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Extremal solutions for third-order nonlinear problems with upper and lower solutions in reversed order

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Abstract

This paper deals with the existence of extremal solutions for the third-order nonlinear boundary value problem

$$-[\phi(u''(t))]' = f(t, u(t)), \quad t \in [a, b],$$

$$u(a) = A, \quad u''(a) = B, \quad u''(b) = C,$$

in the presence of a pair of lower and upper solutions in reversed order.

Here $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism, $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a Carathédory function and $A, B, C \in \mathbb{R}$.

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The proof follows from monotone iterative techniques which are based on suitable anti-maximum principles for adequate operators. To deduce such results, we study some related problems coupled with boundary value conditions of the form

$$p_0u(a) - q_0u'(a) = A,\, p_1u(b) + q_1u'(b) = B, \quad u''(a) = C,$$

and

$$u(a) = A, \quad u'(a) = B, \quad u''(b) = C.$$

MSC: 34B10; 34B15

Keywords: Lower and upper solutions; Third-order nonlinear boundary value problems; ϕ-Laplacian problems

1. Introduction

Third-order equations model an important number of physical problems, as the deflection of a curved beam having a constant or a varying cross-section, three layer beam, electromagnetic waves or gravity-driven flows [11]. The reduction of order or Green's functions and comparison principles [1,2,5,6,8,12–18,20] have been the most used methods to approach this kind of problem, together with different type of boundary value conditions like the periodic, three-point or two-point boundary conditions. Nonlinear boundary conditions coupled with the method of lower and upper solutions have been studied in [3,7,10,19], but in these cases the lower solution is less or equal to the upper one.

In this paper, we assume that the lower solution is not above the upper one. This case has been studied in [1,2,13] when function ϕ is the identity and periodic boundary conditions are considered. In both papers is stated a relation between the existence of extremal solutions, lying in a strip defined by a pair of upper and lower solutions given in the reversed order, and the constant sign of the Green's function of a related linear operator.

As the problem considered now does not have a linear part the Green's function technique is not applied. So, to develop the monotone method, we use some comparison results for nonlinear operators following the arguments given for second-order ϕ-Laplacian equations in [4]. The uniqueness of solution for the considered problems is obtained as a fixed point of a contractive operator. This technique is different from [4] and it is similar to the one developed in [9] for second-order ϕ-laplacian difference equations.

We study the one-dimensional nonlinear third-order ϕ-Laplacian equation

$$-[\phi(u''(t))]' = f(t, u(t)), \quad t \in I \equiv [a, b], \tag{P}$$

with the boundary conditions

$$u(a) = A, \quad u''(a) = B, \quad u''(b) = C \tag{1.1}$$

for $A, B, C \in \mathbb{R}$, $f : I \times \mathbb{R} \to \mathbb{R}$ an L^1-Carathéodory function, i.e. $f(t, \cdot)$ is a continuous function for a.e. $t \in I$, $f(\cdot, y)$ is measurable for $y \in \mathbb{R}$ and for every $R > 0$ there is a real-valued function $h_R \in L^1(I)$ such that

$$|f(t, y)| \leqslant h_R(t) \tag{1.2}$$

for a.e. $t \in I$ and for every $y \in \mathbb{R}$ with $|y| \leqslant R$, and ϕ satisfying the following condition:

(H_1) $\phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$ and ϕ^{-1} is locally Lipschitz, that is, for every compact interval $J = [c, d]$ there is $K = K(J) > 0$ such that for all $u, v \in [c, d]$

$$|\phi^{-1}(u) - \phi^{-1}(v)| \leqslant K \quad |u - v|.$$

For this K we define

$$M^* := -\frac{3}{K(b-a)^3} \tag{1.3}$$

and we look for solutions $u \in C^2(I)$ such that $\phi(u'') \in AC(I)$.

In Section 2, for some $M < 0$, uniqueness results are obtained for the auxiliary equation

$$-[\phi(u''(t))]' + Mu(t) = \sigma(t) \quad \text{for } a.e.\ t \in I, \tag{L$_\sigma$}$$

with boundary conditions (1.1) or

$$u(a) = A, \quad u'(a) = B, \quad u''(b) = C \tag{1.4}$$

or

$$p_0u(a) - q_0u'(a) = A, \quad p_1u(b) + q_1u'(b) = B, \quad u''(a) = C, \tag{1.5}$$

whenever constants p_0, p_1, q_0 and q_1 satisfy

$$p_0, p_1, q_0, q_1 \geqslant 0 \quad \text{and} \quad p_0p_1 + p_0q_1 + p_1q_0 > 0. \tag{1.6}$$

Anti-maximum comparison principles for the previous problems are given in Section 3.

For $M \in (M^*, 0)$ and a suitable monotone condition (see assumption (H_2)), the existence of extremal solutions for the problem (P)–(1.1), lying between the lower and the upper solutions considered in reversed order, is proved in Section 4.

In Section 5, an existence result is obtained under weaker assumptions (see (H_2^*)).

2. Uniqueness results

In this section, it is proved that equation (L$_\sigma$) with boundary conditions (1.1), or (1.4), or (1.5), has a unique solution for given $\sigma \in L^1(I)$ and for suitable values of $M < 0$. These results are obtained assuming that ϕ verifies the following Lipschitz condition, which is stronger than (H_1).

(H_1^*) $\phi : R \to R$ is an increasing homeomorphism such that $\phi(0)=0$ and ϕ^{-1} is a K-Lipschitz function, that is, there is $K>0$ such that for all $u, v \in \mathbb{R}$

$$|\phi^{-1}(u)-\phi^{-1}(v)| \leqslant K|u-v|.$$

First we prove the uniqueness result for problem (L_σ)–(1.1).

Proposition 2.1. *Let* $\phi : \mathbb{R} \to \mathbb{R}$ *satisfy assumption* (H_1^*). *If* $M \in (M^*, 0)$ *then, for every* $\sigma \in L^1(I)$, *the problem* (L_σ)–(1.1) *has a unique solution.*

Proof. *Step* 1: Fixed point problem.

One can verify that u is a solution of (L_σ)–(1.1) if and only if,

$$u(t) = A + \int_a^t \left(\theta_w + \int_a^s \phi^{-1}(w(r))\,\mathrm{d}r\right) \mathrm{d}s,$$

with

$$\theta_w = \frac{2}{M(b-a)^2}\left[\phi(C)-\phi(B)-MA(b-a) - \frac{M}{2}\int_a^b (b-\tau)^2\,\phi^{-1}(w(\tau))\,\mathrm{d}\tau + \int_a^b \sigma(s)\,\mathrm{d}s\right], \tag{2.1}$$

and w a fixed point of operator $T_\sigma : C(I) \to C(I)$, defined as

$$T_\sigma(w(t)) = \phi(B) + MA(t-a) + M\theta_w\,\frac{(t-a)^2}{2} + \frac{M}{2}\int_a^t (t-\tau)^2\phi^{-1}(w(\tau))\,\mathrm{d}\tau - \int_a^t \sigma(s)\,\mathrm{d}s.$$

As a consequence, problem (L_σ)–(1.1) is uniquely solvable if and only if the operator T_σ has a unique fixed point.

Step 2: T_σ is a contraction for $M \in (M^*, 0)$ and every $\sigma \in L^1(I)$.

For $w_1, w_2 \in C(I)$ and θ_w given by (2.1), we have that for every $t \in I$

$$\begin{aligned} T_\sigma(w_1(t)) - T_\sigma(w_2(t)) &= M(\theta_{w_1}-\theta_{w_2})\,\frac{(t-a)^2}{2} \\ &\quad + \frac{M}{2}\int_a^t (t-\tau)^2[\phi^{-1}(w_1(\tau))-\phi^{-1}(w_2(\tau))]\,\mathrm{d}\tau \\ &= -\frac{M(t-a)^2}{2(b-a)^2}\int_a^b (b-\tau)^2[\phi^{-1}(w_1(\tau))-\phi^{-1}(w_2(\tau))]\,\mathrm{d}\tau \\ &\quad + \frac{M}{2}\int_a^t (t-\tau)^2[\phi^{-1}(w_1(\tau))-\phi^{-1}(w_2(\tau))]\,\mathrm{d}\tau. \end{aligned}$$

So, by (H_1^*) and (1.3), we have that for all $M \in (M^*, 0)$ it is verified that

$$\begin{aligned}\|T_\sigma w_1 - T_\sigma w_2\|_\infty &\leqslant -\frac{M}{2} K \|w_1 - w_2\|_\infty \left[\frac{(b-a)^3}{3} + \int_a^b (b-\tau)^2 \, \mathrm{d}\tau\right] \\ &\leqslant -M \frac{K(b-a)^3}{3} \|w_1 - w_2\|_\infty < \|w_1 - w_2\|_\infty.\end{aligned}$$

Then T_σ has a unique fixed point and, therefore, (L_σ)–(1.1) has only one solution. □

Similar arguments can be applied to the problem (L_σ)–(1.4) to obtain the following result:

Proposition 2.2. *Let ϕ satisfy assumption (H_1^*). If $M \in (M^*, 0)$ then, for every $\sigma \in L^1(I)$, the problem (L_σ)–(1.4) has a unique solution.*

Proof. Arguing as in the previous proposition, the solution of problem (L_σ)–(1.4) is given by

$$u(t) = A + \int_a^t \left(B + \int_a^s \phi^{-1} v(r)) \, \mathrm{d}r\right) \mathrm{d}s, \tag{2.2}$$

with v the unique fixed point of the operator $T_{1,\sigma} : C(I) \to C(I)$ defined as

$$\begin{aligned}T_{1,\sigma}(v(t)) = \phi(C) &- MA(b-t) - MB \int_t^b (s-a) \, \mathrm{d}s \\ &- M \int_t^b \int_a^s \int_a^\eta \phi^{-1} v(\tau)) \, \mathrm{d}\tau \, \mathrm{d}\eta \, \mathrm{d}s + \int_t^b \sigma(s) \, \mathrm{d}s,\end{aligned} \tag{2.3}$$

for $M \in (M^*, 0)$. □

Now we obtain a similar result for boundary conditions (1.5). Before proving it, we introduce the following constant:

$$\bar{M} = -\frac{12((b-a)p_0 p_1 + p_1 q_0 + p_0 q_1)}{(b-a)^2((b-a)^2 p_0 p_1 + 4(b-a)(p_1 q_0 + p_0 q_1) + 12 q_0 q_1)K}. \tag{2.4}$$

Proposition 2.3. *Suppose that ϕ satisfy assumption (H_1^*). If $M \in (\bar{M}, 0)$ and condition* (1.6) *holds, then, for every $\sigma \in L^1(I)$, the problem (L_σ)–(1.5) has a unique solution.*

Proof. In this case the solutions of problem (L_σ)–(1.5) are given by the expression

$$u(t) = -\int_a^b G(t,s) \phi^{-1}(v(s)) \, \mathrm{d}s + h(t, A, B), \tag{2.5}$$

where

$$G(t,s)=\frac{1}{D}\begin{cases}(p_1(b-t)+q_1)(p_0(s-a)+q_0), & a\leqslant s\leqslant t\leqslant b,\\ (p_0(t-a)+q_0)(p_1(b-s)+q_1), & a\leqslant t\leqslant s\leqslant b,\end{cases}$$

$$h(t,A,B)=\frac{(p_1(b-t)+q_1)A+(p_0(t-a)+q_0)B}{D},$$

$$D=(b-a)p_0p_1+p_1q_0+p_0q_1,$$

and v is a fixed point of the operator $T_{2,\sigma}: C(I)\to C(I)$, defined as

$$T_{2,\sigma}(v(t))=\phi(C)-M\int_a^t\int_a^b G(s,r)\,\phi^{-1}(v(r))\,\mathrm{d}r\,\mathrm{d}s + M\int_a^t h(s,A,B)\,\mathrm{d}s-\int_a^t\sigma(s)\,\mathrm{d}s. \tag{2.6}$$

First, note that if $v_1, v_2\in C(I)$, then

$$T_{2,\sigma}(v_1(t))-T_{2,\sigma}(v_2(t))=-M\int_a^t\int_a^b G(s,r)[\phi^{-1}(v_1(r))-\phi^{-1}(v_2(r))]\,\mathrm{d}r\,\mathrm{d}s.$$

On the other hand, from condition (1.6) we have that $G\geqslant 0$ in $I\times I$. Moreover, using (2.4), we know that

$$\int_a^b\int_a^b G(t,s)\,\mathrm{d}s\,\mathrm{d}t=-\frac{1}{K\bar{M}}.$$

In consequence, using condition (H_1^*) we arrive at

$$\|T_{2,\sigma}v_1-T_{2,\sigma}v_2\|_\infty<\|v_1-v_2\|_\infty.$$

Thus, for $M\in(\bar{M},0)$, the operator $T_{2,\sigma}$ is a contraction and it has only one fixed point. □

Remark 2.1. Note that if we consider the particular case of (1.5)

$$u(a)=A,\quad u'(b)=B,\quad u''(a)=C,$$

then $\bar{M}=M^*$.

3. Anti-maximum comparison principles

In order to apply the monotone method to problem (P)–(1.1) we prove some anti-maximum comparison principles related to the problems referred in the previous section. For problem (L_σ)–(1.4) the following principle holds:

Proposition 3.1. *Let ϕ verify condition (H_1^*) and $M\in(M^*,0)$. If $u_1,u_2\in C^2(I)$, with $\phi(u_1''),\phi(u_2'')\in AC(I)$, are such that*

$$-[\phi(u_1''(t))]'+Mu_1(t)\geqslant-[\phi(u_2''(t))]'+Mu_2(t)\quad\text{for } a.e.\ t\in I \tag{3.1}$$

and

$$u_1(a) \geqslant u_2(a), \;\; u_1'(a) \geqslant u_2'(a), \;\; u_1''(b) \geqslant u_2''(b),$$

then $u_1 \geqslant u_2$, $u_1' \geqslant u_2'$ *and* $u_1'' \geqslant u_2''$ on I.

Proof. Let $\sigma_1, \sigma_2 \in L^1(I)$, such that $\sigma_1 \geqslant \sigma_2$ $a.e.$ in I, and A_1, A_2, B_1, B_2, C_1 and $C_2 \in \mathbb{R}$. Denote by u_1 and u_2 the unique solutions of the problems composed by Eqs. (L_{σ_1}) and (L_{σ_2}), respectively, and by the boundary conditions

$$\begin{aligned} &u_1(a) = A_1 \geqslant A_2 = u_2(a),\\ &u_1'(a) = B_1 \geqslant B_2 = u_2'(a),\\ &u_1''(b) = C_1 \geqslant C_2 = u_2''(b). \end{aligned}$$

Note that the existence of u_1 and u_2 is given by Proposition 2.2. It can be verified, following the same steps used in the previous section, that $w_1 := \phi(u_1'')$ and $w_2 := \phi(u_2'')$ are the unique fixed points of the operators T_{1,σ_i}, $i = 1, 2$, defined in (2.3).

Let

$$\xi_0(t) := \int_t^b \sigma_1(s)\,\mathrm{d}s \geqslant \int_t^b \sigma_2(s)\,\mathrm{d}s =: \zeta_0(t) \quad \text{for all } t \in I.$$

From (2.3), and by recurrence, we obtain

$$\xi_{n+1} := T_{1,\sigma_1}(\xi_n) \geqslant T_{1,\sigma_2}(\zeta_n) := \zeta_{n+1} \quad \text{on } I \;\text{ for all } n \geqslant 1.$$

Since $M \in (M^*, 0)$ the operators T_{1,σ_i}, for $i = 1, 2$, are contractive and so both sequences converge to w_1 and w_2, respectively. The proof finishes by integration. □

For problem (L_σ)–(1.5) we have the following result:

Proposition 3.2. *Assume that condition* (1.6) *holds, function* ϕ *verifies assumption* (H_1^*) *and* $M \in (\bar{M}, 0)$. *Let* $u_1, u_2 \in C^2(I)$ with $\phi(u_1''), \phi(u_2'') \in AC(I)$ *satisfy inequality* (3.1) *and*

$$\begin{aligned} &p_0 u_1(a) - q_0 u_1'(a) \geqslant p_0 u_2(a) - q_0 u_2'(a),\\ &p_1 u_1(b) + q_1 u_1'(b) \geqslant p_1 u_2(b) + q_1 u_2'(b),\\ &u_1''(a) \leqslant u_2''(a). \end{aligned}$$

Then $u_1'' \leqslant u_2''$ and $u_1 \geqslant u_2$ on I.

Proof. Consider two functions $\sigma_1, \sigma_2 \in L^1(I)$ verifying $\sigma_1 \geqslant \sigma_2$ $a.e.$ on I and $A_1, A_2, B_1, B_2, C_1, C_2 \in \mathbb{R}$ be such that u_1 and u_2 are the unique solutions of Eqs. (L_{σ_1}) and (L_{σ_2}), respectively, satisfying

$$\begin{aligned} &p_0 u_1(a) - q_0 u_1'(a) = A_1 \geqslant A_2 = p_0 u_2(a) - q_0 u_2'(a),\\ &p_1 u_1(b) + q_1 u_1'(b) = B_1 \geqslant B_2 = p_1 u_2(b) + q_1 u_2'(b),\\ &u_1''(a) = C_1 \leqslant C_2 = u_2''(a). \end{aligned}$$

In this case $w_1 := \phi(u_1'')$ and $w_2 := \phi(u_2'')$ are the unique fixed points of the operators T_{2,σ_1} and T_{2,σ_2}, respectively, given by (2.6). Defining

$$\xi_0(t) := -\int_a^t \sigma_1(s)\,\mathrm{d}s \leqslant -\int_a^t \sigma_2(s)\,\mathrm{d}s =: \zeta_0(t) \quad \text{for all } t \in I,$$

we have, from (2.6), that

$$\xi_{n+1} = T_{2,\sigma_1}(\xi_n) \leqslant T_{2,\sigma_2}(\zeta_n) = \zeta_{n+1} \quad \text{on } I \ \text{ for all } n \geqslant 1.$$

As T_{2,σ_i} are contractions, for $i = 1, 2$, we conclude that $w_1 \leqslant w_2$ on I.

Taking into account that $G \geqslant 0$ on $I \times I$, from (2.5) it can be deduced that $u_1 \geqslant u_2$ on I. □

By Remark 2.1, we obtain the following corollary:

Corollary 3.1. *Let ϕ verify condition (H_1^*) and $M \in (M^*, 0)$. Let $u_1, u_2 \in C^2(I)$ with $\phi(u_1''), \phi(u_2'') \in AC(I)$. If assumption* (3.1) *and*

$$u_1(a) \geqslant u_2(a), \quad u_1'(b) \geqslant u_2'(b), \quad u_1''(a) \leqslant u_2''(a)$$

hold, then $u_1'' \leqslant u_2''$ and $u_1 \geqslant u_2$ on I.

Finally, for problem (L_σ)–(1.1) we have the comparison principle:

Proposition 3.3. *Let ϕ verify condition (H_1^*) and $M \in (M^*, 0)$. Let $u_1, u_2 \in C^2(I)$ with $\phi(u_1''), \phi(u_2'') \in AC(I)$, satisfying inequality* (3.1) *and*

$$u_1(a) = u_2(a), \tag{3.2}$$

$$u_1''(a) \leqslant u_2''(a), \quad u_1''(b) \geqslant u_2''(b). \tag{3.3}$$

Then $u_1 \leqslant u_2$ on I.

Proof. Assume, by contradiction, that there is $t_0 \in (a, b]$ such that

$$u_1(t_0) > u_2(t_0). \tag{3.4}$$

If $u_1 \geqslant u_2$ on I then, by (3.1),

$$\int_a^b (-[\phi(u_1''(s))]' + [\phi(u_2''(s))]')\,\mathrm{d}s \geqslant \int_a^b M[u_2(s) - u_1(s)]\,\mathrm{d}s$$

and, by (3.3), we obtain the contradiction

$$\begin{aligned} 0 &\geqslant -\phi(u_1''(b)) + \phi(u_2''(b)) + \phi(u_1''(a)) - \phi(u_2''(a)) \\ &\geqslant M \int_a^b [u_2(s) - u_1(s)]\,\mathrm{d}s > 0. \end{aligned}$$

Then there is $t_1 \in (a, b]$ such that

$$u_1(t_1) < u_2(t_1). \tag{3.5}$$

If $u_1'(a) \geqslant u_2'(a)$, by Proposition 3.1, we have that

$$u_1(t) \geqslant u_2(t) \quad \text{for all } t \in I,$$

which is in contradiction with (3.5). Therefore $u_1'(a) < u_2'(a)$. So, by (3.2) and (3.4), there is $\bar{t} \in (a, b)$ such that

$$u_1(\bar{t}) = u_2(\bar{t}) \quad \text{and } u_1'(\bar{t}) \geqslant u_2'(\bar{t}).$$

Applying Proposition 3.1 to $[\bar{t}, b]$ we have

$$u_1(t) \geqslant u_2(t) \quad \text{for all } t \in [\bar{t}, b].$$

Now, applying Corollary 3.1 on $[a, \bar{t}]$, we conclude that

$$u_1(t) \geqslant u_2(t) \quad \text{for all } t \in [a, \bar{t}],$$

and, therefore, a contradiction with (3.5) is obtained. □

4. Extremal solutions

In this section we develop the monotone method for problem (P)–(1.1) in the presence of a pair of lower and upper solutions given in the reversed order, that is, the lower solution α is over the upper solution β. Such functions are defined as follows:

A function $\alpha \in C^2([a, b])$ such that $\phi(\alpha'') \in AC(I)$ is a lower solution of problem (P)–(1.1) if

$$-[\phi(\alpha''(t))]' \leqslant f(t, \alpha(t)) \quad \text{for } a.e.\ t \in I,$$

and

$$\alpha(a) = A,\ \alpha''(a) \geqslant B,\ \alpha''(b) \leqslant C.$$

An upper solution β is defined in the same way by reversing the above inequalities.

To develop the monotone iterative technique, we impose on function f the following one-sided Lipschitz condition:

(H_2) There is $M < 0$ such that $f(t, x) + Mx$ is nonincreasing in x, for a.e. $t \in I$ and $x \in [\beta(t), \alpha(t)]$.

Now we denote

$$[\beta, \alpha] = \{v \in C^2([a, b]) : \beta(t) \leqslant v(t) \leqslant \alpha(t) \text{ for all } t \in I\}.$$

Before proving the main result of this section, we define the following constant $L := L(\alpha, \beta)$ as

$$L = \max \left\{ \begin{array}{l} |B|, |C|, |\phi(B) - \|h_R\|_1|, |\phi(C) + \|h_R\|_1|, \\ \phi^{-1}(\phi(B) + \|h_R\|_1), \phi^{-1}(\phi(C) + \|h_R\|_1), \\ |\phi^{-1}(\phi(B) - \|h_R\|_1)|, |\phi^{-1}(\phi(C) - \|h_R\|_1)| \end{array} \right\}, \tag{4.1}$$

with

$$R = \max\{\|\alpha\|_\infty, \|\beta\|_\infty\},$$

and h_R the L^1-function given by (1.2).

Let $K := K(\alpha, \beta)$ be the Lipschitz constant associated in condition (H_1) with function ϕ in the interval $[-R, R]$. Define

$$\Phi(x) = \begin{cases} \phi(L) + K\,(x - L) & \text{if } x > L, \\ \phi(x) & \text{if } |x| \leqslant L, \\ \phi(-L) + K(x + L) & \text{if } x < -L. \end{cases} \tag{4.2}$$

It is clear that function Φ satisfies condition (H_1^*) for constant K given before. The next result proves the existence of extremal solutions for problem (P)–(1.1).

Theorem 4.1. *Assume that condition* (H_1) *holds and assumption* (H_2) *is fulfilled for* $M \in (M^*, 0)$, *with* $M^* < 0$ *given in* (1.3) *and* $K > 0$ *as in* (4.2). *If there exist a lower solution* α *and an upper solution* β *of problem* (P)–(1.1), *such that* $\alpha \geqslant \beta$ *in I, then there are two sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *in* $[\beta, \alpha]$, *which are, respectively, nonincreasing and nondecreasing and converge uniformly to* $u_{\min}$ *and* $u_{\max}$, *solutions of* (P)–(1.1), *such that*

$$\beta(t) \leqslant u_{\min}(t) \leqslant u_{\max}(t) \leqslant \alpha(t) \quad \text{for all } t \in I.$$

Moreover, these solutions are extremal, that is, any other solution u of (P)–(1.1) *with* $u \in [\beta, \alpha]$ *verifies on I*

$$u_{\min}(t) \leqslant u(t) \leqslant u_{\max}(t).$$

Proof. Let $M^* < 0$ be the constant given in (1.3), $K > 0$ and Φ as in (4.2). Fix $M \in (M^*, 0)$, $\gamma \in [\beta, \alpha]$ and consider the equation

$$-[\Phi(u''(t))]' + Mu(t) = f(t, \gamma(t)) + M\gamma(t), \quad t \in I. \tag{E$_\gamma$}$$

Since Φ satisfies condition (H_1^*) for the constant K, then by (1.3) and Proposition 2.1, problem (E$_\gamma$)–(1.1) has a unique solution u_0. So, by (H_2) and (4.1)

$$\begin{aligned} -[\Phi(u_0''(t))]' + Mu_0(t) &= f(t, \gamma(t)) + M\gamma(t) \\ &\geqslant f(t, \alpha(t)) + M\alpha(t) \geqslant -[\Phi(\alpha''(t))]' + M\alpha(t), \end{aligned}$$

and

$$u_0(a) = A = \alpha(a), \quad u_0''(a) = B \leqslant \alpha''(a), \quad u_0''(b) = C \geqslant \alpha''(b).$$

Then, by Proposition 3.3, $u_0 \leqslant \alpha$ on I.

The arguments are similar to prove $\beta \leqslant u_0$ on I. Consider $\gamma_1, \gamma_2 \in [\beta, \alpha]$ such that $\gamma_1 \leqslant \gamma_2$ and let u_1, u_2 be the unique solutions of (E_{γ_1})–(1.1) and (E_{γ_2})–(1.1), respectively. Then, by (H_2),

$$\begin{aligned} -[\Phi(u_1''(t))]' + Mu_1(t) &= f(t, \gamma_1(t)) + M\gamma_1(t) \\ &\geqslant f(t, \gamma_2(t)) + M\gamma_2(t) = -[\Phi(u_2''(t))]' + Mu_2(t) \end{aligned}$$

and, by Proposition 3.3, $u_1 \leqslant u_2$ in I.

From these two properties, it can be constructed, recursively, a nonincreasing sequence $\{\alpha_n\}$ and a nondecreasing sequence $\{\beta_n\}$ defined as $\alpha_0 = \alpha$ and α_n the unique solution of $(\mathrm{E}_{\alpha_{n-1}})$–(1.1), $\beta_0 = \beta$ and β_n the unique solution of $(\mathrm{E}_{\beta_{n-1}})$–(1.1).

By a standard way, see [3], one can verify that both sequences converge uniformly in $[\beta, \alpha]$, respectively, to the maximal and the minimal solutions of the problem

$$-[\Phi(u''(t))]' = f(t, u(t)) \quad \text{for a.e. } t \in I,$$

together with boundary conditions (1.1).

By integration, one can verify that any solution in $[\beta, \alpha]$ of this problem satisfies

$$|u''(t)| \leqslant L \quad \text{for all } t \in I,$$

therefore $\Phi(u'') = \phi(u'')$ and so the proof is finished. □

5. Existence result

In this section we prove the existence of at least one solution for problem (P)–(1.1) in the presence of a pair of lower and upper solutions given in the reversed order and replacing assumption (H_2) by a weaker one:

(H_2^*) There is $M < 0$ such that, for a.e. $t \in I$ and $x \in [\beta(t), \alpha(t)]$,

$$f(t, \beta(t)) + M\beta(t) \geqslant f(t, x) + Mx \geqslant f(t, \alpha(t)) + M\alpha(t).$$

Theorem 5.1. *Assume that condition* (H_1) *holds and condition* (H_2^*) *is fulfilled for* $M \in (M^*, 0)$, *with* $M^* < 0$ *given by* (1.3), *and* $K > 0$ *as in* (4.2). *If there exist a lower solution* α *and an upper solution* β *of problem* (P)–(1.1), *such that* $\alpha \geqslant \beta$ *in I, then problem* (P)–(1.1) *has at least one solution in* $[\beta, \alpha]$.

Proof. Let $p(t, x) = \max\{\beta(t), \min\{x, \alpha(t)\}\}$, for $t \in I$ and $x \in \mathbb{R}$. Fix $M \in (M^*, 0)$ and define for a. e. $t \in I$ the following function:

$$g(t, x) := f(t, p(t, x)) + Mp(t, x).$$

It is clear that function g is a Carathéodory function for which there is $H \in L^1(I)$ such that

$$|g(t, x)| \leqslant H(t) \quad \text{for a.e. } t \in I \text{ and all } x \in \mathbb{R}.$$

Consider the equation

$$-[\Phi(u''(t))]' + Mu(t) = g(t, u(t)) \quad \text{for a. e. } t \in I. \tag{5.1}$$

Since Φ satisfies condition (H_1^*) for the constant K and, following the arguments used in the proof of Proposition 2.1, we have that u is a solution of (5.1)–(1.1) if and only if $\Phi(u'')$ is a fixed point of the operator $T : C(I) \to C(I)$, defined as

$$\begin{aligned} T(w(t)) = \Phi(B) + MA(t-a) + M\tau_w \frac{(t-a)^2}{2} \\ + \frac{M}{2}\int_a^t (t-r)^2 \Phi^{-1}(w(r))\,\mathrm{d}r - \int_a^t \sigma_w(s)\,\mathrm{d}s, \end{aligned}$$

with

$$\begin{aligned} \tau_w = \frac{2}{M(b-a)^2} \Bigg[\Phi(C) - \Phi(B) - MA(b-a) \\ - \frac{M}{2}\int_a^b (b-r)^2 \Phi^{-1}(w(r))\,\mathrm{d}r + \int_a^b \sigma_w(s)\,\mathrm{d}s \Bigg], \end{aligned}$$

$$\sigma_w(t) = f(t, p(t, Jw(t))) + Mp(t, Jw(t)).$$

and

$$Jw(t) = A + \int_a^t \left(\tau_w + \int_a^s \Phi^{-1}(w(r))\,\mathrm{d}r \right) \mathrm{d}s.$$

By (H_1^*) there is a positive constant K_1 such that

$$|Tw(t)| \leqslant K_1 + \frac{K\,|M|\,(b-a)^3}{3} \|w\|_\infty \quad \text{for all } t \in I,$$

and, therefore, every solution of the equation

$$w = \lambda T w$$

is bounded independently of $\lambda \in [0, 1)$. As T is a compact operator and $|M| < -M^*$, then by Schauder's fixed point theorem the operator T has at least a fixed point which is a solution for the problem (5.1)–(1.1).

From the definition of g, we know, by (H_2^*), that every solution u of (5.1)–(1.1) satisfies

$$-[\Phi(u''(t))]' + Mu(t) \geqslant -[\Phi(\alpha''(t))]' + M\alpha(t).$$

Thus, from the boundary conditions and Proposition 3.3, we conclude that $u \leqslant \alpha$ on I.

Analogously, it can be proved that $\beta \leqslant u$ on I. Since any solution in $[\beta, \alpha]$ of this problem satisfies

$$|u''(t)| \leqslant L \quad \text{for all } t \in I,$$

we conclude the proof. □

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