IF(ISTATE.EQ.1)THEN
      NEWV(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
     1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDV(JY,JX))+((.5*HJX2(JX)*NIU
     22*OMEG2*XJX4(JX)*KOVH2)*(OLDV(JY,JX+1)+OLDV(JY,JX-1)))+(HJX2(JX)*N
     3IU2*OMEG2*XJX3(JX)*KOVH*(OLDV(JY,JX+1)-OLDV(JY,JX)))+((0.5*R(JY)
     4*SIG2*PSI2*YJY4(JY)*KOVL2)*(OLDV(JY+1,JX)+OLDV(JY-1,JX)))+(R(JY)*S
     5IG2*PSI2*YJY3(JY)*KOVL*(OLDV(JY,JX)-OLDV(JY-1,JX)))+(RHO*H(JX)*S
     6QRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA*XJX2(JX)*KOV4LH*(OLDV(JY+1
     7,JX+1)-OLDV(JY+1,JX-1)-OLDV(JY-1,JX+1)+OLDV(JY-1,JX-1)))-(CRATE*OB
     8(IM)*DISINC)+(R(JY)*TD*DISINC)
      OLDV(JY,JX)=NEWV(JY,JX)
      NEWD(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
     1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDD(JY,JX))+((.5*HJX2(JX)*NIU
     22*OMEG2*XJX4(JX)*KOVH2)*(OLDD(JY,JX+1)+OLDD(JY,JX-1)))+(HJX2(JX)*N
     3IU2*OMEG2*XJX3(JX)*KOVH*(OLDD(JY,JX+1)-OLDD(JY,JX)))+((0.5*R(JY)
     4*SIG2*PSI2*YJY4(JY)*KOVL2)*(OLDD(JY+1,JX)+OLDD(JY-1,JX)))+(R(JY)*S
     5IG2*PSI2*YJY3(JY)*KOVL*(OLDD(JY,JX)-OLDD(JY-1,JX)))+(RHO*H(JX)*S
     6QRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA*XJX2(JX)*KOV4LH*(OLDD(JY+1
```

```
7,JX+1)-OLDD(JY+1,JX-1)-OLDD(JY-1,JX+1)+OLDD(JY-1,JX-1)))
 OLDD(JY,JX)=NEWD(JY,JX)
 NEWC(JY,JX)=MAX(0.0,(NEWA(JY)-OLDV(JY,JX)-OLDD(JY,JX)))
 OLDC(JY,JX)=NEWC(JY,JX)
 NEWI(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDI(JY,JX))+((.5*HJX2(JX)*NIU
22*OMEG2*XJX4(JX)*KOVH2)*(OLDI(JY,JX+1)+OLDI(JY,JX-1)))+(HJX2(JX)*N
3IU2*OMEG2*XJX3(JX)*KOVH*(OLDI(JY,JX+1)-OLDI(JY,JX)))+((0.5*R(JY)
4*SIG2*PSI2*YJY4(JY)*KOVL2)*(OLDI(JY+1,JX)+OLDI(JY-1,JX)))+(R(JY)*S
5IG2*PSI2*YJY3(JY)*KOVL*(OLDI(JY,JX)-OLDI(JY-1,JX)))+(RHO*H(JX)*S
6QRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA*XJX2(JX)*KOV4LH*(OLDI(JY+1
7,JX+1)-OLDI(JY+1,JX-1)-OLDI(JY-1,JX+1)+OLDI(JY-1,JX-1)))
 OLDI(JY,JX)=NEWI(JY,JX)
 NEWCOI(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2
1*PSI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDCOI(JY,JX))+((.5*HJX2(JX)
2*NIU2*OMEG2*XJX4(JX)*KOVH2)*(OLDCOI(JY,JX+1)+OLDCOI(JY,JX-1)))+(HJ
3X2(JX)*NIU2*OMEG2*XJX3(JX)*KOVH*(OLDCOI(JY,JX+1)-OLDCOI(JY,JX)))
4+((0.5*R(JY)*SIG2*PSI2*YJY4(JY)*KOVL2)*(OLDCOI(JY+1,JX)+OLDCOI(JY-
51,JX)))+(R(JY)*SIG2*PSI2*YJY3(JY)*KOVL*(OLDCOI(JY,JX)-OLDCOI(JY-
61,JX)))+(RHO*H(JX)*SQRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA*XJX2(J
7X)*KOV4LH*(OLDCOI(JY+1,JX+1)-OLDCOI(JY+1,JX-1)-OLDCOI(JY-1,JX+1)+O
8LDCOI(JY-1,JX-1)))
 OLDCOI(JY,JX)=NEWCOI(JY,JX)
 IJXCT = MAX(JX,IJXCT)
 IJYCT = MIN(JY,IJYCT)
 ISTATE = 2
 ELSE
 NEWV(JY,JX)=((1-(HJX2(JX)*NIU2*OMEG2*XJX4(JX)*KOVH2)-(R(JY)*SIG2*P
1SI2*YJY4(JY)*KOVL2)-(R(JY)*DISINC))*OLDV(JY,JX))+((.5*HJX2(JX)*NIU
22*OMEG2*XJX4(JX)*KOVH2)*(OLDV(JY,JX+1)+OLDV(JY,JX-1)))+(HJX2(JX)*N
3IU2*OMEG2*XJX3(JX)*KOVH*(OLDV(JY,JX+1)-OLDV(JY,JX)))+((0.5*R(JY)
4*SIG2*PSI2*YJY4(JY)*KOVL2)*(OLDV(JY+1,JX)+OLDV(JY-1,JX)))+(R(JY)*S
5IG2*PSI2*YJY3(JY)*KOVL*(OLDV(JY,JX)-OLDV(JY-1,JX)))+(RHO*H(JX)*S
6QRT(R(JY))*NIU*SIGMA*PSI*YJY2(JY)*OMEGA*XJX2(JX)*KOV4LH*(OLDV(JY+1
7,JX+1)-OLDV(JY+1,JX-1)-OLDV(JY-1,JX+1)+OLDV(JY-1,JX-1)))-(CRATE*OB
8(IM)*DISINC)+(R(JY)*TD*DISINC)
 OLDV(JY,JX) = NEWV(JY,JX)
 NEWD(JY,JX) = 0.0
 OLDD(JY,JX) = NEWD(JY,JX)
 NEWC(JY,JX) = MAX(0.0,(NEWA(JY)-OLDV(JY,JX)-OLDD(JY,JX)))
```

```
      OLDC(JY,JX) = NEWC(JY,JX)
      NEWI(JY,JX) = 0.0
      OLDI(JY,JX) = NEWI(JY,JX)
      NEWCOI(JY,JX) = 0.0
      OLDCOI(JY,JX) = NEWCOI(JY,JX)
      ISTATE = 2
      ENDIF
      OLDA(JY) = NEWA(JY)
 5501 CONTINUE
 5500 CONTINUE
      IF (IJYCT.LE.JYMAX) THEN
      DO 5700 JY=IJYCT,JYMAX-1
      OLDV(JY,JXMIN)=TD
      OLDC(JY,JXMIN)=MAX(0.0,(NEWA(JY)-OLDV(JY,JXMIN)-OLDD(JY,JXMIN)))
 5700 CONTINUE
      END IF
      IF (IJXCT.GT.JXMIN) THEN
      DO 5701 JX=JXMIN,IJXCT
      OLDV(JYMAX,JX)=TD
      OLDD(JYMAX,JX)=0.0
      OLDI(JYMAX,JX)=0.0
      OLDCOI(JYMAX,JX)=0.0
      OLDC(JYMAX,JX)=MAX(0.0,(NEWA(JYMAX)-OLDV(JYMAX,JX)-OLDD(JYMAX,JX))
     1)
 5701 CONTINUE
      END IF
C
      OLDA(JYMIN) = NEWA(JYMIN)
      OLDA(JYMAX) = NEWA(JYMAX)
C
 5000 CONTINUE
 4000 CONTINUE
C    ==============================================================
      VALUE2=OLDV(JYHALF,JXHALF)+OLDI(JYHALF,JXHALF)-((1.0-PVPN)*RLTV)
C
C     IF(ABS(VALUE2).LT.TOLVAL) THEN
C     GOTO 25
C     ELSE
C     ROOT0 = ROOT1
C     ROOT1 =ROOT2
```

```
C     VALUE0 = VALUE1
C     VALUE1 = VALUE2
C     ENDIF
C•10 CONTINUE
C
C  25 CONTINUE
C     RTSEC = ROOT2
      VSEC = VALUE2
C     ====================================================================
C     WRITE(6,5002)RTSEC
C5002 FORMAT(1X,'RTSEC ',F12.9)
      WRITE(6,223)VSEC
  223 FORMAT(1X,'VSEC ',F12.9)
C     ====================================================================
  997 CONTINUE
      WRITE(1,210)
  210 FORMAT(1X,'OLDA(JY): ')
      WRITE (1,211) (OLDA(JY),JY=JYMIN,JYMAX,JYINC)
  211 FORMAT(1X,51(F8.5,2X)/)
      DO 505 JX=JXMIN,JXMAX,JXINC
      WRITE(1,212)JX
  212 FORMAT(1X,'JX ',I5)
      WRITE (1,213)(OLDV(JY,JX),JY=JYMIN,JYMAX,JYINC)
  213 FORMAT(1X,51(F8.5,2X)/)
  505 CONTINUE
      DO 506 JX=JXMIN,JXMAX,JXINC
      WRITE(1,214)JX
  214 FORMAT(1X,'JX ',I5)
      WRITE (1,215)(OLDD(JY,JX),JY=JYMIN,JYMAX,JYINC)
  215 FORMAT(1X,51(F8.5,2X)/)
  506 CONTINUE
      DO 507 JX=JXMIN,JXMAX,JXINC
      WRITE(1,216)JX
  216 FORMAT(1X,'JX ',I5)
      WRITE (1,217)(OLDC(JY,JX),JY=JYMIN,JYMAX,JYINC)
  217 FORMAT(1X,51(F8.5,2X)/)
  507 CONTINUE
      DO 508 JX=JXMIN,JXMAX,JXINC
      WRITE(1,218)JX
  218 FORMAT(1X,'JX ',I5)
```

```
   WRITE (1,219)(OLDI(JY,JX),JY=JYMIN,JYMAX,JYINC)
219 FORMAT(1X,51(F8.5,2X)/)
508 CONTINUE
   DO 509 JX=JXMIN,JXMAX,JXINC
   WRITE(1,220)JX
220 FORMAT(1X,'JX ',I5)
   WRITE (1,221)(OLDCOI(JY,JX),JY=JYMIN,JYMAX,JYINC)
221 FORMAT(1X,51(F8.5,2X)/)
509 CONTINUE
   call clock@(finish)
   write(1,809)atime,adate
809 format(1x,'Starting time was: ',a8,2x,'On ',a8,'(Day/Month/Year)')
   cputime = finish - start
   write(*,999)cputime
999 format(1x,'C.P.U. time used: ',f12.2,' second(s)')
   print *,'At ',time@(),'  On ',edate@()
   WRITE(6,222)OLDA(JYMAX)
222 FORMAT(1X,'OLDA(JYMAX) ',F12.9)
   call beep@
   call beep@
   end□
```

CHAPTER 5

Analysis of the Mortgage Valuation Models Results

5.1. Introduction

In contrast with the securities that are traded on Stock Exchanges or on developed over-the-counter markets, mortgages are far away from being a standardised product. The characteristics of the contracts vary, not only between lenders, but also within each lending institution. Hence, there are a large number of contract specifications which could have been tested. However, with the computer processing power available, an extremely long period time is required in order to generate a complete set of numerical results capable of providing a reasonable picture of the evolution of the valuation functions inherent to each one of the models developed in Chapter 3, under different economic conditions. Therefore, it has been necessary to restrict the number of cases presented to only a few.

A decision was made to present three different contract specifications. Two correspond to repayment mortgages and the other to an endowment. The repayment mortgages differ in accordance with the nature of the corresponding negative incentives to early termination. One includes only an arrangement fee and the other includes simultaneously a (lower) arrangement fee and an early termination penalty. The former is consequently closer to the structure of American mortgages, but, in accordance with what seems to be the rule in the UK, the level of the arrangement fee is lower than the level of "points" that seem to be commonly charged in the US[40].

As mentioned in Chapter 2, the endowment mortgage is a typically British product, without a close counterpart in the US. Therefore, at this level, it was decided to present only a contract based on the common specification proposed for this product by British banks and building societies: a mortgage contract including both an arrangement fee and an early termination penalty.

The type and level of the negative incentives to early termination exert a very significant role in terms of the determination of the equilibrium contract rates, and consequently in terms of the valuation of all mortgage related assets. Later on, special attention will also be given to the value of the insurance coverage that constitutes another crucial factor in the determination of equilibrium combinations.

One point which must be emphasised immediately is the inadequacy of the models designed to value American mortgages adapted to price British mortgage products and vice-versa. The reasons for this are twofold. In the first place, the amount of the arrangement fee ("points" in the US) seems to differ significantly from country to country. In the second place, the insurance coverage that is associated with both products is also different. In the American case the coverage seems to be a simple pre-

[40] There is no published empirical work capable of providing an unquestionable proof that this is the truth. However, contacts made with mortgage and insurance specialists in the UK lead to the conclusion that this is an opinion generally shared among industry professionals.

defined percentage of the value of the debt (see, for instance, Kau et al. 1993a). In the British case, the loss coverage is shared between the insurer and the lender, but the liability of the former is capped to a pre-defined amount that is a function of the difference between the LTV ratio of the loan and an arbitrarily defined "normal" LTV ratio (usually 75%). Obviously, the values of the two products do not coincide.

Both the above mentioned features have direct implications for the determination of the equilibrium contract rates (see equation 3.26). Consequently, even for two mortgages that coincide in every detail except those two, the contract rates would differ and the same would happen with the values of all the underlying assets. In addition, British mortgages also have early termination penalties that affect the exercise of the options by the borrower and consequently the equilibrium contract rates. In conclusion, the comparison between results extracted from models aimed at being used in both of these two countries lacks economic soundness.

Another important point that it is necessary to highlight is the inadequacy of any simple contract rate comparison between repayment and endowment mortgages. Straightforward comparisons between these contracts are meaningless because the basis for the determination of the payments is completely different from one case to the other (see equations 3.9 and 3.10). For instance, a higher contract rate in the case of an endowment mortgage does not necessarily imply a higher monthly payment than that for a repayment mortgage with a lower contract rate. In the former case the monthly payment is the result of a simple multiplication of the contract rate by the amount of the loan and in the latter the monthly instalment includes simultaneously interest and principal (see equation 3.9). Even an analysis of the differences between the values of the monthly payments in both cases would not necessarily lead us to sensible conclusions. Under the framework considered in this thesis, the contracts are in equilibrium, which means that they represent fair deals. Thus, if in the particular case of a certain contract, the borrower pays more than what he would have paid under a different agreement, this is not taken to imply that the former is economically dominated by the latter but that the borrower receives more value in other components of the contract - the options to terminate the loan. Otherwise it would not be possible for both contracts to represent fair deals.

The main aim of this chapter consists of the presentation and discussion of the numerical results provided by the models that were presented in Chapter 3. In order to do that, a basic set of economic parameters was taken into account. The choice was made mainly on the basis of the standard assumptions in the literature (see Buser and Hendershott, 1984; Dunn and McConnell, 1981a,b; Kau et al. 1993b, 1995; Leung and Sirmans, 1990; Stanton, 1995; Stanton and Wallace, 1995b). Table 5.1 presents these values. Unless otherwise is specifically mentioned, this set of economic parameters is assumed to hold.

As all the parameters are fixed at the moment when each solution is achieved, the only mechanism through which it is possible to search for an equilibrium situation is a change in the contract rate. Consequently, under normal circumstances, several attempts were made before reaching a situation where equation 3.26 holds.

Every parameter exercises a double influence on the value of the different mortgage related assets and, consequently, on the value of the mortgage itself. Any change in a parameter used to characterise the economic environment leads to a change in the equilibrium contract rate. Consequently, besides the direct implications that derive from the change of the parameters, the value of each of the mortgage related assets is also influenced by the modification that takes place in the contract rate. This phenomenon has severe implications for the type of empirical work that can be done in the field. Complete repetition of a certain set economic conditions is improbable and consequently, we are faced with different equilibrium rates for similar contracts in different moments in time. A sound empirical test of the implications of changes in the economic environment in terms of the different components of a mortgage contract would consist of observing the evolution of the market value of a mortgage contract during its economic life. In other words, a study of this kind would imply the analysis of the resale market for old mortgages (see Kau et al., 1992, 1995). Unfortunately, no such data seem to be available in the UK.

This chapter is organised as follows: Section 2 provides a graphical representation of the different options inside the grid that is used to represent the state space. The third section provides a brief overview of the valuation functions for each one of the variables under study and a series of figures indicating the behaviour of these functions for different levels of the state variables at the start of the contract. The fourth section presents the framework in which the equilibrium mortgage rates are determined and discusses the results for different types of contracts. Section 5 develops an analysis of the effects induced by changes in the parameters that are used to characterise the economic environment, in terms of the valuation of the assets under study. Section 6 presents an analysis of the consequences induced by changes in the parameters of the contracts in terms of the valuation functions. Section 7 concludes the chapter.

5.2. A Graphical Synthesis of the Structure of the Solution

In order to facilitate an understanding of the way in which numerical results are generated, a general discussion will be given of the evolution of the grid used to represent the state space. Figures 5.1.A to 5.1.D provide a graphical synthesis of the numerical approach used in this work. They are inspired by similar figures presented in Kau et al. (1992) and have been based on the repayment mortgage contract. The state space, ($H \times r$), is compacted into a square, and each figure represents a certain moment in time. Since the solution moves backwards in time, this series of figures proceeds in that order.

Before starting the analysis of the figures it is convenient to spend a few lines redefining the representation of the main variables in order to clarify the presentation. Dropping the corresponding subscripts, the valuation functions will be normally symbolised in this chapter in the following way:

V_B = Value of the mortgage for the borrower;

A = Value of the remaining mortgage payments;

C = Value of the call option to prepay the mortgage;

D = Value of the option to default on the mortgage;

I = Value of the MIG;

CI = Value of the coinsurance.

At the final moment in the life of a mortgage it only makes sense for the borrower to make the last monthly payment if the value of the house, *H*, is higher than the amount to be paid, *MP*. Otherwise, the borrower will default. Figure 5.1.A illustrates this situation. The shadowed column in the left of the picture represents the default region. If the default is not in the borrower's best interests, his only alternative consists in paying. At termination it is not possible for the borrower to prepay, because the loan has reached the last moment in its life. Therefore, there are only two regions inside the state space: the default region, which leads the borrower to exchange the value of the last payment for the house, and the continuation region which implies that the last payment is made.

Figure 5.1.B corresponds to a completely different situation: a moment that is located between payment dates. In this case, it is the default region that does not exist, since, as explained in the third chapter, default only makes sense at a payment date. Otherwise, the borrower loses the service flow for a period of time during which he could enjoy it for free.

The shadowed area, in Figure 5.1.B, corresponds to the prepayment region. Prepayment only becomes a reasonable option if the borrower faces simultaneously high house prices and low interest rates. At high house price levels, the default option becomes less valuable and consequently, the relative attraction of the alternative early termination option (the prepayment option) increases for higher interest rate levels. This is equivalent to saying that the prepayment boundary (free boundary) is positively sloped, given the borrower's predisposition to prepay at higher interest rates when the house prices reach high levels. The prepayment region is bounded in the r dimension by the point that corresponds to $A(r,t) = TD(t)$. If the interest rate reaches values below this level, prepayment is "in the money" (Kau et al., 1992). However, it is important to note that prepayment does not necessarily need to take place immediately. The exercise of the call option to prepay the loan renders the put option to default valueless and a rational borrower takes this fact into consideration. The prepayment region is also bounded in the H dimension by the point in which $H = TD(t)$, since this is a "border" line corresponding to a point in which the appeal of default is equivalent to the appeal of prepayment[41].

Figure 5.2.C illustrates a situation corresponding to an intermediate payment date in the life of the loan. At payment dates, default makes economic sense and so the figure includes simultaneously a prepayment and a default region. The dividing line between both corresponds to the point in which $H = TD(t)$, in which prepayment is as attractive as default, in financial terms. The value r^* corresponds to the highest level of r observed along this boundary. This means that for values of r less than r^*, the loan is automatically terminated either through default (when $H < TD$) or through prepayment (when $H > TD$).

For default to be "in the money", it is necessary that the value of the house, H, is less than the present value of the future mortgage payments, A. However, this is not sufficient for default to take place. As happens with prepayment, exercise of the default option implies the immediate loss of the call option to prepay and, consequently, default tends not to happen immediately. Another important point to note at this level is that higher interest rates reduce the cost of maintaining the loan and lead to default being advantageous only at low levels of H. Consequently, the boundary between the default and the continuation regions is negatively sloped.

[41] Note, however, that default will never take place at this moment in time, independently of the value of the house.

Finally, Figure 5.1.D portrays the situation at the origination of the loan. As in Figure 5.1.B, there is no default region because the moment does not correspond to a payment date. The original combination of state variables corresponds to *[H(0), r(0)]*. If a sudden rise in *H* takes place immediately after the contract is signed, the value of the default option (in future payment dates) also suffers an abrupt decline, eventually leading to a situation in which the contract rate is too high and in which prepayment might become the best alternative to the borrower. An immediate drop in interest rates might likewise lead the borrower to prepay and the mortgage value to decline to the value of the outstanding debt, *TD*.

5.3. Overview of the Different Valuation Functions

Figures 5.2.A to 5.7.C present a graphical image of the value assumed by the valuation functions associated with all the contract specifications under study along the state space at the origination of each contract. All assets are valued in relation to the par value of the house. In each case, the figures were based on a block of results corresponding to *(41x41)* nodes situated in the centre of the corresponding grid, whose dimension is *(51x51)*.

There are two main reasons for displaying these figures: in the first place, they show that the solutions evolve across the state space in a smooth way without any sort of instabilities. In the second place, they indicate that the shape of the graphs makes sense in economic terms.

As would be expected, the main characteristics and the general shape of the valuation functions do not change substantially between contracts. The differences are in the details.

The variable A depends only on r and, therefore, the value of the function is constant along the H axis. As would be expected, A shows an inverse relationship with r. The different equilibrium contract rates associated with each one of the mortgages are the cause of a divergence in the value of the mortgage payments that generates the small divergence in the value of the function across contracts (Figures 5.2.A, 5.2.B and 5.2.C).

The value of the mortgage, V_B, is a complex function of the value of A, C and D. Low levels of house prices tend to increase the value of the default option held by the borrower and consequently to reduce the value of the mortgage contract.

The evolution of interest rates impacts more directly A and C. The effect of the unfolding of these functions produces opposite repercussions on the value of V_B. In the first case the relationship is direct. In contrast, the prepayment option is inversely related to the value of the mortgage. Obviously, C cannot be bigger than A and normally is substantially smaller. Therefore, the relationship between the evolution of the interest rates and the value of the mortgage contract tends to be dictated by the effect caused by this evolution in terms of A. One exception to this principle occurs when the conjugation of low interest rates and high house prices lead to situations in which it becomes preferable for the borrower to prepay the loan. This type of case corresponds to the top section of the graphs presented in Figures 5.3.A, 5.3.B and 5.3.C. As the figures represent values at the origination of the contract, in the case of the repayment mortgage without prepayment penalty (Figure 5.3.A), the prepayment region constitutes a plane levelled at the original value of the loan. In the case of the repayment mortgage with arrangement fee and prepayment penalty (Figure 5.3.B) this surface is again a plane but now levelled slightly higher as a consequence of the inclusion of this latter feature. Finally, in the case of the endowment mortgage (Figure 5.3.C), the top section has a slender concave shape that is not completely evident on the graph (Figure 5.3.D presents a more explicit illustration of this phenomenon in a two dimensional space). This shape is due to the increase in the value of the endowment portfolio that takes place at low levels of the interest rate. In these circumstances the cost of the repayment can be less than the amount borrowed[42].

Another exception to the principle according to which the relationship between interest rates and mortgage values tends to be dictated by the influence of interest rates in the value of the future payments occurs when house prices assume very low levels. Under these circumstances, default is certain to occur at the next payment date and the level of the house price exerts a major influence in the value of the mortgage contract.

Figures 5.4.A, 5.4.B and 5.4.C portray the value of the default option, D. The major source of influence on the value of this option is the relationship between the level of H and the value of the mortgage contract. The value of D is positive in almost all of the subset of state space in which $H < H(0)$. As the increase in the level of r leads to decreases in the value of A and V_B, the value of the default option, whenever positive, tends to be inversely related to r.

[42] Obviously, these situations cannot correspond to equilibrium contracts. Otherwise, it would be possible for the borrower to make an arbitrage profit through the immediate repayment of the mortgage.

Figures 5.5.A to C present the value of the prepayment option, C. The primary determinant of its value is the level of the interest rate. As can be observed, the function only assumes high values for low levels of r that coincide with high levels of house prices. This happens because low house prices tend to generate default and, of course, a defaulted mortgage cannot be prepaid.

Figures 5.6.A to C and 5.7.A to C present the values assumed by the insurance related variables, I and CI along the state space. Both are directly related to the evolution of the default option. At high interest rate levels, the value of the contract is significantly reduced and consequently it is necessary for the house prices to fall to very low levels for the borrower to default and the insurance policy to be exercised. The value of the insurance coverage is capped. Consequently, for low levels of H, the function reaches its maximum level quite quickly. As expected, the coinsurance assumes higher values only after that level is surpassed.

5.4. Equilibrium Contract Rates at the Origination of the Mortgages

Mortgage rates and contract provisions vary widely over time. The economic environment changes continuously and the contract specifications are also subject to frequent readjustment, not only in order to accommodate those changes but also for marketing reasons.

In the equilibrium framework proposed in this thesis, independently of the specific conditions that are present at each moment, a contract can only be acceptable if it represents a fair deal. In order for two economic agents to trade assets freely, it is necessary that neither is able to make any "a priori" profit. In other terms, it is necessary to assure that the borrower is not able to make an instantaneous profit by prepaying the loan at origination and, similarly that the contract is not structured in such a way that allows the lender to make any immediate profit. This is a condition of no arbitrage.

The values assumed by the two state variables considered in the model, *r(0)* and *H(0)*, are known at the origination of each mortgage. Consequently, the identification of an equilibrium contract is an iterative exercise in which, starting with these initial values for the state variables and the functional form specification for the contract provisions, a search is done in order to find a contract rate capable of allowing the mortgage to meet the condition of no-arbitrage for each one of the parties involved[43].

The mortgage contract clauses that exert a more direct influence upon the possibility of early termination by the borrower and, subsequently upon value of the mortgage to the lender are the arrangement fee, ξ, the early termination penalty, π, and the Mortgage Indemnity Guarantee, *I*. Their influence will be analysed in this section.

Both the arrangement fee and the mortgage indemnity guarantee can be treated independently of the other components of the loan (the value of the future payments or the options to prepay or default) because neither of them affects the value of these components (e.g. "sunk costs" for the borrower).

The effects induced by these two types of provisions in terms of equilibrium contract rates are completely different. The inclusion of an arrangement fee only leads

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[43] This process is implemented using a secant iteration technique that iteratively searches for the root of the problem (contract rate) capable of making the equilibrium condition hold. The margin of error is within a £ 10 margin on a £ 100,000 house - the iteration process only stops for errors inferior to 0.0001 of the house value.

to a linear increase in the value of the lender's position in the contract. The influence of the insurance component is not so straightforward. Not only does its value evolve in different ways according to the nature of the underlying contracts but also it does not always evolve in a linear way inside the same contract for different levels of the contract rate (this aspect will be explored with more detail in section 5.4.3).

Rearranging equation 3.26 it is possible to conclude that, at origination, any values c, ξ, π that allow for the next equation to become true correspond to an equilibrium contract:

$$V_B(c, \pi) - (1-\xi)L + I(c,\pi) = 0 \qquad (5.1)$$

5.4.1. Basic Version of the Mortgage Contract

The first configuration of the mortgage contract to be analysed will be a repayment mortgage in which the mortgage indemnity guarantee, the arrangement fee and the early termination penalty do not exist.

Under these circumstances, equation (5.1) will be written as

$$V_B(c) - L = 0 \qquad (5.2)$$

In order for this sort of contract to be viable it would be necessary to reach a situation in which the value of the mortgage to the borrower, V_B, would equal the amount lent, L. As Kau et al. (1995) correctly point out, for this to happen it is necessary that the prepayment region expands in such a way that $(H(0), r(0))$ becomes situated in the prepayment boundary (free-boundary), and immediate prepayment constitutes a possible optimal strategy for the borrower. Table 5.2.A (column V_B - L) and the corresponding Figure 5.8.A illustrate this sort of situation. In this case, the borrower faces a situation of indifference between the alternatives of continuation and immediate repayment. Any increase in the contract rate, c, that corresponds to this initial equilibrium situation generates a peculiar effect. It results in higher present value for future payments to be made by the borrower, but at the same time it also increases the value of the option of early repayment, C, by a similar amount. As these are compensating effects, the borrower repays the loan immediately after taking it. Consequently, despite the possibility of finding contract rates capable of generating fair deals for both borrower and lender, no equilibrium exists, because those contract rates correspond to situations in which the mortgage is immediately terminated.

Adding a prepayment penalty to the mortgage contract, equation (5.2) will be rewritten as:

$$V_B(c,\pi) - L = 0 \qquad (5.3)$$

This particular situation is considered in Table 5.2.B (column V_B - L) and the corresponding Figure 5.8.B. As can be observed, instead of facing an artificial situation of continuous equilibrium, there is only one level of c capable of generating an equilibrium contract.

Finally, considering a fixed rate endowment mortgage that includes a prepayment penalty but no mortgage indemnity guarantee will lead to a similar situation. Table

5.2.C (column V_B - L) and the corresponding Figure 5.8.C provide results for this type of contract.

The introduction of a prepayment penalty allows for the rectification of the "anomalous" situation of artificial equilibrium that exists in the absence of any negative incentive to early termination. As a result of the introduction of this new feature, the borrower faces an additional cost that is translated into an upward move of the line representing the value of the mortgage for the lender in Figures 5.8.B and 5.8.C. The consequence of this move is that there is a single combination of c (d, in the case of an endowment mortgage) and π capable of attaining equilibrium (capable of leading the corresponding function to reach a value of zero).

It is also noteworthy that if the loan to value ratio were 100%, there would be another reason to make the attainment of equilibrium combinations impossible. In that case $L = H$, and consequently equation (5.2) assumes the following form:

$$V_B(c) - H = 0 \qquad (5.4)$$

In this situation, a rational borrower would become indifferent between default and continuation. In spite of that, it is obvious that by defaulting the borrower would lose the service flow of the house from which he could benefit until the first payment was due, if his decision to default were delayed until that moment. Therefore, there can be no contract rate capable of making (5.4) hold. The common imposition of LTV ratios below unity constitutes one of the instruments that helps equilibrium mortgage contract combinations to be reached (for an argument along the same lines, see Kau et al., 1995).

5.4.2. Mortgage Contracts with the Inclusion of Arrangement Fees

The inclusion of an arrangement fee in equation (5.1) will lead to the following result:

$$V_B(c) - (1-\xi)L = 0 \qquad (5.5)$$

Similarly, adding the same arrangement fee to equation (5.3) will result in the following equation:

$$V_B(c,\pi) - (1-\xi)L = 0 \qquad (5.6)$$

In any case the curves represented in Figures 5.8.A, 5.8.B and 5.8.C will shift vertically by the amount of the arrangement fee, ξL. The effect caused by the introduction of the arrangement fee is qualitatively similar to that induced by the early termination penalty. Both constitute costs to the borrower and, consequently, reduce the value that he attributes to the mortgage contract. However, the effects generated by the arrangement fee are more linear, since its amount is fixed and the amount of the early repayment penalty varies in time and space.

This similarity of effects can be observed in the basic version of the mortgage contract: in contrast with the "artificial" continuous equilibrium that takes place when no early termination penalty is considered (see column V_B - L in Table 5.2.A and the corresponding Figure 5.8.A), a single equilibrium combination is reached, in the case of a contract that does not include an early termination penalty, as a result of the change in the value of the mortgage deal produced by the inclusion of an arrangement fee (see column V_B - $(1-\xi)L$ in Table 5.2.A and the corresponding Figure 5.9.A). Tables 5.2.B and 5.2.C (column V_B - $(1-\xi)L$) and the corresponding Figures 5.9.B and 5.9.C illustrate the situation in which both negative incentives to early termination are present, in repayment and endowment mortgage contracts, respectively. As expected, in these cases it is also possible to find single equilibrium combinations.

5.4.3. Mortgage Contracts with the Inclusion of Mortgage Indemnity Guarantees

The effects induced by the inclusion of a mortgage indemnity guarantee are different. Tables 5.3.A and 5.3.B (columns V_B - $L + I$) and the corresponding Figures 5.12.A and 5.12.B give a first insight about the nature of this peculiarity. All these tables and figures are based on versions of repayment and endowment mortgage contracts that do not include arrangement fees or early termination penalties. The line that represents the global value of the lender's position in the deal has a peculiar shape in both the repayment and the endowment contracts (see Figures 5.12.A[44] and 5.12.B).

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[44] Figure 5.12.A corresponds to a detailed version of Figure 5.10.A, particularly conceived with the aim of showing the peculiarities induced by the introduction of MIGs, in terms of the evolution of the repayment mortgage contract rates.

After a first equilibrium is reached, the value of the lender's position in the deal becomes positive for a while and afterwards declines until a situation of continuous equilibrium is reached. The main reason for this is the singular relationship that exists between the MIG and the contract rate. The evolution of the MIG is clearly non-monotone. This phenomenon is closely related to the evolution of the value of the default option, D, whose behaviour not only changes from contract to contract but also changes inside the same contract for different levels of c (d, in the endowment mortgage case) and different areas of the state space.

For a repayment or an endowment mortgage without any prepayment penalty or arrangement fee, initial increases in the contract rate tend to be translated into increases in the value of the MIG, since the value of A becomes higher and subsequently the probability of default increases. However, similar changes at higher levels of the contract rate, c (or d), tend to reduce the value of I because they increase the likelihood of an early repayment by the borrower (see Tables 5.3.A and 5.3.B). Therefore, the situation of continuous equilibrium that is reached for high levels of c (d, in the case of an endowment mortgage) corresponds to a case in which the prepayment region has expanded in such a way that includes the original contract rate. Under these circumstances, insurance has no value and, consequently, prepayment constitutes an optimal course of action for the borrower. In contrast, when the first (isolated) equilibrium situation is reached, I has a positive value and, therefore, the contract rate is outside the prepayment region, which means that this case corresponds to a viable equilibrium combination.

If, alternatively, we consider repayment and endowment mortgage contracts that include early termination penalties and MIGs but do not incorporate arrangement fees, we face a qualitatively different situation. The addition of the early termination penalty, with its inherent benefits for the lender, overshadows the effects induced by the evolution of the contract rate in the value of the MIG, leading to situations in which unique equilibrium combinations exist (see column $V_B - L + I$ in Tables 5.2.B and 5.2.C and the corresponding Figures 5.10.B and 5.10.C).

5.4.4. Full Mortgage Contract

The case where all the common mortgage contract features are simultaneously considered is illustrated by column $V_B - (1-\xi)L + I$ in Tables 5.2.A to C and the corresponding Figures 5.11.A to C. As expected, the simultaneous consideration of arrangement fees, early termination penalties and MIGs always leads to situations of unique equilibrium combinations. All those contractual features generate a net benefit to the lender and, consequently, the equilibrium combinations are now reached at slightly lower levels of the contract rate.

Before moving to other subjects, it is desirable to analyse in more detail the relationship between the contract rate, the negative incentives to early termination and the MIG. The lender can manipulate the combinations of the latter contractual features in order to reach different contract rates for loans that are equally attractive in economic terms. This is in some way reflected in the large panoply of contractual specifications that sometimes can be found for the same type of contract from the major mortgage providers in the British market (magazines specialising in mortgage lending provide a large number of examples of this phenomenon).

Assuming a full mortgage contract with the arrangement fee as the only negative incentive to early repayment, the equilibrium contract rate will be a function of the level assumed by this contractual feature. A curve will exist in a (c, ξ) space mapping all the equilibrium combinations[45]. Figures 5.13.A and 5.13.B give an idea of the evolution of the corresponding curve, in the case of repayment and endowment mortgages, respectively.

The consideration of the prepayment penalty, π, adds another dimension to the puzzle. Tables 5.4.A and 5.4.B report equilibrium contract rates for repayment and endowment mortgages under different combinations of arrangement fees and prepayment penalties. Considering both negative incentives to early redemption, the equilibrium combinations constitute a surface in the (c, ξ, π) space. Figures 5.14.A and 5.14.B illustrate this fact and give an intuitive idea of the trade-offs that exist between the contract rate, the arrangement fee and the early termination penalty.

5.5. Effects Induced by Changes in the Economic Environment

As noted in the first section of this chapter, the economic environment is characterised in the present work through the set of parameters given in Table 5.1. This section presents an analysis of the effects induced by changes in these parameters.

[45] In the case of an endowment mortgage, in a (d, ξ) space.

5.5.1. Volatility of the State Variables

In order to examine the effects of the risk created by changes in the state variables a series of numerical results will be presented in which, for a previously determined equilibrium contract rate, the risk parameters will be changed. This will allow an examination of the partial effects induced by changes in the volatilities of both state variables. In a second stage, the global effect will be analysed based on equilibrium values for different combinations of σ and ν.

5.5.1.1. Interest Rate Volatility

Interest rate volatility affects the values of all mortgage-related assets that are subject to evaluation in this work. Tables 5.5.A to C and the corresponding Figures 5.15.A to C present the simulations that were performed in order to analyse this subject for each one of the contracts under study.

In the first place, the value of the future mortgage payments increases with the increase of interest rate volatility. This apparently unanticipated relationship is caused by the nature of the process that underlies the valuation of expected cash-flows. Two changes of the same magnitude (in absolute terms), but opposite signs, in the value of the discount rate that will be used to determine the present value of a future cash-flow, will result in changes of different magnitude in the present value itself. The gains observed in case of a fall in the discount rate will surpass the losses generated by an increase of the same magnitude in the discount rate. In other words, the additional likelihood of the interest rate attaining unexpectedly high and low levels, which emanates from an increment in σ, results in an increase in the expected value of future payments due to Jensen's inequality. Obviously, this type of effect is not altered in any way by the type of contract that is under scrutiny and so the differences in the value of A that can be observed between Tables 5.5.A, 5.5.B and 5.5.C are dictated by the different contract rates that are used in each case.

All the other mortgage-related assets that are subject to evaluation in this work represent present values of future cash-flows and are directly or indirectly affected by A. Consequently, this type of effect is relevant for all of them. However, its importance will not always be decisive in terms of the final relationship between the evolution of the variable and the evolution of σ.

In the case of the early repayment option, C, an increase in σ, will tend to increase the likelihood that the contract will reach the prepayment region instead of reaching the continuation and the default regions. The size of each of these regions can also change as a result of the change in the interest rate. Additionally, C is also a function of A. Therefore, as a result of all these influences, the value of the prepayment option tends naturally to move in direct relationship with the changes in σ (see Figures 5.15.A to C).

The impact of σ in the relative size of the different regions inside the grid favours a negative relationship between σ and D. However, the value of the default option is directly related to the value of A (see equation 3.6). This factor leads the impact of the changes in C in terms of D to be overturned. As we can see in Tables 5.5.A to C and on Tables 5.15.A to C, the value of the default option tends to move in direct relationship with σ independently of the type of contract.

The insurance related mortgage assets I and CI constitute a clear example of the fact that the Jensen's inequality effect is not always capable of dictating the relationship between the present values of expected cash-flows and the evolution of the interest rate volatility. In both cases the values of the assets tend to move in inverse relation to the evolution of the interest rate volatility. For this to happen something must overturn the effect generated by Jensen's inequality. Increases in interest rate volatility tend to be translated into higher levels of prepayment and consequently a smaller number of default paths. Additionally, in both repayment contracts considered here, in contrast with D and C, I and CI do not benefit from the evolution in the value of A, since the values of the loss and the corresponding indemnity depend on the outstanding balance. This is almost entirely determined by the value of the unpaid principal and not by the value of the future payments, A. In the endowment mortgage contract the situation is different, since the value of the outstanding debt is inversely related to the value of the zero coupon bond portfolio, whose value is directly related to σ. However, the corresponding effects are largely restricted by the high levels of the early repayment penalties associated with the endowment mortgage contract and are clearly dominated by the effects induced by σ in terms of probability of the contract to reach the prepayment region. Consequently, both variables, I and CI, tend to hold a negative relationship with σ in all the contracts under study.

As a result of all those influences, the value of the lender's position also evolves in inverse relationship with σ, for any of the three contracts studied in this work. There are two main reasons for this: in the first place, the positive effect on the value of A is more than offset by the evolution of the joint option to terminate the loan $(C+D)$ - the value of V_B, which is the overall result of the progress of these variables, evolves in a negative relationship with σ. In the second place, the value of the MIG, I, also presents a negative relationship with σ, providing an additional incentive to the final nature of the overall result.

5.5.1.2. House Price Volatility

The partial effects induced by changes in house price volatility are significantly different from those generated by interest rate variation. The first reason for this is related to the valuation of the future payments, A. As was mentioned in the previous chapters, the valuation of A obeys a degenerated version of equation (3.4) in which the only state variable is r. This means that the valuation of A is completely independent of H and consequently its evolution is not constrained in any way by the corresponding volatility, ν.

Under these circumstances, the relationship between ν and V_B is dictated by the way in which the joint option to terminate the mortgage, $(C+D)$, relates to the house price variation. In the case of default, D, the numerical results for all the three contracts under scrutiny point towards a strong direct relationship between this variable and ν (see column D in Tables 5.6.A to C and Figures 5.16.A to C). This is explained by the fact that increases in ν tend to create a relatively higher likelihood for the contract to reach the default region instead of the prepayment and the continuation regions. A direct result of this phenomenon is the tendency for C to decline with increases in ν (see column C in Tables 5.6.A to C and Figures 5.16.A to C). Therefore, changes in ν tend to produce opposite effects in the components of the joint option to terminate the loan. According to our numerical results, V_B presents an inverse relationship with ν. Thus, the effects of the evolution of D tend to dominate the effects of the evolution of C at this level.

In contrast with the previous case, the insurance related variables, I and CI, present a direct relationship with v. The house price volatility impacts default much more directly than prepayment and the values of the insurance-related assets are a direct function of the probability of default and the expected amount of the corresponding losses. Consequently, the evolution of I and CI is very much in line with the evolution of D (see columns I and CI in Tables 5.6.A to C and Figures 5.16.A to C).

5.5.1.3. Global Effects Induced by the Volatilities of the State Variables

The total effects induced by changes in the volatility parameters are portrayed in Tables 5.7.A, 5.7.B and 5.7.C. An analysis of these tables, within each level of LTV ratio, gives an insight on the fundamental aspects at this level.

The effects induced by the interest rate volatility are more or less straightforward. Increases in σ lead to the growth of A and simultaneously tend to generate increases in D and C. However, as mentioned in the previous sub-section, I tends to move in the opposite direction of the movement in σ. Consequently, the overall result in terms of the position of the lender depends on the global effect induced in terms of the evolution of V_B (dictated by the relationship between the increases registered by A and those observed by D and C) and the evolution in I. Both of them tend to decrease with increases in σ. As a result, in these circumstances, in order to reach an equilibrium it is necessary to increase the contract rate and consequently the value of A to compensate for those declines in I and increases in $C+D$.

Changes in v involve complex relationships and make the analysis more intricate. Increases in house price volatility, tend to correspond to increases in D and decreases in C. The joint effect of these changes tends to be translated into a reduction of V_B. Simultaneously, I tends to increase with v. The magnitude of impact tends to be similar at both levels, with the change in I slightly surpassing that in V_B. In order to compensate for this small increase in the value of the lender's position, it is necessary to adjust the contract rate. Therefore, increases in v tend to be accompanied by slight reductions in the coupon rate.

Another relevant aspect, at this level, is related to the value of both options to terminate the loan. According to the numerical results presented in Tables 5.7.A to C, the value of the prepayment option tends to exceed the value of the default option, with the exception of the cases in which the LTV ratio reaches very high levels. This is in line with the results reported in Kau et al. (1995) for the American case.

5.5.2. Different Levels of Loan-to-Value Ratios

The analysis of effects induced by changes in the LTV ratio of the mortgage loans are based on the contents of Tables 5.7.A to C (vertical dimension).

For a certain house price, a rise in the LTV ratio corresponds to an increase in the amount of the loan. Therefore, A tends naturally to grow with the LTV ratio as a result of the increase in the amount of the loan.

A rise in the relation between the amount of the loan and the original value of the house tends to increase the probability of default, since in these circumstances it is easier to reach situations in which the value of the outstanding debt surpasses the value of the house. This effect is in some way counterbalanced by the evolution of the value of the prepayment option. The increase in the value of loan and, consequently, in the value of the outstanding debt at each moment in time also tends to raise the prospects of prepayment and consequently to contribute to a reduction in the size of the default area. However, according to the numerical results presented in Tables 5.7.A to C the former effect always dominates the latter, so that the value of the default option is positively related to the LTV.

In fact, the evolution of the value of the prepayment option is not so straightforward as the previous paragraph might suggest. For low LTV levels, C tends to evolve in direct relationship with the value of A (and the LTV itself). However, for higher levels of LTV, the probability of default and the default region seems to expand so dramatically that the value of the prepayment option declines.

As expected in this sort of circumstance, the insurance related variables, I and CI, follow the evolution of D, and consequently exhibit a direct relationship with the LTV ratio.

The overall effect generated by changes in the LTV ratio is dictated by the evolution of V_B relative to the evolution of the mortgage insurance policy, I. For low levels of LTV, the magnitude of the change tends to be slightly superior in I (positive effect) than in V_B (negative effect). Consequently, the contract rate tends also to be slightly reduced with the increase of the LTV ratio. In contrast, for higher levels of LTV, the sharp increase in the value of the default option leads the relative magnitude of those effects to change, causing a mild increase in the coupon rate.

It is important to emphasise that the relative magnitudes of the changes in the values of V_B and I are always similar, so the effects in terms of the coupon rate are minimal, especially in the case of the endowment mortgage contract.

5.5.3. Changes in the Relationship Between the Spot Rate and the Long Term Average of the Interest Rate Process

Tables 5.8.A to C try to capture the effects induced by different types of yield curves in terms of the value of the mortgage-related assets.

The approach used in this work to capture different yield curve shapes consisted of changing the initial level assumed by the spot interest rate, $r(0)$, while holding constant the steady state spot interest rate, θ, for all the runs that underlie the construction of the tables.

There are three main levels at which the effects induced by different types of yield curves can be analysed. In the first place, it is important to look at the impact produced by changes in the yield curve in cases in which all the other parameters are held constant. According to our numerical solutions, higher levels of $r(0)$ tend to be related to decreases in V_B and in I. Both these effects lead to the reduction of $V_B - (1-\xi)L + I$. As a consequence, in order to reach equilibrium, it is necessary for the coupon rate to attain higher levels. Therefore, there is a direct relationship between the evolution of the level of the initial spot rate and the contract rate for a fixed θ. This relationship helps to understand the rationale that lies behind the evolution of D. The default option, D, is negatively influenced by the increase in the size of the prepayment region that happens as a natural consequence of the consideration of higher spot interest rates. However, at the same time, an increase in the coupon rate also takes place, leading to the rise of A. As previously mentioned, D has a direct relationship with A. This effect overshadows the influence induced by the evolution of the prepayment region in terms of the value of D, and provides the rationale for the direct relationship between D and the initial level of the spot rate.

A further important aspect is the effect induced by increases in σ for different slopes of the yield curve. The increase in the interest rate volatility is directly related to the evolution of D and, especially, of C. Both of these contribute to the reduction in the value of the mortgage to the borrower, V_B. Besides this, it is also necessary to take into account that I is inversely related to σ. The overall effect induced by these changes is a significant reduction in the value of $V_B - (1-\xi)L + I$ as a result of increases in σ, for all types of yield curves and contracts under study. In order to reach equilibrium combinations, it is necessary to compensate for all these adverse effects in the value of the lender's position by increasing the contract rate. This means that, for higher levels of σ, the coupon rate tends to increase and normally to reach levels that are superior to θ. Even in cases where the original level of the spot interest rate was clearly below (8%) the long term average of the interest rate, the equilibrium contract rate is above θ, when $\sigma = 10\%$, for all contracts under study.

Finally, the last aspect to mention is the joint effect produced by increases in ν and changes in the slope of the yield curve. As noted in sub-section 5.5.1.3, the effects induced by increases in ν in terms of the evolution of D and C are of opposite nature and partially tend to compensate each other. Consequently, the influence of house price volatility in terms of the equilibrium contract rates is moderate. As the positive effect generated by increases in ν in terms of D tends to be of a higher magnitude than the combined effects produced in terms of C and I, the overall result tends to be translated into a slight reduction of the coupon rate as a result of the increase of ν, for all the yield curve environments under study.

5.5.4. Changes in the Reversion Coefficient

The level of the reversion coefficient plays a major role in the determination of the real variability of the interest rates that is assumed in the model. A high level of κ implies a quick return of the interest rate to its long term mean (steady state), θ, and consequently an effective level of $r(t)$ that tends to be close to θ even when the disturbance in the interest rate process, σ, is relatively high. In other words, with all the other parameters constant, an increase in κ has an effect equivalent to that of a reduction in the interest rate risk.

Consequently, the numerical results for all the contracts under scrutiny reveal that the equilibrium contract rate moves in the opposite direction to κ, independently of the nature of the term structure (see Tables 5.9.A, 5.9.B and 5.9.C). As a result, the value of the future payments, A, also moves in the reverse direction to the changes registered in κ.

The effects on D and C are more subtle. Both these variables represent contingent claims on the value of an asset whose main component is A. Consequently, the evolution of the contract rate represents a major factor in the explanation of their behaviour. The reductions in the contract rate that are associated with increases in κ and consequent reductions in A, exert a determinant influence towards the reduction of D and C. In the case of C, this trend is in some way reinforced by the natural reduction in the likelihood that the contract reaches the prepayment region as a consequence of decreases in the variability of interest rates associated with higher levels of κ.

For the repayment mortgage contracts, the insurance related variables, I and CI, tend to evolve in direct relation with κ and in inverse relation to D. As mentioned previously, in this type of contract, the amount of the insurance claims is not directly dependent on A but on the value of the outstanding debt, which is independent of $r(t)$. Consequently, when κ increases, the main factor in the explanation of their behaviour seems to be the increase in the likelihood of the contract being in the default region instead of the prepayment and continuation regions.

The results for the endowment mortgage contract are different. For the high levels of interest rate variability associated with a null κ, the MIG assumes high levels. The possibility of a "wild" oscillation in interest rates results, in the first place, in a high level of C. However, the possibility of this wide variation in interest rates also affects significantly the value of A and the likelihood of default, resulting in an high level of I and CI. Consequently, the initial reduction of interest rate variability associated with the increase of κ from 0 to 25% produces a decline in the value of the insurance related variables. For lower levels of interest rate variability, associated with higher κ, the effect of the negative incentives to prepayment becomes more efficient and the increase in the likelihood of the contract reaching the default region seems to become as significant here as in the case of the repayment mortgage. Consequently, I and CI tend to evolve in direct relationship with the reversion coefficient, for these higher levels of κ.

Finally, it is important to note the relationship between the evolution of the equilibrium contract rates and the different types of yield curve. As would be expected, given the previous remarks, the contract rates tend to increase in direct relation with the level of the spot rate, $r(0)$.

5.5.5. Changes in the Correlation Coefficient Between Both State Variables

Tables 5.10.A, 5.10.B and 5.10.C present the effects induced by a change in the correlation coefficient between both state variables, ρ. Increases in ρ tend to be translated into increases in the value of default. In the first place, this phenomenon implies a direct relationship between the evolution of r and H. This is equivalent to saying that when r is high default tends not to happen because of the low level of V_B. However, when r is low, the reduced levels of H will make default more likely. Besides this, when D is high the likelihood of prepayment tends to be lower. Therefore, both factors generate a tendency not only for D to move in direct relationship with ρ, but also for C to move in an opposite direction.

Under these circumstances, it is normal for the values of the MIG and the coinsurance to follow the trend registered by D and move in direct relationship with ρ.

The magnitude of the movement in D is greater than the one registered by C. Consequently, V_B tends to decrease. As the movements in I are minimal, this is equivalent to saying that, with an increase of ρ, $V_B-(1-\xi)L+I$ is reduced. In these circumstances, the contract rate needs to be increased in order for equilibrium to be reached.

5.5.6. Repayment Versus Endowment

In the introductory section of the present chapter, the problems associated with the comparisons between coupon rates from different types of mortgage contracts were mentioned. This sub-section tries to provide some degree of comparison between repayment and endowment mortgage contracts without incurring those problems.

Table 5.11 presents values of the different mortgage-related assets in proportion to the par value of the house for both the repayment and the endowment mortgage contracts (with arrangement fees and early termination penalties), given a relevant range of LTV ratios.

The more relevant aspect to mention is related to the relative weight of the value of the prepayment option, C in each contract.

The endowment mortgage contract that is modelled in the present work includes an early termination penalty that is clearly more punishing than the one included in the repayment mortgage contract under study. The main reason for this disparity is related to what seems to be a common practice in the British mortgage market aimed at reducing the value of the prepayment option held by endowment mortgage borrowers, in order to reach endowment coupon rates capable of competing with the repayment coupon rates. In spite of this disparity, in the endowment mortgage case, C always represents an higher proportion of the original value of the house. As the relative weights of the D and I do not differ significantly between contracts, it is necessary to find a counterbalancing effect capable of compensating for that difference and allowing the contract to reach its equilibrium. This is provided by an increase in the value of the monthly payments, A. Therefore, the main structural differences in terms of the relative weights of the different components of the repayment and endowment mortgage contracts are registered by A and C, whose relative weights tend to be higher in the endowment mortgage case.

5.6. *Effects Induced by Changes in Contract Specifications*

The level of the equilibrium coupon rate depends on the features that are included in the mortgage contract. Therefore, the value of the different mortgage-related assets is also dependent on the way in which these provisions are taken into account in the modelling exercise.

The present section addresses this phenomenon at three levels: the impact of the contract rate on the value of the lender's position, the relationship between coupon rates and arrangement fees, and the connection between the early termination penalties and coupon rates.

5.6.1. *The Relationship Between the Value of the Lender's Position and the Coupon Rate*

The main contractual features inherent to mortgage loans are the embedded options held by the borrower to terminate the contract and the MIG. The contractual specifications that can be achieved combining these features in different ways are summarised in Table 5.12[46]:

Table 5.12. Variants of the Mortgage Contract According to the Provisions Included

Contract	Formulation	Description of the Features
M_1	$A-(1-\xi)L$	Non-Defaultable and Non-Prepayable Mortgage Loan
M_2	$A+I-(1-\xi)L-D$	Insured Non-Prepayable Mortgage Loan
M_3	$A-(1-\xi)L-D$	Non-Insured Collateralized Mortgage Loan (Defaultable)
M_4	$A+I-(1-\xi)L-D-C$	Insured, Prepayable and Defaultable Mortgage Loan
M_5	$A-(1-\xi)L-C$	Non-Insured and Non-Defaultable Mortgage Loan (Prepayable)
M_6	$A-(1-\xi)L-C-D$	Non-Insured, Prepayable and Defaultable Mortgage Loan

Figures 5.17.A to C portray the relationship between the value of the lender's position and the contract rate for each one of these variants. The parameters that underlie those figures are the same that were used to generate the data presented in Tables 5.2.A to C.

[46] There are two additional combinations not considered in Table 5.12: *i)* insured, non-defaultable and non-prepayable mortgage loans and *ii)* insured, non-defaultable but prepayable mortgage loans. Neither of these variants is relevant, since the MIG only makes sense on contracts where the risk of default exists.

A first element to note is that the flat section of the curves that represent the value of the lender's position in contracts M_4, M_5 and M_6 corresponds to the region where the contract rate is so high that prepayment takes place immediately. In this case *D* and *I* are valueless, so M_4, M_5 and M_6 share the same value.

In the case of a repayment mortgage contract with arrangement fee and without early termination penalty, Figure 5.17.A clearly shows that, with the exception of that flat section, the equilibrium combinations always differ in accordance with the features that are included in the modelling exercise. Figures 5.17.B and 5.17.C reveal the same pattern in the case of repayment and endowment mortgage contracts that include simultaneously arrangement fees and early termination penalties.

Knowing that each contractual feature has a non-negative value, facilitates the study of the relationships between the values of the contracts. As *D* is not only non-negative but also not inferior to *I* (the value of the capped default insurance cannot be higher than the value of the option to default) it is possible to assert the following relationships:

$$M_3 \leq M_2 \leq M_1 \tag{5.7}$$

and

$$M_6 \leq M_4 \leq M_5 \tag{5.8}$$

For the particular group of parameters that underlie Figures 5.17.A to C (see Tables 5.2.A to C) the value of the prepayment option is always higher than the value of the default option ($C > D$). Consequently, for this specific set of parameters:

$$M_6 \leq M_4 \leq M_5 \leq M_3 \leq M_2 \leq M_1 \quad (5.9)$$

The change of the equilibrium combinations as a function of the contractual provisions included in the modelling process reveals that the common procedure of avoiding the explicit consideration of one of the embedded options to terminate the loan, in order to simplify the modelling process, may potentially lead to equivocal values for the equilibrium coupon rate, the different contract provisions and ultimately the whole mortgage contract. In conclusion, it is worthwhile to model the mortgage contract in a way that includes as many features as possible in order to get values for the mortgage-related assets capable of providing a reasonable picture of the idiosyncrasies of the contract.

5.6.2. The Relationship Between Arrangement Fees and Contract Rates

Figures 5.18.A to C illustrate the relationship between arrangement fees and contract rates for the three contracts under scrutiny. Each curve represents the level of the arrangement fee that is necessary to generate an equilibrium combination at a certain level of the coupon rate.

Figures 5.18.A to C can be understood as the result of a linear transformation of the data that underlie Figures 5.17.A to C in which the value of the lender's position is re-expressed in terms of the level of the arrangement fee that would be necessary to generate an equilibrium situation. Negative values of the lender's position need to be compensated by positive values of the arrangement fee and vice-versa. Consequently, as compared with expression (5.9), the relationships between the values of the different versions of the mortgage contract are now reversed:

$$M_6 \geq M_4 \geq M_5 \geq M_3 \geq M_2 \geq M_1 \quad (5.10)$$

A significant detail to note is that the distance between curves is not constant. As in some cases the distances between curves correspond to a single contractual feature (i.e. the value of default can be given either by $M_1 - M_3$ or by $M_5 - M_6$, alternatively the value of prepayment can be given by $M_3 - M_6$, $M_2 - M_4$ and $M_1 - M_5$), this means that the addition/exclusion of the underlying provision has different implications in terms of arrangement fee in accordance with the level of the contract rate.

Another important aspect to mention is that, with the exception of the areas of very low and very high coupon rates, the curves corresponding to the six different combinations of contractual features do not coincide. This is equivalent to saying that, for a certain level of arrangement fee, the equilibrium coupon rate diverges from case to case, as a function of the contractual provisions that are included in the mortgage contract.

In summary, once again, our numerical results suggest that mortgage modelling exercises should avoid the exclusion of contractual provisions in order to prevent misleading conclusions.

5.6.3. The Relationship Between Early Termination Penalties and Contract Rates

Figures 5.19.A and 5.19.B present the relationship between the market value of the early termination penalty and the coupon rate for repayment and endowment mortgage contracts, respectively.

As happens with the arrangement fee, the curves representing mortgages with different contractual features, diverge on significant areas of the state space. Additionally, the values of the different contracts continue to observe the relationships established in equation (5.10). In other words, the early redemption penalty affects the value of the mortgage contracts in a way somewhat similar to the arrangement fee, in spite of its different nature. Consequently, the conclusions on this respect are analogous to those that were extracted in the previous sub-section.

5.7. Conclusion

This chapter presents and analyses the numerical solutions of a fixed rate endowment mortgage contract and two versions of the fixed-rate repayment mortgage contract that differ in the content of the negative incentives to prepayment.

In the first place, an analysis of the equilibrium combinations for several versions of the three mortgage contracts under study was undertaken and the conditions and the type of evolution expected from equilibrium contract rates were determined. In the following section numerical solutions were found for the variables associated with each contract in order to analyse the partial and global effects induced by changes in the main parameters associated with the evolution of the economic environment. In spite of the complexity of the product and the intricate relationships that exist between the variables, it is possible to find a rationale for the results based on the underlying economic knowledge.

In the last section the implications induced by changes in the contract specifications were considered. The corresponding numerical results suggest that model specifications that exclude important mortgage contractual features might produce misleading results and consequently should be avoided or, at least, used cautiously.

Table 5.1. Base Values

LIST OF PARAMETERS	CONTRACT		
	Repayment Mortgage Arrangement Fee and Without Early Termination Penalty	Repayment Mortgage With Arrangement Fee and Early Termination Penalty	Endowment Mortgage With Arrangement Fee and Early Termination Penalty
1. ECONOMIC ENVIRONMENT:			
. Spot interest rate, r(0)	10%	10%	10%
. Long term average of the interest rate (steady state rate), θ	10%	10%	10%
. Speed of reversion, κ	25%	25%	25%
. House service flow, δ	7.5%	7.5%	7.5%
. Correlation coefficient, ρ	0	0	0
2. CONTRACT (1)			
. Maturity, η	300 months	300 months	300 months
. Value of the house at origination, H	£ 100 000	£ 100 000	£ 100 000
. Arrangement fee, ξ	1%	0.5%	0.5%
. Early termination penalty, π	-	1%	15%

(1) The insurance coverage formula is constant and corresponds to what was discribed in Chapter 3.

Table 5.2.A. Changes in the Contract Rate

(Repayment Mortgage With Arrangement Fee and Without an Early Termination Penalty) (£)

Contract Rate	Future Payments	Mortgage	Default	Prepayment	Insurance	Coinsurance	Value of the Mortgage to the Lender			
(c)	(A)	(V)	(D)	(C)	(I)	(CI)	V_B-L	V_B-(1-ξ)L	V_B-L+I	V_B-(1-ξ)L+I
7.0%	73188	72958	27	203	62	15	-22042	-21092	-21980	-21030
8.0%	79922	79562	98	262	158	40	-15438	-14488	-15280	-14330
9.0%	86900	85912	307	680	313	78	-9088	-8138	-8775	-7825
10.0%	94097	90909	766	2422	399	100	-4091	-3141	-3692	-2742
10.9%	100894	93643	1522	5729	409	102	-1357	-407	-948	2
12.0%	109062	94954	2316	11793	378	94	-46	904	332	1282
13.0%	116788	95001	11	21787	1	0	1	951	2	952
14.0%	124651	94998	0	29653	0	0	-2	948	-2	948
15.0%	132632	94996	0	37635	0	0	-4	946	-4	946

Underlying the construction of this table are the following parameters: ξ, the arrangement fee, is 1%; $r(0)$, the spot interest rate and θ, the long term average of the interest rate, are 10%; σ, the interest rate volatility and ν, the house price volatility, are 5%; δ, the house service flow, is 7.5%; and the correlation coefficient between the two state variables, ρ, is 0.

Table 5.2.B. Changes in the Contract Rate

(Repayment Mortgage With Arrangement Fee and Early Termination Penalty) (£)

Contract Rate	Future Payments	Mortgage	Default	Prepayment	Insurance	Coinsurance	Value of the Mortgage to the Lender			
(c)	(A)	(V)	(D)	(C)	(I)	(CI)	V_B-L	V_B-(1-ξ)L	V_B-L+I	V_B-(1-ξ)L+I
7.0%	73188	72961	27	200	65	16	-22039	-21564	-21974	-21499
8.0%	79922	79583	99	241	171	43	-15417	-14942	-15246	-14771
9.0%	86900	86034	320	546	357	89	-8966	-8491	-8609	-8134
10.0%	94097	91253	849	1995	489	122	-3747	-3272	-3258	-2783
10.8%	100131	93959	1672	4500	569	142	-1041	-566	-472	3
12.0%	109062	95736	2513	10813	516	129	736	1211	1252	1727
13.0%	116788	95950	2283	18555	352	88	950	1425	1302	1777
14.0%	124651	95946	0	28705	0	0	946	1421	946	1421
15.0%	132632	95948	0	36683	0	0	948	1423	948	1423

Underlying the construction of this table are the following parameters: ξ, the arrangement fee, is 0.5%; π, the early repayment penalty is 1%; *r(0)*, the spot interest rate and θ, the long term average of the interest rate, are 10%; σ, the interest rate volatility and ν, the house price volatility, are 5%; δ, the house service flow, is 7.5%; and the correlation coefficient between the two state variables, ρ, is 0.

Table 5.2.C. Changes in the Contract Rate

(Endowment Mortgage With Arrangement Fee and Early Termination Penalty)

(£)

Contract Rate	Future Payments	Mortgage	Default	Prepayment	Insurance	Coinsurance	Value of the Mortgage to the Lender			
(d)	(A)	(V)	(D)	(C)	(I)	(CI)	V_B-L	V_B-(1-ξ)L	V_B-L+I	V_B-(1-ξ)L+I
7.7%	66638	66450	7	181	25	7	-28550	-28075	-28525	-28050
8.8%	75841	75591	45	205	105	27	-19409	-18934	-19304	-18829
9.9%	85065	84468	237	359	293	73	-10532	-10057	-10239	-9764
10.9%	94307	91298	847	2162	421	105	-3702	-3227	-3281	-2806
11.7%	100874	94044	1828	5001	480	120	-956	-481	-476	-1
13.1%	112840	96079	3086	13676	429	107	1079	1554	1508	1983
14.2%	122127	96211	79	25851	9	2	1211	1686	1220	1695
15.2%	131426	96207	0	35219	0	0	1207	1682	1207	1682

Underlying the construction of this table are the following parameters: ξ, the arrangement fee, is 0.5%; π, the early repayment penalty is 15%; *r(0)*, the spot interest rate and θ, the long term average of the interest rate, are 10%; σ, the interest rate volatility and ν, the house price volatility, are 5%; δ, the house service flow, is 7.5%; and the correlation coefficient between the two state variables, ρ, is 0.

Table 5.3.A. Effects Induced by Changes in Contract Rate

(Repayment Mortgage Without Arrangement Fee and Early Termination Penalty) (£)

Contract Rate	Future Payments	Mortgage	Default	Prepayment	Insurance	Coinsurance	Mortgage Value
(*c*)	(A)	(V)	(D)	(C)	(I)	(CI)	V_B-L+I
9.0%	86900	85312	307	680	313	78	-9375
10.0%	94097	90909	766	2422	399	100	-3692
11.0%	101492	93808	1666	6018	426	107	-766
11.5%	105256	94542	1875	8839	379	95	-79
11.6%	105799	94614	1972	9214	385	96	-1
12.0%	109062	94954	2316	11793	378	94	332
13.0%	116788	95001	11	21787	1	0	2
14.0%	124651	94998	0	29653	0	0	-2
15.0%	132632	94996	0	37635	0	0	-4
16.0%	140714	94998	0	45716	0	0	-2
17.0%	148886	95000	0	53886	0	0	0

The calculations that underlie this table were done using the following parameters: 10% spot interest rate, $r(0)$, and long term mean of the interest rate process, θ; 5% interest rate and house price volatilities, σ and ν; 7.5% house service flow, δ; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Table 5.3.B. Effects Induced by Changes in the Contract Rate

(Endowment Mortgage Without Arrangement Fee and Early Termination Penalty) (£)

Contract Rate	Future Payments	Mortgage	Default	Prepayment	Insurance	Coinsurance	Mortgage Value
(*d*)	(A)	(V)	(D)	(C)	(I)	(CI)	V_B-L+I
9.9%	85065	84020	219	826	216	54	-10764
10.9%	94307	90367	687	3253	309	77	-4324
12.0%	103566	93764	1946	7857	290	72	-946
12.7%	109894	94735	2551	12608	263	66	-2
13.1%	112840	94987	2160	15695	210	53	197
14.2%	122127	94996	13	27118	1	0	-3
15.2%	131426	94996	0	36430	0	0	-4
16.3%	140735	94998	0	45737	0	0	-2
17.4%	150053	94997	0	55056	0	0	-3
18.5%	159378	94990	0	64388	0	0	-10

The calculations that underlie this table were done using the following parameters: 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 5% interest rate and house price volatilities, σ and ν; 7.5% service flow, δ; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Table 5.4.A. Trade-Off Between Arrangement Fee, Early Termination Penalty and Contract Rate

(Repayment Mortgage)

Prepayment Penalty (π)	Arrangement Fee (ξ)			
	0.000	0.005	0.010	0.015
0.00	11.57%	11.16%	10.92%	10.70%
0.01	11.02%	10.82%	10.64%	10.48%
0.02	10.75%	10.61%	10.46%	10.33%
0.05	10.34%	10.25%	10.16%	10.07%

Table 5.4.B. Trade-Off Between Arrangement Fee, Early Termination Penalty and Contract Rate

(Endowment Mortgage)

Prepayment Penalty (π)	Arrangement Fee (ξ)			
	0.000	0.005	0.010	0.015
0.00	12.74%	12.29%	12.00%	11.79%
0.05	12.34%	12.04%	11.82%	11.64%
0.10	12.08%	11.85%	11.67%	11.50%
0.15	11.89%	11.69%	11.53%	11.38%

Underlying the construction of these tables are the following parameters: $r(0)$, the spot interest rate, and θ, the long term average of the interest rate, are 10%; σ, the interest rate volatility and ν, the house price volatility, are 5%; δ, the house service flow, is 7.5%; and the correlation coefficient between the two state variables, ρ is 0.

Table 5.5.A. Interest Rate Variation

(Repayment Mortgage With Arrangement Fee and Without an Early Termination Penalty) (£)

SIGMA	Future Payments	Mortgage	Default	Prepayment	Insurance	Coinsurance
(σ)	(A)	(V_B)	(D)	(C)	(I)	(CI)
5.0%	100894	93643	1522	5729	409	102
7.5%	101732	92526	1604	7603	348	87
10.0%	102897	91420	1737	9740	311	78
12.5%	104341	90388	2094	11860	303	76

The calculations that underlie this table were done using the following parameters: 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 5% house price volatility, ν; 7.5% service flow, δ; 1% arrangement fee, ξ; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Table 5.5.B. Interest Rate Variation

(Repayment Mortgage With Arrangement Fee and Early Termination Penalty) (£)

SIGMA	Future Payments	Mortgage	Default	Prepayment	Insurance	Coinsurance
(σ)	(A)	(V_B)	(D)	(C)	(I)	(CI)
5.0%	100131	93959	1672	4500	569	142
7.5%	100963	92782	1788	6394	488	122
10.0%	102119	91650	2146	8324	461	115
12.5%	103553	90586	2731	10236	475	119

The calculations that underlie this table were done using the following parameters: 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 5% house price volatility, ν; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 1% early termination penalty, π; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Table 5.5.C. Interest Rate Variation

(Endowment Mortgage With Arrangement Fee and Early Termination Penalty) (£)

SIGMA (σ)	Future Payments (A)	Mortgage (V_B)	Default (D)	Prepayment (C)	Insurance (I)	Coinsurance (CI)
5.0%	100874	94044	1828	5001	480	120
7.5%	102089	92778	1859	7453	374	94
10.0%	103775	91604	2249	9923	334	84
12.5%	105880	90524	2757	12600	312	78

The calculations that underlie this table were done using the following parameters: 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 5% house price volatility, ν; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 15% early termination penalty, π; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Table 5.6.A. House Price Variation

(Repayment Mortgage With Arrangement Fee and Without an Early Termination Penalty) *(£)*

NU	Future Payments	Mortgage	Default	Prepayment	Insurance	Coinsurance
(v)	(A)	(V_B)	(D)	(C)	(I)	(CI)
5.0%	100894	93643	1522	5729	409	102
7.5%	100894	93073	2932	4889	1060	265
10.0%	100894	92146	4481	4267	2124	531
12.5%	100894	91028	6251	3615	3116	780

The calculations that underlie this table were done using the following parameters: 10% spot interest rate, $r(0)$, and long term mean of the interest rate process, θ; 5% interest rate volatility, σ; 7.5% service flow, δ; 1% arrangement fee, ξ; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Table 5.6.B. House Price Variation

(Repayment Mortgage With Arrangement Fee and Early Termination Penalty) (£)

NU	Future Payments	Mortgage	Default	Prepayment	Insurance	Coinsurance
(ν)	(A)	(V_B)	(D)	(C)	(I)	(CI)
5.0%	100131	93959	1672	4500	569	142
7.5%	100131	93300	3018	3814	1260	315
10.0%	100131	92280	4489	3362	2375	594
12.5%	100131	91088	6188	2856	3428	858

The calculations that underlie this table were done using the following parameters: 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 5% interest rate volatility, σ; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 1% early termination penalty, π; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Table 5.6.C. House Price Variation

(Endowment Mortgage With Arrangement Fee and Early Termination Penalty) (£)

NU (ν)	Future Payments (A)	Mortgage (V_B)	Default (D)	Prepayment (C)	Insurance (I)	Coinsurance (CI)
5.0%	100874	94044	1828	5001	480	120
7.5%	100874	93405	3136	4334	1094	274
10.0%	100874	92396	4605	3873	2202	551
12.5%	100874	91222	6371	3281	3190	799

The calculations that underlie this table were done using the following parameters: 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 5% interest rate volatility, σ; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 15% early termination penalty, π; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Table 5.7.A. The Combined Effects of Changes in LTV Ratios and House Price and Interest Rate Volatilities

(Repayment Mortgage With Arrangement Fee and Without an Early Termination Penalty)

(£)

LTV	Int. Rate Volatility	Eq.Contract Rate (*c*)		Future Payments (A)		Default (D)		Prepayment (C)		Insurance (I)		Coinsurance (CI)	
		ν = 5%	ν=10%	ν = 5%	ν=10%	ν = 5%	ν=10%	ν = 5%	ν=10%	ν = 5%	ν=10%	ν = 5%	ν=10%
80%	σ = 5%	10.95%	10.93%	85152	85026	23	617	5929	5434	6	220	2	129
	σ = 10%	12.22%	12.19%	95114	94916	20	552	15894	15278	2	111	1	67
85%	σ = 5%	10.94%	10.88%	90407	89972	84	1291	6207	5100	30	566	8	152
	σ = 10%	12.22%	12.13%	101059	100393	96	1484	16818	15100	11	347	3	94
90%	σ = 5%	10.93%	10.85%	95650	95055	347	2480	6322	4613	118	1137	29	285
	σ = 10%	12.20%	12.05%	106841	105709	363	3728	17440	13680	68	801	17	202
95%	σ = 5%	10.92%	10.83%	100894	100207	1522	4348	5729	3942	409	2132	102	533
	σ = 10%	12.18%	12.00%	112652	111227	2485	6946	16429	11879	315	1642	79	411
100%	σ = 5%	10.91%	10.96%	106104	106592	4636	7804	3776	3190	1304	3408	326	852
	σ = 10%	12.26%	12.06%	119223	117587	14943	12846	6292	8928	1017	3187	254	797

The calculations that underlie this table were done using the following parameters: 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ 7.5% service flow, δ; 1% arrangement fee, ξ; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Table 5.7.B. The Combined Effects of Changes in LTV Ratios and House Price and Interest Rate Volatilities

(Repayment Mortgage With Arrangement Fee and Early Termination Penalty)

(£)

LTV	Int. Rate	Eq. Contract Rate *(c)*		Future Payments (A)		Default (D)		Prepayment (C)		Insurance (I)		Coinsurance (CI)	
	Volatility	v = 5%	v=10%	v = 5%	v=10%	v = 5%	v=10%	v = 5%	v=10%	v = 5%	v=10%	v = 5%	v=10%
80%	σ = 5%	10.86%	10.85%	84598	84514	26	693	4976	4483	8	260	3	175
	σ = 10%	12.03%	12.02%	93829	93764	27	682	14204	13609	3	135	1	95
85%	σ = 5%	10.85%	10.79%	89819	89405	104	1431	5181	4092	36	692	9	193
	σ = 10%	12.02%	11.92%	99671	98947	160	1737	14952	13087	17	456	4	129
90%	σ = 5%	10.83%	10.77%	94961	94541	389	2627	5192	3682	161	1316	40	331
	σ = 10%	12.00%	11.86%	105396	104340	422	4042	15517	11742	93	992	23	250
95%	σ = 5%	10.82%	10.77%	100131	99776	1672	4394	4500	3230	569	2373	142	593
	σ = 10%	11.97%	11.85%	110973	110069	2764	7626	14127	9838	451	1917	113	480
100%	σ = 5%	10.86%	10.86%	105724	105763	5166	7588	2632	2509	1567	3826	392	957
	σ = 10%	12.09%	11.90%	117788	116238	14708	12616	5058	7723	1477	3605	369	901

The calculations that underlie this table were done using the following parameters: 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 1% early termination penalty, π; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Table 5.7.C. The Combined Effects of Changes in LTV Ratios and House Price and Interest Rate Volatilities

(Endowment Mortgage With Arrangement Fee and Early Termination Penalty) (£)

LTV	Int. Rate	Eq. Contract Rate (*c*)		Future Payments (A)		Default (D)		Prepayment (C)		Insurance (I)		Coinsurance (CI)	
	Volatility	ν = 5%	ν=10%	ν = 5%	ν=10%	ν = 5%	ν=10%	ν = 5%	ν=10%	ν = 5%	ν=10%	ν = 5%	ν=10%
80%	σ = 5%	11.69%	11.69%	84951	84939	27	723	5339	4870	6	255	2	150
	σ = 10%	12.94%	12.93%	95875	95797	30	693	16248	15631	2	129	1	74
85%	σ = 5%	11.69%	11.66%	90255	90012	100	1465	5614	4592	32	624	8	173
	σ = 10%	12.94%	12.88%	101893	101399	138	1842	17185	15364	13	383	3	107
90%	σ = 5%	11.68%	11.66%	95482	95307	407	2685	5668	4253	140	1186	35	298
	σ = 10%	12.95%	12.88%	107773	107412	476	4084	17823	14560	80	782	20	198
95%	σ = 5%	11.69%	11.67%	100874	100668	1828	4560	5001	3781	480	2206	120	552
	σ = 10%	12.94%	12.91%	113880	113589	3184	8078	16520	12574	358	1591	89	398
100%	σ = 5%	11.82%	11.83%	107378	107476	5917	8373	3207	3107	1243	3502	311	875
	σ = 10%	13.31%	12.98%	123323	120255	20016	14258	4765	9696	962	3195	240	799

The calculations that underlie this table were done using the following parameters: 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 15% prepayment penalty, π; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Table 5.8.A. The Effects of theRelationship Between the Spot Rate - *r(0)* - and the Long Term Average of the Interest Rate - θ

(Repayment Mortgage With Arangement Fee and Without an Early Termination Penalty)

(£)

r(0)	Int. Rate Volatility (σ)	Eq. Contract Rate (c)		Future Payments (A)		Default (D)		Prepayment (C)		Insurance (I)		Coinsurance (CI)	
		v = 5%	v=10%	v = 5%	v=10%	v = 5%	v=10%	v = 5%	v=10%	v = 5%	v=10%	v = 5%	v=10%
8%	5%	9.72%	9.68%	97275	96977	1206	3951	2564	1605	551	2634	138	659
10%		10.92%	10.83%	100894	100207	1522	4348	5729	3942	409	2132	102	533
12%		12.40%	12.35%	106173	105759	2160	5211	10336	8116	377	1624	94	406
8%	10%	10.67%	10.46%	106593	104845	2415	6495	10454	6714	431	2413	108	604
10%		12.18%	12.00%	112652	111227	2485	6946	16429	11879	315	1642	79	411
12%		13.78%	13.68%	118845	118096	2948	8261	22146	17045	301	1266	75	316

The calculations that underlie this table were done using the following parameters: 8%, 10% and 12% spot interest rate, *r(0)*; 10% long term mean of the interest rate process, θ; 7.5% service flow, δ; 1% arrangement fee, ξ; 95% LTV ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Table 5.8.B The Effects of the Relationship Between the Spot Rate - *r(0)* - and the Long Term Average of the Interest Rate - θ

(Repayment Mortgage With Arrangement Fee and Early Termination Penalty)

(£)

r(0)	Int Rate Volatility (σ)	Eq. Contract Rate (*c*)		Future Payments (A)		Default (D)		Prepayment (C)		Insurance (I)		Coinsurance (CI)	
		ν = 5%	ν=10%	ν = 5%	ν=10%	ν = 5%	ν=10%	ν = 5%	ν=10%	ν = 5%	ν=10%	ν = 5%	ν=10%
8%	5%	9.70%	9.68%	97057	96905	1328	4030	1923	1260	715	2911	179	728
10%		10.82%	10.77%	100131	99776	1672	4394	4500	3230	569	2373	142	593
12%		12.22%	12.21%	104819	104762	2322	5480	8499	6618	531	1856	133	464
8%	10%	10.52%	10.37%	105389	104137	2625	6738	8563	5624	626	2751	157	689
10%		11.97%	11.85%	110973	110069	2764	7626	14127	9838	451	1917	113	480
12%		13.53%	13.48%	116950	116510	2867	8540	19944	14945	384	1504	96	376

The calculations that underlie this table were done using the following parameters: 8%, 10% and 12% spot interest rates, *r(0)*; 10% long term mean of the interest rate process, θ; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 1% early termination penalty, π; 95% LTV ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Table 5.8.C. The Effects of the Relationship Between the Spot Rate - $r(0)$ - and the Long Term Average of the Interest Rate - θ

(Endowment Mortgage With Arrangement Fee and Early Termination Penalty)

(£)

$r(0)$	Int. Rate	Eq. Contract Rate (c)		Future Payments (A)		Default (D)		Prepayment (C)		Insurance (I)		Coinsurance (CI)	
	Volatility (σ)	ν = 5%	ν=10%	ν = 5%	ν=10%	ν = 5%	ν=10%	ν = 5%	ν=10%	ν = 5%	ν=10%	ν = 5%	ν=10%
8%		10.65%	10.63%	97052	96893	1329	4020	1843	1206	654	2863	163	717
10%	5%	11.69%	11.67%	100874	100668	1828	4560	5001	3781	480	2206	120	552
12%		13.12%	13.15%	107177	107433	2676	5974	10369	8504	398	1568	99	392
8%		11.51%	11.40%	106859	105833	3224	6943	9633	6854	526	2492	131	625
10%	10%	12.94%	12.91%	113880	113589	3284	8078	16520	12574	358	1591	89	398
12%		14.54%	14.62%	121326	121993	3394	9844	23703	18752	292	1126	73	282

The calculations that underlie this table were done using the following parameters: 8%,10% and 12% spot interest rates, $r(0)$; 10% long term mean of the interest rate process, θ; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 15% early termination penalty, π; 95% LTV ratio; and a correlation coefficient, ρ, of 0 The insurance coverage assumes the standard form defined in Chapter 3.

Table 5.9.A. Effects Induced by Changes in the Reversion Coefficient (κ)

(Repayment Mortgage With Arrangement Fee and Without an Early Termination Penalty) (£)

r(0)/θ	KAPPA (κ)	Contract Rate (*c*)	Value of Payments (A)	Default (D)	Prepayment (C)	Insurance (I)	Coinsurance (CI)
8% - 10%	0%	11.40%	143808	18541	33144	1914	481
	25%	10.46%	104845	6495	6714	2413	604
	50%	10.22%	98836	4496	2829	2539	635
	75%	10.16%	96956	3784	1689	2574	644
	100%	10.15%	96192	3464	1244	2566	641
10% - 10%	0%	13.78%	146848	16986	37062	1253	314
	25%	12.00%	111227	6946	11879	1642	411
	50%	11.20%	102989	4846	6099	2012	503
	75%	10.85%	99821	4224	3700	2155	539
	100%	10.66%	98260	3854	2579	2228	557
12% - 10%	0%	15.99%	146766	14227	39519	1028	257
	25%	13.68%	118096	8261	17045	1266	316
	50%	12.62%	110186	6115	11433	1404	351
	75%	11.91%	105377	4904	8078	1657	414
	100%	11.45%	102297	4380	5743	1868	467

The calculations that underlie this table were done using the following parameters: 8%, 10% and 12% spot interest rate, *r(0)*; 10% long term mean of the interest rate process, θ; 10% interest rate volatility, σ; 10% house price volatility, ν 7.5% service flow, δ; 1% arrangement fee, ξ; 95% LTV ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Table 5.9.B. Effects Induced by Changes in the Reversion Coefficient (κ)

(Repayment Mortgage With Arrangement Fee and Early Termination Penalty) (£)

r(0)/θ	KAPPA (κ)	Contract Rate (*c*)	Value of Payments (A)	Default (D)	Prepayment (C)	Insurance (I)	Coinsurance (CI)
8% - 10%	0%	11.27%	142453	21741	28388	2197	553
	25%	10.37%	104137	6738	5624	2751	689
	50%	10.18%	98519	4574	2263	2846	712
	75%	10.14%	96833	3839	1325	2856	714
	100%	10.15%	96203	3523	990	2832	708
10% - 10%	0%	13.57%	144868	18803	33094	1544	387
	25%	11.85%	110069	7626	9838	1917	480
	50%	11.08%	102050	5009	4832	2310	578
	75%	10.75%	99113	4236	2808	2456	614
	100%	10.59%	97707	3850	1860	2527	632
12% - 10%	0%	15.75%	144744	15925	35598	1299	325
	25%	13.48%	116510	8540	14945	1504	376
	50%	12.39%	108484	6420	9217	1687	422
	75%	11.71%	103874	5090	6189	1927	482
	100%	11.27%	100982	4375	4240	2163	541

The calculations that underlie this table were done using the following parameters: 8%, 10% and 12% spot interest rates, *r(0)*; 10% long term mean of the interest rate process, θ;10% interest rate volatility, σ; 10% house price volatility, ν; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 1% early termination penalty, π; 95% LTV ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Table 5.9.C. Effects Induced by Changes in the Reversion Coefficient (k)

(Endowment Mortgage With Arrangement Fee and EarlyTermination Penalty) (£)

r(0)/θ	KAPPA (κ)	Contract Rate (*d*)	Value of Payments (A)	Default (D)	Prepayment (C)	Insurance (I)	Coinsurance (CI)
8% - 10%	0%	13.72%	162927	18395	52601	2591	734
	25%	11.40%	105833	6943	6854	2492	625
	50%	11.07%	98662	4600	2261	2725	682
	75%	11.01%	96691	3805	1137	2777	695
	100%	11.02%	96075	3497	822	2771	693
10% - 10%	0%	15.82%	163097	13526	56839	1796	476
	25%	12.91%	113589	8078	12574	1591	398
	50%	11.99%	103450	5237	5763	2077	519
	75%	11.61%	99656	4346	3063	2280	570
	100%	11.42%	97874	3881	1844	2377	594
12% - 10%	0%	17.80%	160290	14762	52330	1322	337
	25%	14.62%	121993	9844	18752	1126	282
	50%	13.42%	112084	7262	11635	1336	334
	75%	12.67%	106291	5631	7751	1615	404
	100%	12.16%	102357	4629	5120	1914	479

The calculations that underlie this table were done using the following parameters: 8%,10% and 12% spot interest rates, *r(0)*; 10% long term mean of the interest rate process, θ; 10% interest rate volatility, σ; 10% house price volatility, ν; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 15% early termination penalty, π; 95% LTV ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Table 5.10.A. Effects Induced by Changes in the Correlation Coefficient (ρ)

(Repayment Mortgage With Arrangement Fee and Without an Early Termination Penalty) **(£)**

RHO	Contract Rate	Value of Payments	Default	Prepayment	Insurance	Coinsurance
(ρ)	(*c*)	(A)	(D)	(C)	(I)	(CI)
-0.20	11.97%	110994	5844	12628	1527	382
-0.15	11.98%	111072	6111	12467	1551	388
-0.10	11.99%	111149	6390	12291	1575	394
-0.05	11.99%	111165	6662	12073	1613	404
0.00	12.00%	111227	6946	11879	1642	411
0.05	12.01%	111305	7188	11736	1674	419
0.10	12.02%	111345	7461	11532	1702	426
0.15	12.02%	111407	7746	11332	1730	433
0.20	12.03%	111407	8026	11088	1756	439

The calculations that underlie this table were done using the following parameters:10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 10% interest rate volatility, σ; 10% house price volatility, ν; 7.5% service flow, δ; 1% arrangement fee, ξ; and 95% LTV ratio. The insurance coverage assumes the standard form defined in Chapter 3.

Table 5.10.B Effects Induced by Changes in the Correlation Coefficient (ρ)

(Repayment Mortgage With Arrangement Fee and Early Termination Penalty) (£)

RHO (ρ)	Contract Rate (c)	Value of Payments (A)	Default (D)	Prepayment (C)	Insurance (I)	Coinsurance (CI)
-0.20	11.81%	109748	6429	10575	1788	448
-0.15	11.82%	109825	6751	10368	1815	454
-0.10	11.83%	109903	7048	10185	1849	463
-0.05	11.84%	109981	7344	10001	1883	471
0.00	11.85%	110069	7626	9838	1917	480
0.05	11.86%	110137	7904	9663	1954	489
0.10	11.87%	110215	8181	9494	1991	498
0.15	11.88%	110269	8444	9320	2024	506
0.20	11.89%	110331	8711	9147	2054	514

The calculations that underlie this table were done using the following parameters: 10% spot interest rate, $r(0)$, and long term mean of the interest rate process, θ; 10% interest rate volatility, σ; 10% house price volatility, ν; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 1% early termination penalty, π; 95% LTV ratio. The insurance coverage assumes the standard form defined in Chapter 3.

Table 5.10.C. Effects Induced by Changes in the Correlation Coefficient (ρ)

(Endowment Mortgage With Arrangement Fee and Early Termination Penalty) (£)

RHO (ρ)	Contract Rate (d)	Value of Payments (A)	Default (D)	Prepayment (C)	Insurance (I)	Coinsurance (CI)
-0.20	11.66%	112826	6828	13057	1576	395
-0.15	11.68%	113016	7126	12952	1585	397
-0.10	11.70%	113160	7441	12793	1590	398
-0.05	11.72%	113350	7739	12686	1591	398
0.00	11.74%	113589	8078	12574	1591	398
0.05	11.76%	113779	8414	12426	1591	398
0.10	11.78%	113922	8699	12290	1596	399
0.15	11.79%	114084	9034	12115	1597	400
0.20	11.81%	114208	9357	11922	1602	401

The calculations that underlie this table were done using the following parameters: 10% spot interest rate, $r(0)$, and long term mean of the interest rate process, θ; 10% interest rate volatility, σ; 10% house price volatility, ν; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 15% early termination penalty, π; and 95% LTV ratio. The insurance coverage assumes the standard form defined in Chapter 3.

Table 5.11. The Relative Weight of the Different Financial Assets in the Mortgage Contract: Repayment Versus Endowment

(In Relation to the Par Value of the House)

LTV	SIGMA	NU	Repayment Mortgage					Endowment Mortgage				
	(σ)	(ν)	(A)	(D)	(C)	(I)	(CI)	(A)	(D)	(C)	(I)	(CI)
80%	5%	5%	0.846	0.000	0.050	0.000	0.000	0.850	0.000	0.053	0.000	0.000
		10%	0.845	0.007	0.045	0.003	0.002	0.849	0.007	0.049	0.003	0.002
	10%	5%	0.938	0.000	0.142	0.000	0.000	0.959	0.000	0.162	0.000	0.000
		10%	0.938	0.007	0.136	0.001	0.001	0.958	0.007	0.156	0.001	0.001
85%	5%	5%	0.898	0.001	0.052	0.000	0.000	0.903	0.001	0.056	0.000	0.000
		10%	0.894	0.014	0.041	0.007	0.002	0.900	0.015	0.046	0.006	0.002
	10%	5%	0.997	0.002	0.150	0.000	0.000	1.019	0.001	0.172	0.000	0.000
		10%	0.989	0.017	0.131	0.005	0.001	1.014	0.018	0.154	0.004	0.001
90%	5%	5%	0.950	0.004	0.052	0.002	0.000	0.955	0.004	0.057	0.001	0.000
		10%	0.945	0.026	0.037	0.013	0.003	0.953	0.027	0.043	0.012	0.003
	10%	5%	1.054	0.004	0.155	0.001	0.000	1.078	0.005	0.178	0.001	0.000
		10%	1.043	0.040	0.117	0.010	0.003	1.074	0.041	0.146	0.008	0.002
95%	5%	5%	1.001	0.017	0.045	0.006	0.001	1.009	0.018	0.050	0.005	0.001
		10%	0.998	0.044	0.032	0.024	0.006	1.007	0.046	0.038	0.022	0.006
	10%	5%	1.110	0.028	0.141	0.005	0.001	1.139	0.032	0.165	0.004	0.001
		10%	1.101	0.076	0.098	0.019	0.005	1.136	0.081	0.126	0.016	0.004

The calculations that underlie this table were done using the following parameters: 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 1% and 15% early termination penalty, π, in the repayment and endowment mortgages, respectively; 95% LTV ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.
Legend: Future Payments (A); Default (D); Prepayment (C); Insurance (I); Coinsurance (CI).

Figure 5.1.A. Evolution of the Options to Terminate the Loan Across the State Space in Different Moments During the Life of the Loan

i) Termination of the Loan

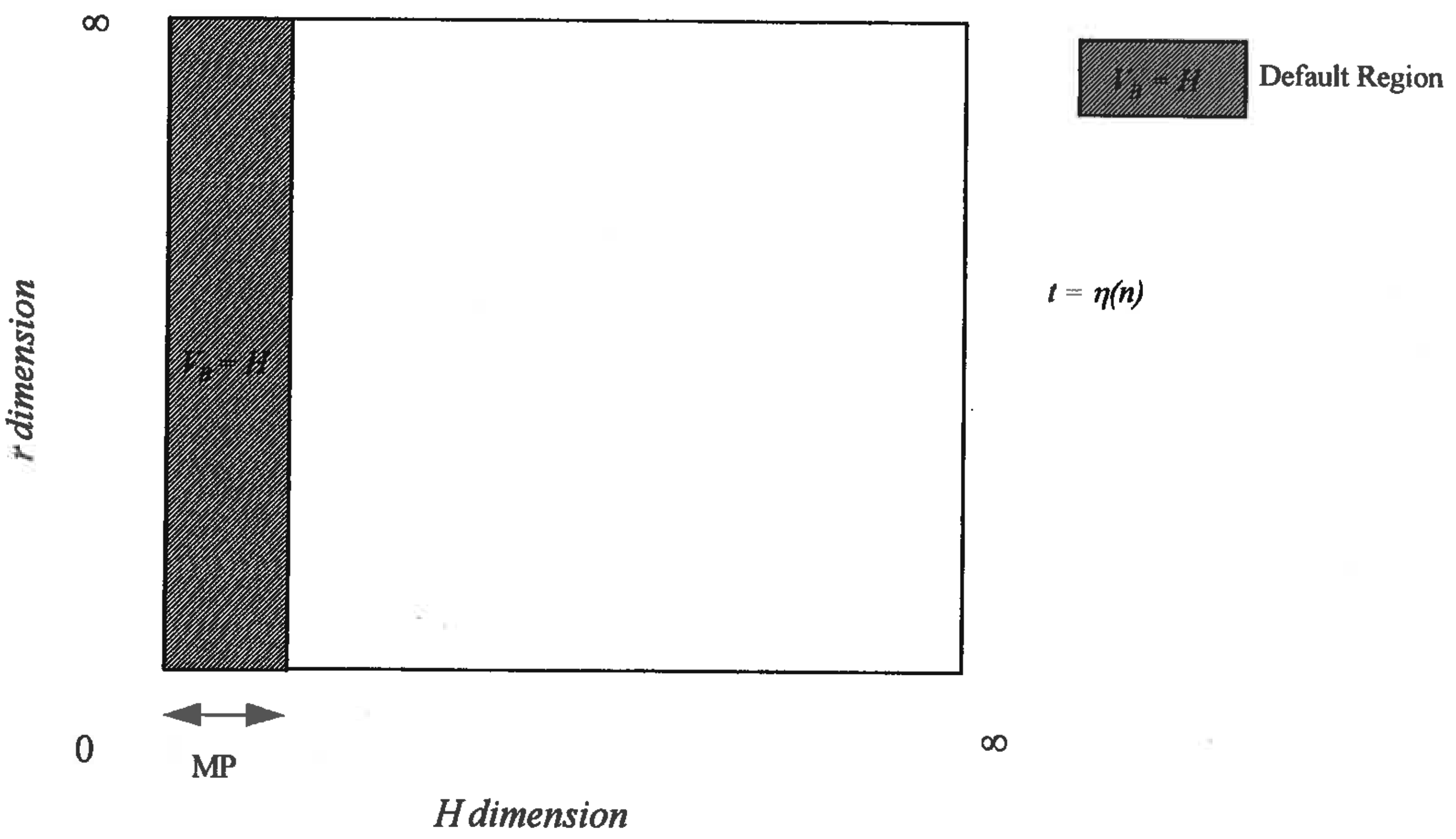


Figure 5.1.B. Evolution of the Options to Terminate the Loan Across the State Space in Different Moments During the Life of the Loan

ii) Moments Other than Monthly Payment Dates

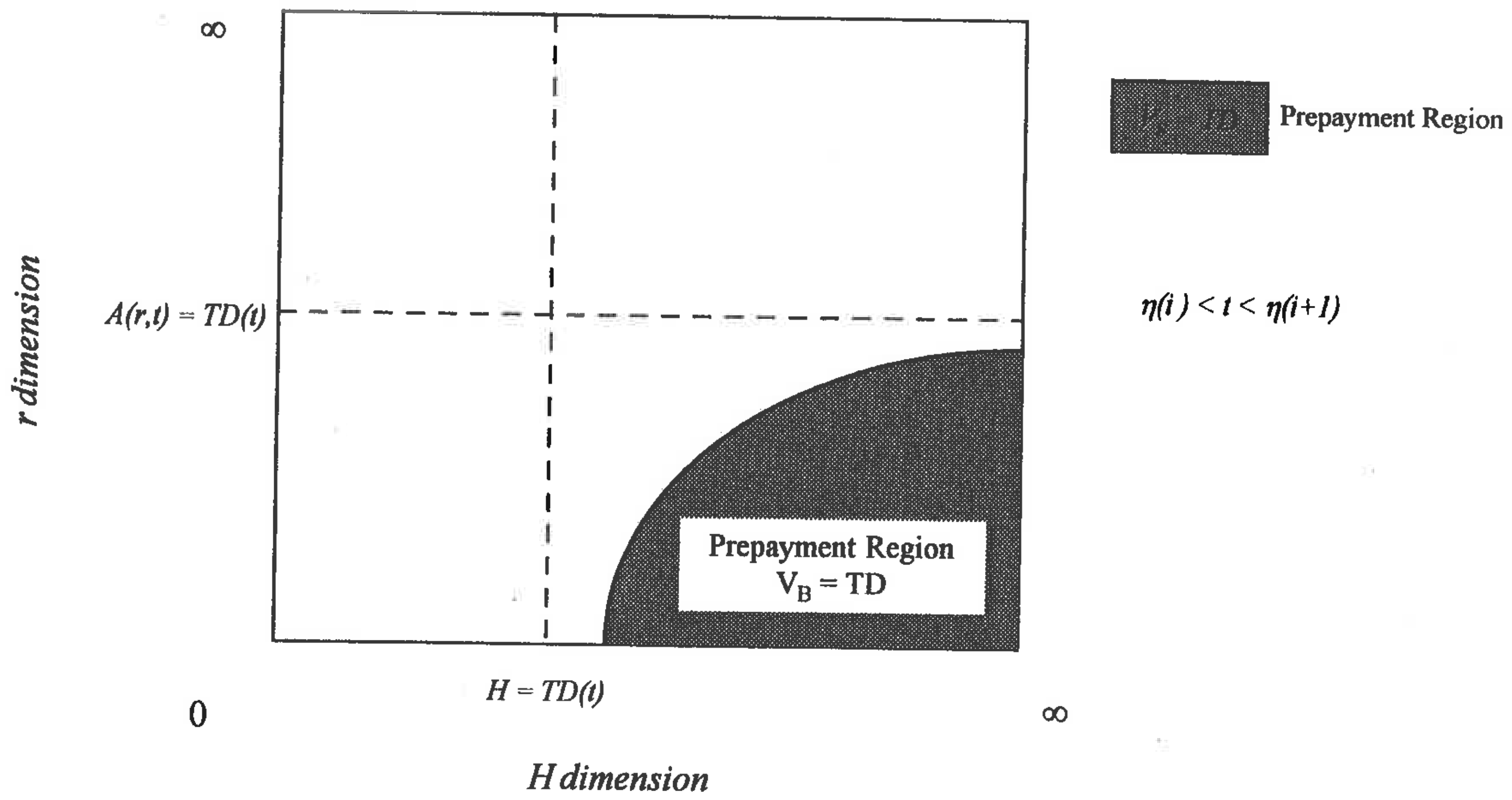


Figure 5.1.C. Evolution of the Options to Terminate the Loan Across the State Space in Different Moments During the Life of the Loan

iii) Monthly Payment Dates

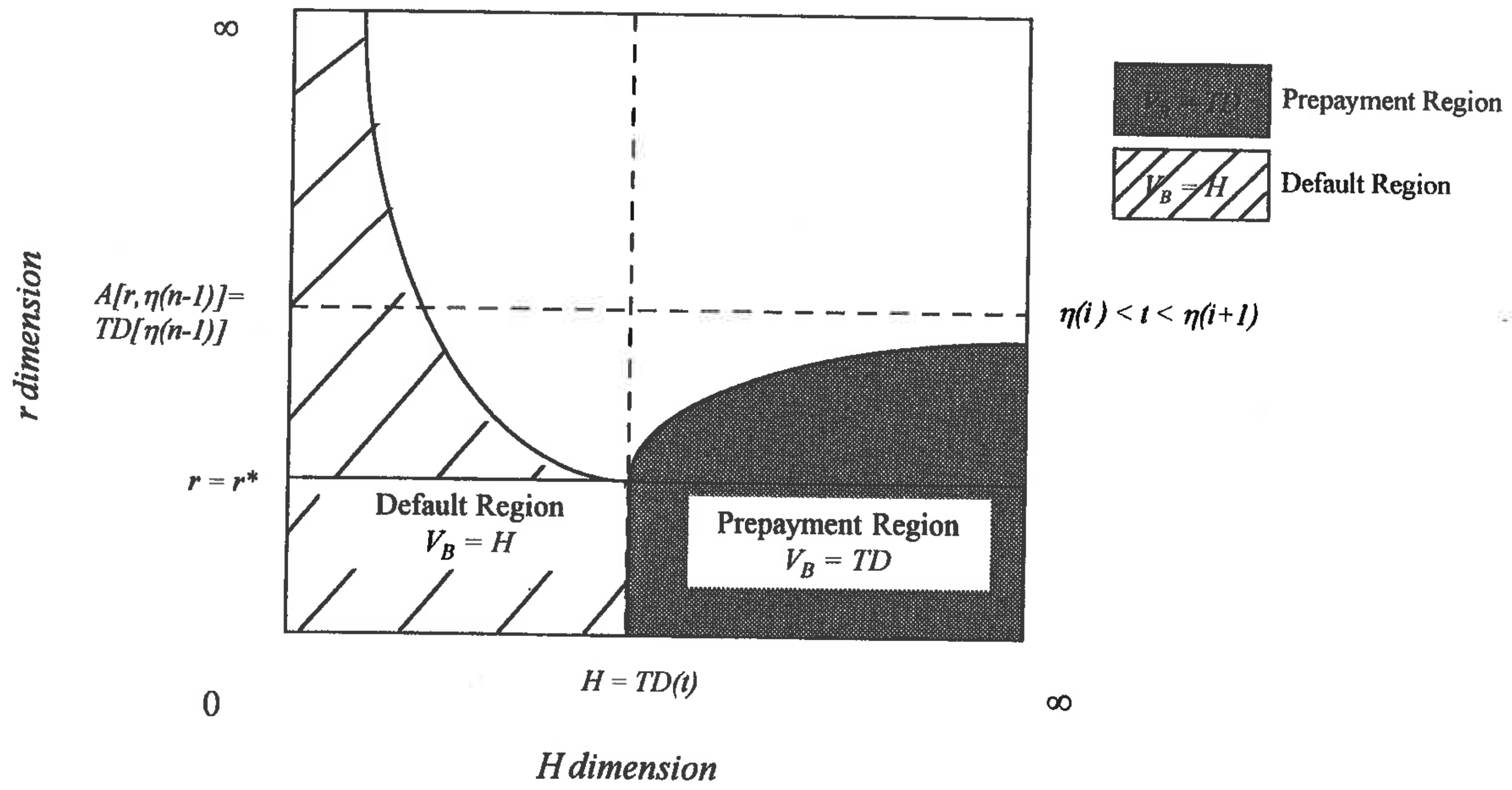


Figure 5.1.D. Evolution of the Options to Terminate the Loan Across the State Space in Different Moments During the Life of the Loan

iv) Origination of the Loan

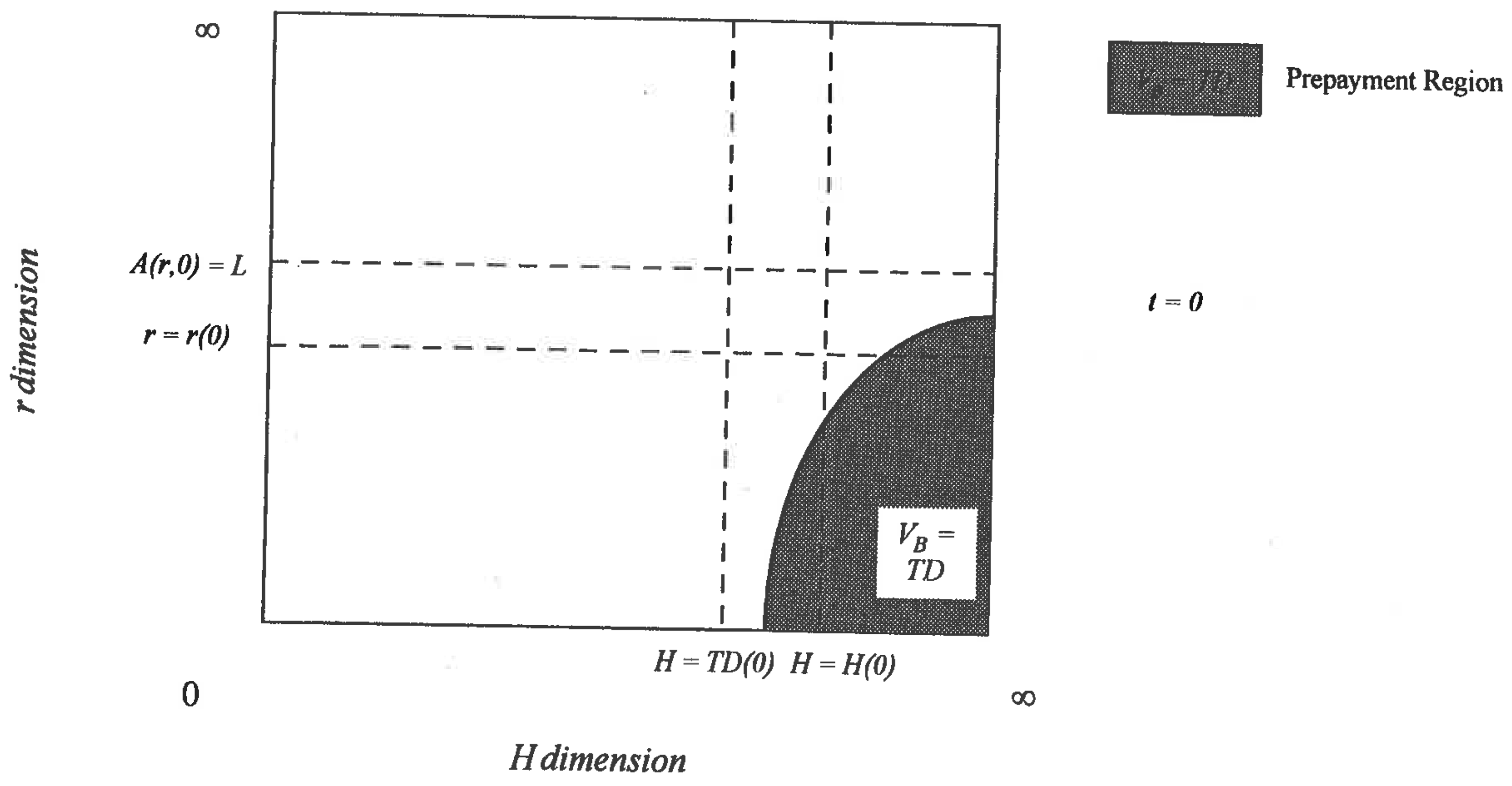
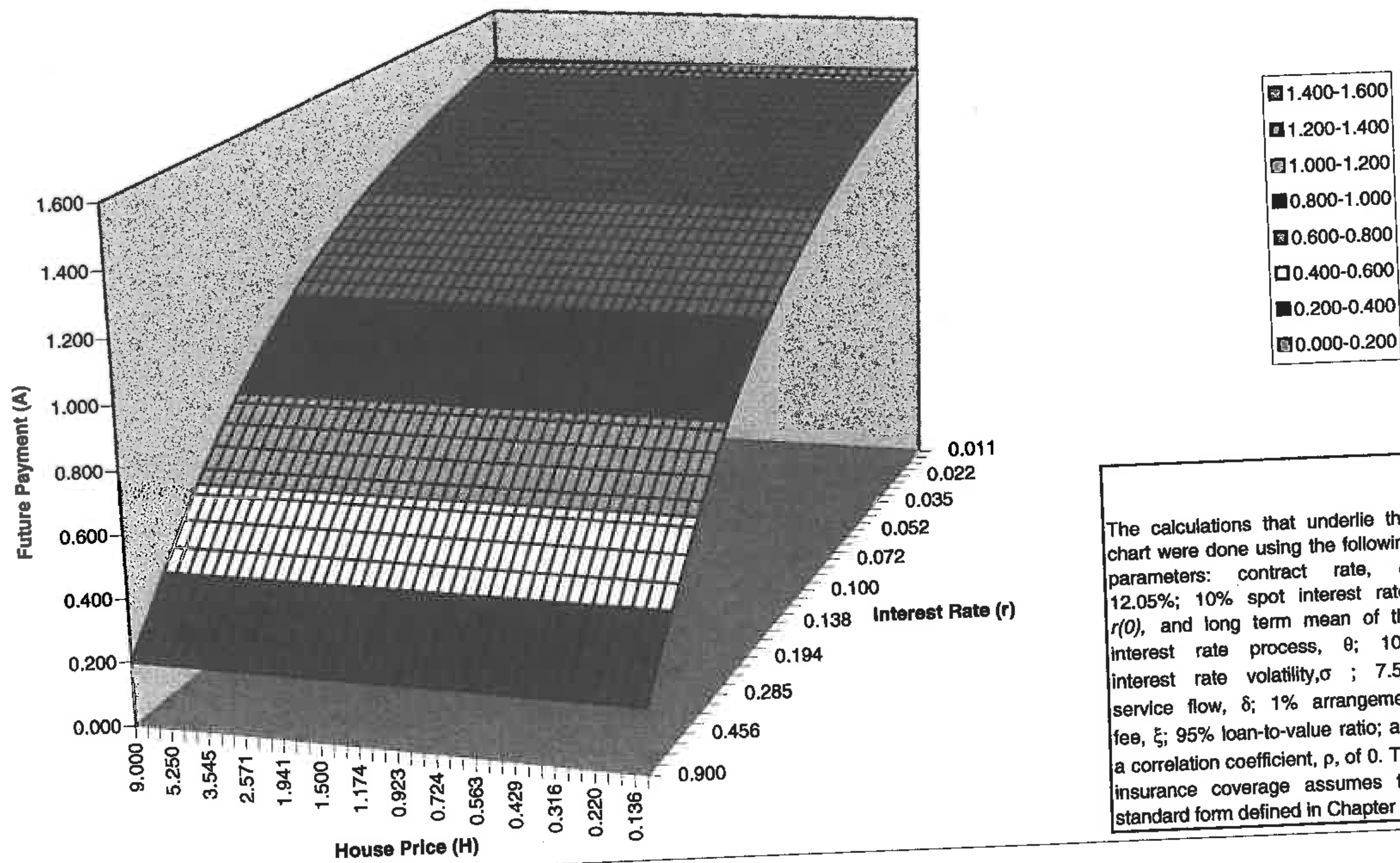


Figure 5.2.A. Value of Future Payments (A)

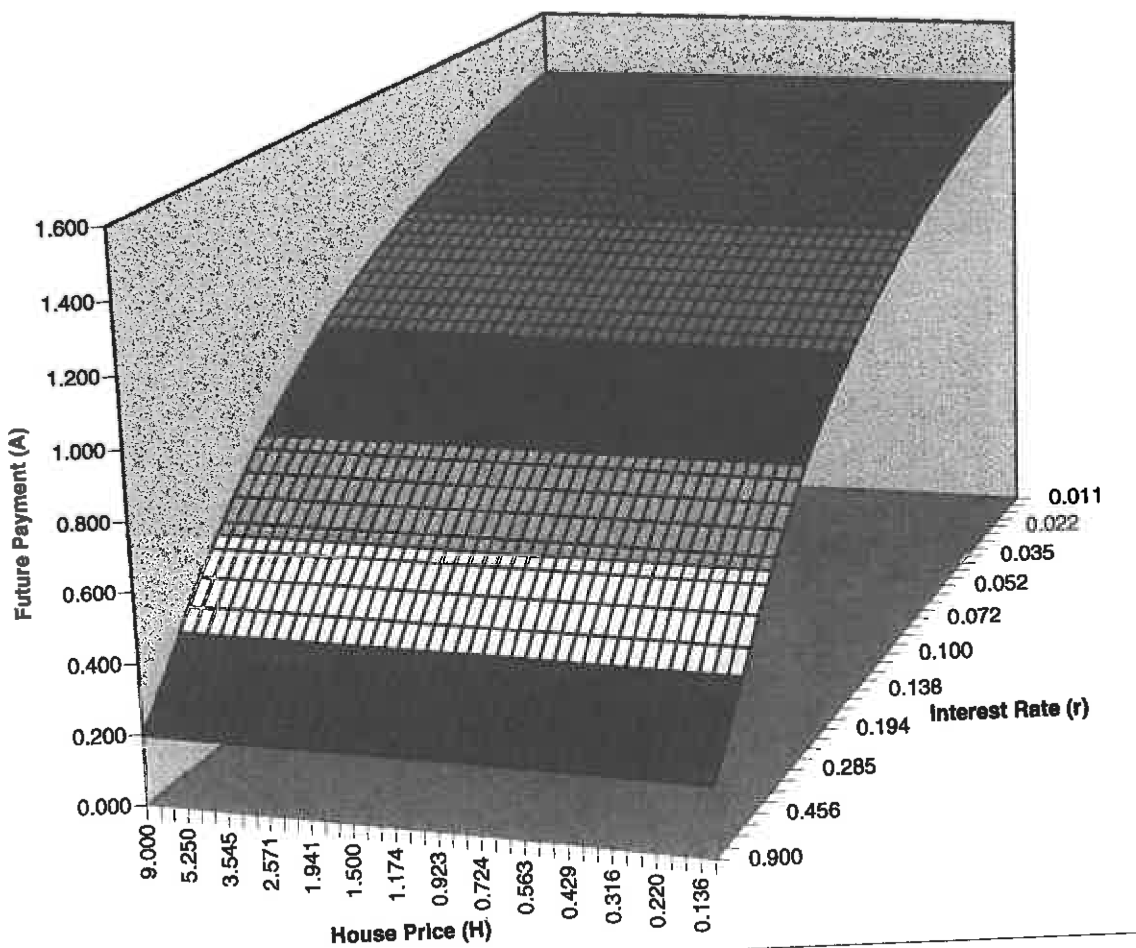
(Repayment Mortgage Without an Early Termination Penalty)



The calculations that underlie this chart were done using the following parameters: contract rate, c 12.05%; 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 10% interest rate volatility, σ ; 7.5% service flow, δ; 1% arrangement fee, ξ; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Figure 5.2.B. Value of Future Payments (A)

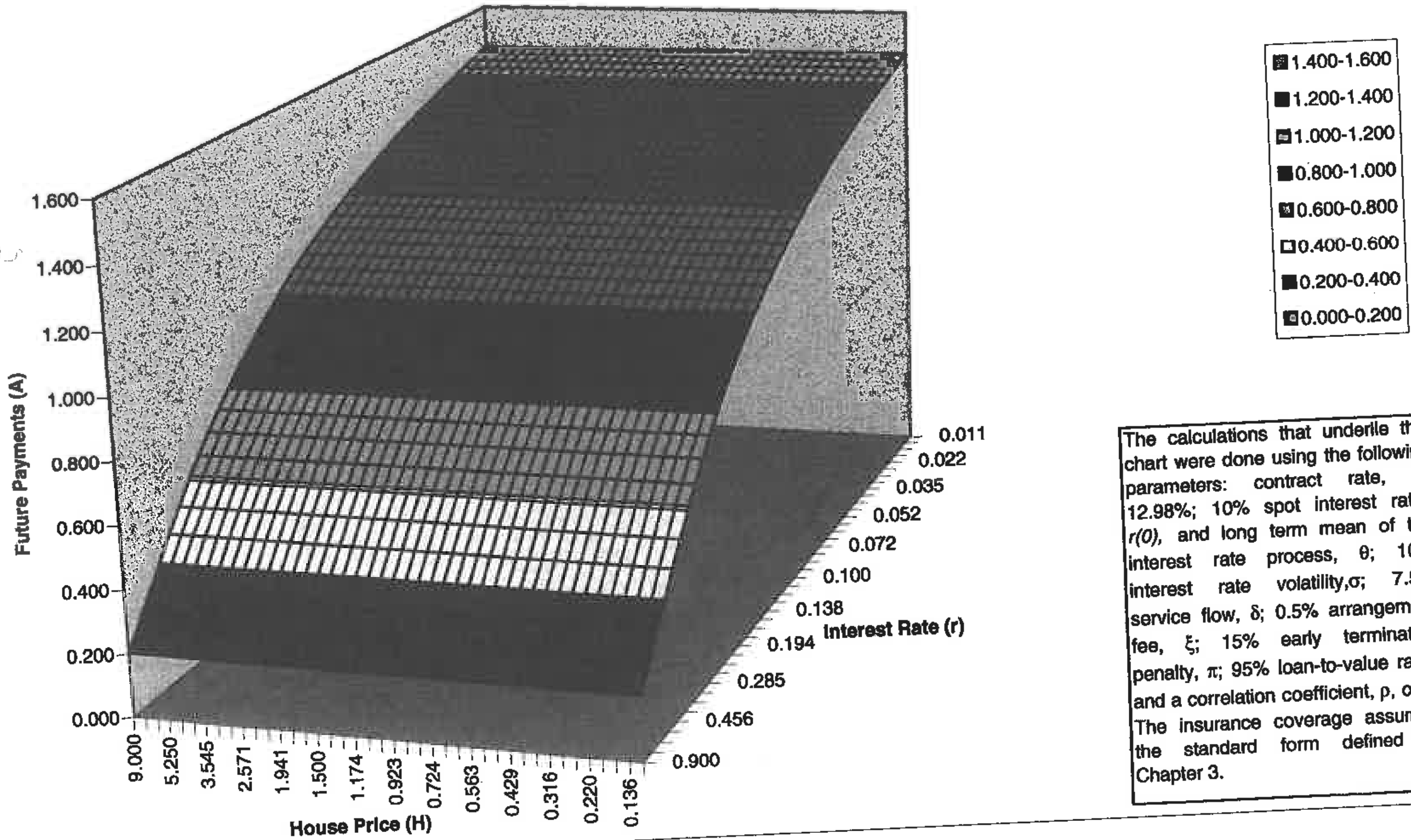
(Repayment Mortgage With an Early Termination Penalty)



The calculations that underlie this chart were done using the following parameters: contract rate, c, 11.89%; 10% spot interest rate, $r(0)$, and long term mean of the interest rate process, θ; 10% interest rate volatility, σ; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 1% early termination penalty, π; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Figure 5.2.C. Value of Future Payments (A)

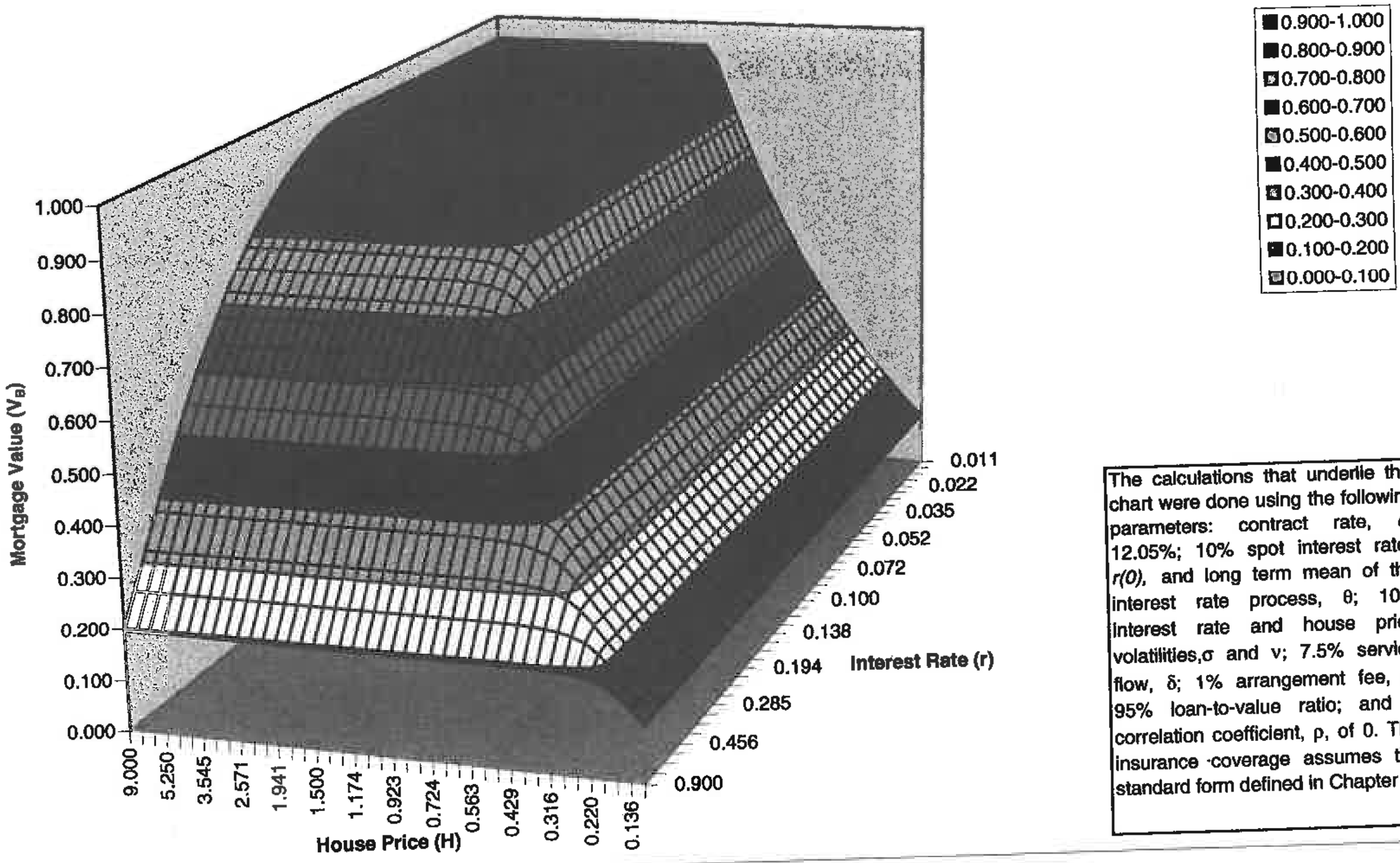
(Endowment Mortgage With an Early Termination Penalty)



The calculations that underlie this chart were done using the following parameters: contract rate, d 12.98%; 10% spot interest rate, $r(0)$, and long term mean of the interest rate process, θ; 10% interest rate volatility, σ; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 15% early termination penalty, π; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Figure 5.3.A. Mortgage Value (V_B)

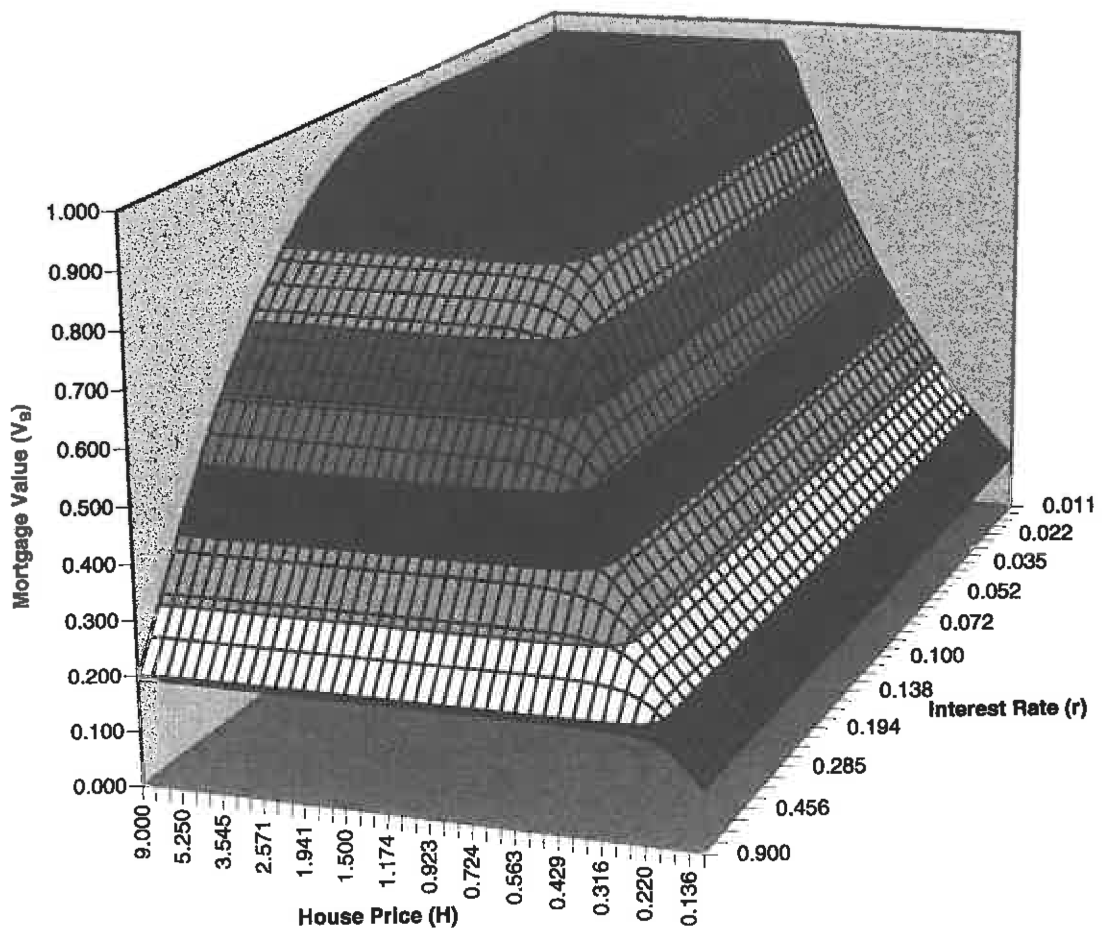
(Repayment Mortgage Without an Early Termination Penalty)



The calculations that underlie this chart were done using the following parameters: contract rate, *c* 12.05%; 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 10% interest rate and house price volatilities,σ and ν; 7.5% service flow, δ; 1% arrangement fee, ξ 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance ·coverage assumes the standard form defined in Chapter 3.

Figure 5.3.B. Mortgage Value (V_B)

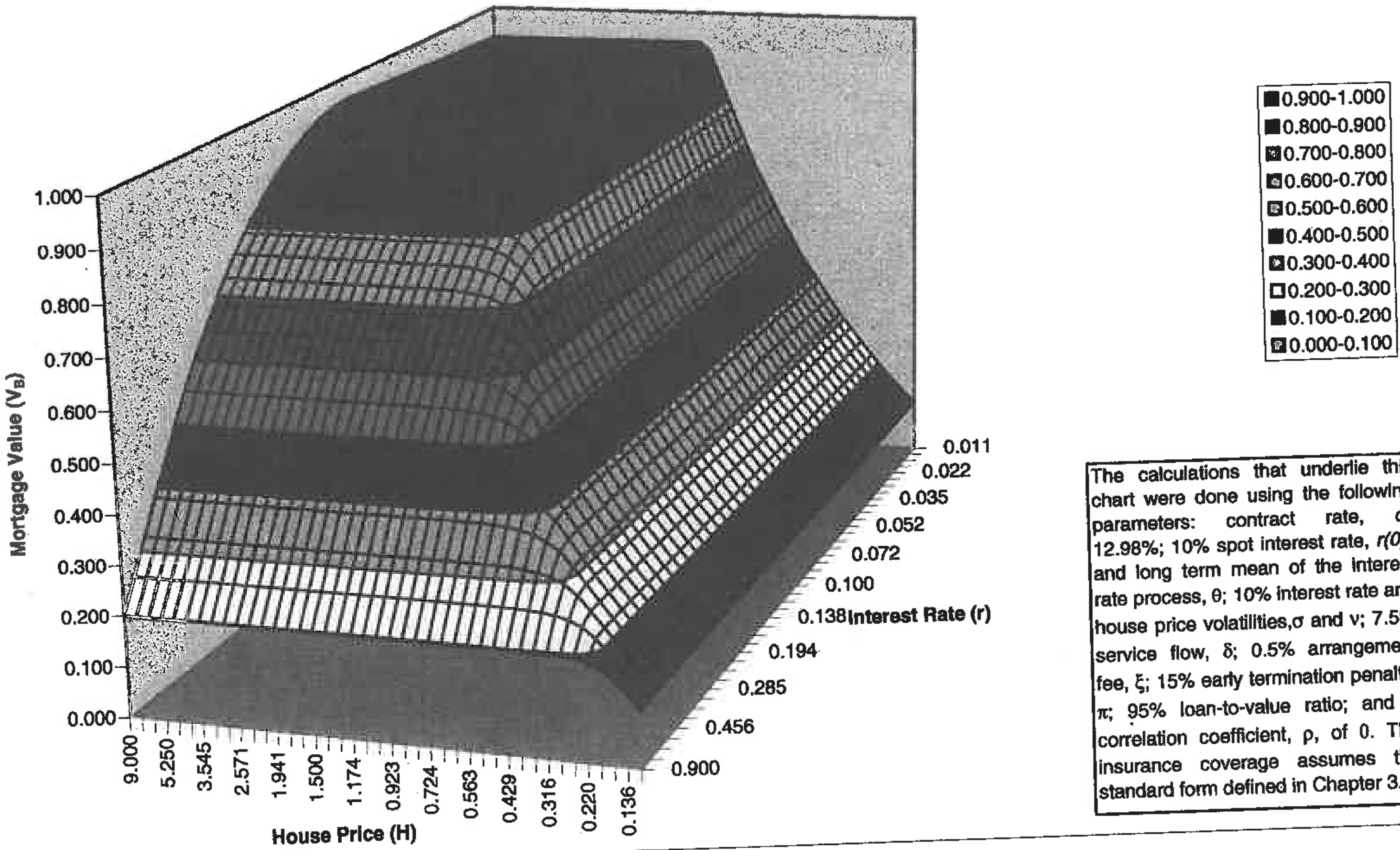
(Repayment Mortgage With an Early Termination Penalty)



The calculations that underlie this chart were done using the following parameters: contract rate, c 11.89%; 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 10% interest rate and house price volatilities, σ and ν; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 1% early termination penalty, π 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

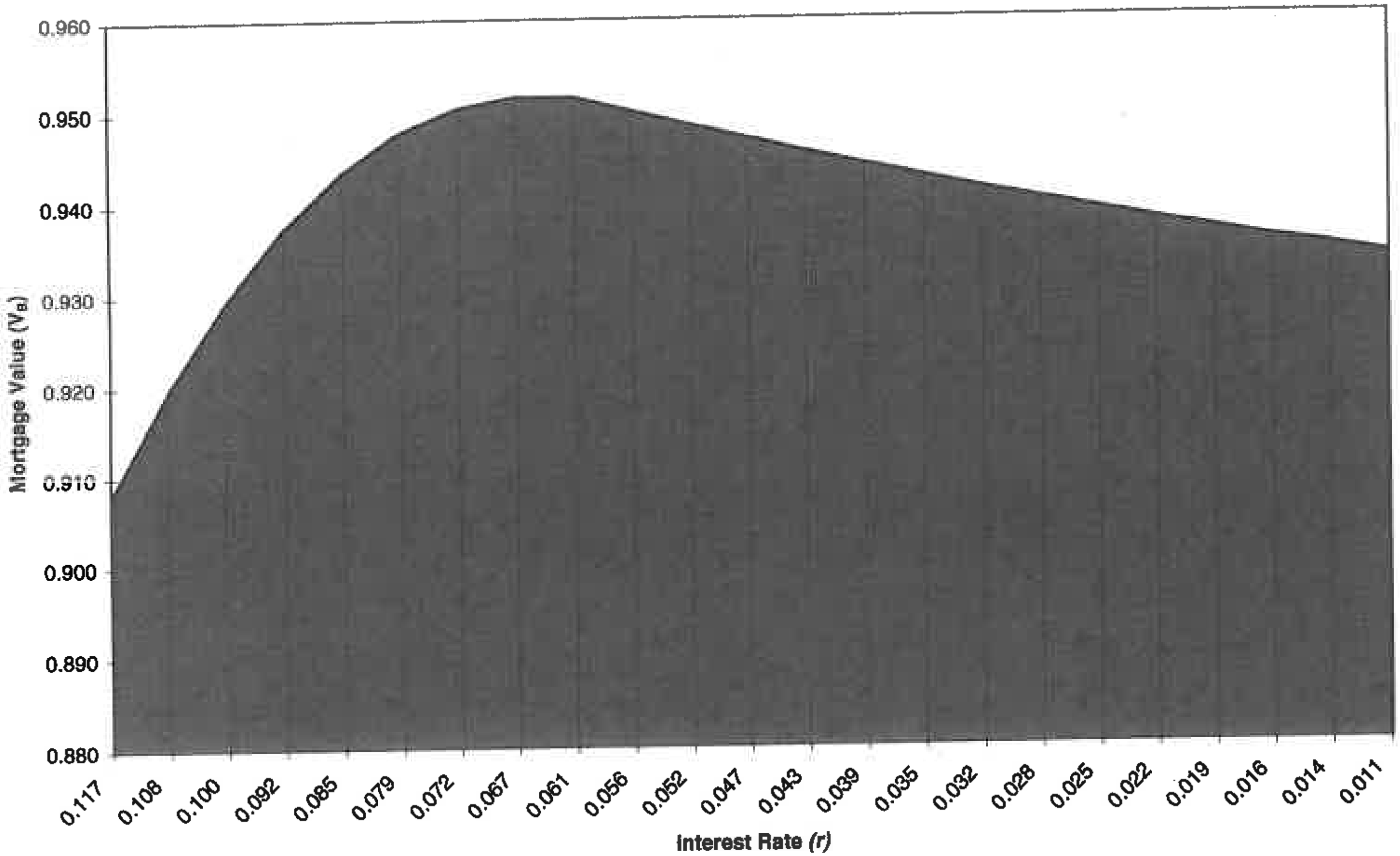
Figure 5.3.C. Mortgage Value (V_B)

(Endowment Mortgage With an Early Termination Penalty)



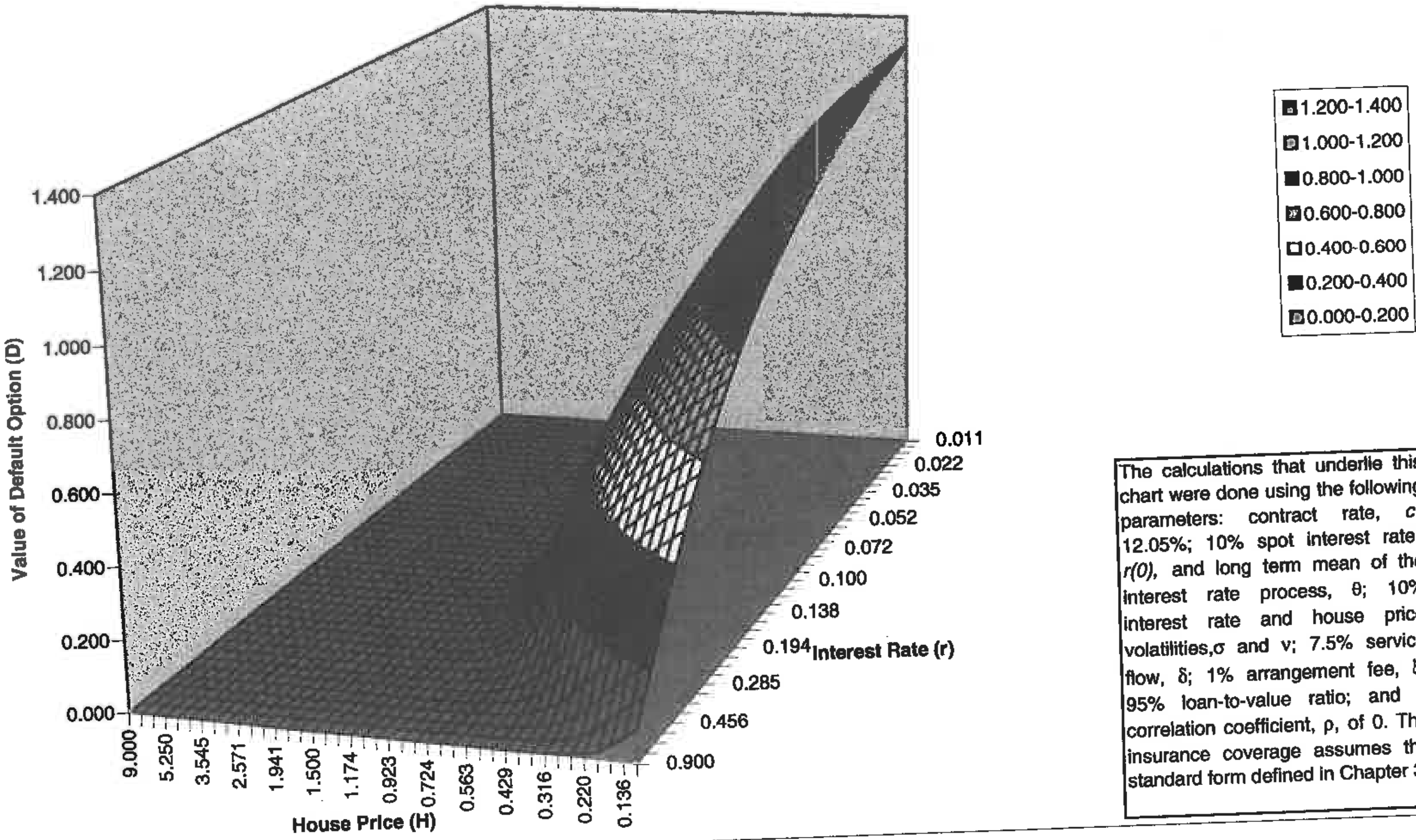
The calculations that underlie this chart were done using the following parameters: contract rate, *d*, 12.98%; 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 10% interest rate and house price volatilities,σ and ν; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 15% early termination penalty, π; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Figure 5.3.D. Evolution of the Mortgage Value (V_B) for Low Levels of the Spot Interest Rate
(Endowment Mortgage With an Early Termination Penalty)



The calculations that underlie this chart were done using the following parameters: contract rate, δ 12.98%; 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ ; 10% interest rate and house price volatilities, σ and ν ; 7.5% service flow, δ ; 0.5% arrangement fee, ξ ; 15% early termination penalty, π ; 95% loan-to-value ratio; 0 correlation coefficient, ρ ; and a house price, H, of 1. The insurance coverage assumes the standard form defined in Chapter 3.

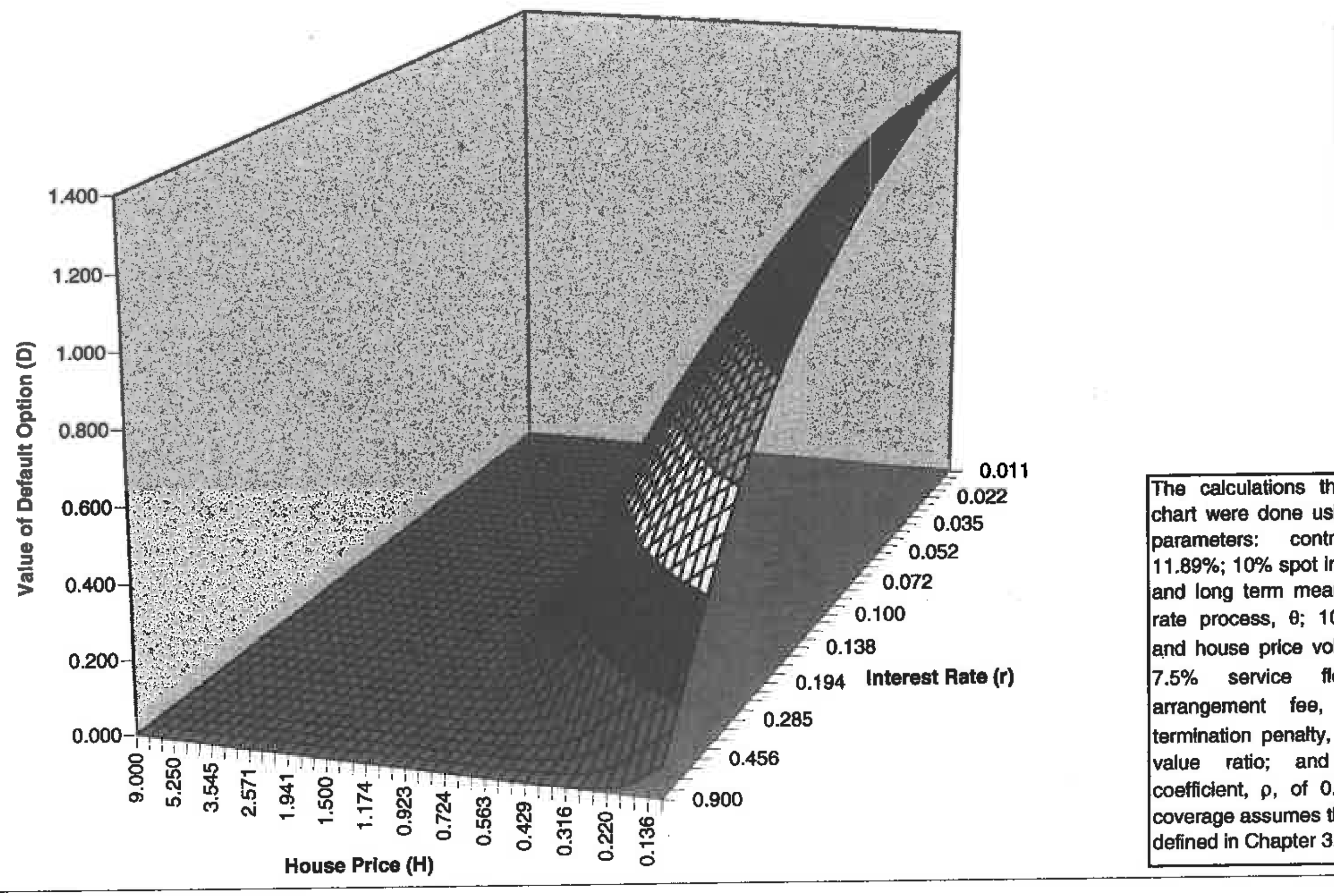
Figure 5.4.A. Default Option (D)
(Repayment Mortgage Without an Early Termination Penalty)



The calculations that underlie this chart were done using the following parameters: contract rate, c 12.05%; 10% spot interest rate, $r(0)$, and long term mean of the interest rate process, θ; 10% interest rate and house price volatilities, σ and ν; 7.5% service flow, δ; 1% arrangement fee, ξ 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Figure 5.4.B. Default Option (D)

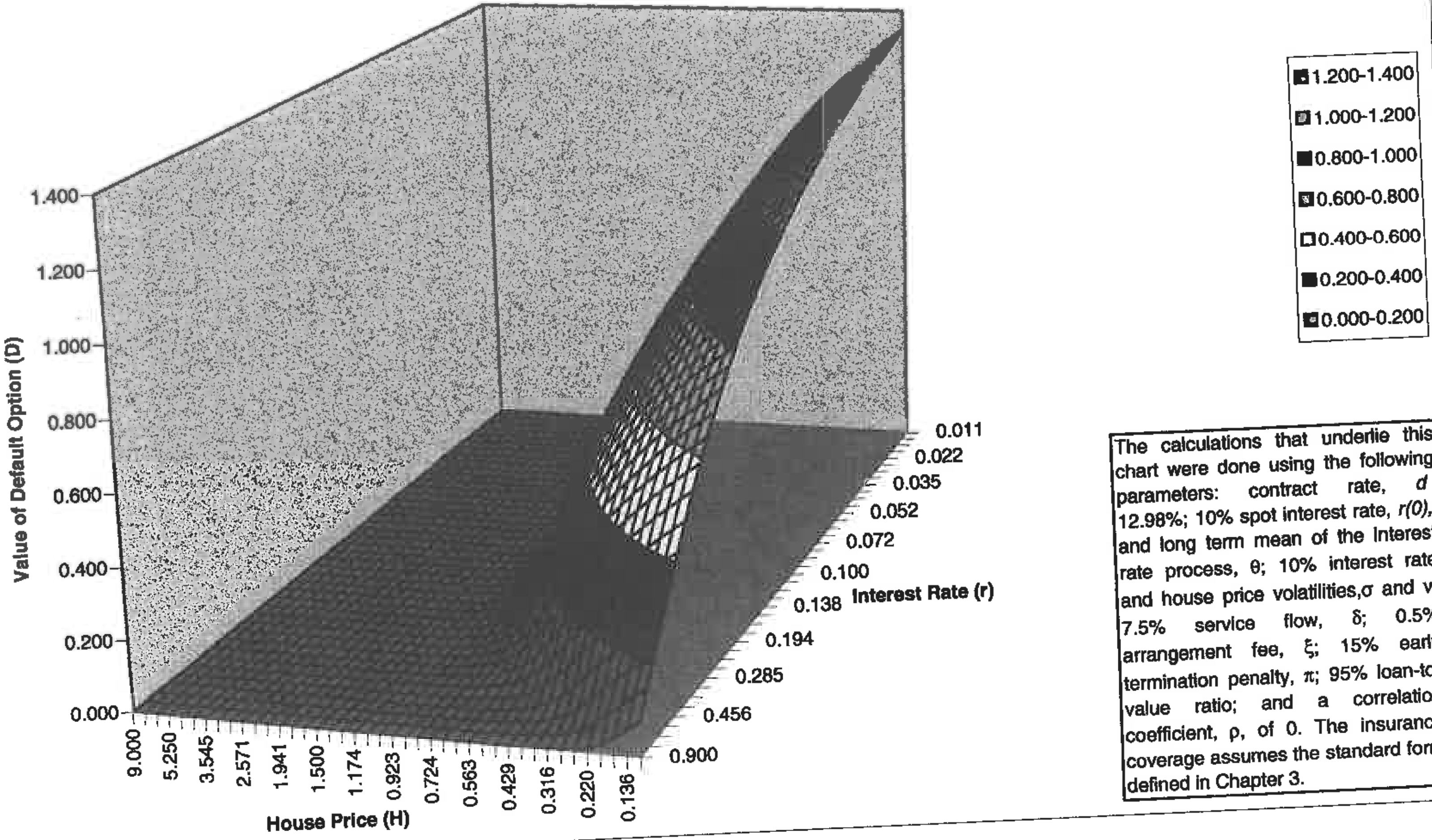
(Repayment Mortgage With an Early Termination Penalty)



The calculations that underlie this chart were done using the following parameters: contract rate, *c* 11.89%; 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 10% interest rate and house price volatilities, σ and ν; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 1% early termination penalty, π; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Figure 5.4.C. Default Option (D)

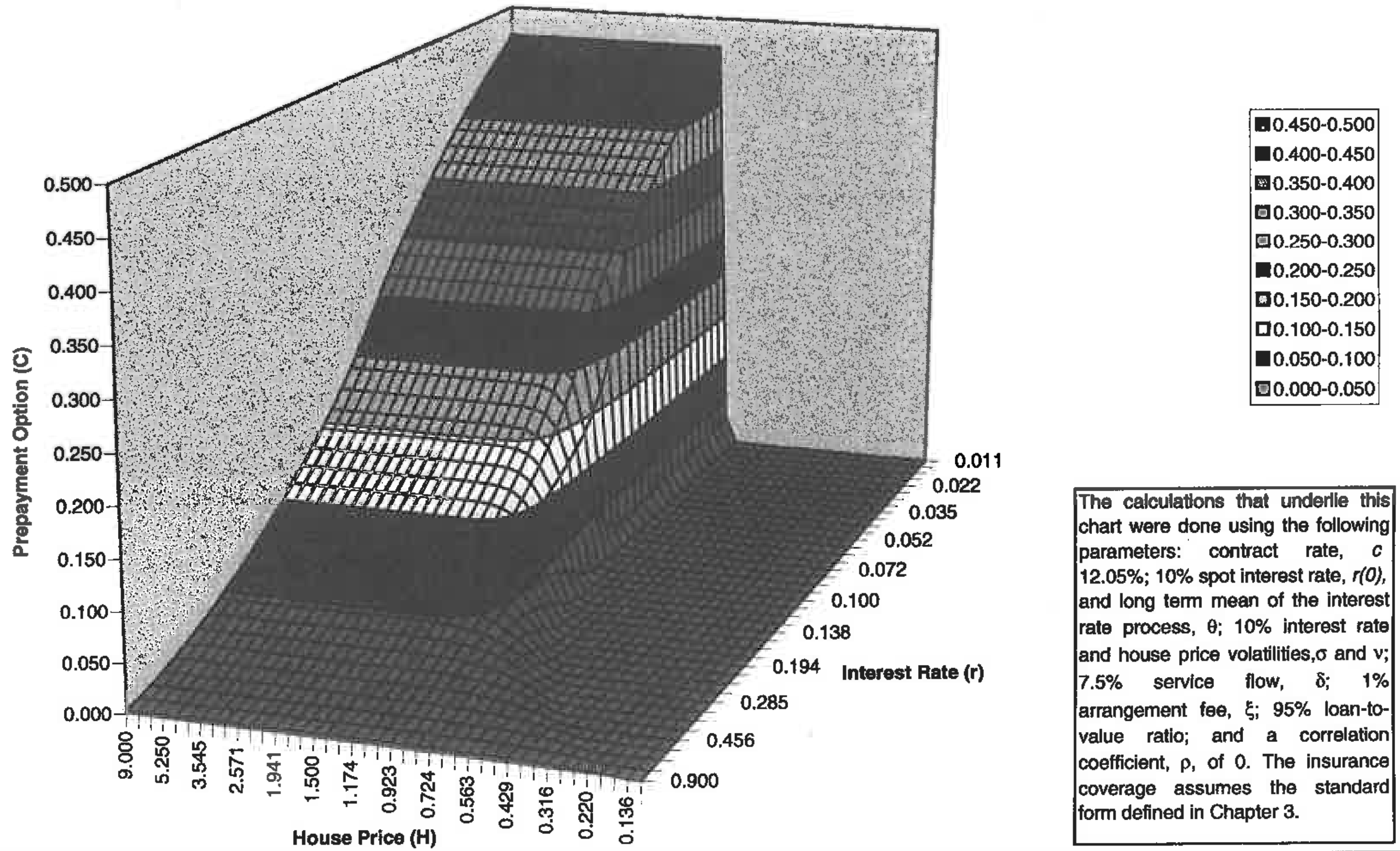
(Endowment Mortgage With an Early Termination Penalty)



The calculations that underlie this chart were done using the following parameters: contract rate, *d* 12.98%; 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 10% interest rate and house price volatilities,σ and ν 7.5% service flow, δ; 0.5% arrangement fee, ξ; 15% early termination penalty, π; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

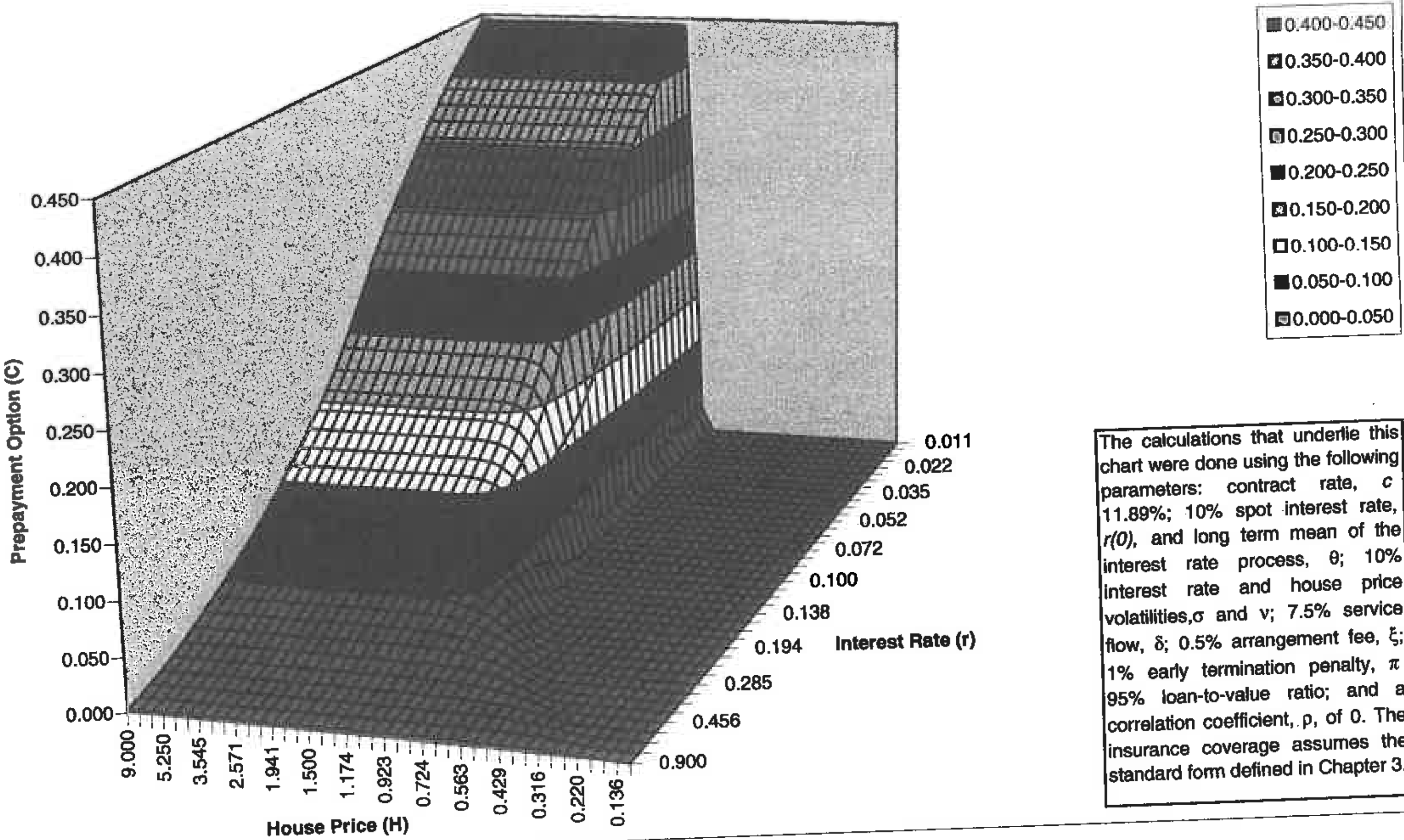
Figure 5.5.A. Value of Prepayment Option (C)

(Repayment Mortgage Without an Early Termination Penalty)



The calculations that underlie this chart were done using the following parameters: contract rate, c 12.05%; 10% spot interest rate, $r(0)$, and long term mean of the interest rate process, θ; 10% interest rate and house price volatilities, σ and v; 7.5% service flow, δ; 1% arrangement fee, ξ; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

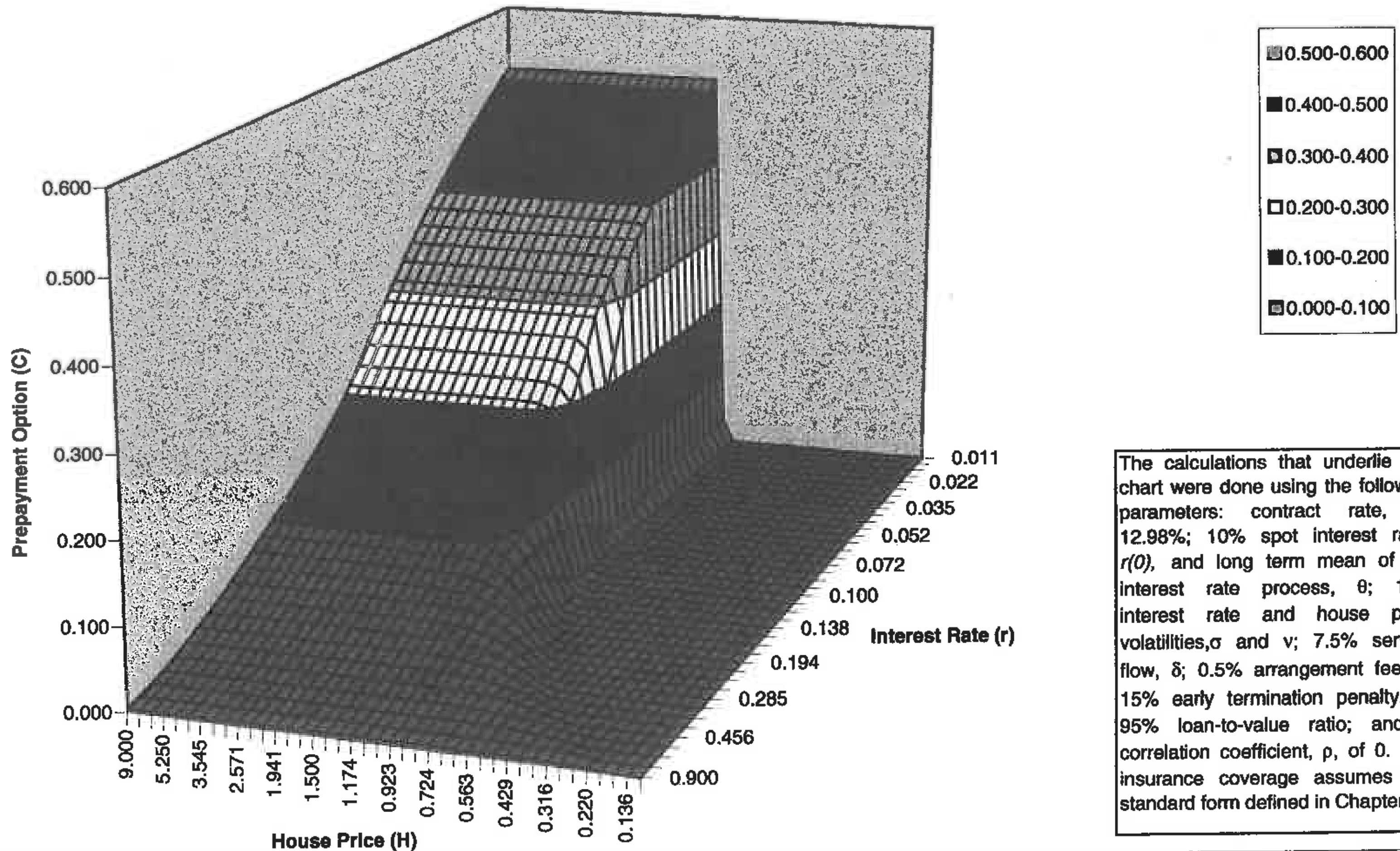
Figure 5.5.B. Value of Prepayment Option (C)
(Repayment Mortgage With an Early Termination Penalty)



The calculations that underlie this chart were done using the following parameters: contract rate, c 11.89%; 10% spot interest rate, $r(0)$, and long term mean of the interest rate process, θ; 10% interest rate and house price volatilities, σ and ν; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 1% early termination penalty, π 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

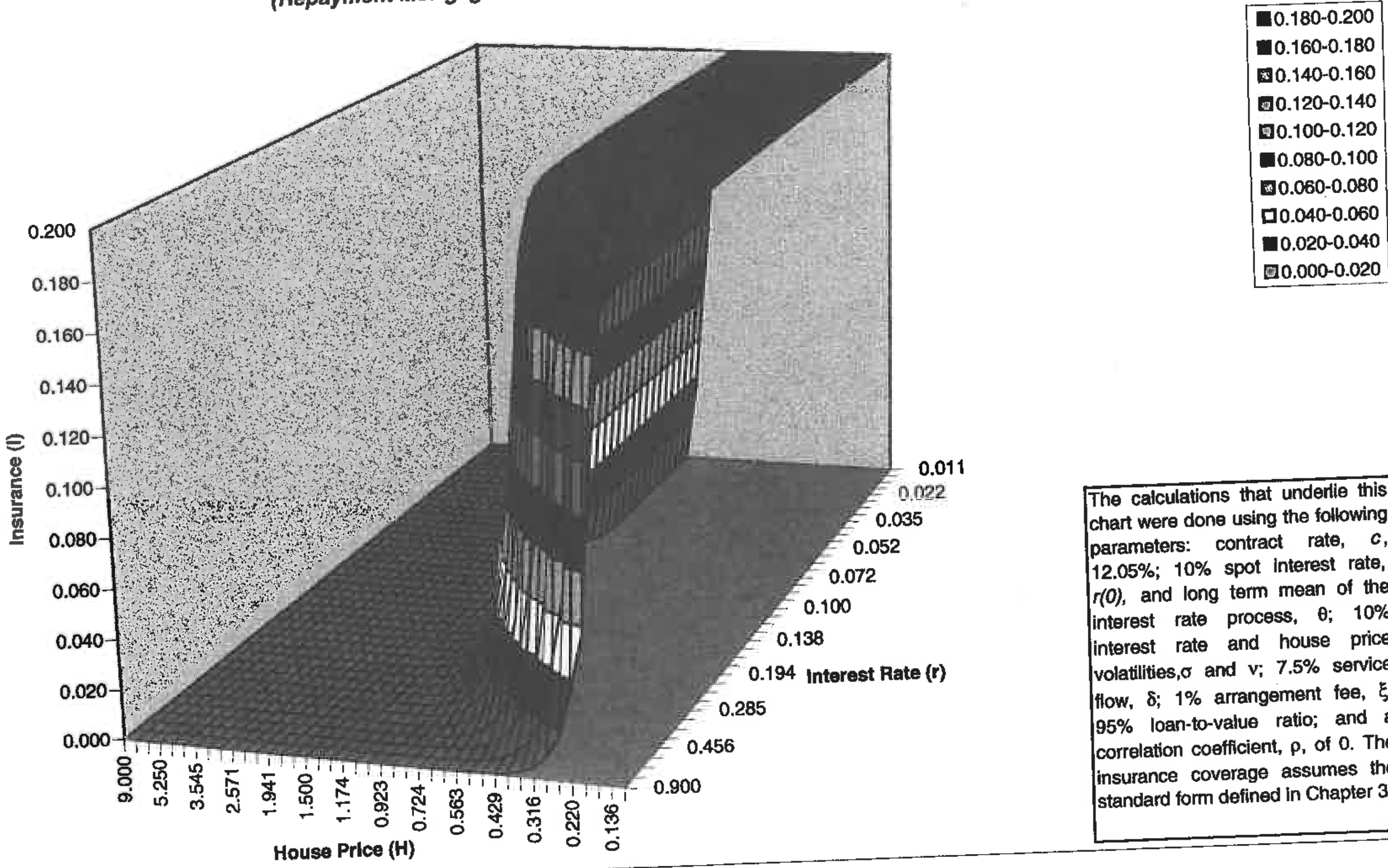
Figure 5.5.C. Value of Prepayment Option (C)

(Endowment Mortgage With an Early Termination Penalty)



The calculations that underlie this chart were done using the following parameters: contract rate, *d*, 12.98%; 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 10% interest rate and house price volatilities,σ and ν; 7.5% service flow, δ; 0.5% arrangement fee, ξ 15% early termination penalty, π 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

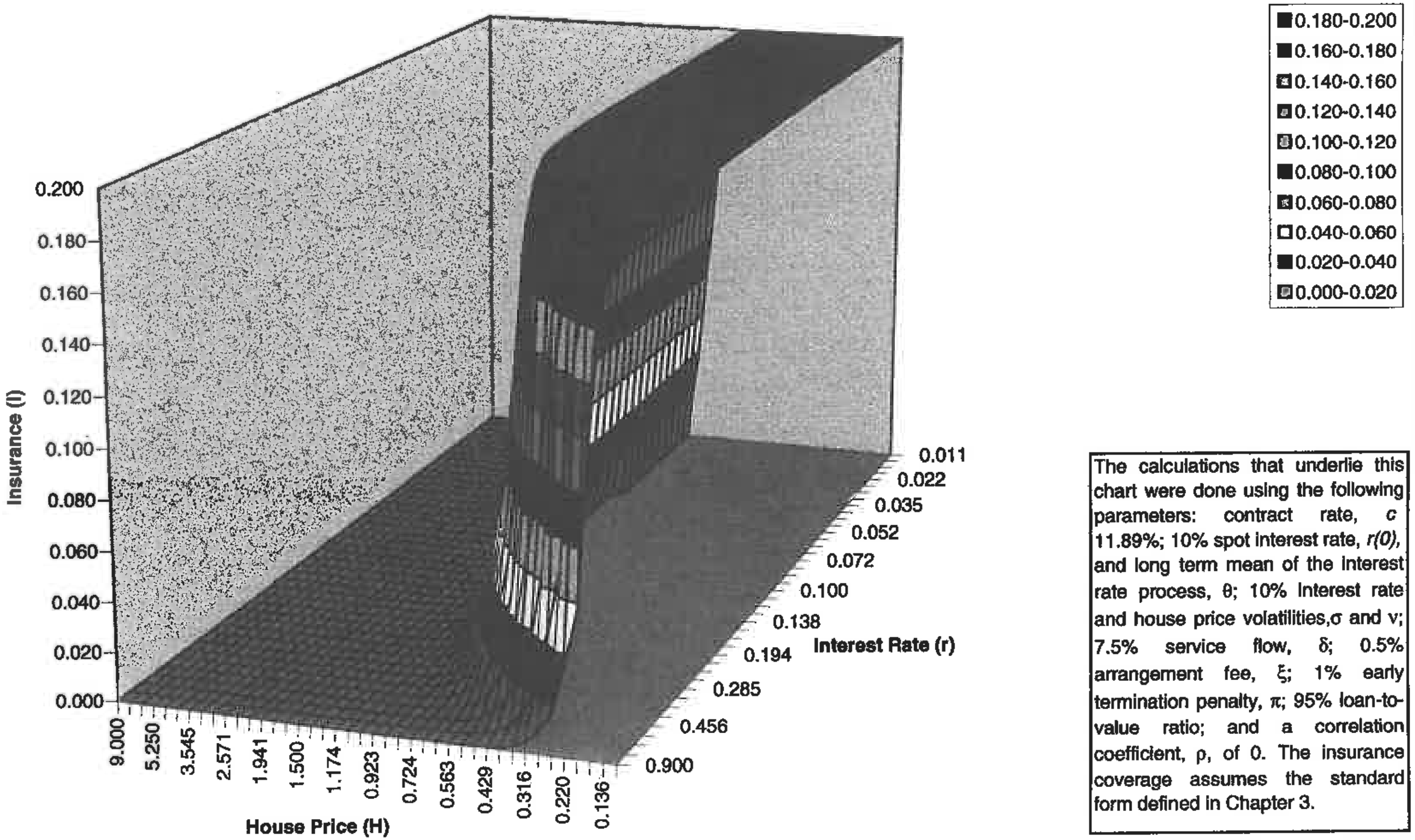
Figure 5.6.A. Value of Insurance Coverage (I)
(Repayment Mortgage Without an Early Termination Penalty)



The calculations that underlie this chart were done using the following parameters: contract rate, c, 12.05%; 10% spot interest rate, $r(0)$, and long term mean of the interest rate process, θ; 10% interest rate and house price volatilities, σ and v; 7.5% service flow, δ; 1% arrangement fee, ξ; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

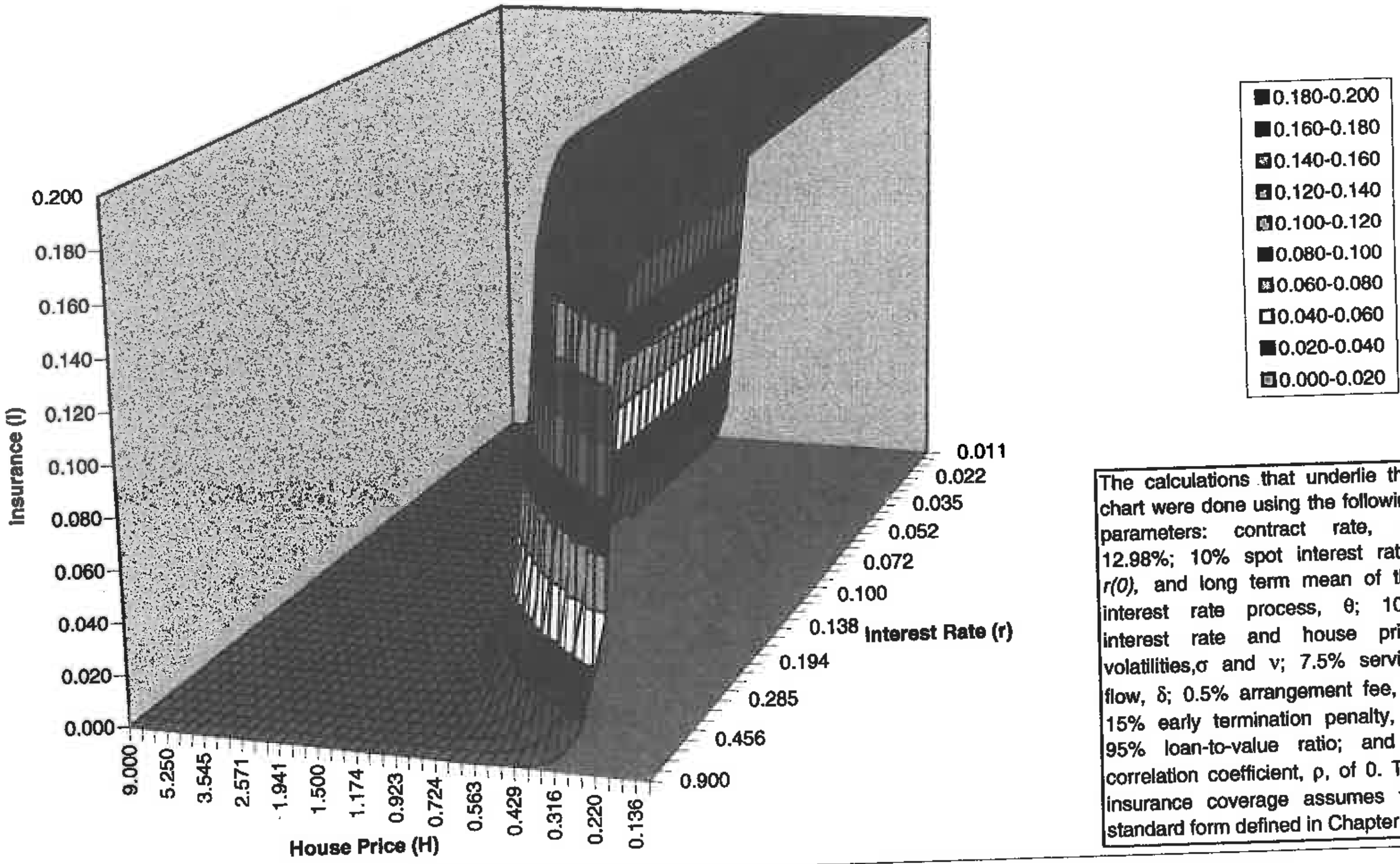
Figure 5.6.B. Value of Insurance Coverage (I)

(Repayment Mortgage With an Early Termination Penalty)



The calculations that underlie this chart were done using the following parameters: contract rate, c 11.89%; 10% spot interest rate, $r(0)$, and long term mean of the interest rate process, θ; 10% interest rate and house price volatilities, σ and v; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 1% early termination penalty, π; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

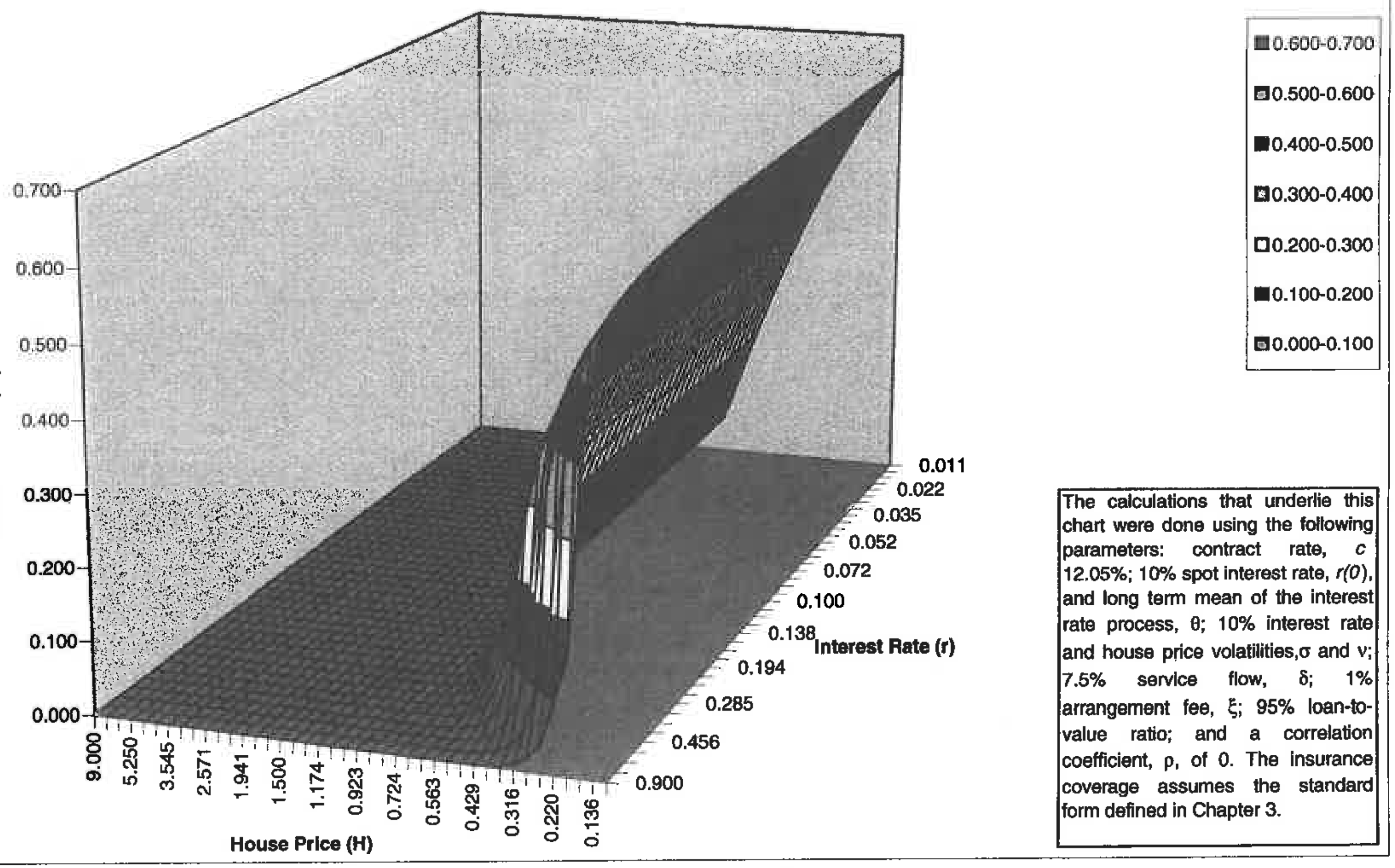
Figure 5.6.C. Value of Insurance Coverage (I)
(Endowment Mortgage With an Early Termination Penalty)



The calculations that underlie this chart were done using the following parameters: contract rate, *d* 12.98%; 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 10% interest rate and house price volatilities,σ and ν; 7.5% service flow, δ; 0.5% arrangement fee, ξ 15% early termination penalty, p; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

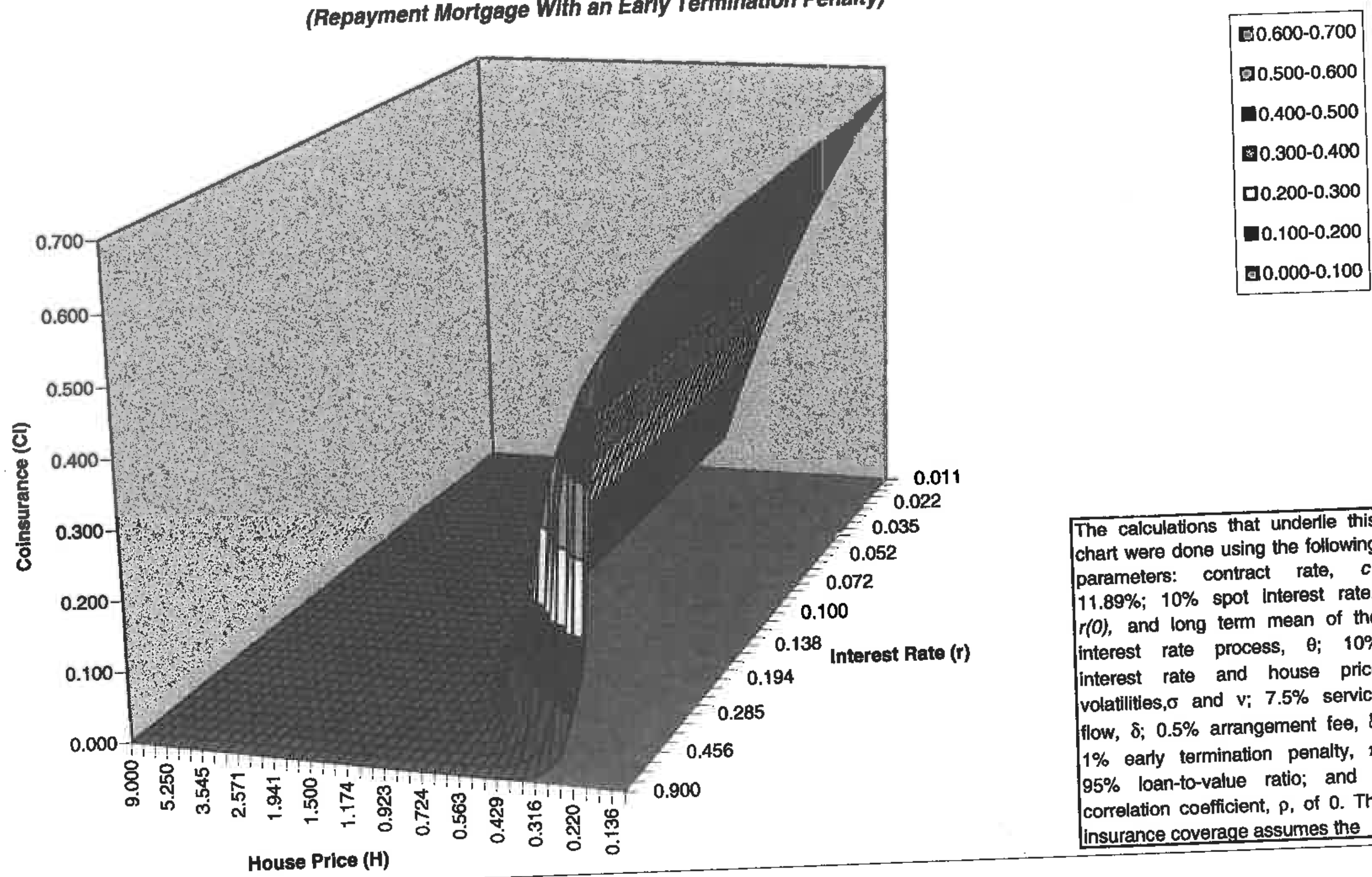
Figure 5.7.A. Value of Coinsurance (CI)

(Repayment Mortgage Without an Early Termination Penalty)



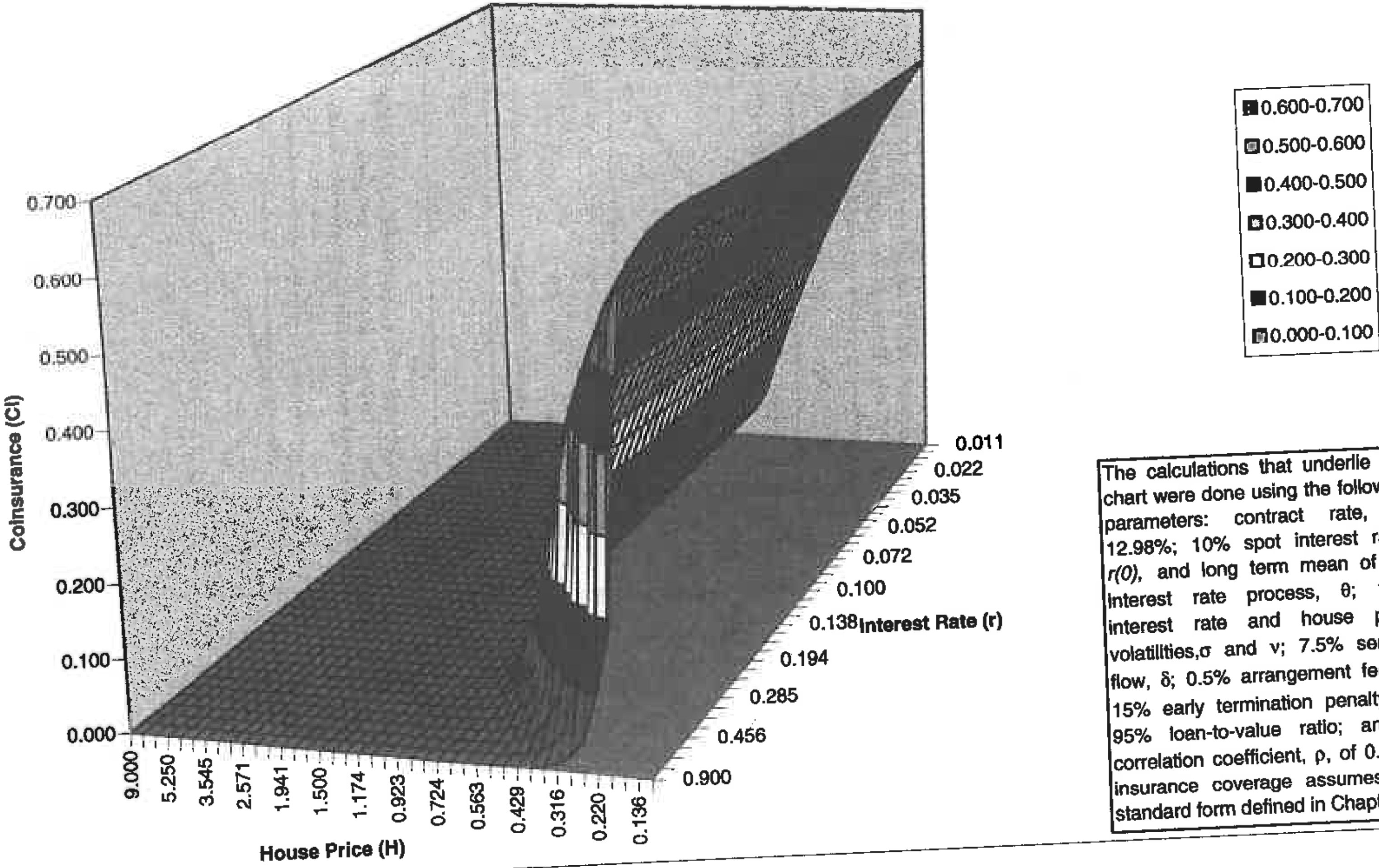
The calculations that underlie this chart were done using the following parameters: contract rate, c 12.05%; 10% spot interest rate, $r(0)$, and long term mean of the interest rate process, θ; 10% interest rate and house price volatilities, σ and ν; 7.5% service flow, δ; 1% arrangement fee, ξ; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Figure 5.7.B. Value of Coinsurance (CI)
(Repayment Mortgage With an Early Termination Penalty)



The calculations that underlie this chart were done using the following parameters: contract rate, c 11.89%; 10% spot interest rate, $r(0)$, and long term mean of the interest rate process, θ; 10% interest rate and house price volatilities, σ and v; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 1% early termination penalty, π; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the

Figure 5.7.C. Value of Coinsurance (CI)
(Endowment Mortgage With an Early Termination Penalty)



The calculations that underlie this chart were done using the following parameters: contract rate, d 12.98%; 10% spot interest rate, $r(0)$, and long term mean of the interest rate process, θ; 10% interest rate and house price volatilities, σ and v; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 15% early termination penalty, π; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Figure 5.8.A. Mortgage Value (Mortgage Contract Without Mortgage Indemnity Guarantee and Arrangement Fee)
Repayment Mortgage Without an Early Termination Penalty

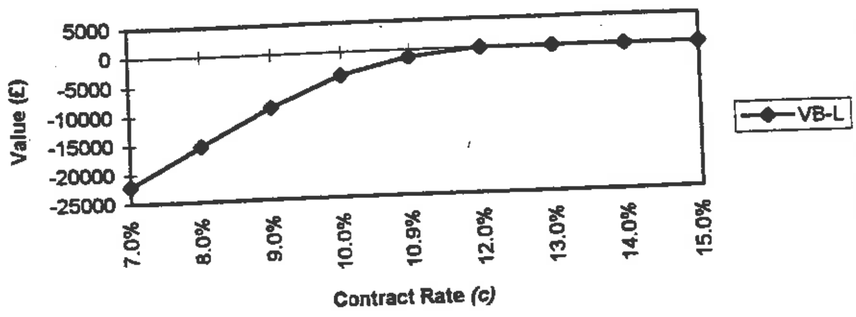


Figure 5.8.B. Mortgage Value (Mortgage Contract Without Mortgage Indemnity Guarantee and Arrangement Fee)
Repayment Mortgage With an Early Termination Penalty

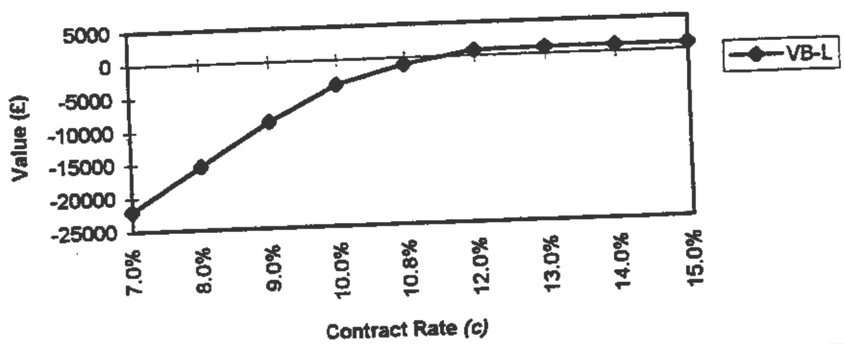
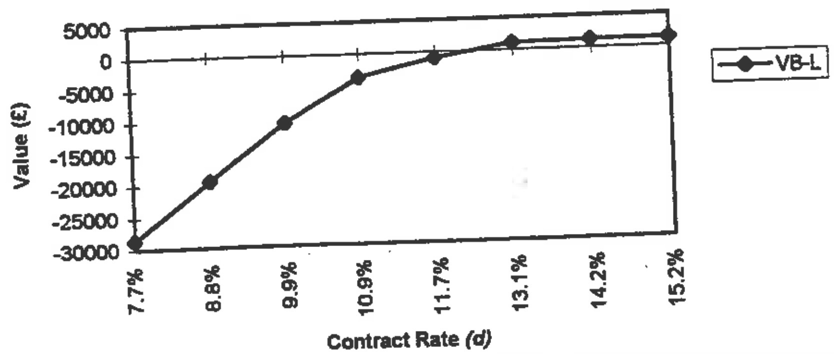
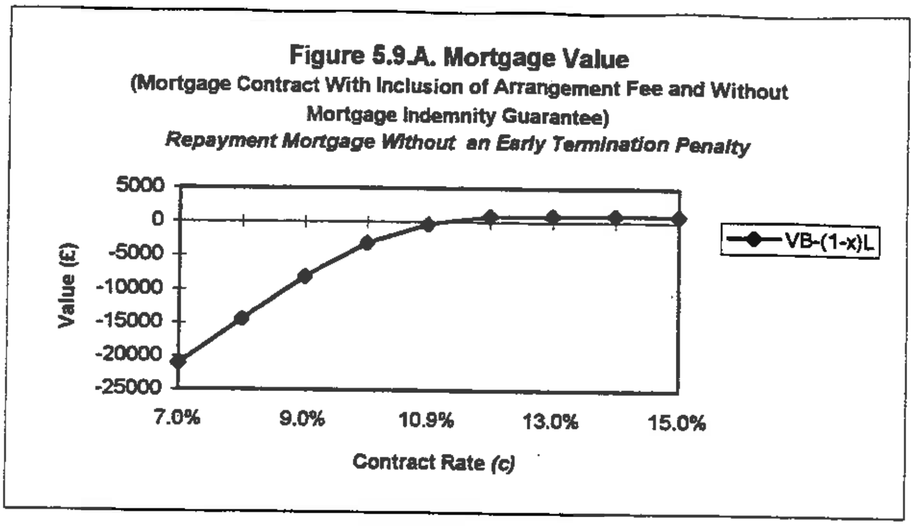


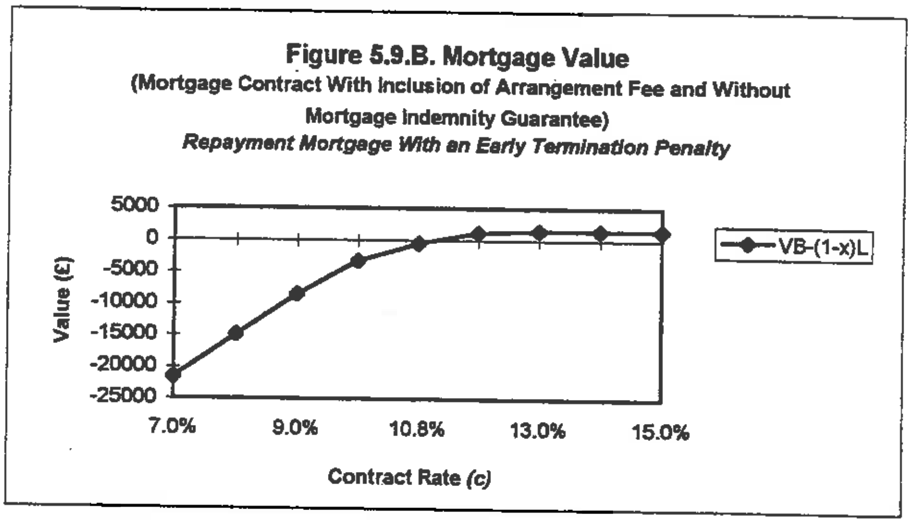
Figure 5.8.C. Mortgage Value (Mortgage Contract Without Mortgage Indemnity Guarantee and Arrangement Fee)
Endowment Mortgage With an Early Termination Penalty

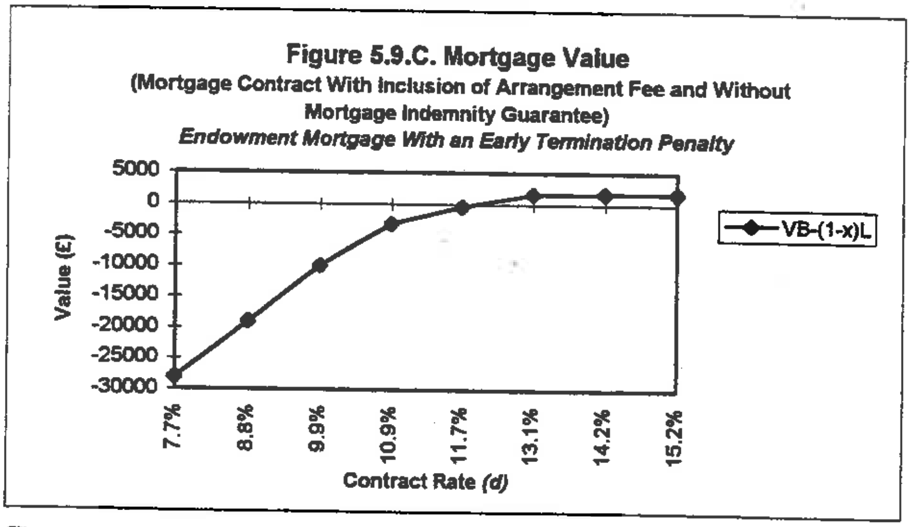


The parameters that were considered in the construction of these charts are the same that underlie the results reported in tables 5.2.A., 5.2.B. and 5.2.C. respectively.

Note: Due to software constraints the value of the mortgage to the borrower, normally represented by V_B, is here represented by VB.







The parameters that were considered in the construction of these charts are the same that underlie the results reported in tables 5.2.A., 5.2.B. and 5.2.C. respectively.

Note: Due to software constraints the value of the mortgage to the borrower, normally represented by V_B, is here represented by VB. Similarly the arrangement fee, normally represented by ξ, is here represented by x.

Figure 5.10.A. Mortgage Value
(Mortgage Contract With Inclusion of a Mortgage Indemnity Guarantee and Without Arrangement Fee)
Repayment Mortgage Without an Early Termination Penalty

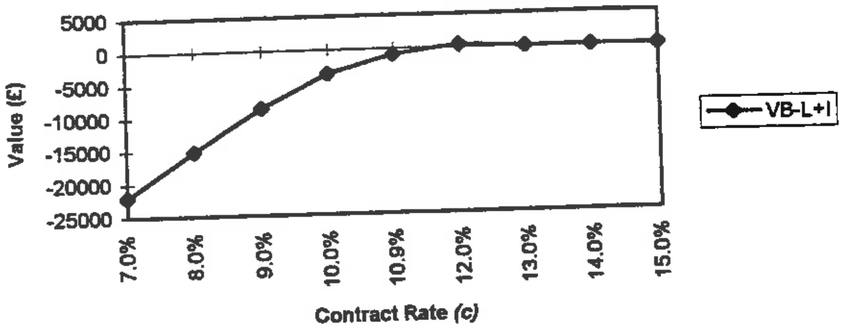


Figure 5.10.B. Mortgage Value
(Mortgage Contract With Inclusion of a Mortgage Indemnity Guarantee and Without Arrangement Fee)
Repayment Mortgage With an Early Termination Penalty

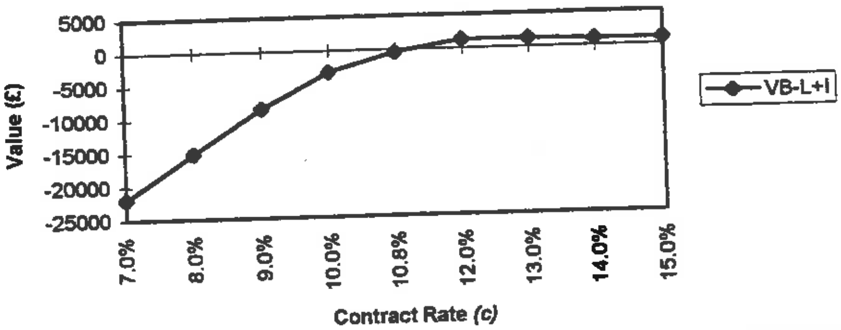
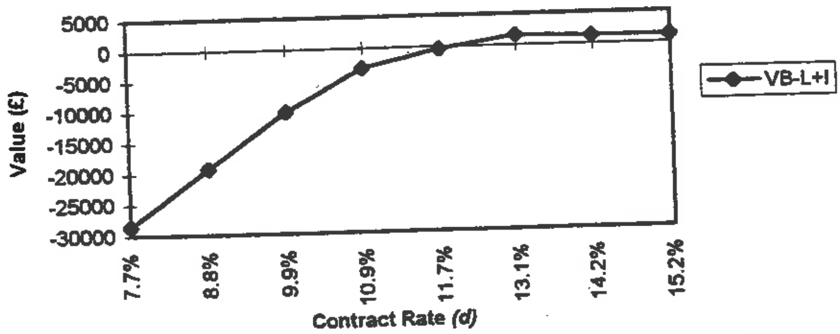
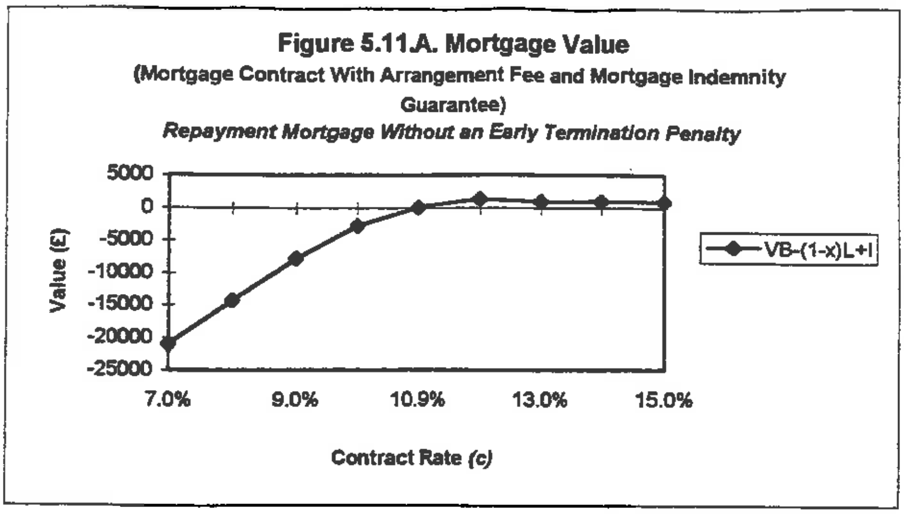


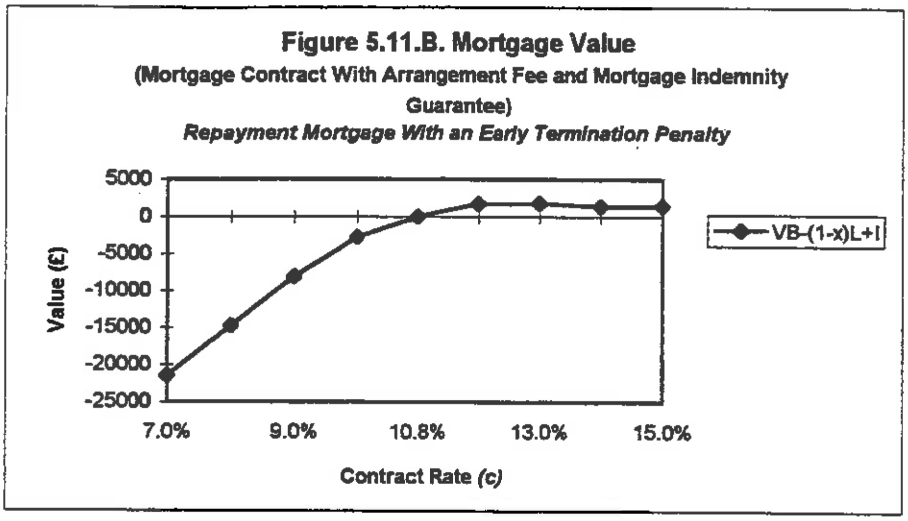
Figure 5.10.C. Mortgage Value
(Mortgage Contract With Inclusion of a Mortgage Indemnity Guarantee and Without Arrangement Fee)
Endowment Mortgage With an Early Termination Penalty

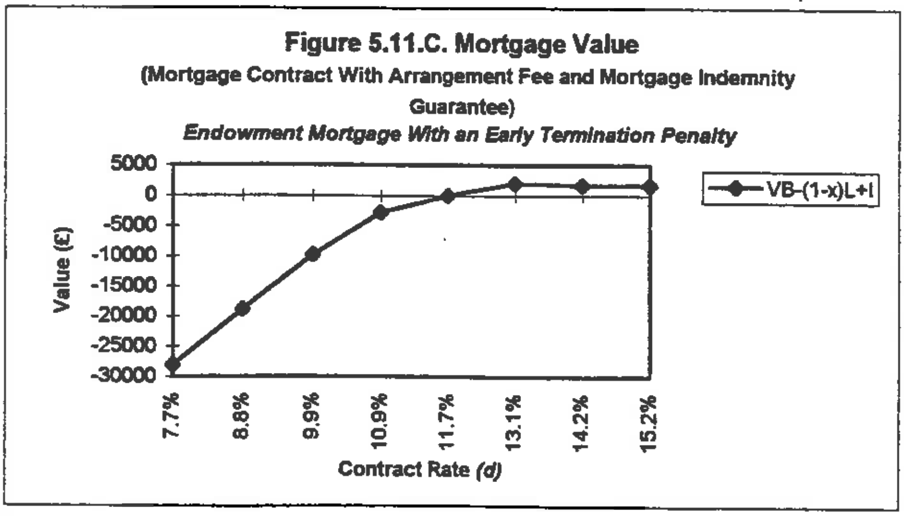


The parameters that were considered in the construction of these charts are the same that underlie the results reported in tables 5.2.A., 5.2.B. and 5.2.C. respectively.

Note: Due to software constraints the value of the mortgage to the borrower, normally represented by V_B, is here represented by VB.



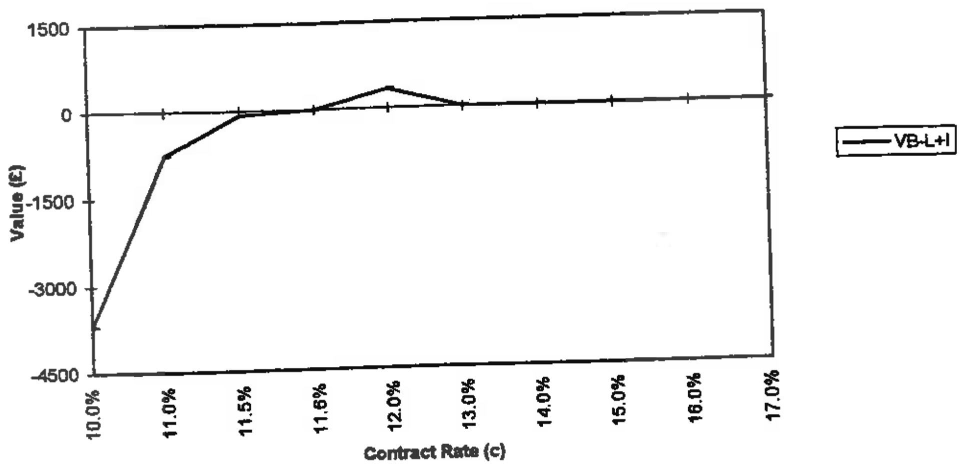




The parameters that were considered in the construction of these charts are the same that underlie the results reported in tables 5.2.A., 5.2.B. and 5.2.C. respectively.

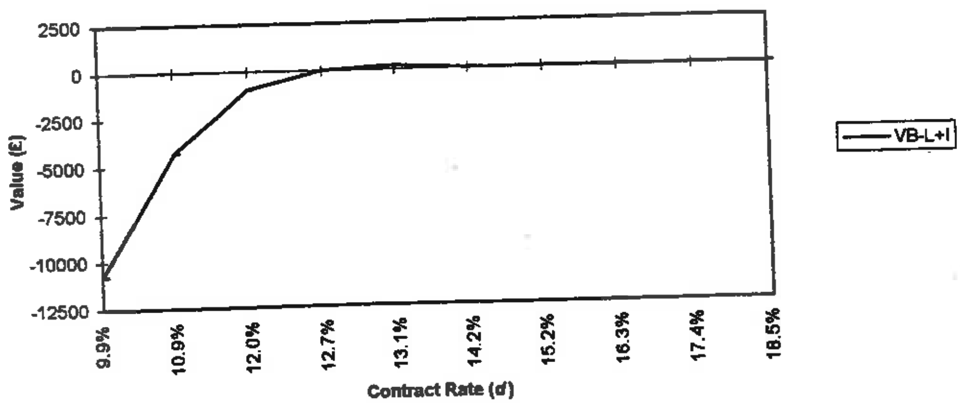
Note: Due to software constraints the value of the mortgage to the borrower, normally represented by V_B, is here represented by VB. Similarly the arrangement fee, normally represented by ξ, is here represented by x.

Figure 5.12.A. Mortgage Value (Mortgage Contract With a Mortgage Indemnity Guarantee)
Repayment Mortgage Without Arrangement Fee and Early Termination Penalty



The calculations that underlie this chart were done using the following parameters: 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 5% interest rate and house price volatilities, σ and v; 7.5% service flow, δ; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.
Note: Due to software constraints the value of the mortgage to the borrower, normally represented by V_B, is here represented by VB .

Figure 5.12.B. Mortgage Value (Mortgage Contract With a Mortgage Indemnity Guarantee)
Endowment *Mortgage Without Arrangement Fee and Early Termination Penalty*



The calculations that underlie this chart were done using the following parameters: 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 5% interest rate and house price volatilities, σ and v; 7.5% service flow, δ; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.
Note: Due to software constraints the value of the mortgage to the borrower, normally represented by V_B, is here represented by VB .

Figure 5.13.A. Equilibrium Contract Rates in a *c*, ξ, space

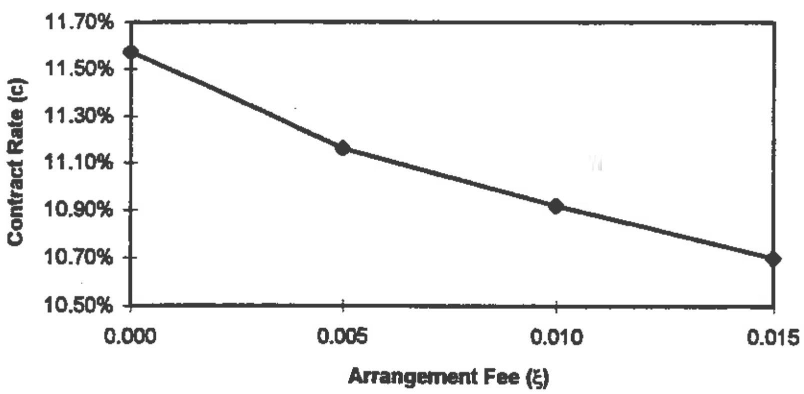
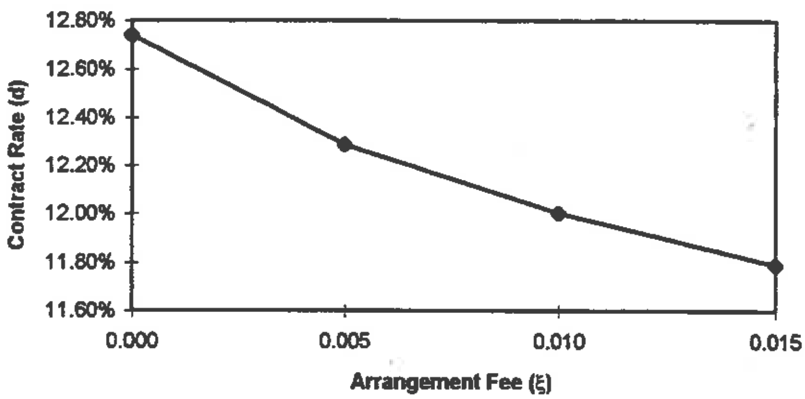


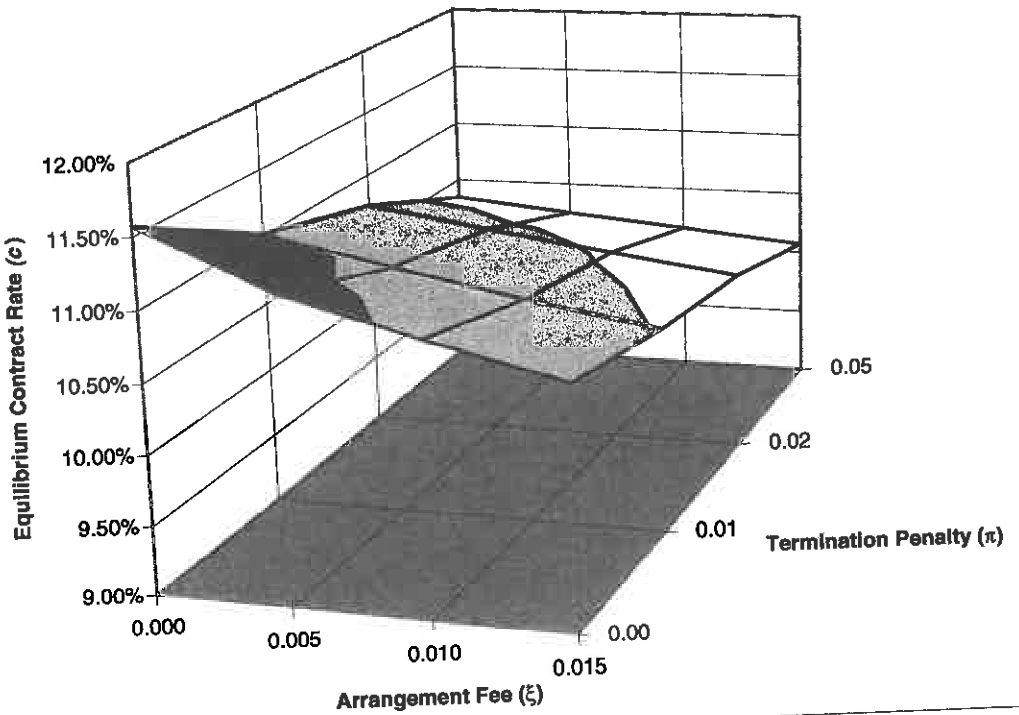
Figure 5.13.B. Equilibrium Contract Rates in a *d*, ξ, space



Underlying the construction of these charts are the following parameters: *r(0)* the spot interest rate, and θ, the long term average of the interest rate, are 10%; σ, the interest rate volatility and ν, the house price volatility, are 5%; δ the house service flow, is 7.5%; and the correlation coefficient between the two state variables, ρ is 0.

Figure 5.14.A. Trade-Off Between Arrangement Fee, Early Termination Penalty and Contract Rate

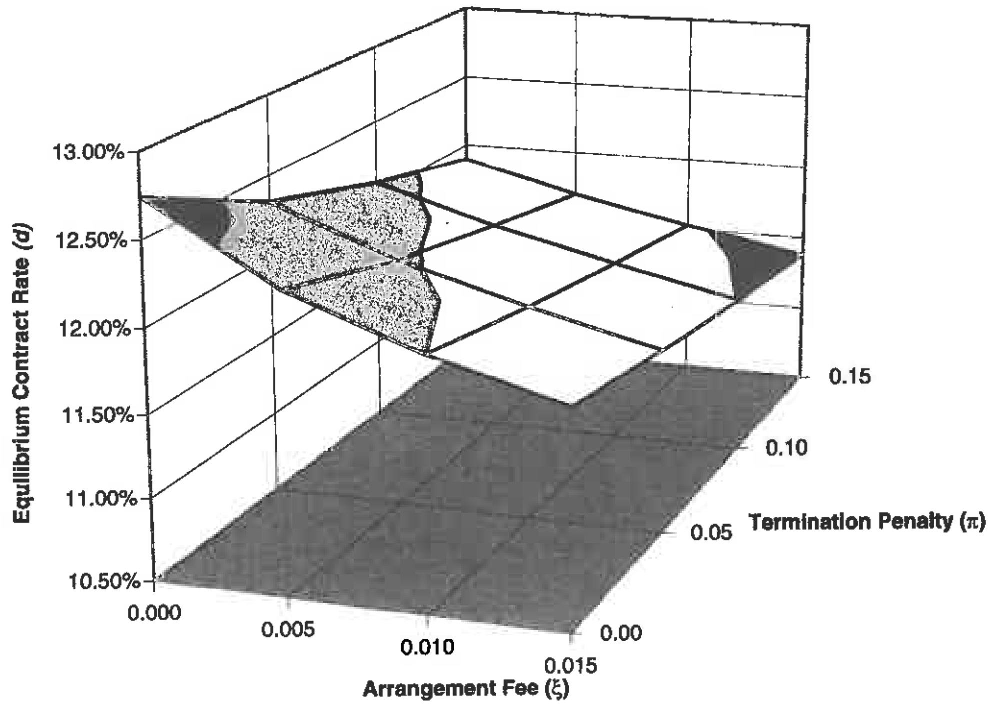
(Repayment Mortgage)



Underlying the construction of this chart are the following parameters: $r(0)$, the spot interest rate, and θ, the long term average of the interest rate, are 10%; σ, the interest rate volatility, and ν, the house price volatility, are 5%; δ, the house service flow, is 7.5%. The correlation coefficient between the two state variables, ρ is 0.

Figure 5.14.B. Trade-Off Between Arrangement Fee, Early Termination Penalty and Contract Rate

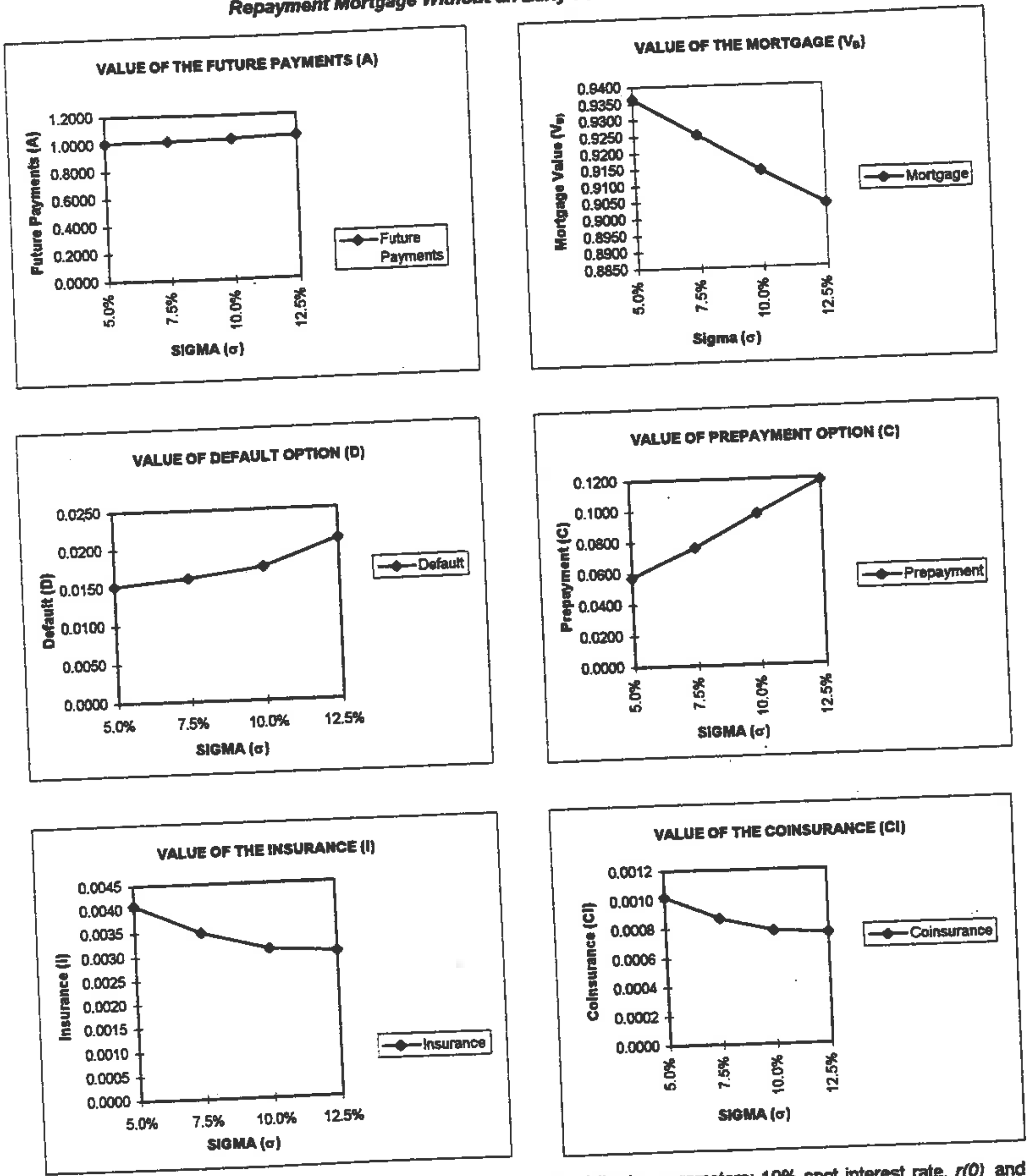
(Endowment Mortgage)



Underlying the construction of this chart are the following parameters: $r(0)$, the spot interest rate, and θ, the long term average of the interest rate, are 10%; σ, the interest rate volatility, and ν, the house price volatility, are 5%; δ, the house service flow, is 7.5%. The correlation coefficient between the two state variables, ρ is 0.

Figure 5.15.A. Effects of Changes in Interest Rate Volatility on the Value of Mortgages and Mortgage Related Assets (In Relaton to Par)

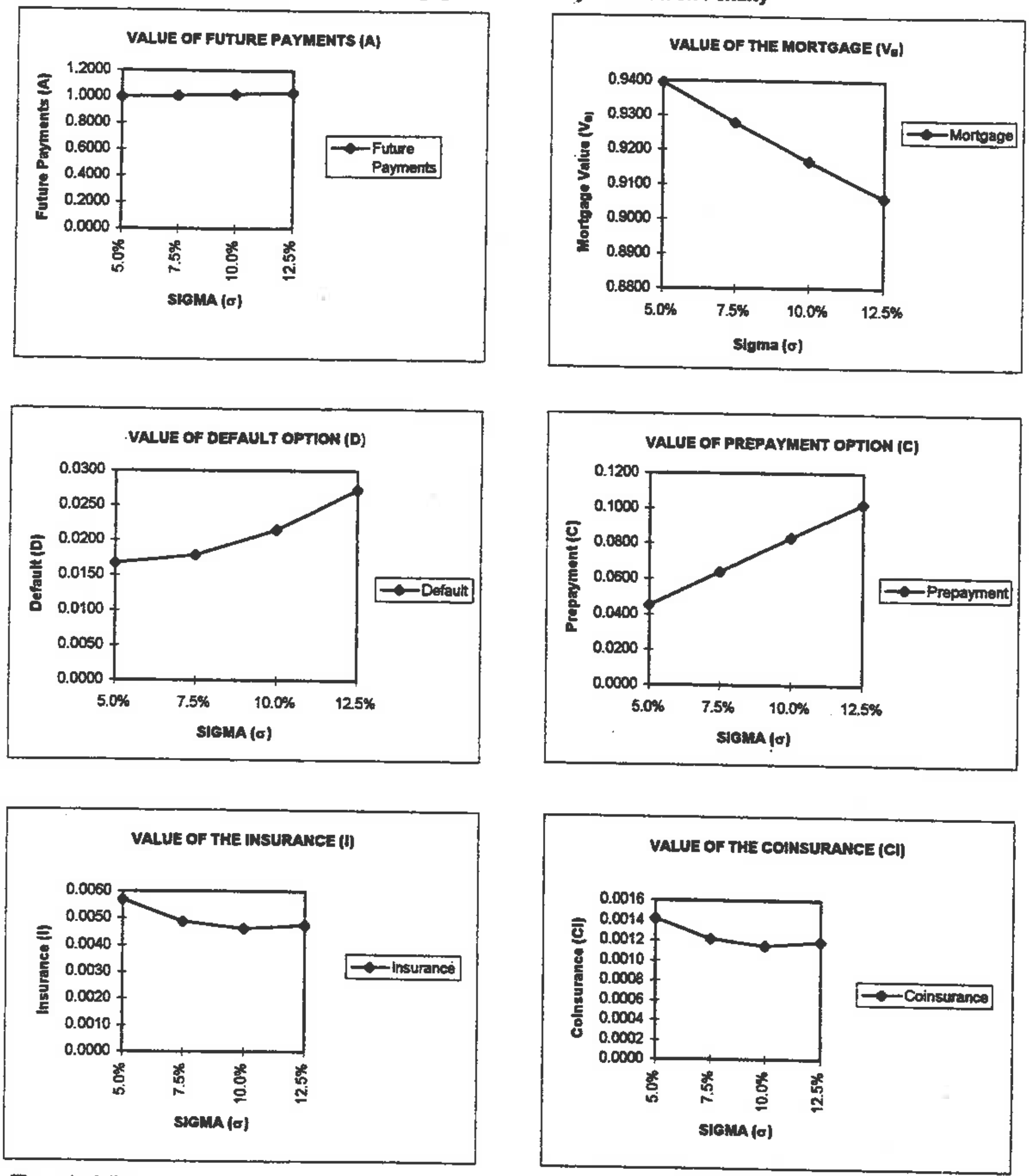
Repayment Mortgage Without an Early Termination Penalty



The calculations that underlie these charts were done using the following parameters: 10% spot interest rate, $r(0)$ and long term mean of the interest rate process, θ; 5% house price volatility, v; 7.5% service flow, δ; 1% arrangement fee, ξ 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Figure 5.15.B. Effects of Changes in Interest rate Volatility on the Value of Mortgages and Mortgage Related Assets (In Relation to Par)

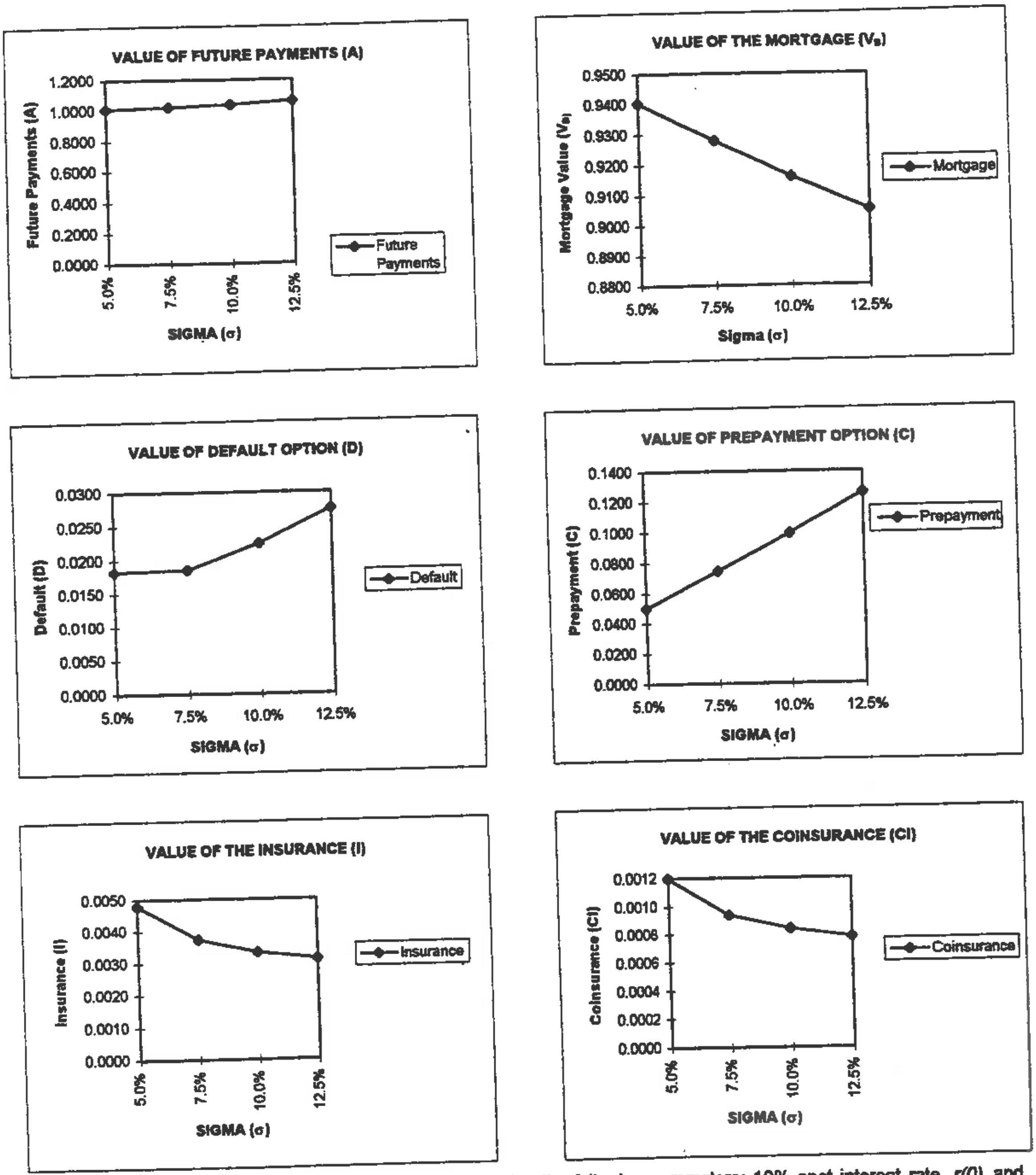
Repayment Mortgage With an Early Termination Penalty



The calculations that underlie these charts were done using the following parameters: 10% spot interest rate, $r(0)$ and long term mean of the interest rate process, θ; 5% house price volatility, ν; 7.5% service flow, δ; 0.5% arrangement fee, ξ 95% loan-to-value ratio; 1% early termination penalty, π; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Figure 5.15.C. Effects of Changes in Interest Rate Volatility on the Value of Mortgages and Mortgage Related Assets (In Relation to Par)

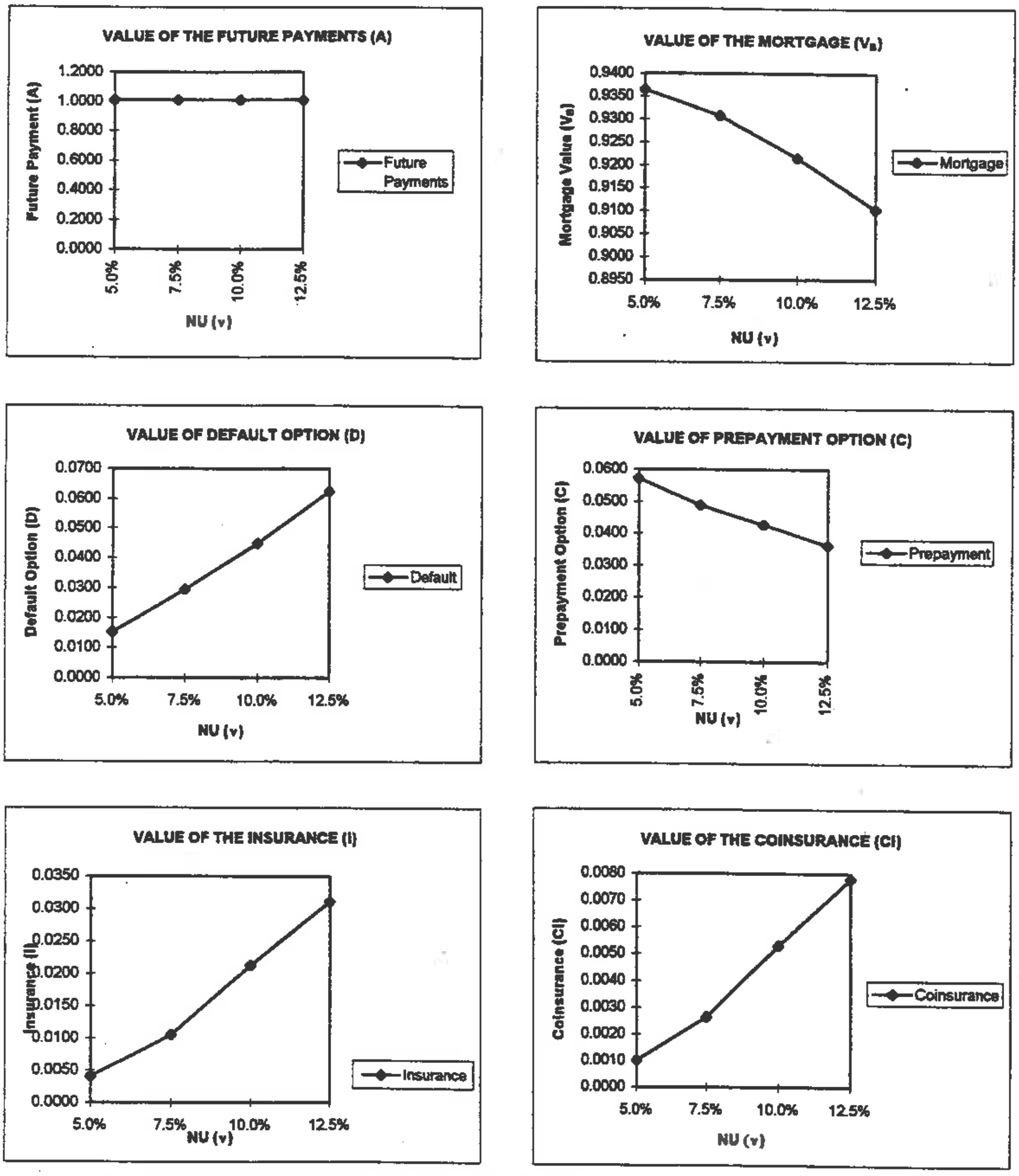
Endowment Mortgage With an Early Termination Penalty



The calculations that underlie these charts were done using the following parameters: 10% spot interest rate, *r(0)* and long term mean of the interest rate process, θ; 5% house price volatility, ν; 7.5% service flow, δ; 0.5% arrangement fee, ξ 95% loan-to-value ratio; 15% early termination penalty, π; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Figure 5.16.A. Effects of Changes in House Price Volatility on the Value of Mortgages and Mortgage Related Assets (In Relation to Par)

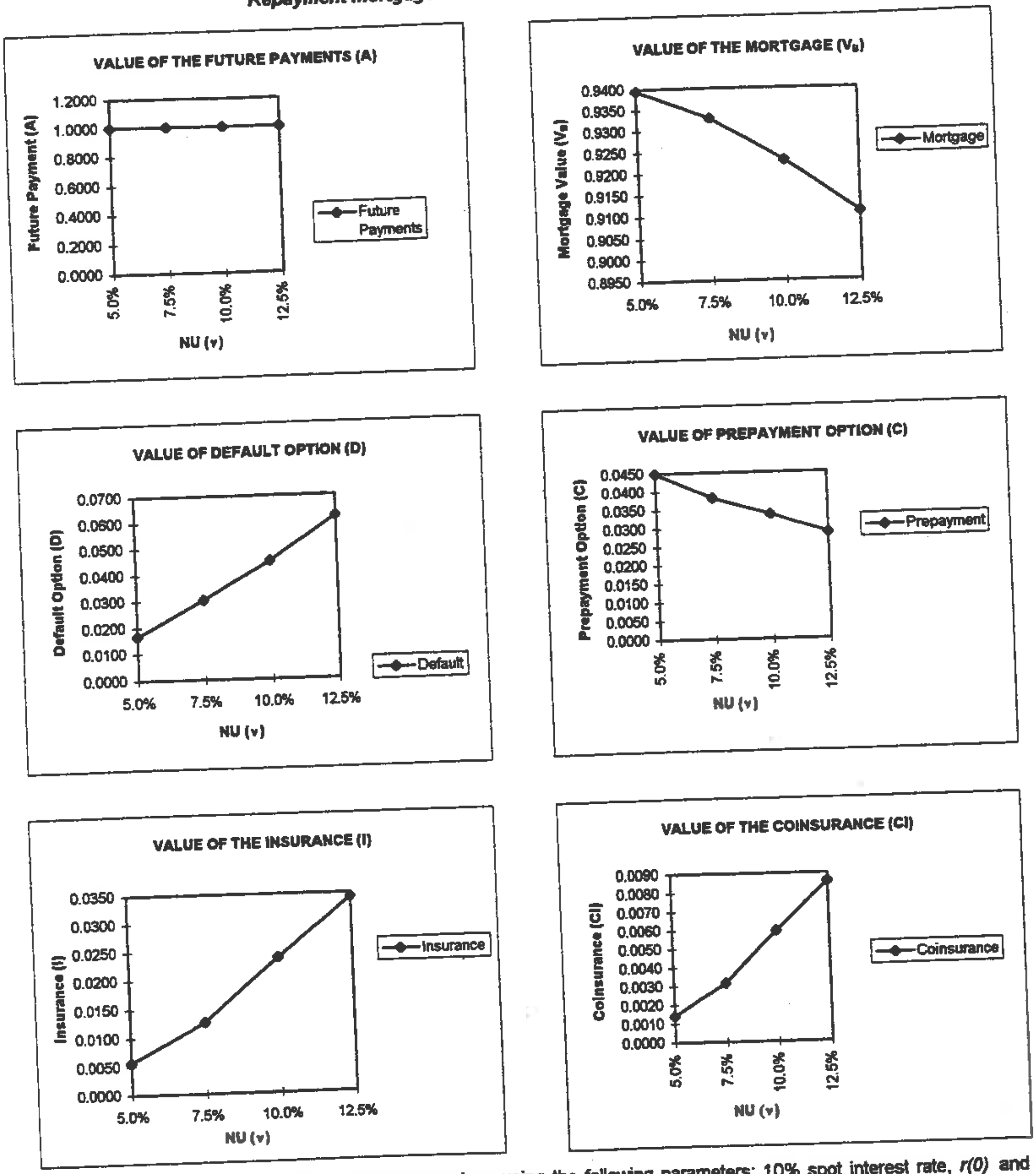
Repayment Mortgage Without an Early Termination Penalty



The calculations that underlie these charts were done using the following parameters: 10% spot interest rate, *r(0)* and long term mean of the interest rate process, θ; 5% interest rate volatility, σ; 7.5% service flow, δ; 1% arrangement fee, ξ 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

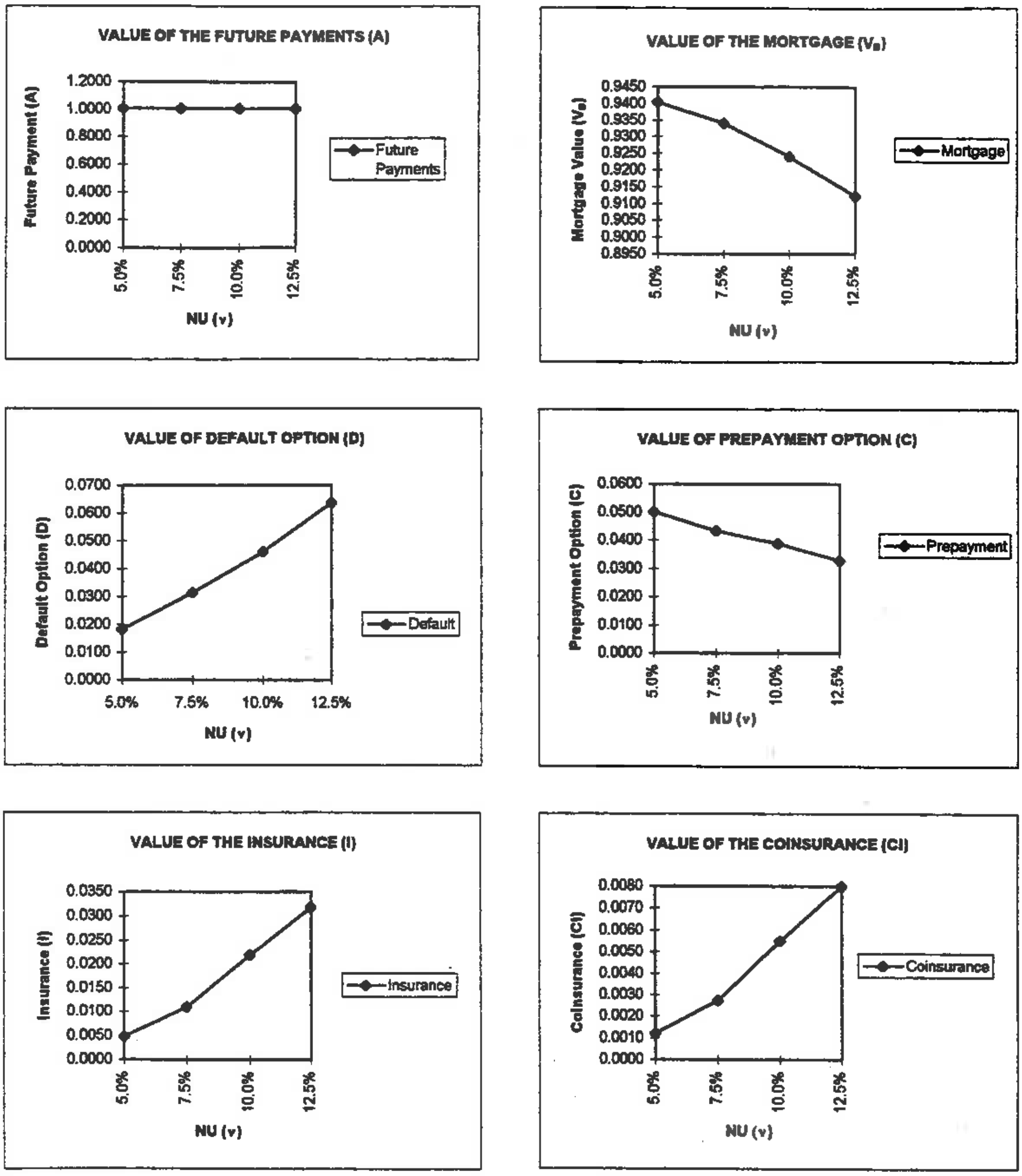
Figure 5.16.B. Effects of Changes in House Price Volatility on the Value of Mortgages and Mortgage Related Assets (In Relation to Par)

Repayment Mortgage With an Early Termination Penalty



The calculations that underlie these charts were done using the following parameters: 10% spot interest rate, *r(0)* and long term mean of the interest rate process, θ; 5% interest rate volatility, σ; 7.5% service flow, δ; 0.5% arrangement fee, ξ 95% loan-to-value ratio; 1% early termination penalty, π; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

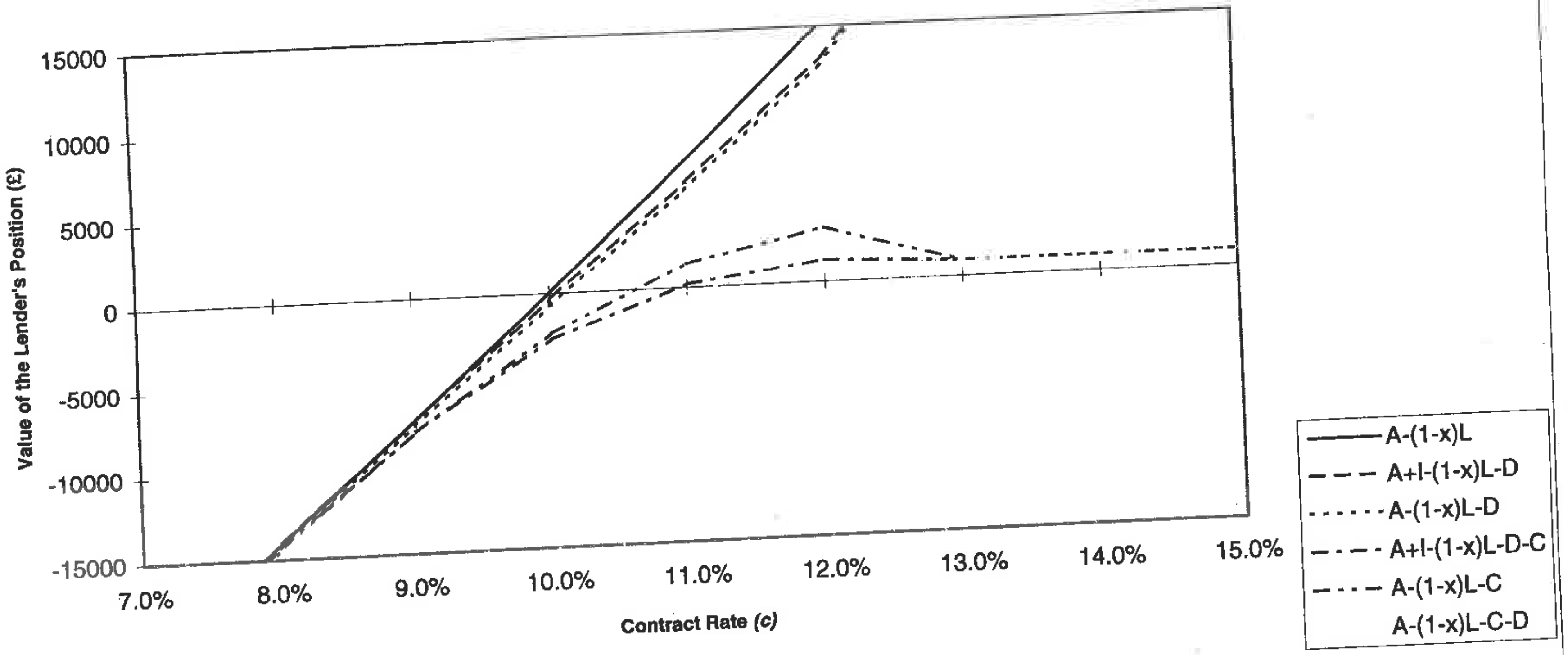
Figure 5.16.C. Effects of Changes in House Price Volatility on the Value of Mortgages and Mortgage Related Assets (In Relation to Par)
Endowment Mortgage With an Early Termination Penalty



The calculations that underlie these charts were done using the following parameters: 10% spot interest rate, *r(0)* and long term mean of the interest rate process, θ; 5% interest rate volatility, σ; 7.5% service flow, δ; 0.5% arrangement fee, ξ; 95% loan-to-value ratio; 15% early termination penalty, π; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Figure 5.17.A. Lender's Position in Contracts Including Different Mortgage Features

Repayment Mortgage With Arrangement Fee and Without Early Termination Penalty

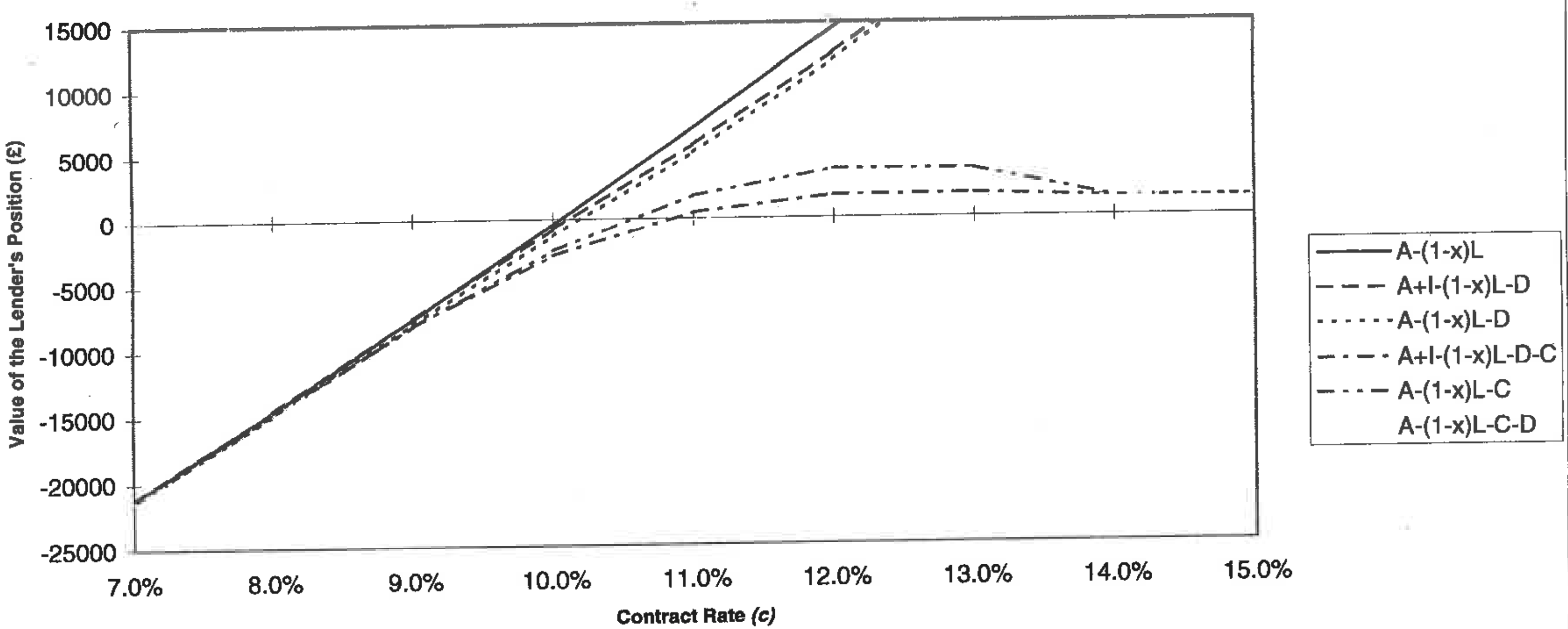


The calculations that underlie this chart were done using the following parameters: 10% spot interest rate, $r(0)$, and long term mean of the interest rate process, θ; 5% interest rate and house price volatilities, σ and ν; 7.5% house service flow, δ; 1% arrangement fee, x; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Note: Due to software constraints, the arrangement fee, normally represented by ξ, is here represented by x.

Figure 5.17.B. Lender's Position in Contracts Including Different Mortgage Features

Repayment Mortgage With Arrangement Fee and Early Termination Penalty

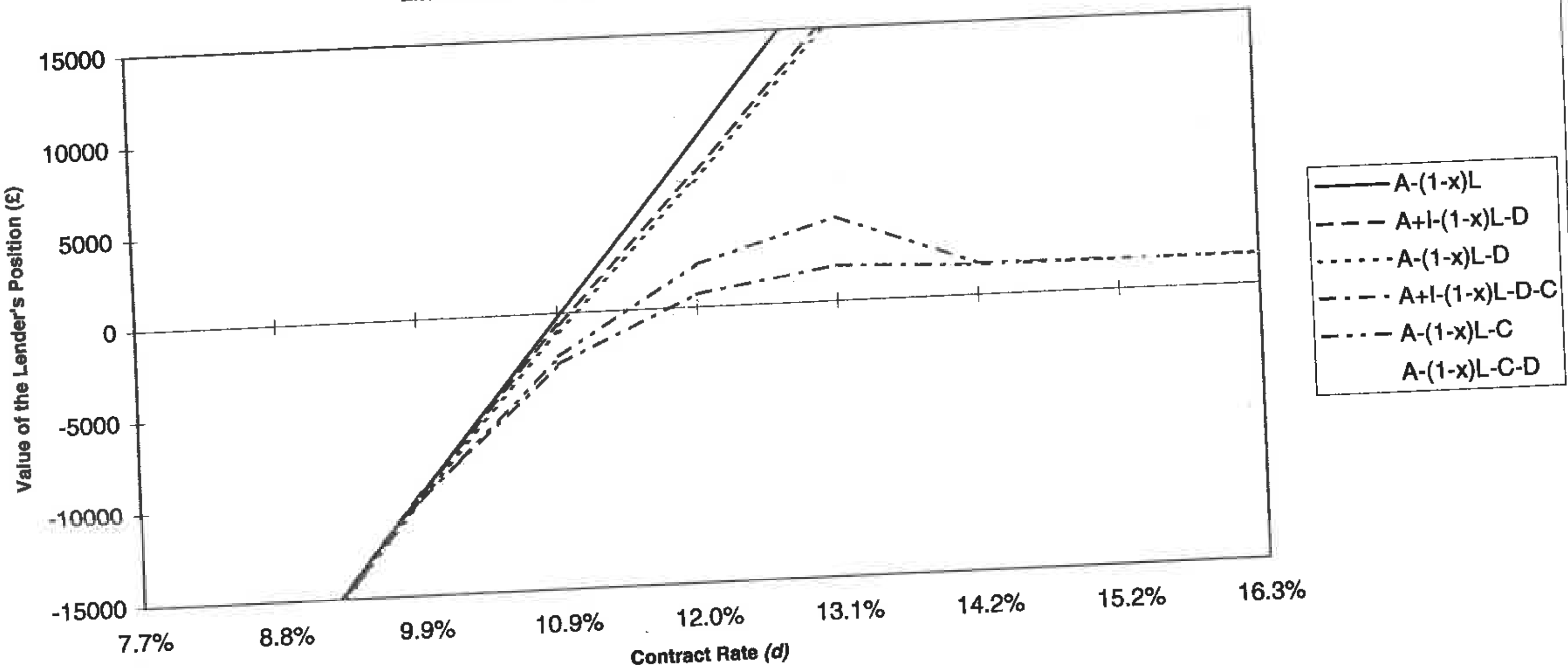


The calculations that underlie this chart were done using the following parameters: 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 5% interest rate and house price volatilities, σ and ν; 7.5% house service flow, δ; 0.5% arrangement fee, x; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Note: Due to software constraints, the arrangement fee, normally represented by ξ, is here represented by x.

Figure 5.17.C Lender's Position in Contracts Including Different Mortgage Features

Endowment Mortgage With Arrangement Fee and Early Termination Penalty

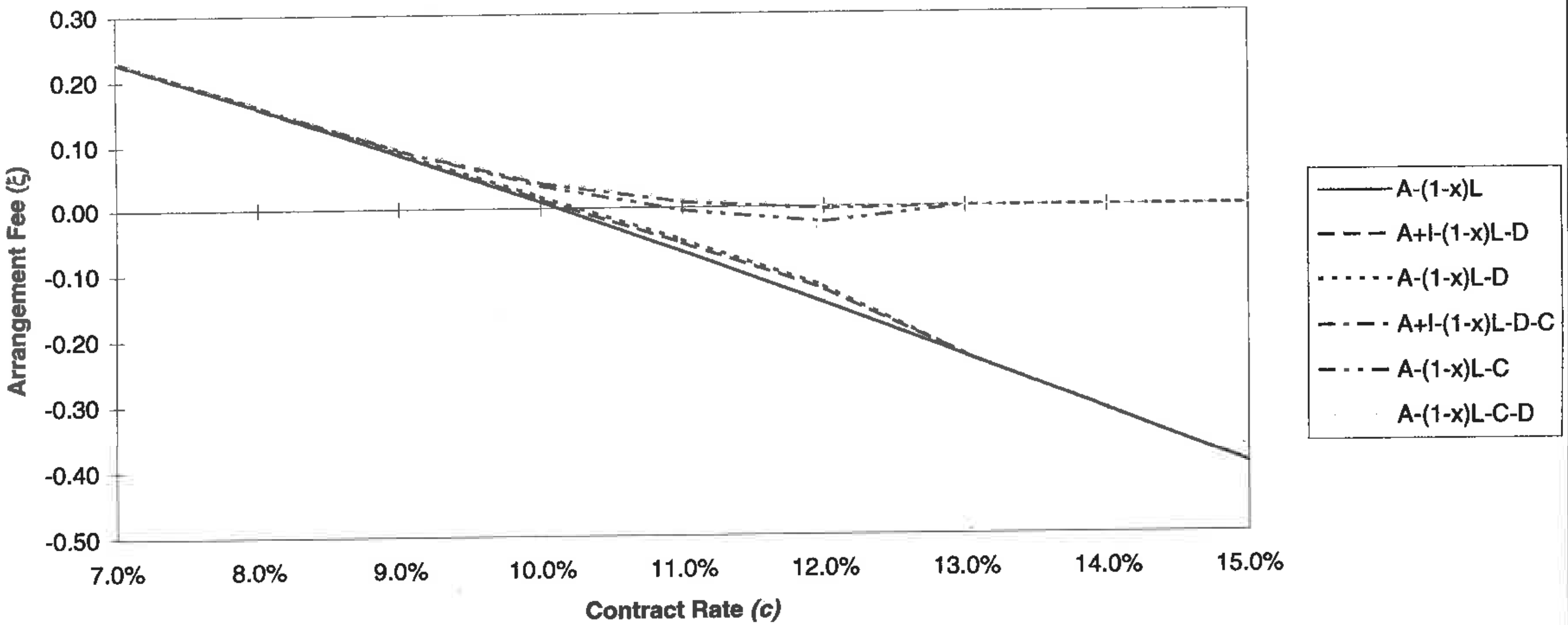


The calculations that underlie this chart were done using the following parameters: 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 5% interest rate and house price volatilities, σ and ν; 7.5% house service flow, δ; 0.5% arrangement fee, x; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Note: Due to software constraints, the arrangement fee, normally represented by ξ, is here represented by x.

Figure 5.18.A. The Relationship Between Contract Rates and Arrangement Fees in Contracts Including Different Mortgage Features

Repayment Mortgage With Arrangement Fee and Without Early Termination Penalty

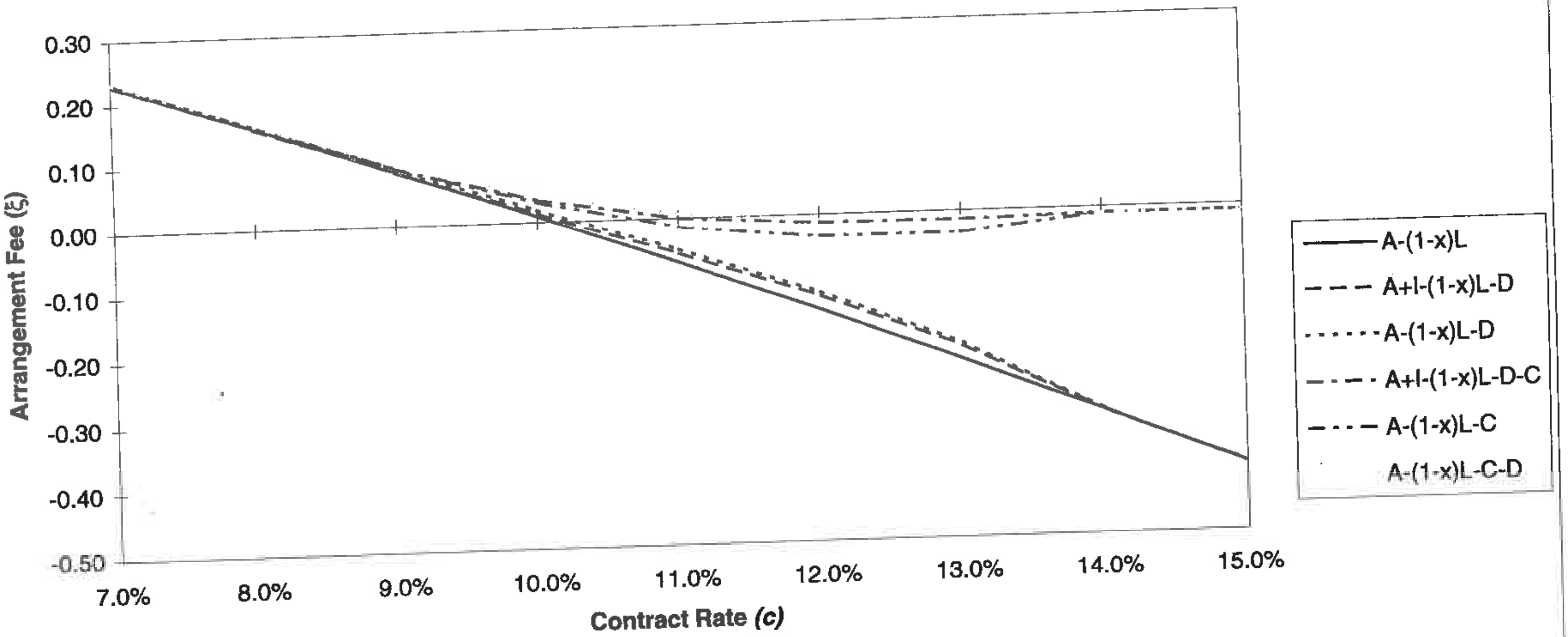


The calculations that underlie this chart were done using the following parameters: 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 5% interest rate and house price volatilities, σ and ν; 7.5% service flow, δ; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Note: Due to software constraints, the arrangement fee, normally represented by ξ, is here represented by x.

Figure 5.18.B. The Relationship Between Contract Rates and Arrangement Fees in Contracts Including Different Mortgage Features

Repayment Mortgage With Arrangement Fee and Early Termination Penalty

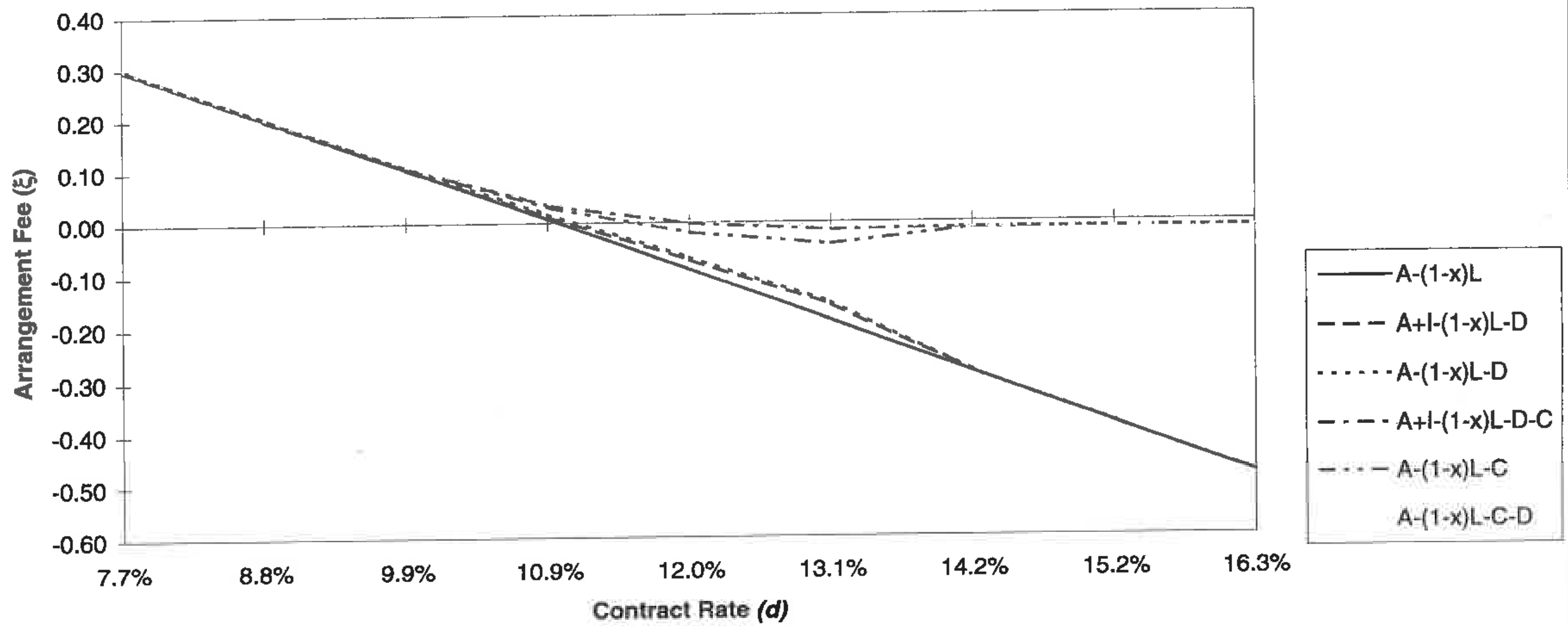


The calculations that underlie this chart were done using the following parameters: 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 5% interest rate and house price volatility, σ and ν; 7.5% service flow, δ; 1% early termination penalty, π; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Note: Due to software constraints, the arrangement fee, normally represented by ξ, is here represented by x.

Figure 5.18.C. The Relationship Between Contract Rates and Arrangement Fees in Contracts Including Different Mortgage Features

Endowment Mortgage With Arrangement Fee and Early Termination Penalty

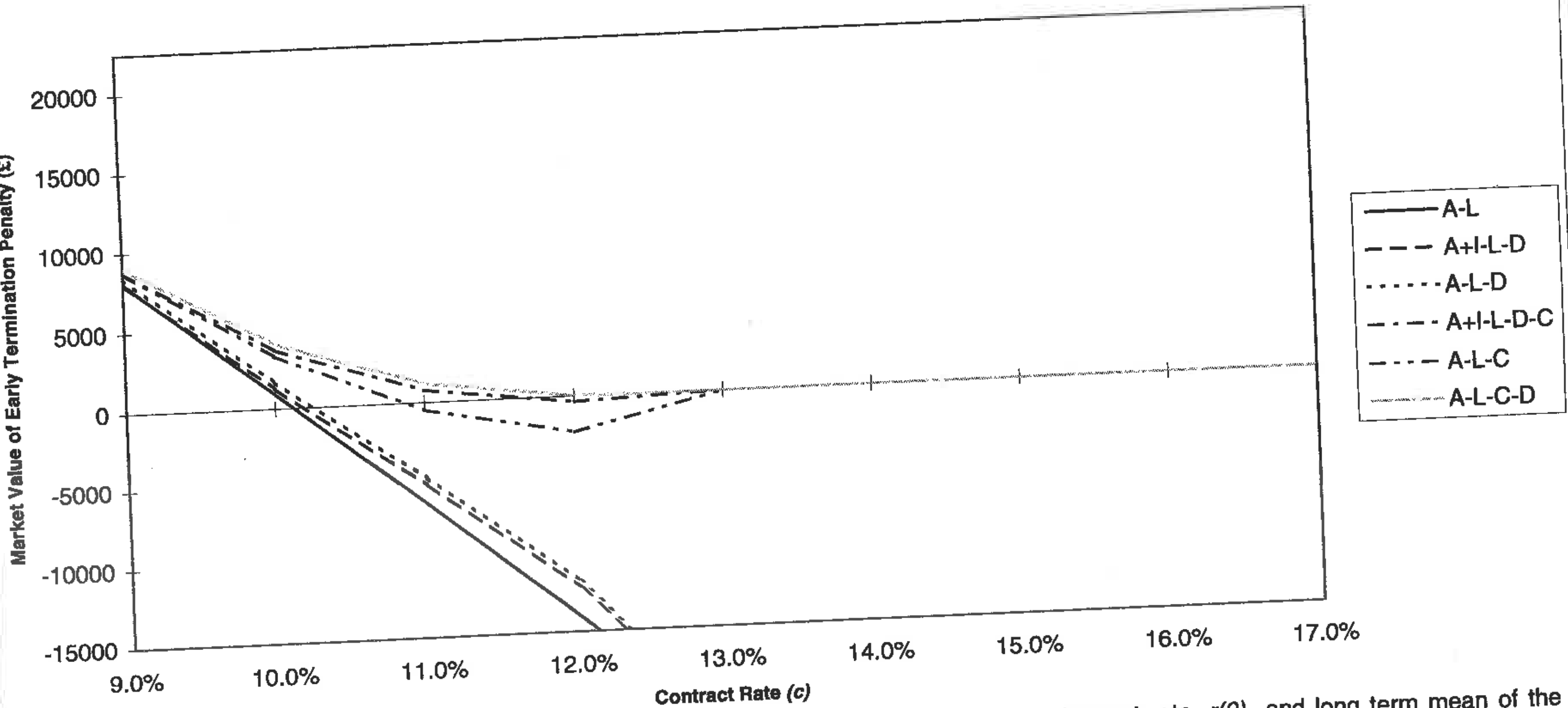


The calculations that underlie this chart were done using the following parameters: 10% spot interest rate, *r(0),* and long term mean of the interest rate process, θ; 5% interest rate and house price volatility, σ and ν; 7.5% service flow, δ; 15% early termination penalty, π; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Note: Due to software constraints, the arrangement fee, normally represented by ξ, is here represented by x.

Figure 5.19.A. The Relationship Between the Market Value of the Early Termination Penalty and the Contract Rate

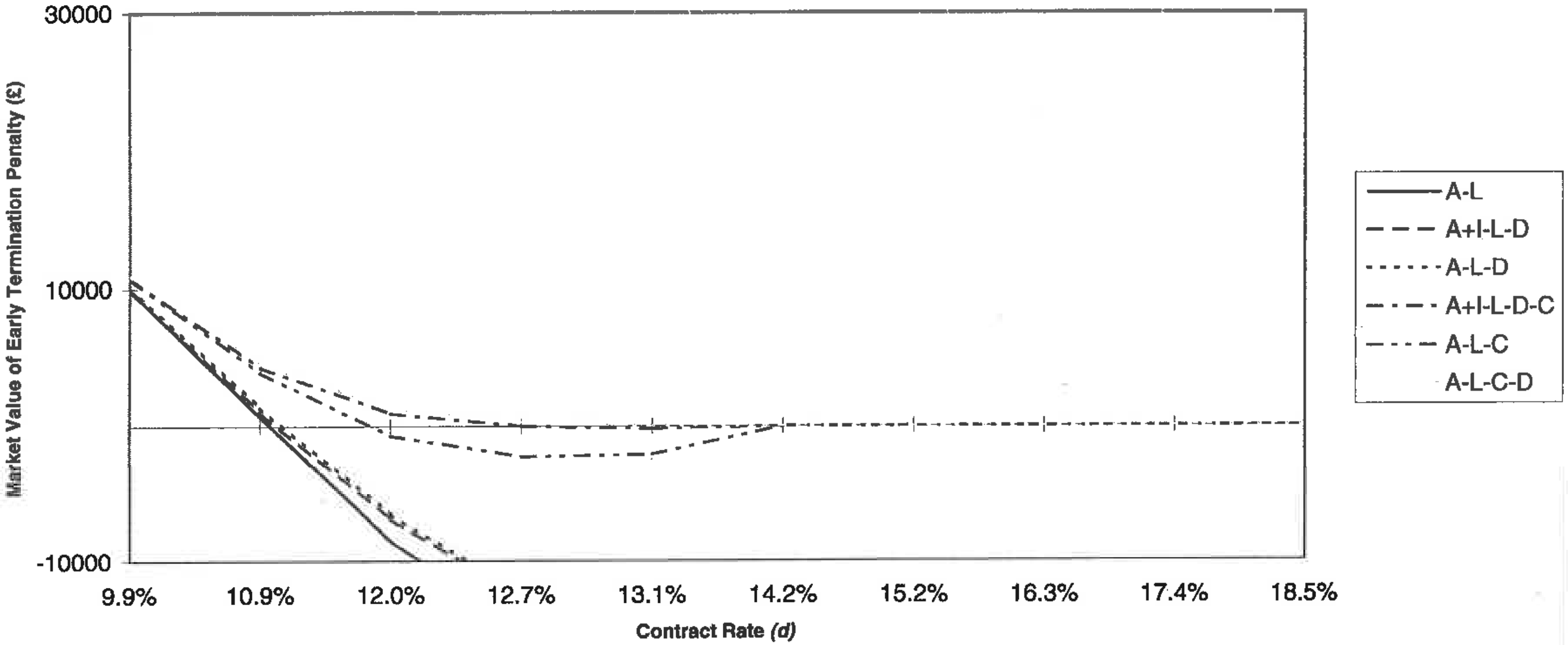
Repayment Mortgage Without Arrangement Fee



The calculations that underlie this chart were done using the following parameters: 10% spot interest rate, *r(0)*, and long term mean of the interest rate process, θ; 5% interest rate and house price volatilities, σ and ν; 7.5% service flow, δ; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

Figure 5.19.B. The Relationship Between the Market Value of the Early Termination Penalty and the Contract Rate

Endowment Mortgage Without Arrangement Fee



The calculations that underlie this chart were done using the following parameters: 10% spot interest rate, *r(0)* and long term mean of the interest rate process, θ; 5% interest rate and house price volatilities, σ and ν; 7.5% service flow, δ; 95% loan-to-value ratio; and a correlation coefficient, ρ, of 0. The insurance coverage assumes the standard form defined in Chapter 3.

CHAPTER 6

Conclusion

6.1. Summary and Conclusions

The present study develops and solves a model for the valuation of fixed rate repayment and "without profits" endowment mortgage contracts whose specifications are based on those that underlie the products traded on the British mortgage market. The corresponding mortgage indemnity guarantees (MIGs) and coinsurance are also evaluated.

The work starts with a brief reference to the economic relevance of the British mortgage market and a synopsis of factors that create dissimilarities between the mortgage contracts available in the US and those traded in the UK. Given these differences, it can be said that US mortgages are distinct financial products from those available in Britain, for whose valuation it is therefore commendable to use different models.

The following chapter starts with an overview of the British mortgage market and the various mortgage instruments available in the UK. Subsequently, a brief review and discussion of the literature on mortgage valuation is presented, giving special emphasis to term structure modelling, the modelling of both early termination options and the valuation of mortgage default insurance. Particular attention is also given to the (lack of) empirical research on mortgage pricing and to the probable future research directions in the field, with the inclusion of a summary of questions and problems that need to be addressed in the near future.

In the light of the literature review, a decision was made to employ a frictionless contingent claims framework. Both early termination options are taken into consideration. In order to keep the final formulation inside the bounds of what is tractable in computational and mathematical terms, suboptimal prepayment behaviour is not considered.

Taking into account the constraints and modelling choices mentioned above, a framework based on the Cox, Ingersoll and Ross (1985a) equilibrium model is developed to evaluate mortgage and mortgage-related assets. A two factor formulation is employed. The spot interest rate and the house price are used as state variables. The corresponding valuation function is a partial differential equation for which there is no closed form solution. Consequently, it is necessary to employ a numerical solution technique. In order to allow for this process to be undertaken, the cash-flow structure of the mortgage contracts is provided. In other words, the work presents formulations for the value of the monthly payments, the value of the borrower's debt in case of early termination, the equilibrium condition, and the terminal conditions on each of the payment dates along the life of the mortgage for both repayment and endowment contracts. Since the value of the different mortgage-related assets is known at termination of the contract, once the boundary conditions are identified, the model is closed and can be solved recursively using backward solution techniques.

The analysis and discussion of the numerical techniques available to solve the problem concluded with the selection of the explicit finite difference method. This method has already been successfully used to solve a problem of similar structure and complexity in the real estate finance field (Kau et al, 1992, 1993a) and since the corresponding algorithm is comparatively easy to program, seems to be particularly appropriate to solve intricate finance problems.

In order to reach a numerical solution, the original PDE was transformed, and its original bi-infinite domain mapped into a unit square. After the conclusion of this phase, the common boundary conditions were formulated. The prepayment free-boundary was also identified and formulated. In order to overcome the difficulties created by this free-boundary condition, the problem was converted in a non-linear PDE with a fixed boundary. Once these steps were fulfilled, difference equations were generated. The complexity of the partial differential equation imposed the use of an "upwind differencing" scheme to preserve the stability of the algorithm. Finally, after the attainment of all the required conditions for the solution of the problem was guaranteed, computer programs in Fortran 77 were developed to implement the numerical solution.

Numerical results were generated for two different specifications of the fixed rate repayment mortgage contract and one specification of the "without profits" fixed rate endowment mortgage contract. In spite of the complex links that exist between the underlying variables, the relationship between the evolution of the parameters used to characterise the economic environment and the value of the different mortgage-related assets is in line with the underlying economic knowledge.

The comparison between the repayment and endowment mortgage contracts whose specifications are considered in this work, leads to the conclusion that value of the call option to prepay the loan is higher in the endowment mortgage case, which implies that, ceteris paribus, in endowment mortgages it is necessary for the borrower to pay more in order to reach an equilibrium combination.

Another aspect that is important to mention is related to the consequences that derive from the addition/exclusion of contractual features inherent to the specification of the mortgage products. Changes in those features lead to different equilibrium rates and different values for the mortgage-related assets. One of the implications of this phenomenon is that modelling exercises based on specifications that ignore some of the most relevant features of the mortgage contracts, like the embedded options to terminate the loan or one of the negative incentives to early termination, might lead to questionable conclusions. This is especially relevant in terms of mortgage valuation research, since models that ignore at least one of the options held by the borrower to bring the contract to an end prior to maturity constitute the bulk of the literature published on the subject during the last decade.

6.2. Contributions of the Research

The main contributions of this research can be summarised in the following terms:

i) The British mortgage products include features that lead to the generation of different cash-flow structures relative to American counterparts. Consequently, even at the repayment mortgage level, the products that are analysed here are in fact different financial products. This study is the first work where this investigation is done under a contingent claims framework that considers both early termination options held by the borrower;

ii) At the level of the mortgage-related insurance products, this is the first time that MIGs are priced using a contingent claims framework that takes into consideration simultaneously the options to prepay and default held by the borrower;

iii) The value of the lender's share in the coverage of the risk of default (coinsurance) is also valued for the first time under such a framework;

iv) This study is the first to value an endowment mortgage product with the explicit consideration of prepayment and default options;

v) At the numerical analysis level, an "upwinding" differencing technique was used in order to assure the smoothness of the solution over all of the grid. The use of this method by numerical analysts is not by any means new, but its use in the solution of finance or real estate related PDEs has not been referred to in the literature.

6.3. Suggestions for Future Research

There are several topics related to this research that might be interesting for future research:

i) The methodology used here to study fixed rate mortgage products can be easily extended to do similar research applied to variable rate mortgages once the necessary computing power is available. For this to became possible it is only necessary to incorporate an additional auxiliary state variable in order to take into account the evolution of the coupon rate, using a methodology for the numerical treatment of backward valuation of path-dependent financial products like those proposed by Kishimoto (1990), Kau et al. (1993a) or Hilliard et al. (1995);

ii) The MIG, in its current form, is a new insurance product that apparently was not studied in detail using a contingent claims approach. The present work takes into consideration its more common version, but it is not particularly focused in the study of the peculiarities of the product. However, a detailed analysis of the consequences, in terms of price and risk associated with changes not only in the value of the coverage and the cap level but also with the main structure of the product, could be conducted based on the present framework;

iii) Once the necessary computing power is available, a study on the valuation of "with profits" endowment mortgages might be considered. In this case it will be necessary to include an additional state variable to represent the additional source of risk;

iv) A comparison between American style mortgage products and British counterparts based on the differences in prices and relative weights of the embedded assets might also be conducted taking into consideration this framework and simultaneously those already available to value the American products (for instance Kau et al. 1992, 1993a)

v) The present work does not take into consideration any sort of transaction costs. An empirical study comparing the evolution of model coupon rates and real coupon rates might shed some light on the size and eventually on the functional form of those types of costs.

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