UNIVERSIDADE DE LISBOA
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A Schur ring approach to supercharacters of adjoint groups and related subgroups

"Documento definitivo"

Doutoramento em Matemática
Especialidade em Álgebra, Lógica e Fundamentos

Tânia Sofia Zaragoza Cotrim Silva

Tese orientada por:
Prof. Doutor Carlos Alberto Martins André
Prof. Doutor Pedro Jorge Santos Freitas

Documento especialmente elaborado para a obtenção do grau de doutor

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Céptico como os cépticos, crente como os crentes.
A metade que avança é crente, a metade que confirma é céptica.
Mas o cientista perfeito é também jardineiro: acredita
que a beleza é conhecimento.

(Gonçalo M. Tavares)

Resumo alargado

A pesquisa dedicada a teoria de grupos evoluiu de forma bastante acentuada durante o século XIX e, até hoje, continua a ser objeto de intenso estudo, graças, em parte, às suas inúmeras aplicações em diversas áreas da matemática, física e em outros ramos do conhecimento. O estudo dedicado à classificação de grupos culminou no desenvolvimento da teoria de representação, uma disciplina que desempenha um papel crucial ao proporcionar uma compreensão mais profunda de grupos específicos ou mesmo de famílias completas de grupos. A teoria de representação pode ser vista como uma ponte que conecta conceitos simples com conceitos complexos, colocando em evidência como os elementos de qualquer grupo podem ser representados através de matrizes.

Um dos principais objetivos da teoria de representação consiste na identificação dos caracteres irredutíveis de um grupo, uma tarefa que, em muitos casos, se revela extremamente desafiadora. Assim, podemos tentar encontrar um conjunto de caracteres do grupo, que, mesmo podendo não se tratarem todos de caracteres irredutíveis, contribuam significativamente para a compreensão da natureza do grupo em estudo. Um exemplo clássico onde podemos encontrar este tipo de dificuldade é a caracterização dos caracteres irredutíveis de grupos unitriangulares finitos, conhecida por ser um problema *selvagem* e, provavelmente, impossível de realizar. André, em [2], e Yan, em [44], lidaram com esta dificuldade, substituindo os caracteres irredutíveis dos grupos por uma família de caracteres, não necessariamente todos irredutíveis, que são constantes num conjunto de uniões de classes de conjugação do grupo. Mais tarde, Diaconis e Isaacs, em [19], apresentaram o conceito geral de uma teoria de supercaracteres para um grupo finito.

Os supercaracteres surgiram sob o nome de *caracteres básicos* relativamente ao grupo unitriangular $U_n(\mathbb{F})$ definido sobre um corpo finito $\mathbb{F}$ de característica p (conferir [2] e [3], ambos de C. André). A construção inicial dos caracteres básicos estava fundamentada no método das órbitas de Kirillov (consultar [31]), conjuntamente com uma construção de David Kazhdan (ver [29]) de caracteres irredutíveis no caso em que $p \geq n$. De destacar que os caracteres básicos foram obtidos como produtos específicos de *caracteres elementares* e essa descrição permitiu a extensão do conceito de supercaracteres aos subgrupos unipotentes dos grupos simpléticos e ortogonais (ver [4], [6], [7]). Posteriormente, Ning Yan (ver [44]) adotou uma abordagem completamente distinta para ampliar os resultados anteriores para o caso mais geral, onde $p \geq n$. Yan utilizou ações à esquerda e à direita do grupo unitriangular $U_n(\mathbb{F})$ na álgebra $\mathfrak{u}_n(\mathbb{F})$, incorporando também as correspondentes ações contragradiente no espaço vetorial dual de $\mathfrak{u}_n(\mathbb{F})$. A abordagem de Yan foi posteriormente utilizada por Persi Diaconis e I. Martin Isaacs para alargar a construção a grupos álgebra arbitrários definidos sobre um corpo finito $\mathbb{F}$ (consultar [19]). Posteriormente, C. André e A. Nicolás, em [5], adaptaram o

trabalho desenvolvido por Diaconis e Isaacs ao caso do grupo adjunto de anéis radicais finitos, destacando, em particular, a sua aplicação a grupos álgebra definidos sobre um anel comutativo finito.

Estudos recentes revelam que enquanto a teoria de representação do grupo simétrico atua como um modelo fundamental para a teoria de representação de grupos finitos, intrinsecamente relacionado com combinatória de partições e *tableaux*, por sua vez, a teoria de supercaracteres do grupo unitriangular apresenta uma estrutura combinatória igualmente rica e complexa, baseada na análise de partições de conjuntos e diagramas de arco (ver, por exemplo, a referência [8] por C. André). Esta identificação abre novas perspetivas e horizontes na pesquisa em teoria de grupos e teoria de representação, prometendo descobertas e avanços que enriquecem ainda mais o nosso entendimento destas áreas do conhecimento da matemática e da física.

A matemática contemporânea, agora impulsionada pelo auxílio da inteligência artificial (IA), está bem posicionada para explorar novas abordagens e promover avanços significativos na nossa compreensão do mundo matemático e científico, ao relacionar mais facilmente diferentes áreas do saber. Sempre que se estabelece uma conexão entre duas áreas distintas do conhecimento, alimentamos a expectativa de descobertas notáveis. No entanto, mesmo quando esses avanços específicos não se materializam, o mero apreço pela beleza inerente a uma nova abordagem pode justificar plenamente o esforço investido. Teorias de supercaracteres distintas, aplicadas a um mesmo grupo finito arbitrário G, estão em bijeção com os anéis centrais de Schur sobre G distintos. É notável que muitos dos teoremas iniciais descritos em [19] já tivessem na verdade sido anteriormente apresentados, sob uma perspetiva distinta, no trabalho de Tamaschke sobre os anéis de Schur, conforme mencionado em [41]. Ainda de destacar que, em [26], Humphries e Johnson levantaram a questão, na terminologia de [19], de identificar que grupos poderiam exibir uma tabela de caracteres idêntica à tabela de supercaracteres de algum grupo abeliano.

A ligação formal entre as teorias dos supercaracteres de um grupo finito G e anéis centrais de Schur sobre G, foi estabelecida de forma rigorosa por A. Hendrickson, em [25]. Utilizando os resultados obtidos por Hendrickson, conseguimos (re)construir os supercaracteres *standard* dos grupos álgebra G, bem como desenvolver a teoria de supercaracteres relacionada a subgrupos de G, compostos por elementos que são fixos por uma involução. Como resultado direto, a demonstração principal apresentada em [9] sofre uma significativa simplificação, destacando a eficácia desta abordagem ao aprofundarmos o nosso conhecimento do papel dos supercarateres no estudo de um grupo finito. Ou seja, utilizando a correspondência entre anéis de Schur centrais e teorias de supercaracteres para grupos finitos, simplificamos a construção da teoria de supercaracteres standard para grupos adjuntos de anéis radicais finitos. Se o anel radical R for de característica ímpar e se estiver munido de uma anti-involução σ, definimos uma teoria de supercaracteres para o subgrupo R°_σ do grupo adjunto R°, constituído pelos elementos fixos através de σ. Esta teoria estende a que foi definida previamente em

[9], quando R é uma álgebra nilpotente sobre um corpo finito.

Ao longo dos seis capítulos deste trabalho, incluímos vários exemplos de modo a clarificar os conceitos apresentados. Os três capítulos iniciais estabelecem os pilares essenciais que servirão como base sólida para o restante desta dissertação.

No primeiro capítulo, iniciamos com a definição da álgebra de grupo de um grupo arbitrário e exploramos diversas propriedades pertinentes à álgebra de grupo complexo de um grupo finito. Ademais, este capítulo proporciona uma visão abrangente sobre as propriedades elementares dos caracteres de grupo, estabelecendo o alicerce necessário para compreender a interligação entre os anéis de Schur e as teorias de supercaracteres.

No segundo capítulo, delineamos os fundamentos dos anéis de Schur, oferecendo uma visão geral que engloba construções pormenorizadas, além de apresentarmos a teoria dos caracteres *Scal* de Tamaschke para um anel de Schur.

O terceiro capítulo introduz uma definição atualizada de uma teoria de supercaracteres aplicável a grupos finitos, equivalente à proposta em [19]. Neste capítulo, também apresentamos uma lista de caracterizações equivalentes, algumas das quais representam contribuições novas e anteriormente não exploradas.

No capítulo subsequente, o quarto, apresentamos a (re)construção da teoria de supercaracteres standard de grupos adjuntos de anéis radicais finitos. Realçamos o facto de que as demonstrações neste capítulo apresentam uma simplificação considerável em relação às apresentadas em [5].

No quinto capítulo, prosseguimos ao estender a construção de uma teoria de supercaracteres para grupos álgebra involutivos, previamente apresentada em [9], para uma teoria de supercaracteres para grupos adjuntos involutivos. Mais uma vez, as provas apresentadas neste capítulo notabilizam-se pela sua substancial simplicidade, em comparação com as de [9].

Por fim, no sexto capítulo, detalhamos diversas ideias para trabalhos futuros. Sugerimos possíveis caminhos de pesquisa que nos parecem promissores de gerar resultados significativos, como o alargamento ao caso dos grupos infinitos ou, por exemplo, a exploração da interligação entre supercaracteres e esquemas de associação.

Palavras-chave: Anéis de Schur, Teoria de Supercaracteres, Grupo Adjunto

Abstract

Using the correspondence between central Schur rings and supercharacter theories for finite groups, we simplify the construction of the standard supercharacter theory for adjoint groups of finite radical rings. If the radical ring R is of odd characteristic and is endowed with an anti-involution σ, we define a supercharacter theory for the subgroup of the adjoint group R°, R°_σ, consisting of elements fixed by σ. This theory extends the one previously defined in [9], when R is a nilpotent algebra over a finite field.

Keywords: Schur rings, Supercharacter Theory, Adjoint Group

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Introduction

Group theory had its big boom in the 19th century, but it remains intensely studied due to its numerous applications in many branches of mathematics, physics, and other fields. The effort put into studying groups led to the emergence of representation theory, which enables us to gain a better understanding of specific groups or entire classes of groups. Representation theory acts as a bridge between simple and complex concepts, as it demonstrates how the elements of any group can be represented as matrices.

One of the main goals of representation theory is to find the irreducible characters of a group. However, this can sometimes be an incredibly challenging task. For instance, the classification of the irreducible characters of the finite unitriangular groups is known to be a *wild* problem. Therefore, it becomes worthwhile to seek a set of characters for the group that, although they may or may not be irreducible, help us better comprehend the essence of the group. This is where the concept of supercharacters arises. In fact, while the representation theory of the symmetric group serves as a fundamental model directly connected to combinatorics of partitions and tableaux, recent years have revealed that the supercharacter theory of the finite unitriangular group boasts a similarly rich combinatorial structure based on set partitions (see [8] by C. André).

Supercharacters were originally introduced under the name of *basic characters* for the unitriangular group $U_n(\mathbb{F})$ defined over a finite field $\mathbb{F}$ of characteristic p (see [2] and [3] by C. André). The original construction of basic characters relied on Kirillov's orbit method (see [31]) and a construction by David Kazhdan (see [29]) of irreducible characters in the case where $p \geq n$. It should be noted that the basic characters were obtained as certain products of *elementary characters*, and this description allowed the notion of a supercharacter to be extended to unipotent subgroups of the symplectic and orthogonal groups (see [4], [6], [7]). Later, Ning Yan (see [44]) used an entirely different method to extend the earlier results to the general case (where $p \geq n$). Yan employed left and right actions of $U_n(\mathbb{F})$ on the algebra $\mathfrak{u}_n(\mathbb{F})$ and the corresponding contragradient actions on the dual vector space of $\mathfrak{u}_n(\mathbb{F})$. Yan's approach has been further extended by Persi Diaconis and I. Martin Isaacs to arbitrary algebraic groups over a finite field $\mathbb{F}$ (see [19]). Subsequently, C. André and A. Nicolás, in [5], adapted the work of Diaconis and Isaacs to the adjoint group of finite radical rings and, in particular, to algebraic groups defined over a finite commutative ring.

Whenever a connection is established between two different areas of knowledge, breakthroughs are expected. However, even when such breakthroughs are not achieved, the beauty of the new approach alone can make the effort worthwhile. With the assistance of AI, mathematicians today can expect these "new approaches" to become increasingly common. Supercharacter theories of an arbitrary finite group G are closely related to central Schur rings over G. Notably, several of the initial theorems in [19] were previously found in a different form in Tamaschke's work on Schur rings [41]. Additionally, related work by Humphries and Johnson in [26] posed the question, in the language of [19], of identifying which groups have a character table identical to the supercharacter table of some abelian group.

The formal connection between supercharacter theories of G and central Schur rings over G was established by A. Hendrickson in [25]. We use Hendrickson's results to (re)construct

the standard supercharacters of algebra groups G, and also the related supercharacter theory of subgroups of G consisting of elements fixed by an involution. Consequently, the main proof in [9] is significantly simplified. Additionally, we extend the results of [19] and [9] to a slightly more general situation where G is the adjoint group of a finite radical ring.

Throughout all chapters, we include examples to illustrate the concepts. The first three chapters lay the fundamental prerequisites for the remainder of the dissertation.

In chapter 1, we start with the definition of the group algebra of an arbitrary group and explore various properties of the complex group algebra of a finite group. Additionally, this chapter provides an overview of several elementary properties of group characters and sets the foundation for establishing the connection between Schur rings and supercharacter theories.

In chapter 2, we provide an overview of the fundamentals of Schur rings, including detailed constructions, and present Tamaschke's $\mathcal{S}$-character theory for a Schur ring.

In chapter 3, we introduce a refreshed definition of a supercharacter theory for a finite group, equivalent to the one in [19]. We also present a list of equivalent characterizations, some of which are new and previously unexplored.

In chapter 4, we (re)construct the standard supercharacter theory for adjoint groups of finite radical rings. The proofs in this chapter are considerably simpler than those presented in [5].

In chapter 5, we extend the construction in [9] of a supercharacter theory for involutive algebra groups to involutive adjoint groups. Once again, the proofs in this chapter are considerably simpler than those in [9].

In chapter 6, we outline some ideas for future work that appear to have a good chance to provide us some nice results.

Chapter 1

Preliminaries

A Schur ring over an arbitrary finite group G is a subalgebra of the group algebra of G that verifies some additional conditions; supercharacters of G are strictly related with idempotents of the group algebra of G; examples of adjoint groups emerge when working with a particular kind of group algebras. Hence, given the relevance of this algebraic structure, we start by reviewing fundamental definitions involving the group algebra and highlight some of the properties that will be useful later on. Then, in the second section of this chapter, we will review definitions and results concerning character theory that will sustain what will be presented further on about supercharacter theories.

1.1 The group algebra

The relevance of the group algebra is well-known, in particular when studying representation theory of groups. Moreover, we can endow the group algebra with several additional operations, making it a very rich algebraic structure that will empower us when working with its elements.

For the most part of this work, G will be a finite group and we will work over the complex field. However, for this first definition, we will consider a more generic situation, knowing that this generalization will be useful in Example 4.2.1 of chapter 4 and on section 1 of chapter 6.

Let G be an arbitrary group (not necessarily finite) and R be a commutative ring with identity.

If $\alpha : G \to R$ is a map, we define the **support of** α by

$$\operatorname{supp}(\alpha) := \{g \in G : \alpha(g) \neq 0\}.$$

Definition 1.1.1. The **group algebra of** G, denoted by $R[G]$, is the R-module consisting of all R-valued functions defined on G and having finite support, endowed with the **convolution product** defined by

$$(\alpha\beta)(g) := \sum_{h \in G} \alpha(h)\beta(h^{-1}g),$$

for all $\alpha, \beta \in R[G]$ and $g \in G$.

It is straightforward to prove that defined this way, $R[G]$ is a R-algebra.

For any finite subset $K \subseteq G$, we consider the **characteristic map**, $\delta_K \in R[G]$, defined for all $g \in G$ by

$$\delta_K(g) := \begin{cases} 1, & \text{if } g \in K \\ 0, & \text{otherwise.} \end{cases}$$

If $K = \{g\}$, for $g \in G$, we will write δ_g instead of δ_K.

Usually, the group algebra $R[G]$ of G is defined as the free R-module with basis G where the multiplication extends in the natural way the multiplication of the group G. However, our definition is equivalent to this; indeed, the mapping $g \to \delta_g$, for $g \in G$, defines an isomorphism of R-algebras. In particular, it follows easily that $R[G]$ is a free R-module with R-basis $\{\delta_g : g \in G\}$.

From now on, unless stated otherwise, let G be a finite group with identity 1 and consider the complex group algebra $\mathbb{C}[G]$. We will always assume that a subalgebra of $\mathbb{C}[G]$ includes the identity of $\mathbb{C}[G]$. Let $\mathrm{Cl}(G)$ denote the set of all conjugacy classes of G. If X is a set, let $|X|$ denote the cardinality of X and let $\mathrm{Part}(X)$ denote the set of all partitions of X.

Remark 1.1.2: Note that δ_1 is the identity of $\mathbb{C}[G]$ and, for all subsets $K, K' \subseteq G$, the following properties are simple to verify:

- $\delta_K + \delta_{K'} = 2\delta_{K \cap K'} + \delta_{K \setminus K'} + \delta_{K' \setminus K}$;
- $(\delta_K \delta_{K'})(g) = |h^{-1}g \in K' : h \in K|$ for all $g \in G$. In particular,

  $$\delta_g \delta_h = \delta_{gh}$$

  for all $g, h \in G$.

Definition 1.1.3. An element $\alpha \in \mathbb{C}[G]$ is called a **class function of** G if

$$\alpha(g) = \alpha(h^{-1}gh),$$

for all $g, h \in G$, or equivalently, if α is constant on each conjugacy class $C \in \mathrm{Cl}(G)$.

Let $\mathrm{cf}(G)$ denote the subalgebra of $\mathbb{C}[G]$ consisting of all class functions of G. A class function is sometimes referred to as a central function because if $\alpha \in \mathrm{cf}(G)$, then

$$\alpha(gh) = \alpha(hg)$$

for all $g, h \in G$.

Consider the usual **Frobenius inner product** defined by

$$\langle \alpha, \beta \rangle := \frac{1}{|G|} \sum_{g \in G} \alpha(g)\overline{\beta(g)},$$

for all $\alpha, \beta \in \mathbb{C}[G]$.

Proposition 1.1.4. *The* ***center of the group algebra***

$$\mathcal{Z}(\mathbb{C}[G]) := \{\alpha \in \mathbb{C}[G] : \alpha\beta = \beta\alpha \ \text{ for all } \beta \in \mathbb{C}[G]\}$$

equals $\mathrm{cf}(G)$*. Moreover, with respect to the Frobenius inner product,* $\{\delta_C : C \in \mathrm{Cl}(G)\}$ *is an orthogonal basis for* $\mathrm{cf}(G)$*.*

Proof. For all $\alpha \in \mathbb{C}[G]$ and all $g, h \in G$,

$$\begin{aligned}(\delta_g \alpha \delta_{g^{-1}})(h) &= \sum_{k\in G} (\delta_g \alpha)(k)\delta_{g^{-1}}(k^{-1}h) \\ &= (\delta_g \alpha)(hg) \\ &= \sum_{k\in G} \delta_g(k)\alpha(k^{-1}hg) \\ &= \alpha(g^{-1}hg).\end{aligned}$$

Therefore, $\alpha \in \mathcal{Z}(\mathbb{C}[G])$ if and only if $\alpha \in \mathrm{cf}(G)$.

Moreover, if $C, D \in \mathrm{Cl}(G)$, then

$$\langle \delta_C, \delta_D \rangle = \begin{cases} \frac{|C|}{|G|}, & \text{if } C = D \\ 0, & \text{otherwise.} \end{cases}$$

□

An element belonging to the center is called a **central** element.

We will now define some more operations on the algebra $\mathbb{C}[G]$, starting with the usual product of functions, with respect to which $\mathbb{C}[G]$ is a commutative algebra.

Definition 1.1.5. For every $\alpha, \beta \in \mathbb{C}[G]$, we define the **Hadamard product** (or **pointwise product**), $\alpha \cdot \beta \in \mathbb{C}[G]$, by

$$(\alpha \cdot \beta)(g) := \alpha(g)\beta(g)$$

for all $g \in G$.

Remark 1.1.6: The following properties are obvious:

- δ_G is the identity of $\mathbb{C}[G]$ with respect to the Hadamard product;
- $\delta_K \cdot \delta_{K'} = \delta_{K\cap K'}$, for all subsets $K, K' \subseteq G$;
- If $\alpha \in \mathbb{C}[G]$ is idempotent with respect to the Hadamard product, then $\alpha = \delta_K$ for some $K \subseteq G$;
- If S is a subalgebra of $\mathbb{C}[G]$ with respect to the Hadamard product, then S is the linear span over $\mathbb{C}$ of characteristic functions of some disjoint subsets of G.

There is still a unary operation that we can define in the group algebra that has a special role in our study. Firstly, we recall the definition of an anti-involution on an arbitrary ring and some related properties.

Definition 1.1.7. Let R be a ring. A map $\star : R \to R$ is called an **anti-involution on** R if:

1. $(x+y)^\star = x^\star + y^\star$;
2. $(xy)^\star = y^\star x^\star$;
3. $(x^\star)^\star = x$.

for all $x, y \in R$. If R has identity we also require that:

4. $1^\star = 1$.

Proposition 1.1.8. *Let R be a ring with anti-involution $\star$ and let $x \in R$. Then the following hold:*

1. $0^\star = 0$;
2. *If x is central, then $x^\star$ is also central;*
3. *If R has identity and x is invertible, then $(x^{-1})^\star = (x^\star)^{-1}$.*

Proof. $0^\star = (0+0)^\star = 0^\star + 0^\star$, which implies $0 = 0^\star$. Also, if x is a central element of R, then $x^\star y = (y^\star x)^\star = (xy^\star)^\star = yx^\star$ for all $y \in R$, and so $x^\star$ is also central. Finally, if $x \in R$ is invertible, then $1 = 1^\star = (xx^{-1})^\star = (x^{-1})^\star x^\star$. □

Example 1.1.9. Let R be a ring with identity and let $\mathcal{M}_n(R)$ denote the set of all square matrices of order n with entries in R. Then $\mathcal{M}_n(R)$ is also a ring with identity. Let $M^\intercal$ denote the transpose of a matrix $M \in \mathcal{M}_n(R)$ and let I_n denote the identity matrix of order n. For all $M, N \in \mathcal{M}_n(R)$, we have:

1. $(M+N)^\intercal = M^\intercal + N^\intercal$;
2. $(MN)^\intercal = N^\intercal M^\intercal$;
3. $(M^\intercal)^\intercal = M$;
4. $I_n^\intercal = I_n$;

and so the transpose defines an anti-involution on $\mathcal{M}_n(R)$.

Example 1.1.10. Let $\overline{z}$ denote the conjugate of the complex number $z \in \mathbb{C}$. Note that, for all $z, w \in \mathbb{C}$, we also have:

1. $\overline{z+w} = \overline{z} + \overline{w}$;
2. $\overline{zw} = \overline{w}\,\overline{z}$;
3. $\overline{\overline{z}} = z$;
4. $\overline{1} = 1$;

and so the conjugate defines an anti-involution on $\mathbb{C}$.

Definition 1.1.11. If S is a $\mathbb{C}$-algebra with anti-involution $\star$ and if, for all $z \in \mathbb{C}$ and all $\alpha \in S$,

$$(z\alpha)^\star = \overline{z}\alpha^\star,$$

then S is called a **$\star$-algebra over $\mathbb{C}$**.

Proposition 1.1.12. *The map $\star : \mathbb{C}[G] \longrightarrow \mathbb{C}[G]$, defined by*

$$\alpha^\star(g) = \overline{\alpha(g^{-1})}$$

for all $\alpha \in \mathbb{C}[G]$ and all $g \in G$, is an anti-involution on $\mathbb{C}[G]$, both for the convolution product and for the Hadamard product. Moreover, $\mathbb{C}[G]$ is a $\star$-algebra over $\mathbb{C}$.

Proof. For all $\alpha, \beta \in \mathbb{C}[G]$, all $g \in G$ and all $z \in \mathbb{C}$:

- $(\alpha+\beta)^\star(g) = \overline{(\alpha+\beta)(g^{-1})} = \overline{\alpha(g^{-1})} + \overline{\beta(g^{-1})} = \alpha^\star(g) + \beta^\star(g)$;
- $$\begin{aligned}(\beta^\star\alpha^\star)(g) &= \sum_{h\in G} \beta^\star(h)\alpha^\star(h^{-1}g) = \sum_{h\in G} \overline{\beta(h^{-1})\alpha(g^{-1}h)} \\ &= \sum_{h\in G} \overline{\alpha(g^{-1}h)\beta(h^{-1})} = \sum_{y\in G} \overline{\alpha(y)\beta(y^{-1}g^{-1})} \\ &= \overline{(\alpha\beta)(g^{-1})} = (\alpha\beta)^\star(g);\end{aligned}$$
- $(\alpha\cdot\beta)^\star(g) = \overline{(\alpha\cdot\beta)(g^{-1})} = \overline{\alpha(g^{-1})}\,\overline{\beta(g^{-1})} = \beta^\star(g)\alpha^\star(g) = (\beta^\star\cdot\alpha^\star)(g)$;
- $(\alpha^\star)^\star(g) = \overline{\alpha^\star(g^{-1})} = \overline{\overline{\alpha(g)}} = \alpha(g)$;
- $\delta_1^\star(g) = \overline{\delta_1(g^{-1})} = \delta_1(g)$;
- $\delta_G^\star(g) = \overline{\delta_G(g^{-1})} = \delta_G(g)$;
- $(z\alpha)^\star(g) = \overline{(z\alpha)(g^{-1})} = \overline{z(\alpha(g^{-1}))} = \overline{z}\alpha^\star(g)$.

□

From now on, unless stated otherwise, $\star$ denotes the anti-involution of $\mathbb{C}[G]$ defined on Proposition 1.1.12.

Note that in virtue of the second point of Proposition 1.1.8, $\mathrm{cf}(G)$ is also a $\star$-algebra over $\mathbb{C}$. This observation together with the next proposition will be helpful for the following sections.

Proposition 1.1.13. *For every $\alpha, \beta \in \mathbb{C}[G]$,*

$$\alpha\alpha^\star = 0 \text{ if and only if } \alpha = 0.$$

Proof. If $\alpha\alpha^\star = 0$, then

$$0 = (\alpha\alpha^\star)(1) = \sum_{h\in G} \alpha(h)\alpha^\star(h^{-1}) = \sum_{h\in G} \alpha(h)\overline{\alpha(h)} = \sum_{h\in G} |\alpha(h)|^2,$$

and thus $\alpha(h) = 0$ for all $h \in G$. The other direction is trivial. □

We end this section with the notion of normality of an algebra that includes clearly all subalgebras of the center of $\mathbb{C}[G]$. Then we finish with two examples that illustrate some of the things we have seen so far.

Definition 1.1.14. A subalgebra S of $\mathbb{C}[G]$ is said to be **normal** if

$$\alpha\alpha^\star = \alpha^\star\alpha$$

for all $\alpha \in S$.

Example 1.1.15. Consider the cyclic group of order 5,

$$C_5 := \langle\sigma\rangle = \{1, \sigma, \sigma^2, \sigma^3, \sigma^4\}.$$

As defined before,

$$\mathbb{C}[C_5] = \mathrm{span}_{\mathbb{C}}\{\delta_1, \delta_\sigma, \delta_{\sigma^2}, \delta_{\sigma^3}, \delta_{\sigma^4}\}.$$

Because C_5 is abelian $\mathbb{C}[C_5]$ is central and, therefore, normal.

Note that

$$\delta^\star_{\{\sigma,\sigma^4\}} = \delta_{\{\sigma,\sigma^4\}} \qquad \text{and} \qquad \delta^\star_{\{\sigma^2,\sigma^3\}} = \delta_{\{\sigma^2,\sigma^3\}}.$$

Moreover,

$$\begin{aligned}\delta_{\{\sigma,\sigma^4\}}\delta_{\{\sigma,\sigma^4\}} &= 2\delta_1 + \delta_{\{\sigma^2,\sigma^3\}}\\ \delta_{\{\sigma^2,\sigma^3\}}\delta_{\{\sigma^2,\sigma^3\}} &= 2\delta_1 + \delta_{\{\sigma,\sigma^4\}}\\ \delta_{\{\sigma,\sigma^4\}}\delta_{\{\sigma^2,\sigma^3\}} &= \delta_{\{\sigma,\sigma^4\}} + \delta_{\{\sigma^2,\sigma^3\}}.\end{aligned}$$

Taking into account remark 1.1.6, it is now obvious that $\mathrm{span}_{\mathbb{C}}\{\delta_1, \delta_{\{\sigma,\sigma^4\}}, \delta_{\{\sigma^2,\sigma^3\}}\}$ is a $\star$-subalgebra of $\mathbb{C}[C_5]$, both for the convolution product and the Hadamard product.

On the other hand, note that, for example,

$$\delta^\star_{\{\sigma,\sigma^2,\sigma^3\}} = \delta_{\{\sigma^2,\sigma^3,\sigma^4\}}.$$

So, $\mathrm{span}_{\mathbb{C}}\{\delta_1, \delta_{\{\sigma,\sigma^2,\sigma^3\}}, \delta_{\{\sigma^4\}}\}$ is not a $\star$-subalgebra of $\mathbb{C}[C_5]$.

Example 1.1.16. Consider the symmetric group of order 3,

$$S_3 := \{(1), (1\,2), (1\,3), (2\,3), (1\,2\,3), (1\,3\,2)\}.$$

It is easy to check that $\mathrm{Cl}(S_3) = \{\{(1)\}, (**), (***)\}$ where $(**) = \{(1\,2), (1\,3), (2\,3)\}$ and $(***) = \{(1\,2\,3), (1\,3\,2)\}$. So

$$\mathrm{cf}(S_3) = \mathrm{span}_{\mathbb{C}}\{\delta_{(1)}, \delta_{(**)}, \delta_{(***)}\}.$$

Note that $\mathbb{C}[S_3]$ is not normal, because, for example,

$$\begin{aligned}(\delta_{(1\,2)} + \delta_{(1\,2\,3)})(\delta_{(1\,2)} + \delta_{(1\,2\,3)})^\star &= (\delta_{(1\,2)} + \delta_{(1\,2\,3)})(\delta_{(1\,2)} + \delta_{(1\,3\,2)}) = 2(\delta_{(1)} + \delta_{(1\,3)})\\ &\neq 2(\delta_{(1)} + \delta_{(2\,3)}) = (\delta_{(1\,2)} + \delta_{(1\,2\,3)})^\star(\delta_{(1\,2)} + \delta_{(1\,2\,3)}).\end{aligned}$$

Moreover, note that, for example,

$$\begin{aligned}(\delta_{\{(1\,2),(1\,2\,3)\}})(\delta_{\{(1\,2),(1\,2\,3)\}}) &= \delta_{(1)} + \delta_{\{(1\,3),(2\,3),(1\,3\,2)\}}\\ (\delta_{\{(1\,3),(2\,3),(1\,3\,2)\}})(\delta_{\{(1\,3),(2\,3),(1\,3\,2)\}}) &= 2\delta_{(1)} + 2\delta_{\{(1\,2),(1\,2\,3)\}} + \delta_{\{(1\,3),(2\,3),(1\,3\,2)\}}\\ (\delta_{\{(1\,2),(1\,2\,3)\}})(\delta_{\{(1\,3),(2\,3),(1\,3\,2)\}}) &= (\delta_{\{(1\,3),(2\,3),(1\,3\,2)\}})(\delta_{\{(1\,2),(1\,2\,3)\}})\\ &= \delta_{(1)} + \delta_{\{(1\,2),(1\,2\,3)\}} + \delta_{\{(1\,3),(2\,3),(1\,3\,2)\}}\end{aligned}$$

So, $\mathrm{span}_{\mathbb{C}}\{\delta_{(1)}, \delta_{\{(1\,2),(1\,2\,3)\}}, \delta_{\{(1\,3),(2\,3),(1\,3\,2)\}}\}$ is a subalgebra of $\mathbb{C}[S_3]$ with respect to the Hadamard and to the convolution product but is not a $\star$-subalgebra of $\mathbb{C}[S_3]$.

On the other hand,

$$\mathrm{span}_{\mathbb{C}}\{\delta_{(1)}, \delta_{(1\,2)}, \delta_{(1\,3)}, \delta_{\{(2\,3),(1\,2\,3),(1\,3\,2)\}}\}$$

is a $\star$-subalgebra of $\mathbb{C}[S_3]$ with respect to the Hadamard product but not with respect to the convolution product.

Finally,

$$\mathrm{span}_{\mathbb{C}}\{\delta_{(1)}, \delta_{\{(1\,2),(1\,3),(2\,3),(1\,2\,3),(1\,3\,2)\}}\}$$

is a $\star$-subalgebra of $\mathbb{C}[S_3]$ with respect to both the Hadamard product and the convolution product as $\mathbb{C}[S_3]$ itself or $\mathrm{cf}(S_3)$.

1.2 Semisimplicity and Characters

Assuring that the subalgebras of $\mathbb{C}[G]$, that we will work with, are semisimple, is fundamental for our study. A semisimple algebra has a unique complete set of central, primitive, pairwise orthogonal idempotents that are in a strict relation with the characters of G. We adapt some well-known results to our particular case in order to simplify the proofs and emphasize the fundamental properties that these algebraic structures have and that allow us to obtain the results we need. We also recall the notion of partition of unity, whose relevance is transverse to several areas of mathematics (for more about this subject see, for example, [21]).

For every ring R, let $\mathcal{J}(R)$ denote the **Jacobson radical of** R. Recall that, by definition, $\mathcal{J}(R)$ is the intersection of all maximal left (equivalently, right) ideals of R. Further, recall that a finite-dimensional algebra A is **semisimple** if and only if $\mathcal{J}(A) = \{0\}$ and in this case A is a direct sum of simple subalgebras of A, whose components are well determinated. For further details about semisimple algebras see, for example, chapter IV of [18].

Theorem 1.2.1. *Every $\star$-subalgebra of $\mathbb{C}[G]$, with respect to the convolution product, is semisimple.*

Proof. Note that $\mathbb{C}[G]$ is an artinian ring since G is finite. Suppose S is a $\star$-subalgebra of $\mathbb{C}[G]$ that is not semisimple. Then there must be a nonzero $\alpha \in \mathcal{J}(S)$ such that $S\alpha$ is a simple left ideal of S contained on $\mathcal{J}(S)$. Note that $S\alpha$ must be a nilpotent ring. By the simplicity of $S\alpha$, $(S\alpha)(S\alpha)$ equals zero or $S\alpha$. If $(S\alpha)(S\alpha) = S\alpha$ then every power of $S\alpha$ is different from zero, which contradicts the nilpotency of $S\alpha$. Therefore, $(S\alpha)(S\alpha)$ must be zero and, in particular, because $\alpha^\star \in S$, $\alpha\alpha^\star\alpha = 0$. So,

$$\alpha\alpha^\star\alpha\alpha^\star = (\alpha\alpha^\star)(\alpha\alpha^\star)^\star = 0$$

and applying two times the first point of Proposition 1.1.13 we conclude that α must be zero. □

We can require the normality of a subalgebra of $\mathbb{C}[G]$ instead of the closure of the operation $\star$ to guarantee that the subalgebra, with respect to the convolution product, is semisimple.

Theorem 1.2.2. *Every normal subalgebra of $\mathbb{C}[G]$, with respect to the convolution product, is semisimple.*

Proof. Suppose S is a normal subalgebra of $\mathbb{C}[G]$. It is enough to prove that S has no nonzero nilpotent elements, as $\mathcal{J}(S)$ is nilpotent. Let $\alpha \in S$ be nonzero. Then $\alpha\alpha^\star \neq 0$, by Proposition 1.1.13, and so $(\alpha\alpha^\star)(\alpha\alpha^\star)^\star \neq 0$, by the same reason. Since $\alpha\alpha^\star = \alpha^\star\alpha$, because S is normal, we conclude that

$$\begin{aligned}(\alpha\alpha)(\alpha\alpha)^\star &= \alpha\alpha(\alpha^\star\alpha^\star) = \alpha(\alpha\alpha^\star)\alpha^\star \\ &= \alpha(\alpha^\star\alpha)\alpha^\star = (\alpha\alpha^\star)(\alpha\alpha^\star)^\star \neq 0,\end{aligned}$$

and thus $\alpha\alpha \neq 0$, again by Proposition 1.1.13. This is valid for every power of α and so α is not nilpotent. □

Note that every subalgebra of $\mathbb{C}[G]$, with respect to the Hadamard product, has no nonzero nilpotent elements, so it is semisimple.

In particular, we have proved that $\mathbb{C}[G]$ is semisimple (with respect to both products), which means it is isomorphic to a direct sum of matrix rings over $\mathbb{C}$.

Definition 1.2.3. If A is a finite-dimensional associative algebra with identity over a field $\mathbb{F}$, we say that $\mathcal{E} = \{e_1, \ldots, e_r\} \subseteq A$ is a **partition of unity of** A if it is a complete set of pairwise orthogonal primitive idempotents, that is,

1. $e_i e_i = e_i$ for all $1 \leq i \leq r$ (**idempotent**);
2. $e_i e_j = 0$ for all $1 \leq i, j \leq r$, with $i \neq j$ (**orthogonal**);
3. $e_1 + \ldots + e_r = 1$ (**complete**);
4. e_i is not the sum of two nonzero orthogonal idempotents of A for all $1 \leq i \leq r$ (**primitive**).

Remark 1.2.4: It can be seen that if S is a finite-dimensional subalgebra with identity of $\mathbb{C}[G]$, with respect to the Hadamard product, then S has a unique central partition of unity, Δ_S, with respect to this product. Moreover, there is a unique partition $\mathcal{K} \in \mathrm{Part}(G)$ such that

$$\Delta_S = \{\delta_K : K \in \mathcal{K}\}.$$

For example,

$$\Delta_{\mathbb{C}[G]} = \{\delta_g : g \in G\} \quad \text{and} \quad \Delta_{\mathrm{cf}(G)} := \{\delta_C : C \in \mathrm{Cl}(G)\},$$

are the partitions of unity of $\mathbb{C}[G]$ and $\mathrm{cf}(G)$, respectively, with respect to the Hadamard product.

There is a similar result with respect to the convolution product.

Theorem 1.2.5. *Every finite-dimensional $\star$-subalgebra S of $\mathbb{C}[G]$, with respect to the convolution product, has a unique central partition of unity, $\mathcal{E}_S$, with respect to this product. Moreover, $\mathcal{E}_S$ is a basis for the center of S.*

Proof. Let r be the dimension of the center of S, $\mathcal{Z}(S)$, as a $\mathbb{C}$ vector space, and let

$$\{\alpha_1, \ldots, \alpha_r\}$$

be an orthonormal basis for $\mathcal{Z}(S)$ with respect to the Frobenius inner product. If $\beta \in S$, let $\operatorname{ad}\beta : S \to S$ be the **adjoint function** defined by

$$(\operatorname{ad}\beta)(\alpha) = \beta\alpha$$

for all $\alpha \in S$. Note that if $z_1, \ldots, z_r \in \mathbb{C}$ are such that

$$\sum_{i=1}^{r} z_i(\operatorname{ad}\alpha_i) = 0$$

then, evaluating at $\alpha_j^\star$, for $1 \leq j \leq r$, we get $\sum_{i=1}^{r} z_i(\alpha_i\alpha_j^\star) = 0$. Since

$$\begin{aligned}(\alpha_i\alpha_j^\star)(1) = \sum_{g\in G} \alpha_i(g)\alpha_j^\star(g^{-1}) = \sum_{g\in G} \alpha_i(g)\overline{\alpha_j(g)} \\ = \begin{cases} |G|, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}\end{aligned}$$

necessarily $z_1 = \ldots = z_r = 0$ and $\{\operatorname{ad}\alpha_1, \ldots, \operatorname{ad}\alpha_r\}$ is a linearly independent set of $\operatorname{End}_{\mathbb{C}}(S)$, the algebra of all endomorphisms of S.

The subalgebra S is semisimple which implies that $\operatorname{span}_{\mathbb{C}}\{\operatorname{ad}\alpha_1, \ldots, \operatorname{ad}\alpha_r\}$ is a commutative semisimple subalgebra of $\operatorname{End}_{\mathbb{C}}(S)$ and $\mathcal{Z}(S)$ must be isomorphic to the subalgebra of $\mathcal{M}_r(\mathbb{C})$ consisting of all $r \times r$ diagonal matrices with entries in $\mathbb{C}$. In particular, there must exist a basis for $\mathcal{Z}(S)$, $\{f_1, \ldots, f_r\}$ and scalars $a_{i,j}$, for $1 \leq i, j \leq r$, such that

$$(\operatorname{ad}\alpha_i)(f_j) = a_{i,j}f_j$$

for all $1 \leq i, j \leq r$. Let $1 \leq i, j \leq r$, $f_i = \sum_{k=1}^{r} w_k\alpha_i$ and $f_j = \sum_{k=1}^{r} z_k\alpha_k$, for some scalars $w_1, \ldots, w_r, z_1, \ldots, z_k$. We have

$$f_if_j = f_i\sum_{k=1}^{r} z_k\alpha_k = \sum_{k=1}^{r} z_ka_{k,i}f_i$$

but at the same time

$$f_if_j = \sum_{k=1}^{r} w_k\alpha_kf_j = \sum_{k=1}^{r} z_ka_{k,j}f_j.$$

So, by the linear independence of $\{f_1, \ldots, f_r\}$, we have

$$f_if_j = \begin{cases} \sum_{k=1}^{r} z_ka_{k,i}f_i, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that, for every $1 \leq i \leq r$, there is $z_i \in \mathbb{C}$ such that $e_i = z_i f_i$ and e_i is idempotent. The set

$$\mathcal{E} = \{e_1, \ldots, e_r\}$$

is a basis for $\mathcal{Z}(S)$, so, there are $w_1, \ldots w_r \in \mathbb{C}$ such that

$$1 = w_1 e_1 + \ldots + w_r e_r. \tag{1.1}$$

Multiplying Eq. (1.1) by e_i we conclude that $w_i = 1$ for all $1 \leq i \leq r$. It follows that $\mathcal{E}$ is a set of pairwise orthogonal idempotents of S that add up to 1.

Now, let $1 \leq i \leq r$, and suppose that $e_i = e + f$ where e, f are two orthogonal central idempotents of S. Suppose $e \neq 0$. Then

$$ee_i = e(e + f) = ee + ef = e.$$

Also, there must exist $z \in \mathbb{C}$ such that $ze = e_i$. Therefore

$$e_i = ze = ze^2 = e_i e = e,$$

which implies $f = 0$. It follows that e_i is primitive, as required.

Finally, let $\mathcal{E}' = \{e'_1, \ldots, e'_s\}$ be a central partition of unity of S. Multiplying equation

$$\sum_{j=1}^{s} e'_j = 1 = \sum_{i=1}^{r} e_i \tag{1.2}$$

by e_i, we conclude that $e'_j e_i = e_i$ for some $1 \leq j \leq s$, by the primitivity of each e_i, for all $1 \leq i \leq r$. Moreover, multiplying Eq. (1.2) by e'_j, by the primitivity of e'_j , we have that $e'_j = e'_j e_i = e_i$ which implies $\mathcal{E}' = \mathcal{E}$. □

More generally, it is true that every finite-dimensional semisimple algebra with identity over a field has a unique central partition of unity, but we found the proof chosen more suitable, regarding the work that will be done in the further chapters.

Let $GL(V)$ denote the set of all invertible linear transformations of the vector space V and if $M \in GL(V)$, then $\mathrm{tr}(M)$ represents the trace of M.

Definition 1.2.6. A ($\mathbb{C}$-linear) **representation of** G is a homomorphism of groups

$$\phi : G \longrightarrow GL(V)$$

where V is a finite-dimensional vector space over $\mathbb{C}$, and the function

$$\begin{array}{rccc} \chi_\phi : & G & \longrightarrow & \mathbb{C} \\ & g & \mapsto & \mathrm{tr}(\phi(g)) \end{array}$$

is called the (complex) **character** of G afforded by the representation ϕ. The value $\chi_\phi(1)$ is called the **degree** of χ_ϕ (and equals with the dimension of V).

We say ϕ is an **irreducible representation** of G if V is a simple $\mathbb{C}[G]$-module. A character is said to be **irreducible** if it is afforded by an irreducible representation.

Let $\mathrm{Irr}(G)$ denote the set of all irreducible characters of G. Note that $\delta_G \in \mathbb{C}[G]$ is always a member of $\mathrm{Irr}(G)$, called the **trivial character**.

We recall some elementary properties of characters. If ϕ is a character of G, then it belongs to $\mathrm{cf}(G)$. Moreover, it must be a linear combination of $\mathrm{Irr}(G)$, with non-negative integer coefficients. In fact,

$$\phi = \sum_{\chi \in \mathrm{Irr}(G)} \langle \phi, \chi \rangle \chi$$

for every $\phi \in \mathrm{cf}(G)$.

If $\chi \in \mathrm{Irr}(G)$, then:

- $\chi(g^{-1}) = \overline{\chi(g)}$, for all $g \in G$;
- the map $\overline{\chi} : G \to \mathbb{C}$ defined by $\overline{\chi}(g) = \overline{\chi(g)}$, for all $g \in G$, is also a member of $\mathrm{Irr}(G)$;
- the element of $\mathbb{C}[G]$

$$e_\chi := \frac{\chi(1)}{|G|} \overline{\chi}$$

is a primitive idempotent. Moreover, $\{e_\chi : \chi \in \mathrm{Irr}(G)\}$ equals $\mathcal{E}_{\mathbb{C}[G]}$ and so it is a basis for $\mathrm{cf}(G)$.

If $X \subseteq \mathrm{Irr}(G)$ consider the set

$$\overline{X} := \{\overline{\chi} : \chi \in X\} \subseteq \mathrm{Irr}(G)$$

and the character of G defined by

$$\xi_X := \sum_{\chi \in X} \chi(1)\chi.$$

Note that the map defined by

$$e_X := \sum_{\chi \in X} e_\chi = \frac{1}{|G|} \xi_{\overline{X}} \tag{1.3}$$

is a central idempotent of $\mathbb{C}[G]$, not necessarily primitive, that belongs to $\mathrm{span}_{\mathbb{C}}\{\xi_{\overline{X}}\}$. Let $X(\alpha) \subseteq \mathrm{Irr}(G)$ denote the set of irreducible characters of G constituents of α, when $\alpha \in \mathrm{cf}(G)$.

Lemma 1.2.7. *The mapping $\mathcal{X} \mapsto \mathrm{span}_{\mathbb{C}}\{e_X : X \in \mathcal{X}\}$ defines a one-to-one correspondence between the set of all set partitions of $\mathrm{Irr}(G)$ and the set of all finite-dimensional subalgebras of $\mathrm{cf}(G)$, with respect to the convolution product.*

Proof. Clearly, if $\mathcal{X} \in \mathrm{Part}(\mathrm{Irr}(G))$ then $\mathrm{span}_{\mathbb{C}}\{e_X : X \in \mathcal{X}\}$ is a finite-dimensional subalgebra of $\mathrm{cf}(G)$.

Consider S, a finite-dimensional subalgebra of $\mathrm{cf}(G)$, with respect to the convolution product. From theorem 1.2.1 we know S has a unique central partition of unity, $\mathcal{E}_S$, with respect to this product, which is a linear basis for S. Consider that

$$\mathcal{E}_S = \{\beta_1, \ldots, \beta_r\}.$$

So, writing the elements β's as a linear combination of the idempotents of $\mathcal{E}_{\mathbb{C}[G]}$, each $e_\chi \in \mathcal{E}_{\mathbb{C}[G]}$ must appear in exactly one β_i and thus this shows us there exists a partition $\mathcal{X} \in \mathrm{Part}(\mathrm{Irr}(G))$ such that

$$\{\beta_1, \ldots, \beta_r\} = \{e_X : X \in \mathcal{X}\}.$$

The injectivity comes from the fact that the central partition of unity of a finite-dimensional semisimple subalgebra of $\mathbb{C}[G]$ is unique. □

Note that $\mathcal{E}_{\mathbb{C}[G]}$, $\mathrm{Irr}(G)$ and $\Delta_{\mathrm{cf}(G)}$, are all orthogonal bases of $\mathrm{cf}(G)$, with respect to the Frobenius inner product. Next, we evidence a little bit more the relations between these bases of $\mathrm{cf}(G)$.

Definition 1.2.8. Consider that $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_r\}$ (all distinct) and $\mathrm{Cl}(G) = \{C_1, \ldots, C_r\}$. The **character table** of G is the $r \times r$-matrix M with, for all $1 \leq i, j \leq r$,

$$M_{ij} = \chi_i(C_j),$$

where $\chi_i(C_j)$ denotes the unique image of the elements of the conjugacy class C_j through the irreducible character χ_i.

In other words, the rows of M are indexed by the irreducible characters of G, the columns by the conjugacy classes of G and the ij-entry is the value of the ith-irreducible character on the jth-conjugacy class of G.

Observe that the character table is the transpose of the change of basis matrix from the basis $\mathrm{Irr}(G)$ to the basis $\{\delta_C : C \in \mathrm{Cl}(G)\}$ of $\mathrm{cf}(G)$.

Example 1.2.9. Consider $C_5 = \langle\sigma\rangle$, the cyclic group of order 5, as in Example 1.1.15.

Because C_5 is abelian, all irreducible representations of C_5 must be of degree 1. So, because $\sigma^5 = 1$, the image of σ through an irreducible character of C_5 must be a root of unity of order 5. So, the character table of C_5 turns to be very easy to compute.

Table 1: Character table of C_5

	1	σ	σ^2	σ^3	σ^4
χ_1	1	1	1	1	1
χ_ω	1	ω	ω^2	ω^3	ω^4
χ_{ω^2}	1	ω^2	ω^4	ω	ω^3
χ_{ω^3}	1	ω^3	ω	ω^4	ω^2
χ_{ω^4}	1	ω^4	ω^3	ω^2	ω

where $\omega = e^{2\pi i/5} \in \mathbb{C}$. Moreover,

$$\mathcal{E}_{\mathbb{C}[C_5]} = \left\{\frac{\chi_{\omega^i}}{5} : 0 \leq i \leq 4\right\} \quad \text{and} \quad \Delta_{\mathbb{C}[C_5]}\{\delta_{\sigma^i} : 0 \leq i \leq 5\}$$

are the partitions of unity of $\mathbb{C}[C_5]$ with respect to the convolution product and to the Hadamard product, respectively, both linear basis for this algebra. About the subalgebra

$S = \mathrm{span}_{\mathbb{C}}\{\delta_1, \delta_{\{\sigma,\sigma^4\}}, \delta_{\{\sigma^2,\sigma^3\}}\}$ of $\mathbb{C}[C_5]$, it is easy to check that

$$\mathcal{E}_S = \{\chi_1, \frac{1}{5}(\chi_\omega + \chi_{\omega^4}), \frac{1}{5}(\chi_{\omega^2} + \chi_{\omega^3})\} \quad \text{and} \quad \Delta_S = \{\delta_1, \delta_{\{\sigma,\sigma^4\}}, \delta_{\{\sigma^2,\sigma^3\}}\}$$

are the partitions of unity of S, with respect to the convolution product and the Hadamard product, respectively. Moreover, if we denote by $P_{B\to C}$ the change of basis matrix from basis B to basis C,

$$P_{\mathcal{E}_S\to\Delta_S} = \frac{1}{5}\begin{bmatrix} 1 & 2 & 2 \\ 1 & \omega+\omega^4 & \omega^2+\omega^3 \\ 1 & \omega^2+\omega^3 & \omega+\omega^4 \end{bmatrix}.$$

Example 1.2.10. Consider S_3, the symmetric group of order 3, as in Example 1.1.16. Consider the map

$$\phi : S_3 \longrightarrow GL(\mathbb{C}^4),$$

defined by:

$$\phi((1))=\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \quad \phi((1\,2))=\begin{bmatrix} 1&0&0&0\\0&1&-1&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}, \quad \phi((1\,3))=\begin{bmatrix} 1&0&0&0\\0&-1&0&0\\0&-1&1&0\\0&0&0&-1\end{bmatrix}$$

$$\phi((2\,3))=\begin{bmatrix} 1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&-1\end{bmatrix}, \quad \phi((1\,2\,3))=\begin{bmatrix} 1&0&0&0\\0&-1&1&0\\0&-1&0&0\\0&0&0&1\end{bmatrix}, \quad \phi((1\,3\,2))=\begin{bmatrix} 1&0&0&0\\0&0&-1&0\\0&1&-1&0\\0&0&0&1\end{bmatrix}.$$

It is easy to check that ϕ is a group homomorphism and

$$\chi_\phi(g) = \begin{cases} 4, & \text{if } g = 1 \\ 0, & \text{if } g \in (**) \\ 1, & \text{if } g \in (***), \end{cases}$$

called the **regular character of** S_3.

Observe that χ_ϕ is not an irreducible character because, for example

$$\phi(S_3)\,\mathrm{span}_{\mathbb{C}}\{(1,0,0,0)\} = \mathrm{span}_{\mathbb{C}}\{(1,0,0,0)\}.$$

It is easy to prove that $\mathcal{E}_{\mathbb{C}[S_3]} = \{e_1, e_2, e_3\}$ where

$$e_1 = \frac{1}{6}\left(\delta_{(1)} + \delta_{(1\,2)} + \delta_{(1\,3)} + \delta_{(2\,3)} + \delta_{(1\,2\,3)} + \delta_{(1\,3\,2)}\right) = \frac{1}{6}\left(1\delta_{(1)} + 1\delta_{(**)} + 1\delta_{(***)}\right);$$
$$e_2 = \frac{1}{3}\left(2\delta_{(1)} - \delta_{(1\,2\,3)} - \delta_{(1\,3\,2)}\right) = \frac{1}{3}\left(2\delta_{(1)} + 0\delta_{(**)} - 1\delta_{(***)}\right);$$
$$e_3 = \frac{1}{6}\left(\delta_{(1)} - \delta_{(1\,2)} - \delta_{(1\,3)} - \delta_{(2\,3)} + \delta_{(1\,2\,3)} + \delta_{(1\,3\,2)}\right) = \frac{1}{6}\left(1\delta_{(1)} - 1\delta_{(**)} + 1\delta_{(***)}\right).$$

Moreover,

$$\chi_{e_1} = 1\delta_{(1)} + 1\delta_{(**)} + 1\delta_{(***)} \quad (\textbf{trivial character}),$$
$$\chi_{e_2} = 2\delta_{(1)} + 0\delta_{(**)} - 1\delta_{(***)} \quad (\textbf{standard character}),$$
$$\chi_{e_3} = 1\delta_{(1)} - 1\delta_{(**)} + 1\delta_{(***)} \quad (\textbf{sign character}),$$

are the irreducible characters of S_3 and its character table is

Table 2: Character table of S_3

	(1)	$(**)$	$(***)$
χ_1	1	1	1
χ_2	2	0	-1
χ_3	1	-1	1

Consider the ordered sets

$$\begin{aligned}
\mathcal{E} &= (e_1, e_2, e_3),\\
\mathcal{X} &= (\chi_{e_1}, \chi_{e_2}, \chi_{e_3}) = (6e_1, 3e_2, 6e_3),\\
\Delta &= (\delta_{(1)}, \delta_{(**)}, \delta_{(***)}),\\
\nabla &= (6\delta_{(1)}, 2\delta_{(**)}, 3\delta_{(***)}).
\end{aligned}$$

They are all orthogonal bases of $\mathrm{cf}(S_3)$ with respect to the Frobenius inner product. Observe that $\mathcal{E}$ and ∇ are in fact orthonormal and if we denote by $P_{B\to C}$ the change of basis matrix from basis B to basis C, we have

$$P_{\Delta\to\mathcal{X}} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}^{-1} = \frac{1}{|S_3|}\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -2 \\ 1 & -3 & 2 \end{bmatrix}$$

Following this, can easily write each characteristic map belonging to basis Δ, as a linear combination of the irreducible characters of S_3,

$$\begin{aligned}
\delta_{(1)} &= \frac{1}{6}\chi_{e_1} + \frac{1}{3}\chi_{e_2} + \frac{1}{6}\chi_{e_3},\\
\delta_{(**)} &= \frac{1}{2}\chi_{e_1} + 0\chi_{e_2} - \frac{1}{2}\chi_{e_3},\\
\delta_{(***)} &= \frac{1}{3}\chi_{e_1} - \frac{1}{3}\chi_{e_2} + \frac{1}{3}\chi_{e_3}.
\end{aligned}$$

The matrices $P_{\Delta\to\nabla}$ and $P_{\mathcal{E}\to\mathcal{X}}$ are both diagonal and its entries are the sizes of the conjugacy classes and the dimension of the irreducible representations of S_3, respectively, multiplied by $\frac{1}{|S_3|}$.

In general, the problem of describing supercharacter theories (see chapter 3) for the symmetric group, or equivalently, central Schur rings (see chapter 2) over a symmetric group, is not an easy task. The problem remains open.

Chapter 2

Schur rings

Isaac Schur, when studing permutation groups, found it relevant to explore subrings of the group algebra that have a linear basis coming from certain partitions of the group (see [39]). Later, H. Wielandt developed a theory of these rings and O. Tamaschke wrote about their representation theory. These particular subrings were denominated as Schur rings (see in [43] and [41]). More recently, there are studies about applications of Schur rings in algebraic combinatorics, especially in the study of circulant graphs, evidencing the relations with association schemes, and also in representation theory by the strong connection with supercharacter theories (see [24] and [25]).

2.1 Definition and main properties

Let G be a finite group. Schur rings over G are in fact subalgebras of $\mathbb{C}[G]$, afforded by special partitions of the group. The group algebra itself is a Schur ring and hence the theory of Schur rings can be seen as a generalization of the theory of group algebras. The center of the group algebra is also a Schur ring and the representation theory of Schur rings can also be seen as a generalization of the representation theory of groups.

Most of the results presented here hold for Schur rings defined over more arbitrary fields, but again, we choose to focus ourselves on the complex case.

Definition 2.1.1. A partition $\mathcal{K} \in \mathrm{Part}(G)$ is said to be a **Schur partition of** G if the following conditions hold:

1. $\{1\} \in \mathcal{K}$;
2. $K^{-1} := \{g^{-1} : g \in K\} \in \mathcal{K}, \quad$ for all $K \in \mathcal{K}$;
3. The convolution product of characteristic maps of $\mathcal{K}$ is a linear combination of characteristic maps of $\mathcal{K}$.

If $\mathcal{K}$ is a Schur partition of G, then

$$\mathcal{S}_{\mathcal{K}} := \mathrm{span}_{\mathbb{C}}\{\delta_K : K \in \mathcal{K}\}$$

is the **Schur ring over** G **afforded by** $\mathcal{K}$. We refer to $\{\delta_K : K \in \mathcal{K}\}$ as the **standard basis of** $\mathcal{S}_{\mathcal{K}}$.

Note that if $\mathcal{S}_\mathcal{K}$ is a Schur ring, then it is a subalgebra of $\mathbb{C}[G]$ with respect to the convolution product, with dimension equal to $|\mathcal{K}|$. If $\mathcal{S}_\mathcal{K}$ is central we say that $\mathcal{K}$ is a **central Schur partition of** G. In this case,

$$\mathcal{S}_\mathcal{K} = \text{span}_\mathbb{C}\{\delta_K : K \in \mathcal{K}\} \subseteq \text{cf}(G) = \text{span}_\mathbb{C}\{\delta_C : C \in \text{Cl}(G)\},$$

and each $K \in \mathcal{K}$ must be an union of conjugacy classes of G. Clearly, $\mathbb{C}[G]$ is a Schur ring over G afforded by the Schur partition $\{\{g\} : g \in G\}$. Moreover, note that $\{1\}$ is a member of $\text{Cl}(G)$ and that, if C is a conjugacy class of G, then C^{-1} is also a conjugacy class making it also clear that $\text{cf}(G)$ is a central Schur ring over G afforded by the Schur partition $\text{Cl}(G)$. Observe that Schur rings over a group G are subalgebras of the group algebra $\mathbb{C}[G]$ which also verify some properties that both $\mathbb{C}[G]$ and $\text{cf}(G)$ do verify.

O. Tamaschke, in [41], used a weaker definition for Schur rings, allowing a Schur ring to be a subalgebra of $\mathbb{C}[G]$ without identity, with respect to the convolution product. Nowadays, several authors call those rings quasi-Schur rings (see [24]) or pre-Schur rings (see [36]). K. H. Leung and S. L. Ma, in [35], called unitary Schur rings the Schur rings defined above. Note that, for example, $\{G\} \in \text{Part}(G)$ gives rise to a quasi-Schur ring but not a unitary one.

Proposition 2.1.2. *Let $\mathcal{S}_\mathcal{K}$ be a Schur ring over G afforded by the Schur partition $\mathcal{K}$.*
If $L, M \in \mathcal{K}$, then

$$\delta_L \delta_M = \sum_{K \in \mathcal{K}} c_{L,M,K} \delta_K,$$

for some non-negative integers $c_{L,M,K}$, called the **structure constants of the Schur partition** $\mathcal{K}$.

Proof. $\mathcal{S}_\mathcal{K}$ is a subalgebra of $\mathbb{C}[G]$ with linear basis $\{\delta_K : K \in \mathcal{K}\}$. The convolution product of two δ_K's must be a linear combination of δ_K's. But, recalling remark 1.1.2, the coefficients of the linear combination are the number of times a certain product of elements of G appears in a certain $K \in \mathcal{K}$, so the structure constants must be non-negative integers. □

Note that if $\mathcal{K}$ is a Schur partition, then $\{\delta_K : K \in \mathcal{K}\}$ is the central partition of unity of $\mathcal{S}_\mathcal{K}$, with respect to the Hadamard product. Following this, we present a characterization of Schur rings, due to Muzychuk in [37].

Theorem 2.1.3. *A vector subspace S of $\mathbb{C}[G]$ is a Schur ring over G if and only if it is a finite-dimensional $\star$-subalgebra with identity, both for the convolution product and the Hadamard product.*

Proof. If S is a Schur ring over G, then, by definition, it is a $\star$-subalgebra with identity of $\mathbb{C}[G]$, with respect to the convolution product. Let $\mathcal{K}$ be the associated Schur partition of S. Clearly, $\sum_{K \in \mathcal{K}} \delta_K = \delta_G$, so the identity of $\mathbb{C}[G]$, with respect to the Hadamard product, belongs to S. Moreover, if $\alpha, \beta \in S$ then

$$\alpha \cdot \beta = \left(\sum_{K \in \mathcal{K}} \alpha_K \delta_K\right) \cdot \left(\sum_{K \in \mathcal{K}} \beta_K \delta_K\right) = \sum_{K \in \mathcal{K}} \alpha_K \beta_K \delta_K \in S$$

which proves that S is also a subalgebra of $\mathbb{C}[G]$, with respect to the Hadamard product.

Now, assume that S is a finite-dimensional $\star$-subalgebra with identity, of $\mathbb{C}[G]$, both for the convolution product and the Hadamard product. So S is a semisimple subalgebra for both products. The unique central partition of unity, with respect to the Hadamard product, must be the characteristic maps of some partition of G, $\mathcal{K}$, where necessarily $\{1\} \in \mathcal{K}$, since both δ_G (identity of the Hadamard product) and δ_1 (identity of the convolution product) belong to S. Note that if $K \subseteq G$,

$$\delta_K^\star = \delta_{K^{-1}}.$$

So, if $K \in \mathcal{K}$, then both δ_K and $\delta_{K^{-1}}$ are primitive idempotents of S, with respect to the Hadamard product, which implies $K^{-1} \in \mathcal{K}$. Finally, it is clear that, in this case, we must have $S = \mathrm{span}_{\mathbb{C}}\{\delta_K : K \in \mathcal{K}\}$. □

Schur homomorphisms were studied by Muzychuk in [37]. We present here the definition of this kind of homomorphisms and an immediate consequence that, again, will be useful in the next chapters.

Definition 2.1.4. Let $\mathcal{S}$ and $\mathcal{S}'$ be Schur rings over arbitrary finite groups. A **Schur ring homomorphism from $\mathcal{S}$ to $\mathcal{S}'$** is a map

$$\psi : \mathcal{S} \mapsto \mathcal{S}'$$

that is a $\star$-algebra homomorphism both for the convolution product and the Hadamard product.

Proposition 2.1.5. *If $\psi : \mathcal{S} \mapsto \mathcal{S}'$ is an isomorphism of Schur rings, then ψ maps the standard basis of $\mathcal{S}$ to the standard basis of $\mathcal{S}'$.*

Proof. If $\mathcal{K}$ is the Schur partition associated to $\mathcal{S}$, then $\{\delta_K : K \in \mathcal{K}\}$ is the central primitive partition of unity of S, with respect to the Hadamard product and so

$$\{\psi(\delta_K) : K \in \mathcal{K}\}$$

must be the central primitive partition of unity of S', with respect to the Hadamard product. □

Example 2.1.6. Consider C_5, the cyclic group of order 5, as in Examples 1.1.15 and 1.2.9. The partition of C_5,

$$\mathcal{K}_o = \{\{1\}, \{\sigma, \sigma^3\}, \{\sigma^2, \sigma^4\}\}$$

is not a Schur partition because, for example,

$$\{\sigma, \sigma^3\} \times \{\sigma, \sigma^3\} = \{\sigma, \sigma^2, \sigma^4\}$$

which is not an union of elements of $\mathcal{K}_o$.

In Example 1.1.15, it was verified that $\mathrm{span}_{\mathbb{C}}\{\delta_1, \delta_{\{\sigma,\sigma^4\}}, \delta_{\{\sigma^2,\sigma^3\}}\}$ is a $\star$-subalgebra of $\mathbb{C}[C_5]$, both for the convolution product and the Hadamard product. Taking into account the calculations already done, we can conclude that

$$\mathcal{K}_{C_5^\star} := \{\{1\}, \{\sigma, \sigma^4\}, \{\sigma^2, \sigma^3\}\}$$

is a central Schur partition of C_5 with the following structure constants:

Table 3: Structure constants of $\mathcal{K}_{C_5^\star}$

$K_1 = \{1\}$	$K_2 = \{\sigma, \sigma^4\}$	$K_3 = \{\sigma^2, \sigma^3\}$
$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	$M_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$	$M_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

where the constant c_{K_i,K_j,K_k} is the (i,j)-entry of the matrix M_k. Moreover, $\mathcal{S}_{\mathcal{K}_{C_5^\star}}$ is a central Schur ring over C_5 with dimension 3.

Example 2.1.7. Consider S_3, the symmetric group of order 3, as in Examples 1.1.16 and 1.2.10.

The algebra $\mathrm{cf}(S_3)$, with dimension 3, is a central Schur ring over S_3, associated to the partition of the conjugacy classes of this group. It is easy to check that the morphism $\psi : \mathcal{S}_{\mathcal{K}_{C_5^\star}} \longrightarrow \mathrm{cf}(S_3)$, that maps

$$\delta_{\{\sigma,\sigma^4\}} \mapsto \delta_{(**)} \quad \text{and} \quad \delta_{\{\sigma^2,\sigma^3\}} \mapsto \delta_{(***)},$$

is a $\star$-isomorphism of algebras, with respect to the Hadamard product, but it is not an isomorphism of Schur rings, considering that $\mathrm{cf}(S_3)$ has the following structure constants:

Table 4: Structure constants of $\mathrm{cf}(S_3)$

$K_1 = \{(1)\}$	$K_2 = (**)$	$K_3 = (***)$
$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	$M_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$	$M_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

2.2 Orbit Schur rings

Finding all Schur rings over a group is still an open question for most families of groups. In this next section, we present a construction to obtain Schur rings from the orbits of a subgroup of the automorphism group of G. It is known that not all Schur rings arise this way, but this kind of Schur rings will be of special interest to us.

From now on, unless stated otherwise, when we refer to the product in $\mathbb{C}[G]$ we are considering the convolution product.

Let $\mathrm{Aut}(G)$ denote the group of automorphisms of G. Consider H, a subgroup of $\mathrm{Aut}(G)$. If $g \in G$, we will denote the orbit of g, with respect to H, by

$$H(g) := \{\tau(g) : \tau \in H\} \subseteq G$$

and the set of all orbits of G, with respect to H, will be denoted by

$$\mathcal{O}_H := \{H(g) : g \in G\},$$

which is necessarily a partition of G.

We can extend by linearity each $\tau \in H$ to the group algebra $\mathbb{C}[G]$, where

$$(\tau(\alpha))(g) = \alpha(\tau(g))$$

for all $\alpha \in \mathbb{C}[G]$ and all $g \in G$.

Proposition 2.2.1. *If H is a subgroup of the group of automorphisms of G, then the set of H-fixed points*

$$\mathbb{C}[G]^H := \{\alpha \in \mathbb{C}[G] : \tau(\alpha) = \alpha, \ \textit{for all} \ \tau \in H\}$$

is a Schur ring over G. Moreover, $\mathcal{O}_H$ is a Schur partition of G.

A Schur ring arising from a set of fixed points, as described in the previous proposition, is called an **orbit Schur ring**.

Proof. For all $\tau \in H$, $\tau(1) = 1$, which implies that $H_1 = \{1\}$ belongs to $\mathcal{O}_H$.

Let $g \in G$. Considering the orbit $H(g)$, we have that

$$H(g)^{-1} = \{\tau(g)^{-1} : \tau \in H\} = \{\tau(g^{-1}) : \tau \in H\} = H_{g^{-1}} \in \mathcal{O}_H.$$

Moreover, it is clear that $\{\delta_O : O \in \mathcal{O}_H\}$ is a linearly independent set. Consider

$$\mathcal{S} = \text{span}_{\mathbb{C}}\{\delta_O : O \in \mathcal{O}_H\}.$$

If $\alpha \in \mathcal{S}$ is such that

$$\alpha = \sum_{O \in \mathcal{O}_H} \alpha_O \delta_O$$

with $\alpha_O \in \mathbb{C}$, for all $O \in \mathcal{O}_H$, then note that if $\tau \in H$ and $O \in \mathcal{O}_H$, $\tau(\delta_O) = \delta_O$ and so we have that

$$\tau(\alpha) = \sum_{O \in \mathcal{O}_H} \alpha_O(\tau(\delta_O)) = \sum_{O \in \mathcal{O}_H} \alpha_O \delta_O = \alpha$$

for all $\tau \in H$, which implies that $\alpha \in \mathbb{C}[G]^H$.

On the other hand, if $\alpha \in \mathbb{C}[G]^H$, then $\alpha = \tau\alpha$ for all $\tau \in H$. This implies that $\alpha(g) = \alpha(\tau(g))$, for al $\tau \in H$ and all $g \in G$. So, necessarily,

$$\alpha \in \text{span}_{\mathbb{C}}\{\delta_O : O \in \mathcal{O}_H\}.$$

This finishes the proof. □

Note that the identity map in G is a subgroup of $\text{Aut}(G)$. The orbit Schur ring over G that arises from this subgroup of the automorphisms of G is the all group algebra, $\mathbb{C}[G]$. Moreover, each $g \in G$ can be identified as an automorphism of G, mapping each $h \in G$ into ghg^{-1}. The subgroup of all automorphisms of G of this kind gives rise to $\text{cf}(G)$, which is also an orbit Schur ring over G.

Example 2.2.2. Consider C_5, the cyclic group of order 5, as in Example 2.1.6.

The set $H = \{\text{Id}_{C_5}, {}^{-1}\}$, where Id_{C_5} denotes the identity map of C_5 and ${}^{-1}$ denotes the map that takes each element to its inverse, defined on C_5, is a subgroup of $\text{Aut}(C_5)$ and

$$\mathbb{C}[C_5]^H = \mathcal{S}_{\mathcal{K}_{C_5^\star}},$$

as defined previously, in Example 2.1.6.

2.3 $\mathcal{S}$-character theory

In [41], Tamaschke described a representation and character theory of a Schur ring over G as a generalization of the ordinary representation and character theory of G.

In this section, we present the main definitions and results of the work of Tamaschke on Schur rings, highlighting that, if $\mathcal{S}$ is a central Schur ring over G, then the irreducible $\mathcal{S}$-characters and $\mathcal{S}$-conjugacy classes described by Tamaschke are exactly the supercharacters and superclasses of G, respectively, that we are going to relate, in the next chapter, with the central Schur ring over G.

This way, the $\mathcal{S}$-character theory of a Schur ring studied by Tamaschke can be seen not just as a generalization of the character theory of finite groups, but also as a generalization of supercharacter theories.

From now on, unless stated otherwise, let $\mathcal{S}$ be a Schur ring over G associated to the Schur partition $\mathcal{K}$. Note that, by Theorem 1.2.1, every Schur ring over a finite group is a semisimple algebra. Let $\mathcal{M}_m(\mathbb{C})$ denote the algebra of all $m \times m$ matrices with entries in $\mathbb{C}$.

Definition 2.3.1. Let $\mathcal{S}$ be a Schur ring over G and $\mathcal{K}$ the corresponding Schur partition. We say that a map

$$\phi : \mathcal{S} \longrightarrow \mathcal{M}_m(\mathbb{C})$$

is a **representation of** $\mathcal{S}$ if ϕ is a $\mathbb{C}$-algebra homomorphism. In this case, the map $\chi_\phi : G \to \mathbb{C}$ defined by

$$\chi_\phi(g) = \mathrm{tr}(\phi(\delta_{K_g}))$$

for all $g \in G$, is called the **$\mathcal{S}$-character of** G related to ϕ, where K_g denotes the element of $\mathcal{K}$ that includes g.
We say ϕ is an **irreducible representation of** $\mathcal{S}$ if all matrices of $\phi(\mathcal{S})$ cannot be diagonalizable by the same invertible matrix. An $\mathcal{S}$-character χ_ϕ is called **irreducible** if the representation ϕ is irreducible.

Let $\mathrm{Irr}_{\mathcal{S}}(G)$ denote the set of all irreducible $\mathcal{S}$-characters of G. Any two elements $g, h \in G$ are called **$\mathcal{S}$-conjugate** if

$$\chi(g) = \chi(h)$$

for all $\chi \in \mathrm{Irr}_{\mathcal{S}}(G)$. We will denote the set of all $\mathcal{S}$-conjugacy classes of G by $\mathrm{Cl}_{\mathcal{S}}(G)$. Note that $\mathcal{S}$-conjugacy is an equivalence relation on G. We finish this section recalling the main theorem in [41], emphasizing the relation between the cardinalities of two partitions of unity, namely, the one defined by the convolution product and the one defined by the Hadamard product.

Theorem 2.3.2. *Let $\mathcal{S}$ be a Schur ring over G. If $\chi \in \mathrm{Irr}_{\mathcal{S}}(G)$, then the map $\varepsilon_\chi : G \to \mathbb{C}$, defined by*

$$\varepsilon_\chi(g) := \frac{1}{|G|}\chi(1)\chi(g^{-1})$$

for all $g \in G$, is a central, primitive idempotent of $\mathcal{S}$. Moreover, $\{\varepsilon_\chi : \chi \in \mathrm{Irr}_{\mathcal{S}}(G)\}$ is the central partition of unity of $\mathcal{S}$, with respect to the convolution product, and

$$|\mathrm{Irr}_{\mathcal{S}}(G)| \leq |\mathrm{Cl}_{\mathcal{S}}(G)|.$$

Note that when $\mathcal{S} = \mathrm{cf}(G)$, the $\mathcal{S}$-character theory of Tamaschke is in fact the ordinary character theory of G.

Chapter 3

Supercharacters

The characterization of the irreducible characters of finite unitriangular groups is known to be a wild problem and probably impossible to accomplish. André, in [2], and Yan, in [44], dealt with this difficulty, coarsening the usual irreducible character theory for this groups. In their work, they replaced the irreducible characters by a family of characters, not necessarily irreducible, that are constant on a set of clumped conjugacy classes. Later, Diaconis and Isaacs, in [19], introduced the general concept of a supercharacter theory for a finite group. The main definitions and properties will be presented in this chapter. More recently, Hendrickson, in [25], described the relation between supercharacter theories and central Schur rings, reproduced here as well with slight differences.

3.1 Definition and main properties

There are several ways to define a supercharacter theory for finite groups. The one we choose tries to highlight the relation between supercharacters and central Schur rings.

Definition 3.1.1. Let $\mathcal{K} \in \mathrm{Part}(G)$ and $\mathcal{X} \in \mathrm{Part}(\mathrm{Irr}(G))$ be such that:

1. $|\mathcal{K}| = |\mathcal{X}|$;
2. If $X \in \mathcal{X}$, then the character $\xi_X := \sum_{\chi \in X} \chi(1)\chi$ takes a constant value on each member $K \in \mathcal{K}$.

In this case, for each $X \in \mathcal{X}$, we say that the character ξ_X is a **supercharacter** of G, and each member $K \in \mathcal{K}$ is a **superclass** of G. We also say that the pair $(\mathcal{K}, \mathcal{X})$ is a **supercharacter theory** for G.

Note that $(\mathrm{Cl}(G), \mathrm{Irr}(G))$ and $(\{\{1\}, G \setminus \{1\}\}, \{\{\delta_G\}, \mathrm{Irr}(G) \setminus \{\delta_G\}\})$ are trivial supercharacter theories for G, named the **minimal** one and the **maximal** one, respectively.

Definition 3.1.2. Let $(\mathcal{K}, \mathcal{X})$ be a supercharacter theory for G. An element $\alpha \in \mathbb{C}[G]$ is called a **$\mathcal{K}$-superclass function of** G if

$$\alpha(g) = \alpha(h)$$

for all g, h belonging to the same $K \in \mathcal{K}$.

If $(\mathcal{K}, \mathcal{X})$ is a supercharacter theory for G, then let $\mathrm{scf}_{\mathcal{K}}(G)$ denote the $\mathbb{C}$-vector space of $\mathbb{C}[G]$ consisting of all $\mathcal{K}$-superclass functions of G.

Similarly to what happens in representation theory, the supercharacter table is the transpose of the change of basis matrix from the basis $\{\xi_X : X \in \mathcal{X}\}$ to the basis $\{\delta_K : K \in \mathcal{K}\}$ of $\mathrm{scf}_{\mathcal{K}}(G)$ which, evidently, can play an important role on the study of supercharacter theories.

Definition 3.1.3. Consider that $\mathcal{X} = \{\xi_1, \ldots, \xi_r\}$ and $\mathcal{K} = \{K_1, \ldots, K_r\}$, such that $(\mathcal{K}, \mathcal{X})$ is a supercharacter theory for G. The **supercharacter table of G associated to** $(\mathcal{K}, \mathcal{X})$ is the $r \times r$-matrix $A = (A_{ij})$, with

$$A_{ij} = \xi_i(K_j),$$

for all $i, j \in \{1, \ldots, r\}$. In other words, the rows of A are indexed by the supercharacters of G, the columns by the superclasses of G and the ij-entry is the value of the ith-supercharacter on the jth-superclass of G.

Example 3.1.4. Consider the cyclic group of order 5, C_5, as in examples 1.1.15, 1.2.9, 2.1.6 and 2.2.2. Recalling the character table of C_5, it is easy to see that defining

$$\mathcal{K}_{C_5^\star} := \big\{\{1\}, \{\sigma, \sigma^4\}, \{\sigma^2, \sigma^3\}\big\} \in \mathrm{Part}(C_5)$$
$$\mathcal{X}_{C_5^\star}. = \{\{\chi_1\}, \{\chi_\omega, \chi_{\omega^4}\}, \{\chi_{\omega^2}, \chi_{\omega^3}\}\} \in \mathrm{Part}(\mathrm{Irr}(C_5))$$

we get that $(\mathcal{K}_{C_5^\star}, \mathcal{X}_{C_5^\star})$ is a non-trivial supercharater theory for C_5 with the following supercharacter table:

Table 5: Supercharacter table of $(\mathcal{K}_{C_5^\star}, \mathcal{X}_{C_5^\star})$

	$\{1\}$	$\{\sigma, \sigma^4\}$	$\{\sigma^2, \sigma^3\}$
χ_1	1	1	1
$\chi_\omega + \chi_{\omega^4}$	2	$\omega + \omega^4$	$\omega^2 + \omega^3$
$\chi_{\omega^2} + \chi_{\omega^3}$	2	$\omega^2 + \omega^3$	$\omega + \omega^4$

where $\omega = e^{2\pi i/5} \in \mathbb{C}$.

In [25], Hendrickson classified the supercharacter theories of cyclic p-groups, based in a similar work with Schur rings by Leung and Man in [35]. Following their work, we know that, when p is prime, the cyclic group C_p has as many distinct supercharacter theories as the number of divisors of the positive integer $p-1$. So, in the case we are following, C_5 has exactly 3 distinct supercharacter theories: the trivial ones and the one presented in this example.

Example 3.1.5. It is clear that the symmetric group of order 3, S_3, has only two distinct supercharacter theories, the minimal one and the maximal one. In [15], we can find a classification of finite groups with exactly two supercharacter theories. However, the symmetric group of order 4, S_4, has three non-trivial distinct supercharacter theories. We will present one of them.

Just as in example 1.2.10, we can display the character table of S_4 as the following:

Table 6: Character table of S_4

	$\{(1)\}$	$(**)$	$(**)(**)$	$(*\ *\ *)$	$(*\ *\ **)$
$\chi_{\ydiagram{4}}$	1	1	1	1	1
$\chi_{\ydiagram{3,1}}$	3	1	-1	0	-1
$\chi_{\ydiagram{2,2}}$	2	0	2	-1	0
$\chi_{\ydiagram{2,1,1}}$	3	-1	-1	0	1
$\chi_{\ydiagram{1,1,1,1}}$	1	-1	1	1	-1

If we do a linear combination of the non-linear irreducible characters of S_4, with coefficients equal to the degree of each character, we see that, putting aside the identity of S_4, the character obtained is constant in the set of all even permutations and in the set of all odd permutations. Defining

$$\mathcal{K}_{S_4^{\S}} := \{\{(1)\}, \{\sigma \in S_4 \setminus \{(1)\} : \sigma \text{ is odd }\}, \{\sigma \in S_4 \setminus \{(1)\} : \sigma \text{ is even }\}\} \in \text{Part}(S_4)$$

$$\mathcal{X}_{S_4^{\S}} := \left\{ \{\chi_{\ydiagram{4}}\}, \{\chi_{\ydiagram{3,1}}, \chi_{\ydiagram{2,2}}, \chi_{\ydiagram{2,1,1}}\}, \{\chi_{\ydiagram{1,1,1,1}}\} \right\} \in \text{Part}(\text{Irr}(S_4))$$

we get that $(\mathcal{K}_{S_4^{\S}}, \mathcal{X}_{S_4^{\S}})$ is a non-trivial supercharacter theory for S_4 with the following supercharacter table:

Table 7: Supercharacter table of $(\mathcal{K}_{S_4^{\S}}, \mathcal{X}_{S_4^{\S}})$

	$\{(1)\}$	$(**) \cup (*\ *\ **)$	$(**)(**) \cup (*\ *\ *)$
$\chi_{\ydiagram{4}}$	1	1	1
$3\chi_{\ydiagram{3,1}} + 2\chi_{\ydiagram{2,2}} + 3\chi_{\ydiagram{2,1,1}}$	22	0	-2
$\chi_{\ydiagram{1,1,1,1}}$	1	-1	1

3.2 Correspondence with Schur rings

Diaconis and Isaacs, in [19], presented a list of properties verified by supercharacter theories for finite groups. In this section, we recall those properties and use some of the tools that we have introduced in the previous chapters to prove this results using a different approach from the one we can find in [19].

Lemma 3.2.1. *Let $(\mathcal{K}, \mathcal{X})$ be a supercharacter theory for G. The following hold:*

1. *The set $\mathrm{scf}_{\mathcal{K}}(G) = \mathrm{span}_{\mathbb{C}}\{\delta_K : K \in \mathcal{K}\} = \mathrm{span}_{\mathbb{C}}\{\xi_X : X \in \mathcal{X}\}$ is a subalgebra of $\mathbb{C}[G]$, with respect to the convolution product and to the Hadamard product;*
2. *Each $K \in \mathcal{K}$ is an union of conjugacy classes of G;*
3. *$\{1\} \in \mathcal{K}$;*
4. *$\{\chi_1\} \in \mathcal{X}$, where χ_1 denotes the trivial character of G;*
5. *If $K \in \mathcal{K}$ then $K^{-1} \in \mathcal{K}$;*
6. *There is no other $\mathcal{X}' \in \mathrm{Part}(\mathrm{Irr}(G))$ (or $\mathcal{K} \in \mathrm{Part}(G)$) such that $(\mathcal{K}, \mathcal{X}')$ (resp. $(\mathcal{K}', \mathcal{X})$) is also a supercharacter theory for G.*

Proof. Consider

$$S = \mathrm{span}_{\mathbb{C}}\{\xi_X : X \in \mathcal{X}\}.$$

By lemma 1.2.7 and Eq. (1.3) we know that S is a subalgebra of $\mathbb{C}[G]$ with respect to the convolution product. Moreover,

$$\sum_{X \in \mathcal{X}} \xi_X = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\chi = |G|\delta_1$$

meaning that δ_1, the identity of $\mathbb{C}[G]$ with respect to the convolution product, belongs to S.

By point 2 of definition 3.1.1 we know that

$$S \subseteq \mathrm{span}_{\mathbb{C}}\{\delta_K : K \in \mathcal{K}\}.$$

Note that both sets $\{\delta_K : K \in \mathcal{K}\}, \{\xi_X : X \in \mathcal{X}\} \subseteq \mathbb{C}[G]$ are linearly independent and have the same cardinality. So, we conclude that the two sets are bases of S. Since $S = \mathrm{span}_{\mathbb{C}}\{\delta_K : K \in \mathcal{K}\}$, S is closed with respect to the Hadamard product. Moreover,

$$\sum_{K \in \mathcal{K}} \delta_K = \delta_G$$

meaning that the identity of $\mathbb{C}[G]$ with respect to the Hadamard product, belongs to S.

Each irreducible character is a class function, so

$$S \subseteq \mathrm{span}_{\mathbb{C}}\{\delta_C : C \in \mathrm{Cl}(G)\}.$$

This means S is central and each $K \in \mathcal{K}$ must be a union of conjugacy classes of G.

We have that the identity of $\mathbb{C}[G]$ with respect to the convolution product, δ_1, belongs to $S = \mathrm{span}_{\mathbb{C}}\{\delta_K : K \in \mathcal{K}\}$, so necessarily $\{1\} \in \mathcal{K}$. On the other hand, the identity of $\mathbb{C}[G]$ with

respect to the Hadamard product, δ_G, also belongs to $S = \mathrm{span}_{\mathbb{C}}\{\xi_X : X \in \mathcal{X}\}$, which implies that the singleton with the trivial character is necessarily an element of $\mathcal{X}$.

Note that

$$\xi_X^\star(g) = \overline{\xi_X(g^{-1})} = \sum_{\chi \in X} \overline{\chi(1)\chi(g^{-1})} = \sum_{\chi \in X} \chi(1)\chi(g) = \xi_X(g)$$

for all $g \in G$ and all $X \in \mathcal{X}$, which means S is a $\star$-subalgebra of $\mathbb{C}[G]$. Since S is a $\star$-subalgebra of $\mathbb{C}[G]$, then

$$\delta_K^\star = \delta_{K^{-1}}$$

belongs to S, which implies $K^{-1} \in \mathcal{K}$, for all $K \in \mathcal{K}$.

By remark 1.2.4 and theorem 1.2.5, $\mathcal{X}$ and $\mathcal{K}$ are unique in this conditions, depending on each other. □

As a direct consequence of the previous lemma, we have proved the following theorem.

Theorem 3.2.2. *Let $(\mathcal{K}, \mathcal{X})$ be a supercharacter theory for G. Then $\mathcal{S}_{\mathcal{K}}$ is a central Schur ring and*

1. *The set $\{e_X : X \in \mathcal{X}\} \subseteq \mathbb{C}[G]$ is the central partition of unity of $\mathcal{S}_{\mathcal{K}}$, with respect to the convolution product, and the mapping $\xi_X \mapsto \overline{\xi_X}$ defines an automorphism of the Schur ring $\mathcal{S}_{\mathcal{K}}$;*
2. *The set $\{\delta_K : K \in \mathcal{K}\} \subseteq \mathbb{C}[G]$ is the central partition of unity of $\mathcal{S}_{\mathcal{K}}$, with respect to the Hadamard product, and the mapping $\delta_K \mapsto \delta_{K^{-1}}$ defines an automorphism of the Schur ring $\mathcal{S}_{\mathcal{K}}$.*

Note that we have $\mathrm{scf}_{\mathcal{K}}(G) = \mathcal{S}_{\mathcal{K}}$.

The following proposition is a direct consequence of theorem 2.1.3 combined with lemma 3.2.1.

Proposition 3.2.3. *Let $\mathcal{K} \in \mathrm{Part}(G)$ and $\mathcal{X} \in \mathrm{Part}(\mathrm{Irr}(G))$. The following are equivalent:*

1. *$(\mathcal{K}, \mathcal{X})$ is a supercharacter theory for G;*
2. *$\mathcal{S}_{\mathcal{K}}$ is a central Schur ring;*
3. *$\mathrm{span}_{\mathbb{C}}\{\xi_X : X \in \mathcal{X}\}$ is a subalgebra of $\mathrm{cf}(G)$ (with identity) with respect to the Hadamard product;*
4. *$\mathrm{span}_{\mathbb{C}}\{\delta_K : K \in \mathcal{K}\}$ is a subalgebra of $\mathrm{cf}(G)$ (with identity) with respect to the convolution product;*
5. *$|\mathcal{K}| = |\mathcal{X}|$ and $\mathrm{span}_{\mathbb{C}}\{\xi_X : X \in \mathcal{X}\}$ is the set of all $\mathcal{K}$-superclass functions of G.*

It is clear now that the deep connections between Schur rings and supercharacter theories give us additional tools to work with both, and a broader view of what happens in the cases where we pick a *good* partition of the group, in order to obtain a Schur ring or a supercharacter theory.

We finish this section highlighting also the relation, now obvious, between supercharacters and the $\mathcal{S}$-characters, developed by Tamaschke, in [41], and presented in section 2.3.

Theorem 3.2.4. *Let $(\mathcal{K}, \mathcal{X})$ be a supercharacter theory for G. Then*

$$\mathrm{Irr}_{\mathcal{S}_{\mathcal{K}}}(G) = \{\xi_X : X \in \mathcal{X}\}$$

i.e., the supercharacters of G of this supercharacter theory are precisely the irreducible $\mathcal{S}_{\mathcal{K}}$-characters of G, and

$$\mathrm{Cl}_{\mathcal{S}_{\mathcal{K}}}(G) = \mathcal{K}$$

i.e, the set of all $\mathcal{S}_{\mathcal{K}}$-conjugacy classes of G equals $\mathcal{K}$.

Chapter 4

Adjoint groups

In [19], Diaconis and Isaacs, constructed a standard supercharacter theory for algebra groups extending what André and Yan did with unitriangular groups in [2] and [44], respectively. In [5], André and Nicolás generalized this construction to adjoint groups of finite radical rings. In this chapter, we present the main definitions and properties involving this family of groups and then we use a Schur ring approach to construct the standard supercharacter theory for these groups. As we shall see, the tools acquired by the relation between supercharacteres and Schur rings will considerably simplify the proofs of the main results.

4.1 Definitions and main properties

Definition 4.1.1. If R is a ring (not necessarily with an identity), then the **adjoint semigroup of** R is the set R endowed with the multiplication

$$x \times y := x + y + xy$$

for all $x, y \in R$. The subgroup of the adjoint semigroup consisting of all (two-sided) invertible elements is called the **adjoint group of** R and is denoted by R°.

Note that this definition of the adjoint group turns the 0 of the ring R the identity of R°. To easily distinguish between the product in R and the product in the adjoint semigroup, we can consider an element of the adjoint semigroup as the formal sum $1 + x$, with $x \in R$. We will use this convention to write the elements of R°, and note that this way the product of two elements $x, y \in R^\circ$ arises naturally from the distributive law:

$$(1 + x) \times (1 + y) = 1 + x + y + xy.$$

With this way to work with the elements of R°, the identity of the adjoint group turns to be the formal sum $1 = 1 + 0$.

The case where R is a radical ring is of special interest for our purpose. The relation between a radical ring and its adjoint group has been studied by several authors (see, for instance, [1] and [42]).

Definition 4.1.2. Let R be a ring.

- If every element of R is nilpotent then we call R a **nil ring**, that is, for each $x \in R$ there is a natural number k such that $x^k = 0$.
- If there is a natural number k such that $R^k = 0$ then we call R a **nilpotent ring**; here R^k denotes the set of all sums of k-fold products of elements of R.
- If $R = J(R)$ then we call R a **radical ring**.

Recall that in an artinian ring the Jacobson radical equals the set of all nilpotent elements of the ring. Therefore, it is straightforward to check that when R is finite these three definitions are equivalent, and, in this case, $R = R^\circ$.

4.2 The standard Schur partition of adjoint groups

From now on, unless stated otherwise, let R be a finite radical ring. In this situation, the adjoint group of R , R°, is the **algebra group** associated with R, as defined by Diaconis and Isaacs in section 3 of [19]. Let R^+ denote the additive abelian group associated with the radical ring R.

Inspired by the work of Kirillov in [31] but mainly by the work of André in [2] and Yan in [44], the coadjoint orbit method for algebra groups was adapted by Diaconis and Isaacs in section 3 of [19]. This was done from the natural action of $R^\circ \times R^\circ$ on the additive group, R^+, of the radical ring R. In this section we approach this construction using a slightly different method. Namely, we prove that the orbit Schur ring over R^+ that arises from this action is isomorphic to a Schur ring over R°. In the proofs of the theorems, we use the tools acquired in the previous chapters. This new way of approaching this problem will simplify considerably the proofs and clarify some of the properties.

Consider the bijective map $\nu : R^\circ \to R^+$ given by

$$\nu(g) := g - 1$$

for all $g \in R^\circ$. This bijection can be extended by linearity to an isomorphism of $\mathbb{C}$-vector spaces $\nu : \mathbb{C}[R^\circ] \to \mathbb{C}[R^+]$ where

$$\nu(\alpha) = \sum_{g \in R^\circ} \alpha(g)\delta_{\nu(g)} \tag{4.1}$$

for all $\alpha \in \mathbb{C}[R^\circ]$. Despite mapping the standard basis of the Schur ring over R°, $\mathbb{C}[R^\circ]$, to the standard basis of the Schur ring over R^+, $\mathbb{C}[R^+]$, ν is not necessarily an isomorphism of Schur rings. However, we will see that it restricts to an isomorphism of Schur rings.

Consider the action of $R^\circ \times R^\circ$ on R^+ by automorphisms of R^+, defined by

$$(g, h) \odot x := gxh^{-1} \tag{4.2}$$

for all $g, h \in R^\circ$ and for all $x \in R^+$. Again, we can extend by linearity this action to the group algebra $\mathbb{C}[R^+]$ where

$$((g, h) \odot \alpha)(x) := \alpha(g^{-1}xh)$$

for all $g, h \in R^\circ$, for all $\alpha \in \mathbb{C}[R^+]$ and for all $x \in R^+$. Let

$$\mathcal{K}^+ := \{R^\circ x R^\circ : x \in R^+\}$$

denote the set of orbits of this action of $R^\circ \times R^\circ$ on R^+. By Proposition 2.2.1,

$$\mathcal{S}_{\mathcal{K}^+} = \{\alpha \in \mathbb{C}[R^+] : (g,h) \odot \alpha = \alpha \text{ for all } g, h \in R^\circ\}$$

is a central Schur ring with standard basis $\{\delta_K : K \in \mathcal{K}^+\}$, that we will call the **standard Schur ring over** R^+. We will call the supercharacter theory for R^+ associated with $\mathcal{K}^+$, $(\mathcal{K}^+, \mathcal{X}^+)$, the **standard supercharacter theory for** R^+.

For all $x \in R^+$, define the subset of R°

$$K_x := \{1 + gxh^{-1} : g, h \in R^\circ\},$$

and note that

$$\nu(\delta_{K_x}) = \delta_{R^\circ x R^\circ} = \delta_{\nu(K_x)}, \tag{4.3}$$

meaning that ν and δ commute. Moreover, notice that

$$gK_x = g + \nu(K_x) \tag{4.4}$$

for all $g \in R^\circ$ and for all $x \in R^+$, and that

$$\nu(gh) = g\nu(h) + \nu(g) \tag{4.5}$$

for all $g, h \in R^\circ$.

Theorem 4.2.1. *The set*

$$\mathcal{K}^\circ := \{K_x : x \in R^+\}$$

is a central Schur partition of R°.

Proof. First, the set

$$K_0 = \{1 + g0h^{-1} : g, h \in R^\circ\} = \{1\}$$

belongs to $\mathcal{K}^\circ$. Secondly, for all $x \in R^+$,

$$\begin{aligned}(1 - x(1+x)^{-1})(1+x) &= 1 + x - x(1+x)^{-1} - x(1+x)^{-1}x \\ &= 1 + x - x(1+x)^{-1}(1+x) = 1,\end{aligned}$$

and thus

$$(1+x)^{-1} = 1 - x(1+x)^{-1} \in K_{-x}.$$

Therefore, for all $g, h \in R^\circ$,

$$(1+gxh)^{-1} = 1 + g(-x)(h(1+gxh)^{-1}) \in K_{-x}.$$

It follows that

$$K_x^{-1} = K_{-x} \tag{4.6}$$

belongs to $\mathcal{K}^\circ$, for all $x \in R^+$. Finally, let

$$\mathcal{S}_{\mathcal{K}^\circ} := \operatorname{span}_{\mathbb{C}}\{\delta_K : K \in \mathcal{K}^\circ\}.$$

Following equalities 4.1 and 4.3, we conclude that the operators ν and δ commute and that $\nu(\mathcal{S}_{\mathcal{K}^\circ}) = \mathcal{S}_{\mathcal{K}^+}$. Consider now a left action of R° on R^+ by setting

$$g \cdot x := gx + \nu(g)$$

for all $g \in R^\circ$ and for all $x \in R^+$. Observe that, following 4.5,

$$\begin{aligned} g \cdot (h \cdot x) &= g \cdot (hx + \nu(h)) \\ &= ghx + g\nu(h) + \nu(g) = ghx + \nu(gh) = (gh) \cdot x \end{aligned}$$

for all $g, h \in R^\circ$ and for all $x \in R^+$, and so the axiom of compatibility is verified. Moreover, we have

$$g \cdot \nu(h) = \nu(gh) \tag{4.7}$$

for all $g, h \in R^\circ$. This action extends by linearity to the group algebra $\mathbb{C}[R^+]$, so that $\mathbb{C}[R^+]$ becomes a left $\mathbb{C}[R^\circ]$-module where

$$\alpha \cdot \nu(\beta) = \nu(\alpha\beta) \tag{4.8}$$

for all $\alpha, \beta \in \mathbb{C}[R^\circ]$. Moreover, it is clear that $\nu : \mathbb{C}[R^\circ] \to \mathbb{C}[R^+]$ is an isomorphism of left $\mathbb{C}[R^\circ]$-modules where $\mathbb{C}[R^\circ]$ is naturally considered as the left regular $\mathbb{C}[R^\circ]$-module.

We have that

$$\begin{aligned} \nu(\delta_K \delta_{K'}) &= \delta_K \cdot \nu(\delta_{K'}) && \text{(by 4.8)} \\ &= \delta_K \cdot \delta_{\nu(K')} && \text{(by 4.3)} \\ &= \sum_{g \in K} \sum_{h \in K'} \delta_g \cdot \delta_{\nu(h)} = \sum_{g \in K} \sum_{h \in K'} \delta_{\nu(gh)} && \text{(by 4.7)} \\ &= \sum_{g \in K} \delta_{\nu(gK')} = \sum_{g \in K} \delta_{\nu(g)+\nu(K')} && \text{(by 4.4)} \\ &= \sum_{g \in K} \delta_{\nu(g)} \delta_{\nu(K')} = \delta_{\nu(K)} \delta_{\nu(K')} && \text{(by 1.1.2)} \end{aligned}$$

for all $K, K' \in \mathcal{K}^\circ$.

We conclude that $\mathcal{S}_{\mathcal{K}^\circ}$ is a subalgebra of $\mathbb{C}[R^\circ]$, with respect to the convolution product. Moreover, ν defines by restriction a ring isomorphism $\nu : \mathcal{S}_{\mathcal{K}^\circ} \to \mathcal{S}_{\mathcal{K}^+}$. Since $\nu(\delta_K) = \delta_{\nu(K)}$ for all $K \in \mathcal{K}^\circ$, we conclude that this isomorphism is indeed a Schur ring isomorphism.

It follows that $\mathcal{K}^\circ$ is a central Schur partition of R°. For the last assertion, it is enough to recall Proposition 2.1.2. □

Definition 4.2.2. The Schur ring $\mathcal{S}_{\mathcal{K}^\circ}$ will be referred as the **standard Schur ring over** R°. The supercharacter theory $(\mathcal{K}^\circ, \mathcal{X}^\circ)$ of R° associated with the central Schur ring $\mathcal{S}_{\mathcal{K}^\circ}$ will be referred to as the **standard supercharacter theory of** R°. For each $X \in \mathcal{X}^\circ$, the character ξ_X will be referred to as a **standard supercharacter of** R° and the sets $K \in \mathcal{K}^\circ$ will be referred to as the **standard superclasses of** R°.

Theorem 4.2.3. *Let R be a finite radical ring and let ξ be a standard supercharacter of the adjoint group R°. Then there exists $\vartheta \in \mathrm{Irr}(R^+)$ such that*

$$\xi(g) = \sum_{\vartheta' \in R^\circ \vartheta R^\circ} \vartheta'(\nu(g))$$

for all $g \in R^\circ$.

Proof. As seen in the second section of chapter 1, if G is a finite group and $X \subseteq \mathrm{Irr}(G)$ then $e_X = \sum_{\chi \in X} \frac{\chi(1)}{|G|}\overline{\chi}$. Then by corollary 3.2.2, $\{e_X : X \in \mathfrak{X}^\circ\}$ and $\{e_{X'} : X' \in \mathfrak{X}^+\}$ are the central partition of unity of $\mathcal{S}_{\mathcal{K}^\circ}$ and $\mathcal{S}_{\mathcal{K}^+}$, respectively, with respect to the convolution product. In virtue of the central Schur ring isomorphism of Theorem 4.2.1 we conclude that $\{\nu(e_X) : X \in \mathfrak{X}^\circ\}$ must be the central partition of unity of $\mathcal{S}_{\mathcal{K}^+}$ with respect to the convolution product, and so there is a bijection $\nu : \mathfrak{X}^\circ \to \mathfrak{X}^+$ such that

$$\nu(e_X) = e_{\nu(X)}$$

for all $X \in \mathfrak{X}^\circ$. It follows that $\{e_{\nu(X)} : X \in \mathfrak{X}^\circ\}$ is a basis of $\mathcal{S}_{\mathcal{K}^+}$ which is a central Schur ring over R^+ with respect to the action defined by equality 4.2. Note also that R^+ is abelian so all characters of $\mathrm{Irr}(R^+)$ are of degree 1 and they form a basis for $\mathbb{C}[R^+]$. So, for each $X \in \mathfrak{X}^\circ$ there must be an irreducible character $\vartheta \in \mathrm{Irr}(R^+)$ such that

$$e_{\nu(X)} = R^\circ e_\vartheta R^\circ = e_{R^\circ \vartheta R^\circ}.$$

If $\vartheta \in \mathrm{Irr}(R^+)$, let X_ϑ denote the unique element of $\mathfrak{X}^\circ$ such that $\nu(e_{X_\vartheta}) = e_{R^\circ \vartheta R^\circ}$.

It follows that, for every $X_\vartheta \in \mathfrak{X}^\circ$,

$$\begin{aligned}\nu\left(\frac{1}{|R^\circ|}\overline{\xi_{X_\vartheta}}\right) &= \nu(e_{X_\vartheta}) = e_{R^\circ \vartheta R^\circ} \\ &= \sum_{\vartheta' \in R^\circ \vartheta R^\circ} e_{\vartheta'} = \frac{1}{|R^+|} \sum_{\vartheta' \in R^\circ \vartheta R^\circ} \overline{\vartheta'}\end{aligned}$$

Therefore

$$\xi_{X_\vartheta}(g) = \sum_{\vartheta' \in R^\circ \vartheta R^\circ} \vartheta'(\nu(g)) = \frac{|R^\circ \vartheta R^\circ|}{|R^\circ|^2} \sum_{h_1, h_2 \in R^\circ} \vartheta(h_1 \nu(g) h_2) \tag{4.9}$$

for all $g \in R^\circ$. □

An induced character of a linear character is called a **monomial character**. Standard supercharacters of R° are usually defined as monomial characters; see for example [19]. Consider the $\mathbb{C}[R^\circ]$-submodule

$$\mathcal{V}_\vartheta = \mathbb{C}[R^\circ] \cdot e_\vartheta$$

of $\mathbb{C}[R^+]$ generated by e_ϑ for $\vartheta \in R^+$ arbitrary. Given $\vartheta \in \mathrm{Irr}(R^+)$ and $g \in R^\circ$, we define an

action of R° on $\mathrm{Irr}(R^+)$ by the formula $g\vartheta(x) = \vartheta(g^{-1}x)$ for all $x \in R^+$. Note that

$$\begin{aligned}
g \cdot e_\vartheta &= \frac{1}{|R^+|} \sum_{x \in R^+} \overline{\vartheta(x)} \delta_{gx+\nu(g)} \\
&= \frac{1}{|R^+|} \sum_{h \in R^\circ} \overline{\vartheta(\nu(h))} \delta_{\nu(gh)} \\
&= \frac{1}{|R^+|} \sum_{h \in R^\circ} \overline{\vartheta(\nu(g^{-1}h)} \delta_{\nu(h)} \\
&= \frac{1}{|R^+|} \sum_{h \in R^\circ} \overline{\vartheta(g^{-1}\nu(h) + \nu(g^{-1}))} \delta_{\nu(h)} \\
&= \frac{1}{|R^+|} \sum_{h \in R^\circ} \overline{\vartheta(g^{-1}\nu(h))\vartheta(\nu(g^{-1}))} \delta_{\nu(h)} \\
&= \overline{\vartheta(\nu(g^{-1}))} \frac{1}{|R^+|} \sum_{h \in R^\circ} \overline{g\vartheta(\nu(h))} \delta_{\nu(h)} \\
&= \overline{\vartheta(\nu(g^{-1}))} e_{g\vartheta}
\end{aligned}$$

for all $g \in R^\circ$, and thus $\mathcal{V}_\vartheta$ is linearly spanned by the set $\{e_{g\vartheta} : g \in R^\circ\}$. Since R° permutes transitively the one-dimensional subspaces $\mathbb{C}e_{g\vartheta} \subseteq \mathcal{V}_\vartheta$ for all $g \in R^\circ$, following the characterization of induced character in [18], we conclude that $\mathcal{V}_\vartheta$ is isomorphic to the induced $\mathbb{C}[R^\circ]$-module $\mathrm{Ind}_{L_\vartheta}^{R^\circ}(\mathbb{C}e_\vartheta)$ where

$$L_\vartheta = \{g \in R^\circ : g\alpha = \alpha \text{ for all } \alpha \in \mathbb{C}e_\vartheta\}$$

is the stabilizer of $\mathbb{C}e_\vartheta$ in R°. Consider also the map $\varsigma_\vartheta : L_\vartheta \to \mathbb{C}$ defined by

$$\varsigma_\vartheta(g) = \vartheta(\nu(g))$$

for all $g \in L_\vartheta$. We have the following result.

Proposition 4.2.4. *Let $\vartheta \in \mathrm{Irr}(R^+)$ be arbitrary. Then ς_ϑ is a linear character of L_ϑ and $\xi_{X_\vartheta} = n_\vartheta \mathrm{Ind}_{L_\vartheta}^{R^\circ}(\varsigma_\vartheta)$ where $n_\vartheta = \frac{|R^\circ \vartheta R^\circ|}{|R^\circ \vartheta|}$.*

Proof. Firstly, we note that for every $g \in R^\circ$, we have $g \cdot e_\vartheta = \overline{\vartheta(\nu(g^{-1}))} e_{g\vartheta} \in \mathbb{C}e_\vartheta$ if and only if $g\vartheta = \vartheta$, and hence $L_\vartheta = \{g \in R^\circ : g\vartheta = \vartheta\}$. On the other hand, we have $\nu(g^{-1}) = -g^{-1}\nu(g)$ for all $g \in R^\circ$, and thus

$$\vartheta(\nu(g^{-1})) = \vartheta(-g^{-1}\nu(g)) = (g\vartheta)(-\nu(g)) = \overline{\vartheta(\nu(g))}$$

for all $g \in L_\vartheta$. It follows that

$$g \cdot e_\vartheta = \overline{\vartheta(\nu(g^{-1}))} e_{g\vartheta} = \vartheta(\nu(g)) e_{g\vartheta}$$

for all $g \in L_\vartheta$, and so ς_ϑ is in fact the linear character of L_ϑ afforded by the one-dimensional $\mathbb{C}[L_\vartheta]$-module $\mathbb{C}e_\vartheta$.

For the final assertion, we observe that the supercharacter ξ_{X_ϑ} of R° is afforded by the $\mathbb{C}[R^\circ]$-submodule $\mathbb{C}[R^\circ]e_{X_\vartheta}$ of $\mathbb{C}[R^\circ]$. Furthermore, we deduce that

$$\nu(ge_{X_\vartheta}) = g \cdot \nu(e_{X_\vartheta}) = g \cdot e_{R^\circ \vartheta R^\circ}$$

for all $g \in R^\circ$, and thus the isomorphism $\nu : \mathbb{C}[R^\circ] \to \mathbb{C}[R^+]$ induces by restriction an isomorphism of $\mathbb{C}[R^\circ]$-modules $\nu : \mathbb{C}[R^\circ]e_{X_\vartheta} \to \mathbb{C}[R^+]e_{R^\circ \vartheta R^\circ}$. On the other hand, it is straightforward to check that the $\mathbb{C}[R^\circ]$-module $\mathbb{C}[R^\circ] \cdot e_{R^\circ \vartheta R^\circ}$ decomposes as the direct sum

$$\mathbb{C}[R^\circ] \cdot e_{R^\circ \vartheta R^\circ} = \bigoplus_{\tau \in \mathcal{T}} \mathbb{C}[R^\circ] \cdot e_\tau = \bigoplus_{\tau \in \mathcal{T}} \mathcal{V}_\tau$$

where $\mathcal{T}$ is a complete set of representatives of the right R°-orbits which are contained in $R^\circ \vartheta R^\circ$. Furthermore, it is not hard to check that, for every $h \in R^\circ$, the mapping $e_\vartheta \mapsto e_{\vartheta h}$ induces an isomorphism of $\mathbb{C}[R^\circ]$-modules $\mathcal{V}_\vartheta \to \mathcal{V}_{\vartheta h}$, and so $\mathcal{V}_\tau \cong_{R^\circ} \mathcal{V}_\vartheta$ for all $\tau \in \vartheta R^\circ$.

Since $\mathbb{C}[R^\circ] \cdot e_{R^\circ \vartheta R^\circ}$ affords the supercharacter ξ_{X_ϑ} and $\mathcal{V}_\vartheta \cong_{R^\circ} \mathrm{Ind}_{L_\vartheta}^{R^\circ}(\mathbb{C}e_\vartheta)$ affords the induced character $\mathrm{Ind}_{L_\vartheta}^{R^\circ}(\varsigma_\vartheta)$, we conclude that

$$\xi_{X_\vartheta} = |\mathcal{T}| \, \mathrm{Ind}_{L_\vartheta}^{R^\circ}(\varsigma_\vartheta),$$

and the result follows because $|\mathcal{T}| = \frac{|R^\circ \vartheta R^\circ|}{|R^\circ \vartheta|}$. □

Remark 4.2.5: We should mention that the supercharacter of R° associated with $\vartheta \in \mathrm{Irr}(R^+)$ is usually defined as the induced character $\mathrm{Ind}_{L_\vartheta}^{R^\circ}(\varsigma_\vartheta)$. We also observe that, in virtue of Eq. (4.9), we have

$$\mathrm{Ind}_{L_\vartheta}^{R^\circ}(\varsigma_\vartheta)(g) = \frac{|R^\circ \vartheta|}{|R^\circ \vartheta R^\circ|} \sum_{\vartheta' \in R^\circ \vartheta R^\circ} \vartheta'(\nu(g))$$

for all $g \in R^\circ$.

4.2.1 Example: The augmentation ideal

Let p be a prime and let $\mathbb{F}$ be a finite field of characteristic p.

Definition 4.2.6. The **augmentation map of** $\mathbb{F}[G]$, $\mathrm{aug} : \mathbb{F}[G] \to \mathbb{F}$, is defined by

$$\mathrm{aug}(x) := \sum_{g \in G} x(g)$$

for all $x \in \mathbb{F}[G]$. The kernel of the augmentation map is the **augmentation ideal of** $\mathbb{F}[G]$, $\mathcal{I}(\mathbb{F}[G])$, that is,

$$\mathcal{I}(\mathbb{F}[G]) := \{x \in \mathbb{F}[G] : \mathrm{aug}(x) \equiv 0 \mod p\}.$$

If G is a finite p-group, $\mathcal{I}(\mathbb{F}[G])$ is an example of a finite radical ring (it equals the Jacobson radical of $\mathbb{F}[G]$ in this case), with basis $\{\delta_g - \delta_1 : g \in G \backslash \{1\}\}$. Its adjoint group is of the form

$$\mathcal{I}(\mathbb{F}[G])^\circ = \{x \in \mathbb{F}[G] : \mathrm{aug}(x) \equiv 1 \mod p\}.$$

In particular, if we consider the cyclic group of order p,

$$C_p := \langle \omega \rangle = \{1, \omega, \ldots, \omega^{p-1}\}$$

and $\mathbb{F} = \{0, 1, \ldots, p-1\}$ the field with p elements, then the augmentation ideal of $\mathbb{F}[C_p]$ is

$$\mathcal{I}(\mathbb{F}[C_p]) = \text{span}_{\mathbb{F}}\{\delta_{\omega^i} - \delta_1 : 1 \leq i \leq p-1\}.$$

Note that $(\delta_\omega - \delta_1)^p \equiv 0 \mod p$, and $\mathcal{B} = \{(\delta_\omega - \delta_1)^i : 1 \leq i \leq p-1\}$ is also a basis of $\mathcal{I}(\mathbb{F}[C_p])$.

If $x \in \mathcal{I}(\mathbb{F}[C_p])$, let x_k denote the k-th coordinate of x in basis $\mathcal{B}$. Assume that $x_0 = 0$ for all $x \in \mathcal{I}(\mathbb{F}[C_p])$. This way, the convolution product on $\mathcal{I}(\mathbb{F}[C_p])$ simplifies to

$$(xy)_k = \sum_{i=0}^{k} x_i y_{k-i}$$

for all $x, y \in \mathcal{I}(\mathbb{F}[C_p])$ and $0 \leq k \leq p-1$. Moreover, note that $(xy)_0 = (xy)_1 = 0$.

In this case, $\mathcal{I}(\mathbb{F}[G])^\circ$ is the exponential group of $\mathcal{I}(\mathbb{F}[G])^+$, so it is straightforward to prove that

$$\mathcal{I}(\mathbb{F}[G])^\circ = \left\{ \prod_{i=1}^{p-1} ((\delta_\omega - \delta_1)^i + \delta_1)^{a_i} : a_i \in \mathbb{F} \right\}$$

and it is isomorphic to the direct product $C_p \times \ldots \times C_p$ $p-1$ times.

When $p = 3$, if (x_1, x_2) is the set of coordinates of $x \in \mathbb{F}[C_3]$ on basis $\mathcal{B}$, the Cayley table for the convolution product of the finite radical ring $\mathcal{I}(\mathbb{F}[C_3])$ is:

Table 8: Cayley table for the convolution product in $\mathcal{I}(\mathbb{F}[C_3])$

$\times$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,0)$	$(2,1)$	$(2,2)$
$(0,0)$	0	0	0	0	0	0	0	0	0
$(0,1)$	0	0	0	0	0	0	0	0	0
$(0,2)$	0	0	0	0	0	0	0	0	0
$(1,0)$	0	0	0	$(0,1)$	$(0,1)$	$(0,1)$	$(0,2)$	$(0,2)$	$(0,2)$
$(1,1)$	0	0	0	$(0,1)$	$(0,1)$	$(0,1)$	$(0,2)$	$(0,2)$	$(0,2)$
$(1,2)$	0	0	0	$(0,1)$	$(0,1)$	$(0,1)$	$(0,2)$	$(0,2)$	$(0,2)$
$(2,0)$	0	0	0	$(0,2)$	$(0,2)$	$(0,2)$	$(0,1)$	$(0,1)$	$(0,1)$
$(2,1)$	0	0	0	$(0,2)$	$(0,2)$	$(0,2)$	$(0,1)$	$(0,1)$	$(0,1)$
$(2,1)$	0	0	0	$(0,2)$	$(0,2)$	$(0,2)$	$(0,1)$	$(0,1)$	$(0,1)$

The adjoint group of this augmentation ideal, $\mathcal{I}^\circ(\mathbb{F}[C_3])$, is isomorphic to the direct product $C_3 \times C_3$ and isomorphic to the additive group of the radical ring $\mathcal{I}(\mathbb{F}[C_3])$.

For all $0 \leq g_1, g_2 \leq 2$, the element $(\delta_\omega)^{g_1}((\delta_\omega - \delta_1)^2 + \delta_1)^{g_2} \in \mathcal{I}^\circ(\mathbb{F}[C_3])$ will be denoted by (g_1, g_2). This adjoint group has the following character table, where $\omega = e^{\frac{2\pi i}{3}} \in \mathbb{C}$.

Table 9: Character table of $\mathcal{I}^\circ(\mathbb{F}[C_3])$ (and $\mathcal{I}^+(\mathbb{F}[C_3])$ and $C_3 \times C_3$)

	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,0)$	$(2,1)$	$(2,2)$
$\chi_{1,1}$	1	1	1	1	1	1	1	1	1
$\chi_{1,\omega}$	1	1	1	ω	ω	ω	ω^2	ω^2	ω^2
χ_{1,ω^2}	1	1	1	ω^2	ω^2	ω^2	ω	ω	ω
$\chi_{\omega,1}$	1	ω	ω^2	1	ω	ω^2	1	ω	ω^2
$\chi_{\omega,\omega}$	1	ω	ω^2	ω	ω^2	1	ω^2	1	ω
χ_{ω,ω^2}	1	ω	ω^2	ω^2	1	ω	ω	ω^2	1
$\chi_{\omega^2,1}$	1	ω^2	ω	1	ω^2	ω	1	ω^2	ω
$\chi_{\omega^2,\omega}$	1	ω^2	ω	ω	1	ω^2	ω^2	ω	1
χ_{ω^2,ω^2}	1	ω^2	ω	ω^2	ω	1	ω	1	ω^2

For all $1 \leq k \leq 2$, consider the set $(k,*) := \{(k,0),(k,1),(k,2)\} \subseteq \mathcal{I}^\circ(\mathbb{F}[C_3])$. It is straighforward to check that the standard supercharacter theory for $\mathcal{I}^\circ(\mathbb{F}[C_3])$ is given by the partitions:

$$\mathcal{K}_3^\circ := \{\{(0,0)\},\{(0,1)\},\{(0,2)\},(1,*),(2,*)\} \in \mathrm{Part}(\mathcal{I}^\circ(\mathbb{F}[C_3]))$$
$$\mathcal{X}_3^\circ := \{\{\chi_{1,1}\},\{\chi_{1,\omega}\},\{\chi_{1,\omega^2}\},\{\chi_{\omega,\omega^i} : 0{\leq}i{\leq}2\},\{\chi_{\omega^2,\omega^i} : 0{\leq}i{\leq}2\}\} \in \mathrm{Part}(\mathrm{Irr}(\mathcal{I}^\circ(\mathbb{F}[C_3])))$$

and the standard supercharacter table of $\mathcal{I}^\circ(\mathbb{F}[C_3])$ is:

Table 9: Supercharacter table of $(\mathcal{K}_3^\circ, \mathcal{X}_3^\circ)$

	$(0,0)$	$(0,1)$	$(0,2)$	$(1,*)$	$(2,*)$
$\chi_{1,1}$	1	1	1	1	1
$\chi_{1,\omega}$	1	1	1	ω	ω^2
χ_{1,ω^2}	1	1	1	ω^2	ω
$\chi_{\omega,1} + \chi_{\omega,\omega} + \chi_{\omega,\omega^2}$	3	3ω	$3\omega^2$	0	0
$\chi_{\omega^2,1} + \chi_{\omega^2,\omega} + \chi_{\omega^2,\omega^2}$	3	$3\omega^2$	3ω	0	0

We can also easily compute the structure constants of the standard Schur ring of $\mathcal{I}^\circ(\mathbb{F}[C_3])$. Recall that the constant c_{K_i,K_j,K_k} is the (i,j)-entry of the matrix M_k.

Table 10: Structure constants of $\mathcal{S}_{\mathcal{K}_3^\circ}$

$K_1 = \{(0,0)\}$	$K_2 = (0,1)$	$K_3 = (0,2)$	$K_4 = (1,*)$	$K_5 = (2,*)$
$M_1 = \begin{bmatrix} 1&0&0&0&0 \\ 0&0&1&0&0 \\ 0&1&0&0&0 \\ 0&0&0&0&3 \\ 0&0&0&3&0 \end{bmatrix}$	$M_2 = \begin{bmatrix} 0&1&0&0&0 \\ 1&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&3 \\ 0&0&0&3&0 \end{bmatrix}$	$M_3 = \begin{bmatrix} 0&0&1&0&0 \\ 0&1&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&0&3 \\ 0&0&0&3&0 \end{bmatrix}$	$M_4 = \begin{bmatrix} 0&0&0&1&0 \\ 0&0&0&1&0 \\ 0&0&0&1&0 \\ 1&1&1&0&0 \\ 0&0&0&0&3 \end{bmatrix}$	$M_5 = \begin{bmatrix} 0&0&0&0&1 \\ 0&0&0&0&1 \\ 0&0&0&0&1 \\ 0&0&0&3&0 \\ 1&1&1&0&0 \end{bmatrix}$

4.2.2 Example: The unitriangular group

Again, let p be a prime and let $\mathbb{F}$ be a finite field of characteristic p.

Definition 4.2.7. The set

$$\mathfrak{u}_n(\mathbb{F}) := \left\{ \begin{bmatrix} 0 & x_{1,2} & \dots & x_{1,n} \\ 0 & 0 & \dots & x_{2,n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix} : x_{i,j} \in \mathbb{F} \right\}$$

consisting of all strictly upper-triangular $n \times n$ matrices with coefficients in $\mathbb{F}$ is a finite radical ring. The adjoint group of $\mathfrak{u}_n(\mathbb{F})$ is called the **unitriangular group** and denoted by $U_n(\mathbb{F})$, that is,

$$U_n(\mathbb{F}) := I_n + \mathfrak{u}_n(\mathbb{F})$$

where I_n denotes the identity matrix of order n.

The unitriangular group $U_3(\mathbb{F}_2)$ is isomorphic to the dihedral group of order 8, which is known to have 9 distinct supercharacter theories. A classification of the supercharacter theories for dihedral groups of even order can be found in [34]. The Cayley table for the product of the finite radical ring $\mathfrak{u}_3(\mathbb{F}_2)$ is as follows:

Table 11: Cayley table for the product in $\mathfrak{u}_3(\mathbb{F}_2)$

$\cdot$	$\begin{bmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & * & * \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
$\begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}$	0	0
$\begin{bmatrix} 0 & 1 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}$	0	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

with $* \in \{0, 1\}$. The standard supercharacter theory for this adjoint group is given by the partitions:

$$\mathcal{K}^{\circ}_{3,2} := \left\{ \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 & 1 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 & 1 & * \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right\} \right\} \in \mathrm{Part}(U_3(\mathbb{F}_2))$$

$$\mathcal{X}^{\circ}_{3,2} := \left\{ \{\xi_1\}, X\left(\xi_{X_{\left[\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]}}\right), X\left(\xi_{X_{\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right]}}\right), X\left(\xi_{X_{\left[\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right]}}\right), X\left(\xi_{X_{\left[\begin{smallmatrix} 1 & * & 1 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{smallmatrix}\right]}}\right) \right\} \in \mathrm{Part}(\mathrm{Irr}(U_3(\mathbb{F}_2)))$$

and the standard supercharacter table is as follows:

Table 12: Supercharacter table of $(\mathcal{K}^{\circ}_{3,2}, \mathcal{X}^{\circ}_{3,2})$

	$\left\{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right\}$	$\left\{\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\right\}$	$\left\{\begin{bmatrix}1&1&*\\0&1&0\\0&0&1\end{bmatrix}\right\}$	$\left\{\begin{bmatrix}1&0&*\\0&1&1\\0&0&1\end{bmatrix}\right\}$	$\left\{\begin{bmatrix}1&1&*\\0&1&1\\0&0&1\end{bmatrix}\right\}$
ξ_1	1	1	1	1	1
$\xi_{X_{\left\{\begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix}\right\}}}$	1	1	1	-1	-1
$\xi_{X_{\left\{\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}\right\}}}$	1	1	-1	-1	1
$\xi_{X_{\left\{\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}\right\}}}$	1	1	-1	1	-1
$\sum \xi_{X_{\left\{\begin{bmatrix}1&*&1\\0&1&*\\0&0&1\end{bmatrix}\right\}}}$	4	-4	0	0	0

Again, we can easily compute the structure constants of the standard Schur ring of $U_3(\mathbb{F}_2)$ being as follows:

Table 13: Structure constants of $\mathcal{S}_{\mathcal{K}^{\circ}_{3,2}}$

$K_1 = \left\{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right\}$	$K_2 = \left\{\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\right\}$	$K_3 = \left\{\begin{bmatrix}1&1&*\\0&1&0\\0&0&1\end{bmatrix}\right\}$	$K_4 = \left\{\begin{bmatrix}1&0&*\\0&1&1\\0&0&1\end{bmatrix}\right\}$	$K_5 = \left\{\begin{bmatrix}1&1&*\\0&1&1\\0&0&1\end{bmatrix}\right\}$
$M_1 = \begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&2&0&0\\0&0&0&2&0\\0&0&0&0&2\end{bmatrix}$	$M_2 = \begin{bmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&2&0&0\\0&0&0&2&0\\0&0&0&0&2\end{bmatrix}$	$M_3 = \begin{bmatrix}0&0&1&0&0\\0&0&1&0&0\\1&1&0&0&0\\0&0&0&0&2\\0&0&0&2&0\end{bmatrix}$	$M_4 = \begin{bmatrix}0&0&0&1&0\\0&0&0&1&0\\0&0&0&0&2\\1&1&0&0&0\\0&0&2&0&0\end{bmatrix}$	$M_5 = \begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&2&0\\0&0&2&0&0\\1&1&0&0&0\end{bmatrix}$

Chapter 5

Involutive adjoint groups

In [27, Lemma 13.8], Isaacs recalls a result due to Glauberman:

Lemma 5.0.1 (Glauberman)**.** *Let S and G be two finite groups with coprime orders, and where, at least, one is solvable. Let S act on G and let S and G both act on a set K such that G is transitive on K and*

$$\sigma \cdot (g \cdot k) = (\sigma \cdot g) \cdot (\sigma \cdot k)$$

for all $\sigma \in S$, all $g \in G$ and all $k \in K$. Then S fixes a point on K.

In [9], André, Freitas and Neto use Glauberman's lemma to develop a supercharacter theory for involutive algebra groups, that is, groups associated with finite-dimensional nilpotent $\mathbb{F}$-algebras with involution, where $\mathbb{F}$ is a finite field of odd characteristic p. Here we generalize and simplify the results obtained using the relations between supercharacters and Schur rings. Glauberman's lemma will be fundamental in this chapter, taking into account it garantees that σ-invariant superclasses have at least one fixed element, where σ is an involution on the adjoint group R° built from an anti-involution defined on the finite radical ring R.

5.1 Definitions and main properties

Let R be a finite radical ring of odd characteristic endowed with an anti-involution σ, as defined in chapter 1.

Proposition 5.1.1. *The map $\sigma^\circ : R^\circ \longrightarrow R^\circ$ defined by*

$$\sigma^\circ(g) = 1 + \sigma(\nu(g^{-1}))$$

for all $g \in R^\circ$ is an involution on the adjoint group R°, that is:

- $\sigma^\circ(gh) = \sigma^\circ(g)\sigma^\circ(h)$;
- $\sigma^\circ(\sigma^\circ(g)) = g$.

for all $g, h \in R^\circ$.

Proof. Firstly,

$$\begin{aligned}
\sigma^\circ(gh) &= 1 + \sigma(\nu(h^{-1}g^{-1})) = 1 + \sigma(h^{-1}\nu(g^{-1}) + \nu(h^{-1})) && \text{(by Eq.(4.5))} \\
&= 1 + \sigma((h^{-1} - 1)\nu(g^{-1}) + \nu(g^{-1}) + \nu(h^{-1})) \\
&= 1 + \sigma(\nu(g^{-1}))\sigma(\nu(h^{-1})) + \sigma(\nu(g^{-1})) + \sigma(\nu(h^{-1})) && \text{(by Definition 1.1.7)} \\
&= (1 + \sigma(\nu(g^{-1})))(1 + \sigma(\nu(h^{-1}))) \\
&= \sigma^\circ(g)\sigma^\circ(h)
\end{aligned}$$

for all $g, h \in R^\circ$. Secondly,

$$\sigma^\circ(\sigma^\circ(g)) = 1 + \sigma(\nu(\sigma^\circ(g^{-1}))) = 1 + \sigma(\sigma(\nu(g))) = 1 + \nu(g) = g$$

for all $g \in R^\circ$. □

Note that, $(\sigma^\circ(g))^{-1} = \sigma^\circ(g^{-1})$ for all $g \in R^\circ$, and

$$\sigma(gxh) = \sigma^\circ(h^{-1})\sigma(x)\sigma^\circ(g^{-1}). \tag{5.1}$$

for all $g, h \in R^\circ$ and all $x \in R^+$.

Definition 5.1.2. The subgroup of the adjoint group R°, defined by

$$R^\circ_\sigma := \{g \in R^\circ : \sigma^\circ(g) = g\},$$

is called the **involutive adjoint group of R with respect to σ.**

In theorem 4.2.1, we proved that

$$\mathcal{K}^\circ = \{K_x : x \in R^+\}$$

is a central Schur partition of R°, where, for each $x \in R^+$, $K_x = \{1 + gxh^{-1} : g, h \in R^\circ\}$.

Lemma 5.1.3. *Let $K \in \mathcal{K}^\circ$. Then $\sigma^\circ(K) = K$ if and only if $K \cap R^\circ_\sigma \neq \emptyset$.*

Proof. The set $S = \{1, \sigma^\circ\}$ with the composition of functions is an abelian group with order coprime with the order of the direct product $G = R^\circ \times R^\circ$. Consider $K_x \in \mathcal{K}$ such that $\sigma^\circ(K_x) = K_x$.

Note that, by Eq.(4.6),

$$(1 + \sigma(x))^{-1} = \sigma^\circ(1 + x) \in K_{-\sigma(x)}$$

and so $K_x = K_{-\sigma(x)}$.

The set S acts on G by

$$\sigma^\circ \cdot (g, h) = (\sigma^\circ(h), \sigma^\circ(g))$$

for all $(g, h) \in G$, and acts on K_x by

$$\sigma^\circ \cdot (1 + a) = 1 - \sigma(a)$$

for all $1 + a \in K_x$. Moreover, G acts transitively on K_x by

$$(g, h) \cdot (1 + a) = 1 + gah^{-1}$$

for all $(g, h) \in G$ and all $1 + a \in K_x$.

Note that, following Eq. (5.1),

$$\begin{aligned}(\sigma^\circ \cdot (g, h)) \cdot (\sigma^\circ \cdot (1 + a)) &= (\sigma^\circ(h), \sigma^\circ(g)) \cdot (1 - \sigma(a)) \\ &= 1 - \sigma^\circ(h)\sigma(a)\sigma^\circ(g^{-1}) \\ &= 1 - \sigma(gah^{-1}) \\ &= \sigma^\circ \cdot ((g, h) \cdot (1 + a))\end{aligned}$$

for all $(g,h) \in G$ and all $1+a \in K_x$. It follows that this actions verify the conditions of Glauberman's lemma 5.0.1 and so there must be an element $1+a$ of K_x such that

$$1+a = \sigma^\circ \cdot (1+a) = 1-\sigma(a).$$

Recall that R is of odd characteristic and so 2 is invertible in R. Let $b \in R^+$ be the unique element on R^+ such that $2b = a$. Note that, $\sigma(b) = -b$, $(1+b)$ and $(1-b)^{-1}$ commute, $(1+b)(1-b)^{-1} = 1+2b(1-b)^{-1}$ belongs to $K_a = K_x$ and

$$\sigma^\circ((1+b)(1-b)^{-1}) = (1+b)(1-b)^{-1}.$$

It folllows that $K_x \cap R_\sigma^\circ \neq \emptyset$.

Conversely, if $K \cap R_\sigma^\circ \neq \emptyset$ then there is $g \in K$ such that $\sigma^\circ(g) = g$. Regarding what we have just seen above, we also know that there must be $1+a \in K$ such that $\sigma(a) = -a$. It follows that

$$\sigma^\circ(K) = \sigma^\circ(1+R^\circ aR^\circ) = 1-R^\circ\sigma(a)R^\circ = 1+R^\circ aR^\circ = K$$

for all $K \in \mathcal{K}^\circ$ that has at least one element fixed by σ°. □

Consider the subgroup of R^+

$$R_\sigma^+ := \{x \in R^+ : \sigma(x) = -x\}.$$

Note that the map ν defined in 4.5 is not necessarily a bijection between R_σ° and R_σ^+ but following what was observed in the previous proof, we obtain the following result.

Corollary 5.1.4. *The* **Cayley transform** *defined by*

$$\begin{array}{rccc} \Phi: & R^+ & \longrightarrow & R^\circ \\ & x & \mapsto & (1+x)(1-x)^{-1} \end{array}$$

is a bijection, its inverse is

$$\begin{array}{rccc} \Psi: & R^\circ & \longrightarrow & R^+ \\ & 1+2x & \mapsto & x(1+x)^{-1} \end{array}$$

and

$$\Phi(R_\sigma^+) = R_\sigma^\circ.$$

5.2 The standard Schur ring of involutive adjoint groups

We are now able to present the main result of this chapter. We extend the result in [9, Theorem 3.8], using a distinct approach which will considerably simplify the proof. The construction in [9] relies strongly on Glauberman's theory, as described in chapter 13 of [27], and depends on the supercharacters of R° that have a linear constituent.

Theorem 5.2.1. *The set*

$$\mathcal{K}_\sigma^\circ := \{K \cap R_\sigma^\circ : K \in \mathcal{K}^\circ \text{ and } \sigma^\circ(K) = K\}$$

is a central Schur partition of R_σ°.

Proof. By lemma 5.1.3 we know that if $g \in R_\sigma^\circ$ then $\sigma^\circ(K_{\nu(g)}) = K_{\nu(g)}$ which implies that $\mathcal{K}_\sigma^\circ$ is a partition of R_σ°. It is also clear that each element of $\mathcal{K}_\sigma^\circ$ is necessarily a union of conjugacy classes of R_σ°. It remains to check the three conditions for a partition to be a Schur partition.

First, $\{1\} \in \mathcal{K}^\circ$ and $\sigma^\circ(1) = 1$ so $\{1\} \in \mathcal{K}_\sigma^\circ$. Second, note that $(K \cap R_\sigma^\circ)^{-1} = K^{-1} \cap R_\sigma^\circ$ and clearly if $\sigma^\circ(K) = K$ then $\sigma^\circ(K^{-1}) = K^{-1}$. Finally, we can extend by linearity the involution σ° to the group algebra $\mathbb{C}[R^\circ]$ by defining for all $\alpha \in \mathbb{C}[R^\circ]$

$$\sigma^\circ(\alpha) = \sum_{g \in R^\circ} \alpha(g)\delta_{\sigma^\circ(g)} \in \mathbb{C}[R^\circ].$$

Following proposition 2.2.1, we know that

$$\mathbb{C}[R^\circ]_\sigma := \{\alpha \in \mathbb{C}[R^\circ] : \sigma^\circ(\alpha) = \alpha\} = \mathrm{span}_{\mathbb{C}}\{\delta_{\{g,\sigma^\circ(g)\}} : g \in R^\circ\}$$

is a subalgebra of $\mathbb{C}[R^\circ]$. Note that, if $\alpha \in \mathbb{C}[R^\circ]_\sigma$ then

$$\begin{aligned}\alpha &= \frac{1}{2}\left(\alpha + \sigma^\circ(\alpha)\right) = \frac{1}{2}\left(\sum_{g \in R^\circ} \alpha(g)\delta_g + \sum_{g \in R^\circ} \alpha(g)\delta_{\sigma^\circ(g)}\right) \\ &= \sum_{g \in R_\sigma^\circ} \alpha(g)\delta_g + \frac{1}{2}\sum_{\substack{g \in R^\circ \\ g \notin R_\sigma^\circ}} \alpha(g)\delta_{\{g,\sigma^\circ(g)\}}.\end{aligned}$$

Let $\alpha, \beta \in \mathbb{C}[R^\circ]_\sigma$ such that

$$\alpha \in \mathrm{span}_{\mathbb{C}}\{\delta_g : g \in R_\sigma^\circ\} \quad \text{and} \quad \beta \in \mathrm{span}_{\mathbb{C}}\{\delta_{\{g,\sigma^\circ(g)\}} : g \in R^\circ \text{ and } g \notin R_\sigma^\circ\}.$$

We have that

$$\begin{aligned}\alpha\beta(g) &= \sum_{h \in R^\circ} \alpha(h)\beta(h^{-1}g) = \left(\sum_{h \in R_\sigma^\circ} \alpha(h)\beta(h^{-1}g) + \sum_{\substack{h \in R^\circ \\ h \notin R_\sigma^\circ}} \alpha(h)\beta(h^{-1}g)\right) \\ &= \begin{cases} 0, & \text{if } g \in R_\sigma^\circ \\ \sum\limits_{h \in R_\sigma^\circ} \alpha(h)\beta(h^{-1}g), & \text{if } g \in R^\circ \text{ and } g \notin R_\sigma^\circ \end{cases}\end{aligned}$$

and

$$\begin{aligned}\beta\alpha(g) &= \sum_{h \in R^\circ} \beta(h)\alpha(h^{-1}g) = \left(\sum_{h \in R_\sigma^\circ} \beta(h)\alpha(h^{-1}g) + \sum_{\substack{h \in R^\circ \\ h \notin R_\sigma^\circ}} \beta(h)\alpha(h^{-1}g)\right) \\ &= \begin{cases} 0, & \text{if } g \in R_\sigma^\circ \\ \sum\limits_{\substack{h \in R^\circ \\ h \notin R_\sigma^\circ}} \beta(h)\alpha(h^{-1}g), & \text{if } g \in R^\circ \text{ and } g \notin R_\sigma^\circ \end{cases}\end{aligned}$$

for all $g \in R^\circ$. It follows that

$$\mathcal{N}(R^\circ) := \mathrm{span}_{\mathbb{C}}\{\delta_{\{g,\sigma^\circ(g)\}} : g \notin R_\sigma^\circ\}$$

is a two-sided ideal of $\mathbb{C}[R^\circ]_\sigma$. Moreover,

$$\mathbb{C}[R^\circ]_\sigma = \mathrm{span}_{\mathbb{C}}\{\delta_g : g \in R_\sigma^\circ\} \oplus \mathrm{span}_{\mathbb{C}}\{\delta_{\{g,\sigma^\circ(g)\}} : g \notin R_\sigma^\circ\} = \mathbb{C}[R_\sigma^\circ] \oplus \mathcal{N}(R^\circ),$$

and the projection

$$\begin{array}{rccc} \pi: & \mathbb{C}[R^\circ]_\sigma & \longrightarrow & \mathbb{C}[R^\circ_\sigma] \\ & \alpha+\beta & \mapsto & \alpha \end{array}$$

for all $\alpha \in \mathbb{C}[R^\circ_\sigma]$ and all $\beta \in \mathcal{N}(R^\circ)$, is an algebra homomorphism.

If $K \in \mathcal{K}^\circ$ is such that $\sigma^\circ(K) = K$ then

$$\pi(\delta_K) = \delta_{K \cap R^\circ_\sigma}.$$

So, if $C_i, C_j \in \mathcal{K}^\circ_\sigma$ then there are unique $K_i, K_j \in \mathcal{K}^\circ$ such that $C_i = K_i \cap R^\circ_\sigma$ and $C_j = K_j \cap R^\circ_\sigma$ and, by the second item of proposition 2.1.2,

$$\begin{aligned} \delta_{C_i}\delta_{C_j} &= \pi(\delta_{K_i})\pi(\delta_{K_j}) = \pi(\delta_{K_i}\delta_{K_j}) \\ &= \pi\left(\sum_{K\in\mathcal{K}^\circ} c_{i,j}^K \delta_K\right) \\ &= \pi\left(\sum_{\substack{K\in\mathcal{K}^\circ \\ \sigma^\circ(K)=K}} c_{i,j}^K \left(\delta_{K\cap R^\circ_\sigma} + \delta_{K\setminus R^\circ_\sigma}\right) + \sum_{\substack{K\in\mathcal{K}^\circ \\ \sigma^\circ(K)\neq K}} c_{i,j}^K \delta_K\right) \\ &= \sum_{\substack{K\in\mathcal{K}^\circ \\ \sigma^\circ(K)=K}} c_{i,j}^K \delta_{K\cap R^\circ_\sigma} \end{aligned}$$

with $c_{i,j}^K \in \mathbb{Z}_0^+$. This implies that

$$\delta_{C_i}\delta_{C_j} \in \mathrm{span}_{\mathbb{C}}\{\delta_{K\cap R^\circ_\sigma} : K \in \mathcal{K} \text{ and } \sigma^\circ(K) = K\}.$$

It follows that $\mathcal{S}_{\mathcal{K}^\circ_\sigma} = \mathrm{span}_{\mathbb{C}}\{\delta_C : C \in \mathcal{K}^\circ_\sigma\}$ is a subalgebra of $\mathbb{C}[R^\circ_\sigma]$. □

Definition 5.2.2. The Schur ring $\mathcal{S}_{\mathcal{K}^\circ_\sigma}$ will be referred as the **standard Schur ring over** R°_σ. The supercharacter theory $(\mathcal{K}^\circ_\sigma, \mathcal{X}^\circ_\sigma)$ of R°_σ associated with the central Schur ring $\mathcal{S}_{\mathcal{K}^\circ_\sigma}$ will be referred to as the **standard supercharacter theory of** R°_σ. For each $X \in \mathcal{X}^\circ$, the character ξ_X will be referred to as a **standard supercharacter of** R°_σ and the sets $K \in \mathcal{K}^\circ_\sigma$ will be referred to as the **standard superclasses of** R°_σ.

Recall that ν defines a Schur ring isomorphism $\nu : \mathcal{S}_{\mathcal{K}^\circ} \to \mathcal{S}_{\mathcal{K}^+}$.

The Cayley transform can be extended by linearity to an isomorphism of $\mathbb{C}$-vector spaces $\Phi : \mathbb{C}[R^+] \to \mathbb{C}[R^\circ]$ where

$$\Phi(\alpha) = \sum_{x\in R^+_\sigma} \alpha(x)\delta_{\Phi(a)}$$

for all $\alpha \in \mathbb{C}[R^+]$. Note that $\Phi(\delta_{\nu(K)}) = \delta_{1+2\nu(K)}$ for all $K \in \mathcal{K}^\circ$. Since the mapping $x \mapsto 2x$, for all $x \in R^+$, extends naturally to an algebra automorphism of $\mathbb{C}[R^+]$ and since $\mathcal{S}_{\mathcal{K}^+}$ is a Schur ring, we easily conclude that

$$\delta_{2\nu(K)}\delta_{2\nu(K')} = \sum_{K''\in\mathcal{K}^\circ} c_{K,K'}^{K''}\delta_{2\nu(K'')},$$

where $c_{K,K'}^{K''}$ are non-negative integers, for all $K, K' \in \mathcal{K}^\circ$. Recall that $c_{K,K'}^{K''}$ are the structure constants both for $\mathcal{S}_{\mathcal{K}^+}$ and $\mathcal{S}_{\mathcal{K}^\circ}$. Therefore,

$$\begin{aligned}\Phi\left(\delta_{\nu(K)}\delta_{\nu(K')}\right) &= \Phi\left(\sum_{K''\in\mathcal{K}^\circ} c_{K,K'}^{K''}\delta_{\nu(K'')}\right)\\ &= \sum_{K''\in\mathcal{K}^\circ} c_{K,K'}^{K''}\delta_{\Phi(\nu(K''))}\\ &= \sum_{K''\in\mathcal{K}^\circ} c_{K,K'}^{K''}\delta_{1+2\nu(K'')}\\ &= \left(\delta_{1+2\nu(K)}\right)\left(\delta_{1+2\nu(K')}\right) = \Phi(\delta_{\nu(K)})\Phi(\delta_{\nu(K'')})\end{aligned}$$

for all $K, K' \in \mathcal{K}^\circ$, and so also the linear map $\Phi : \mathbb{C}[R^+] \to \mathbb{C}[R^\circ]$ defines by restriction an isomorphism of Schur rings $\Phi : \mathcal{S}_{\mathcal{K}^+} \to \mathcal{S}_{\mathcal{K}^\circ}$.

Consider the action of R° on R_σ^+ by automorphisms of R_σ^+, defined by

$$g \odot x := gx\sigma^\circ(g^{-1})$$

for all $g \in R^\circ$ and all $x \in R_\sigma^+$. Again, we can extend by linearity this action on the group algebra $\mathbb{C}[R_\sigma^+]$ where

$$(g \odot \alpha)(x) := \alpha(g^{-1}x\sigma^\circ(g))$$

for all $g \in R^\circ$, all $\alpha \in \mathbb{C}[R_\sigma^+]$ and all $x \in R_\sigma^+$.

Let

$$\mathcal{K}_\sigma^+ := \{R^\circ \odot x : x \in R_\sigma^+\}$$

denote the set of orbits of this action of R° on R_σ^+.

By proposition 2.2.1,

$$\mathcal{S}_{\mathcal{K}_\sigma^+} = \{\alpha \in \mathbb{C}[R_\sigma^+] : g \odot \alpha = \alpha \text{ for all } g \in R^\circ\}$$

is an orbit Schur ring over R_σ^+ with standard basis $\{\delta_C : C \in \mathcal{K}_\sigma^+\}$.

Note that $\Phi(\delta_{R^\circ \odot x}) = \delta_{K_{2x}\cap R_\sigma^\circ}$ for all $x \in R_\sigma^+$. It follows that the Cayley transform defines by restriction an isomorphism of Schur rings $\Phi : \mathcal{S}_{\mathcal{K}_\sigma^+} \to \mathcal{S}_{\mathcal{K}_\sigma^\circ}$. With analogous arguments to the ones used on the proof of theorem 4.2.3, we prove the following result.

Theorem 5.2.3. *If ξ is a standard supercharacter of R_σ° then there exists $\vartheta \in \mathrm{Irr}(R_\sigma^+)$ such that*

$$\xi(g) = \sum_{\vartheta' \in R^\circ \odot \vartheta} \vartheta'(\Psi(g))$$

for all $g \in R_\sigma^\circ$.

5.2.1 Example: The classical anti-involution

Let p be a prime, let $\mathbb{F}$ be a finite field of characteristic p and let G be a finite p-group, as in example 4.2.1.

Definition 5.2.4. The **classical anti-involution of** $\mathbb{F}[G]$, $* : \mathbb{F}[G] \to \mathbb{F}[G]$, is defined by

$$x^*(g) = x(g^{-1})$$

for all $x \in \mathbb{F}[G]$ and all $g \in G$. An element x of $\mathbb{F}[G]$ is said to be **symmetric** if

$$x^* = x$$

and we will denote by $\Sigma(\mathbb{F}[G])$ the set of all symmetric elements of $\mathbb{F}[G]$.

Classical anti-involution and symmetric elements were studied by several authors (see for example [14] and [32]). The set

$$\mathcal{I}(\mathbb{F}[G])^\circ_* := \Sigma(\mathbb{F}[G]) \cap \mathcal{I}(\mathbb{F}[G])^\circ$$

is the involutive adjoint group of the finite radical ring $\mathcal{I}(\mathbb{F}[G])$, with respect to the classical anti-involution.

Consider the cyclic group of order 5

$$C_5 := \langle \sigma \rangle = \{1, \sigma, \sigma^2, \sigma^3, \sigma^4\}$$

and let $\mathbb{F}_5$ be the field with 5 elements. Consider the two bases of the vector space of dimension 4, $\mathcal{I}(\mathbb{F}[C_5])$,

$\mathcal{B}_1 = \{\delta_\sigma - \delta_1, \delta_{\sigma^2} - \delta_1, \delta_{\sigma^3} - \delta_1, \delta_{\sigma^4} - \delta_1\}$ and $\mathcal{B}_2 = \{(\delta_\sigma - \delta_1), (\delta_\sigma - \delta_1)^2, (\delta_\sigma - \delta_1)^3, (\delta_\sigma - \delta_1)^4\}$.

The change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$ is given by

$$P_{\mathcal{B}_1 \to \mathcal{B}_2} := \begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Observing that

$$P_{\mathcal{B}_1 \to \mathcal{B}_2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \text{ and } P_{\mathcal{B}_1 \to \mathcal{B}_2} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

we conclude that

$$\mathcal{I}(\mathbb{F}_5[C_5])^\circ_* = \{((\delta_\sigma - \delta_1)^2 + \delta_1)^a((\delta_\sigma - \delta_1)^3 + \delta_1)^{-a}((\delta_\sigma - \delta_1)^4 + \delta_1)^b : a, b \in \mathbb{F}_5\}.$$

The irreducible character of $\mathcal{I}(\mathbb{F}_5[C_5])^\circ_*$, $\chi_{\varepsilon^i, \varepsilon^j}$, is defined by

$$\chi_{\varepsilon^i, \varepsilon^j}(((\delta_\sigma - \delta_1)^2 + \delta_1)^a((\delta_\sigma - \delta_1)^3 + \delta_1)^{-a}((\delta_\sigma - \delta_1)^4 + \delta_1)^b) = (\varepsilon^i)^b(\varepsilon^j)^a$$

for all $i, j, a, b \in \mathbb{F}_5$.

In this case, the standard Schur partition of the involutive adjoint group $\mathcal{I}(\mathbb{F}_5[C_5])^\circ_*$ is

$$\mathcal{K}_{\mathcal{S}^\circ_*} = \{\{k_b\} : b \in \mathbb{F}_5\} \cup \{K_a : a \in \mathbb{F}_5 \setminus \{0\}\}$$

where, for each $a \in \mathbb{F}_5 \setminus \{0\}$ and for each $b \in \mathbb{F}_5$,

$$\begin{aligned} k_b :=& b\delta_{\sigma^4} + b\delta_{\sigma^3} + b\delta_{\sigma^2} + b\delta_\sigma + (b+1)\delta_1 \\ &\text{and} \\ K_a :=& \{s\delta_{\sigma^4} + (s-a)\delta_{\sigma^3} + (s-a)\delta_{\sigma^2} + s\delta_\sigma + (s+2a+1)\delta_1 : s \in \mathbb{F}_5\}. \end{aligned}$$

The supercharacter table associated to the central Schur ring $\mathcal{S}^\circ_*$ over the involutive adjoint group $\mathcal{I}(\mathbb{F}_5[C_5])^\circ_*$, where $\varepsilon = e^{\frac{2\pi i}{5}} \in \mathbb{C}$, is

Table 14: Supercharacter table of $\mathcal{S}_{\mathcal{K}^\circ_*}$

	$\{k_0\}$	$\{k_1\}$	$\{k_2\}$	$\{k_3\}$	$\{k_4\}$	K_1	K_2	K_3	K_4
$\chi_{1,1}$	1	1	1	1	1	1	1	1	1
$\chi_{1,\varepsilon}$	1	1	1	1	1	ε	ε^2	ε^3	ε^4
χ_{1,ε^2}	1	1	1	1	1	ε^2	ε^4	ε	ε^3
χ_{1,ε^3}	1	1	1	1	1	ε^3	ε	ε^4	ε^2
χ_{1,ε^4}	1	1	1	1	1	ε^4	ε^3	ε^2	ε
$\sum_{i\in\mathbb{F}} \chi_{\varepsilon,\varepsilon^i}$	5	5ε	$5\varepsilon^2$	$5\varepsilon^3$	$5\varepsilon^4$	0	0	0	0
$\sum_{i\in\mathbb{F}} \chi_{\varepsilon^2,\varepsilon^i}$	5	$5\varepsilon^2$	$5\varepsilon^4$	5ε	$5\varepsilon^3$	0	0	0	0
$\sum_{i\in\mathbb{F}} \chi_{\varepsilon^3,\varepsilon^i}$	5	$5\varepsilon^3$	5ε	$5\varepsilon^4$	$5\varepsilon^2$	0	0	0	0
$\sum_{i\in\mathbb{F}} \chi_{\varepsilon^4,\varepsilon^i}$	5	$5\varepsilon^4$	$5\varepsilon^3$	$5\varepsilon^2$	5ε	0	0	0	0

5.2.2 Example: The symplectic anti-involution

Let p be a prime, let $\mathbb{F}$ be a finite field of characteristic p and cardinality q as in example 4.2.2. Let J_m be the $m \times m$ matrix with $1's$ along the anti-diagonal and 0's elsewhere, let $\Omega := \begin{bmatrix} 0 & J_m \\ -J_m & 0 \end{bmatrix}$ and let $\mathcal{M}_n(\mathbb{F})$ denote the set of all square matrices of order n.

The symplectic group of order $2m$ with entries in $\mathbb{F}$ can be defined by,

$$\mathrm{Sp}_{2m}(\mathbb{F}) := \{g \in \mathcal{M}_{2m}(\mathbb{F}) : g^\intercal \Omega g = \Omega\}$$

where $\Omega := \left[\begin{array}{c|c} 0 & J_m \\ \hline -J_m & 0 \end{array}\right]$. In this case we can write

$$\mathrm{Sp}_{2m}(\mathbb{F}) = \left\{ \left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right] \in \mathcal{M}_{2m} : a^\intercal Jc = (a^\intercal Jc)^\intercal, a^\intercal Jd - c^\intercal Jb = J, b^\intercal Jd = (b^\intercal Jd)^\intercal \right\}$$

where J stands for the matrix J_m. It is well-known that the Sylow p-subgroup of $\mathrm{Sp}_{2m}(\mathbb{F})$ has order q^{m^2}.

Recall that $\mathfrak{u}_n(\mathbb{F})$ denotes the finite radical ring of all strictly upper-triangular $n \times n$ matrices with coeficients in $\mathbb{F}$.

Definition 5.2.5. Let J_m be the $m \times m$ matrix with $1's$ along the anti-diagonal and 0's elsewhere and let $\Omega := \begin{bmatrix} 0 & J_m \\ -J_m & 0 \end{bmatrix}$. The **symplectic anti-involution of** $\mathfrak{u}_{2m}(\mathbb{F})$ is the map $\sigma : \mathfrak{u}_{2m}(\mathbb{F}) \to \mathfrak{u}_{2m}(\mathbb{F})$ defined by

$$\sigma(x) = \Omega^{-1} x^{\intercal} \Omega$$

for all $x \in \mathfrak{u}_{2m}(\mathbb{F})$.

Proposition 5.2.6. *The involutive adjoint group of* $\mathfrak{u}_{2m}(\mathbb{F})$ *with respect to the symplectic anti-involution is a Sylow p-subgroup of the symplectic group* $\mathrm{Sp}_{2m}(\mathbb{F})$.

Proof. We have

$$\mathfrak{u}_{2m}(\mathbb{F})^+_\sigma = \left\{ \left[\begin{array}{c|c} y & z \\ \hline 0 & -Jy^{\intercal}J \end{array}\right] \in \mathfrak{u}_{2m}(\mathbb{F}) : z = Jz^{\intercal}J \right\}$$

where J stands for the matrix J_m. Note that the vector space $\mathfrak{u}_m(\mathbb{F})$ has dimension $\frac{(m-1)m}{2}$ and the vector space of $\{z \in \mathcal{M}_m : z = Jz^{\intercal}J\}$ has dimension $\frac{m(m+1)}{2}$, so $\mathfrak{u}_{2m}(\mathbb{F})^+_\sigma$ has dimension m^2 and cardinality q^{m^2}.

Using the Caley transform in the subgroup $\mathfrak{u}_{2m}(\mathbb{F})^+_\sigma$ we obtain

$$U_{2m}(\mathbb{F})^\circ_\sigma = \left\{ \left[\begin{array}{c|c} \Phi(y) & (\Phi(y)+I)zJ(I+y^{\intercal})^{-1}J \\ \hline 0 & J\Phi(-y^{\intercal})J \end{array}\right] : y \in \mathfrak{u}_m(\mathbb{F}), z \in \mathcal{M}_m(\mathbb{F}), z = Jz^{\intercal}J \right\}$$

where J stands for the matrix J_m. Note that $\Phi(y) + I = 2(I-y)^{-1}$ and

$$(\Phi(y)^{\intercal})J(J\Phi(-y^{\intercal})J) = \Phi(y^{\intercal})\Phi(y^{\intercal})^{-1}J = J$$

and

$$\begin{aligned} ((\Phi(y)+I)zJ(I+y^{\intercal})^{-1}J)^{\intercal} J(J\Phi(-y^{\intercal})J))^{\intercal} &= (J(I+y)^{-1}Jz^{\intercal}(\Phi(y^{\intercal})+I)\Phi(y^{\intercal})^{-1}J)^{\intercal} \\ &= (J(I+y)^{-1}zJ(I+\Phi(y^{\intercal})^{-1})J)^{\intercal} \\ &= (2J(I+y)^{-1}zJ(1+y^{\intercal})^{-1}J)^{\intercal} \\ &= 2J(I+y)^{-1}zJ(1+y^{\intercal})^{-1}J \end{aligned}$$

which proves that $U_{2m}(\mathbb{F})^\circ_\sigma$ is a subgroup of $\mathrm{Sp}_{2m}(\mathbb{F})$ of cardinality q^{m^2}. □

When $m = 2$ and $p = q = 3$, the involutive adjoint group of $\mathfrak{u}_4(\mathbb{F})$ becomes

$$U_4(\mathbb{F})^\circ_\sigma = \left\{ \begin{bmatrix} 1 & a & b & c \\ 0 & 1 & d & b-ad \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{bmatrix} : a, b, c, d \in \mathbb{F} \right\}$$

and the standard Schur partition of the involutive adjoint group $U_{2m}(\mathbb{F})^\circ_\sigma$ is

$$\mathcal{K}_{\mathbb{S}^\circ_\sigma} = \{C_c : c \in \mathbb{F}\} \cup \{B_b : b \in \mathbb{F}\backslash\{0\}\} \cup \{AD_{(a,d)} : a, d \in (\mathbb{F} \times \mathbb{F})\backslash\{(0,0)\}\}$$

where, for each $b \in \mathbb{F}\backslash\{0\}$, for each $c \in \mathbb{F}$ and for each $(a,d) \in (\mathbb{F}\times\mathbb{F})\backslash\{(0,0)\}$,

$$C_c := \left\{ \begin{bmatrix} 1 & 0 & 0 & c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}, B_b := \left\{ \begin{bmatrix} 1 & 0 & b & x \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : x \in \mathbb{F} \right\}, AD_{(a,d)} := \left\{ \begin{bmatrix} 1 & a & x & y \\ 0 & 1 & d & x-ad \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{bmatrix} : x,y \in \mathbb{F} \right\}.$$

For all $i,j,k,l \in \mathbb{F}$ we define

$$\chi_{\varepsilon^i,\varepsilon^j,\varepsilon^k,\varepsilon^l}\left(\begin{bmatrix} 1 & a & b & c \\ 0 & 1 & d & b-ad \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) = (\varepsilon^i)^a(\varepsilon^j)^b(\varepsilon^k)^c(\varepsilon^l)^d$$

for all $a,b,c,d \in \mathbb{F}$.

The supercharacter table associated with the central Schur ring $\mathcal{S}^\circ_\sigma$ over the involutive adjoint group $U_{2m}(\mathbb{F})^\circ_\sigma$, where $\varepsilon = e^{\frac{2\pi i}{3}} \in \mathbb{C}$, is

	C_0	C_1	C_2	B_1	B_2	$AD_{(0,1)}$	$AD_{(0,2)}$	$AD_{(1,0)}$	$AD_{(1,1)}$	$AD_{(1,2)}$	$AD_{(2,0)}$	$AD_{(2,1)}$	$AD_{(2,2)}$
$\chi_{1,1,1,1}$	1	1	1	1	1	1	1	1	1	1	1	1	1
$\chi_{\varepsilon,1,1,1}$	1	1	1	1	1	1	1	ε	ε	ε	ε^2	ε^2	ε^2
$\chi_{\varepsilon^2,1,1,1}$	1	1	1	1	1	1	1	ε^2	ε^2	ε^2	ε	ε	ε
$\chi_{1,1,1,\varepsilon}$	1	1	1	1	1	ε	ε^2	1	ε	ε^2	1	ε	ε^2
$\chi_{\varepsilon,1,1,\varepsilon}$	1	1	1	1	1	ε	ε^2	ε	ε^2	1	ε^2	1	ε
$\chi_{\varepsilon^2,1,1,\varepsilon}$	1	1	1	1	1	ε	ε^2	ε^2	1	ε	ε	ε^2	1
$\chi_{1,1,1,\varepsilon^2}$	1	1	1	1	1	ε^2	ε	1	ε^2	ε	1	ε^2	ε
$\chi_{\varepsilon,1,1,\varepsilon^2}$	1	1	1	1	1	ε^2	ε	ε	1	ε^2	ε^2	ε	1
$\chi_{\varepsilon^2,1,1,\varepsilon^2}$	1	1	1	1	1	ε^2	ε	ε^2	ε	1	ε	1	ε^2
$\sum_{i,l\in\mathbb{F}} \chi_{\varepsilon^i,\varepsilon,1,\varepsilon^l}$	9	9	9	9ε	$9\varepsilon^2$	0	0	0	0	0	0	0	0
$\sum_{i,l\in\mathbb{F}} \chi_{\varepsilon^i,\varepsilon^2,1,\varepsilon^l}$	9	9	9	$9\varepsilon^2$	9ε	0	0	0	0	0	0	0	0
$\sum_{i,j,l\in\mathbb{F}} \chi_{\varepsilon^i,\varepsilon^j,\varepsilon,\varepsilon^l}$	27	27ε	$27\varepsilon^2$	0	0	0	0	0	0	0	0	0	0
$\sum_{i,j,l\in\mathbb{F}} \chi_{\varepsilon^i,\varepsilon^j,\varepsilon^2,\varepsilon^l}$	27	$27\varepsilon^2$	27ε	0	0	0	0	0	0	0	0	0	0

Chapter 6

Topics for future work

Since 2008, when Diaconis and Isaacs formalized the definition of a supercaracter theory for finite groups in [19], more than a hundred papers came out about this topic. There are several paths that we can choose to continue our study. In this chapter we present some of our ideas to continue our work about supercharacters.

6.1 The infinite case

Recall that in definition 1.1 we did not require G to to be a finite group. In this section consider G an arbitrary group, not necessarily finite.

Definition 6.1.1. Let G be an arbitrary group. A partition $\mathcal{K} \in \mathrm{Part}(G)$ is said to be **a partition of finite support** if

$$|K| < \infty$$

for all $K \in \mathcal{K}$.

Bastian, Brewer and Misseldine, in [13], defined Schur rings over arbitrary groups. Following their work, we present a reformulated definition of a Schur partition.

Definition 6.1.2. A partition $\mathcal{K} \in \mathrm{Part}(G)$ is a **Schur partition of** G if:

1. $\mathcal{K}$ is a partition of finite support;
2. $\{1\} \in \mathcal{K}$;
3. $K^{-1} \in \mathcal{K}$ for all $K \in \mathcal{K}$;
4. $\mathrm{span}_{\mathbb{C}}\{\delta_K : K \in \mathcal{K}\}$ is a subalgebra of $\mathbb{C}[G]$.

When $\mathcal{K}$ is a Schur partition of G, $\mathcal{S}_{\mathcal{K}} := \mathrm{span}_{\mathbb{C}}\{\delta_K : K \in \mathcal{K}\}$ is called the **Schur ring over** G **afforded by** $\mathcal{K}$.

With this definition most of the results we presented in chapter 2 hold, however, taking into account the work of André, Gomes and Lochon in [10], the relation between Schur rings and supercharacter theories breaks.

6.1.1 The infinite unitriangular group

In this section we summarize the construction of a supercharacter theory for the infinite unitriangular group that we can find in [10].

Definition 6.1.3. Let G be an arbitrary group (possibly infinite). A complex-value function $\alpha : G \to \mathbb{C}$ is said to be **positive definite normalized** if:

1. $\alpha(1) = 1$;
2. $\alpha^\star = \alpha$;
3. The matrix $H = (H_{ij})$ is a nonnegative Hermitian matrix, where

$$H_{ij} = \alpha(g_i g_j^{-1}),$$

for all finite subsets $\{g_1, \ldots, g_n\} \subseteq G$ and all $1 \leq i, j \leq n$.

Note that in the previous definition we do not require that the function α has to be an element of the group algebra $\mathbb{C}[G]$, as in definition 1.1.

Let p be a prime and $\mathbb{F}$ a field with characteristic p. Recall the definition 4.2.7 of unitriangular group of order n over $\mathbb{F}$. For simplicity, we shall write $U_n = U_n(\mathbb{F})$ and $\mathfrak{u}_n(\mathbb{F}) = \mathfrak{u}_n$.

For every positive integer n, we can naturally identity U_n with the subgroup of U_{n+1}

$$\left\{ \left[\begin{array}{ccc|c} & & & 0 \\ & \mathbf{g} & & \vdots \\ & & & 0 \\ \hline 0 & \ldots & 0 & 1 \end{array}\right] : g \in U_n \right\} \subseteq U_{n+1}.$$

Definition 6.1.4. We call the inductive limit of the infinite chain of unitriangular groups

$$U_1 \subset U_2 \subset U_3 \subset \ldots$$

the **infinite unitriangular group** and denote by U_∞.

Analogously, $\mathfrak{u}_n$ can be identify as a subring of $\mathfrak{u}_{n+1}$. Note that, with this identification, a superclass of U_n is always contained in a superclass of U_{n+1}. Consider

$$\mathcal{K}_\infty := \bigcup_{n \geq 1} \{1 + U_n x U_n : x \in \mathfrak{u}_n\},$$

whose elements we will call the superclasses of U_∞. Consider also $\mathcal{X}_\infty$ the set of all $\mathcal{K}_\infty$-superclass functions and positive defined normalized on U_∞ whose elements we will call supercharacters of U_∞.

The pair $(\mathcal{K}_\infty, \mathcal{X}_\infty)$ is what we can naturally call the **standard supercharacter theory for** U_∞. Note that each superclass of U_∞ is an infinite set and the supercharacters are not necessarily elements of $\mathbb{C}[U_\infty]$. In [10], it is observed that there is no bijection between superclasses and supercharacters, the supercharacters being in a higher number than the superclasses. Moreover, following the well-known parametrization of the superclasses of U_n by coloured set partitions of a set with n elements, in [10] it is shown how superclasses of U_∞ can be parametrized by colored set partitions of the set of natural numbers, $\mathbb{N}$.

6.1.2 Set partitions of $[n]$

In this section we present the definition of set partition and its representation with arc diagrams.

For all natural n we define

$$[n] := \{1, \ldots, n\} \quad \text{and} \quad [[n]] := \{(i, j) : 1 \leq i < j \leq n\}.$$

Definition 6.1.5. A **set partition** π **of** $[n]$ is a family of nonempty sets, called **blocks**, whose disjoint union is $[n]$. We write

$$\pi = B_1 \ldots B_l \vdash [n]$$

where $B_1, \ldots, B_l$ are the blocks of π ordered by increasing value of the smallest element in the block.

Part($[n]$) will represent the **set of all set partitions of** $[n]$.

Set partitions have a natural representation using arc diagrams which sketch the combinatorial object as a labeled graph and are useful to detect certain patterns.

Definition 6.1.6. In the **arc diagram representation of a set partition** $\pi \vdash [n]$, a row of dots is labeled from 1 to n and the pair $(i, j) \in [[n]]$ is said to be an **arc of** π if i and j occur in the same block B of π and there is no $k \in B$ with $i < k < j$ i.e., a block of π, $B = \{a_1, \ldots, a_i\}$, ordered $a_1 < \ldots < a_i$, is represented by the set of arcs

$$\{(a_1, a_2), \ldots, (a_{j-1}, a_j)\}$$

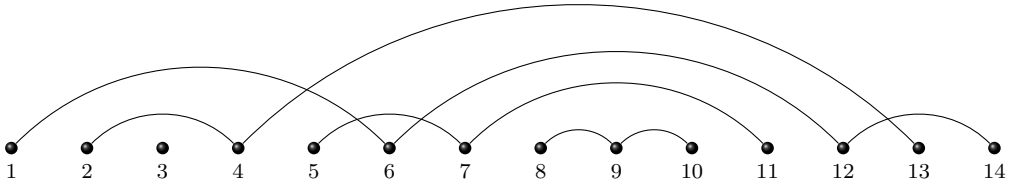
which are always drawn above the row of dots.

We will denote by $\mathcal{D}(\pi)$ the **set of all arcs of the set partition** π and a subset $D \subseteq [[n]]$ is said to be a **basic subset** if there is a set partition $\pi \vdash [n]$ such that $D = \mathcal{D}(\pi)$.

Example 6.1.7. Consider the set partition

$$\pi = \{1, 6, 12, 14\}\{2, 4, 13\}\{3\}\{5, 7, 11\}\{8, 9, 10\} \vdash [14].$$

The arc diagram representation of π is



and $\mathcal{D}(\pi) = \{(1,6),(6,12),(12,14),(2,4),(4,13),(5,7),(7,11),(8,9),(9,10)\}$.

The connected components of the arc diagram representation of a set partition are precisely the set partition blocks. Observe also that the set partitions of $[n]$ are in one-to-one correspondence with the basic subsets of $[[n]]$.

For every $(i,j) \in [[n]]$ we detone by $e_{i,j} \in \mathfrak{u}_n$ the matrix with 1 in the $(i,j)-$th entry and 0's elsewhere, i.e.,

$$e_{i,j} := \begin{pmatrix} & & & j & & & \\ & & & \uparrow & & & \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \rightarrow i$$

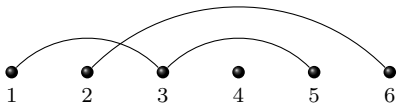
If $\pi \vdash [n]$ consider the matrix

$$e_\pi = \sum_{(i,j)\in\mathcal{D}(\pi)} e_{i,j}.$$

Example 6.1.8. Consider the set partition

$$\pi = \{1,3,5\}\{2,6\}\{4\} \vdash [6].$$

The arc diagram representation of π is



and $\mathcal{D}(\pi) = \{(1,3),(2,6),(3,5)\}$, so

$$e_\pi = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathfrak{u}_6(\mathbb{F}_2).$$

Consider the standard $\mathbb{F}$-basis $\{e_{i,j} : (i,j) \in [[n]]\}$ of the vector space $\mathfrak{u}_n$. For every $x \in \mathfrak{u}_n$ we have

$$x = \sum_{(i,j)\in[[n]]} x_{i,j}e_{i,j}.$$

Observe that for all $g, h \in U_n$,

$$ge_{i,j}h = \begin{pmatrix} & & & \overset{j}{\uparrow} & & & \\ 0 & \dots & 0 & g_{1,j} & g_{1,j}h_{i,j+1} & \dots & g_{1,j}h_{i,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & g_{i-1,j} & g_{i-1,j}h_{i,j+1} & \dots & g_{i-1,j}h_{i,n} \\ 0 & \dots & 0 & 1 & h_{i,j+1} & \dots & h_{i,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \rightarrow i$$

and so, if $x = \sum_{(i,j)\in[[n]]} x_{i,j}e_{i,j}$ the correspondent superclass of U_n is of the form

$$\left\{ I_n + \sum_{(i,j)\in[[n]]} x_{i,j}(ge_{i,j}h) : g, h \in U_n \right\}$$

which means that we can parametrise the superclasses of U_n using the matrices

$$t^{-1}e_\pi t$$

where $\pi \vdash [n]$ and t is an invertible diagonal matrix of order n, which we can see as a coloring of the set partitions of $[n]$.

We can see a set partition of $[n]$ as a set partition of $[n+1]$ by adding one block consisting only in the element $n+1$. This way we can parametrize the superclasses of U_∞ using the set

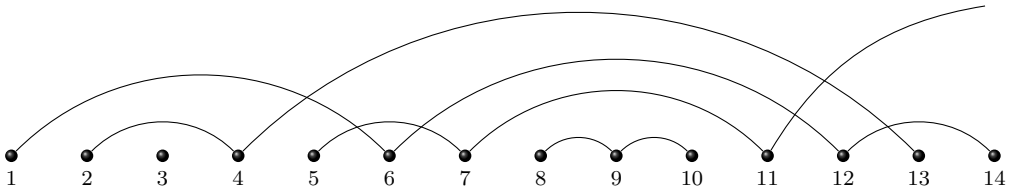
$$\mathrm{Part}(\mathbb{N}) := \bigcup_{n\in\mathbb{N}} \mathrm{Part}([n])$$

associated with a coloring for each set partition.

Recently, Burrill, Elizalde, Mishna and Yen in [16] generalized arc diagrams by permitting **open arcs**.

Definition 6.1.9. From the arc diagram representation of a set partition π we can construct an **open arc diagram** by choosing at least one block B of π to add an arc with left endpoint at the highest number of B and with no right endpoint. The open arc diagram associated to π that has just one open arc and is associated to the i-th block of π will be denoted by π_i.

Example 6.1.10. The set partition of example 6.1.7 has 5 blocks so there are $2^5 - 1$ open diagrams associated to π. The open arc diagram π_4 is the following:



With this attempt at summarizing some of the work that has been done with supercharacters and Schur over an arbitrary group G, we leave here some open questions:

- Can we relate Schur rings of arbitrary groups, as defined in [13] with supercharacter theories for those groups?
- What would be a Schur ring associated with the standard supercharater theory of the infinite unitriangular group that we find in [10]?
- Can we parametrize all supercharacteres of U_∞ using the definition of open arc diagrams?

6.2 t-periodic antiautomorphisms

In chapter 5 we constructed a supercharacter theory for the σ-subgroup of fixed points in the adjoint group of a finite radical ring endowed with an anti-involution σ. Can we generalize this construction when the radical ring R is endowed with a more generic anti-automorphism?

Let R be a finite radical with characteristic p, coprime with t.

Definition 6.2.1. The map

$$\sigma : R \longrightarrow R$$

is a **t-periodic antiautomorphism on** R if it satisfies the following conditions:

1. $\sigma(x+y) = \sigma(x) + \sigma(y)$;
2. $\sigma(xy) = \sigma(y)\sigma(x)$;
3. t is the smaller positive even integer such that $\sigma^t(x) = x$;

for all $x, y \in R$.

Note that an anti-involution defined on R is a particular case of a t-periodic antiautomorphism, where $t = 2$.

6.3 Association schemes

The theory of association schemes arose in statistics (see for example [12]) and its close relation with Schur rings is well known. Naturally there is also a close relation with supercharacter theories for finite groups.

Definition 6.3.1. Let G be a finite group and $\mathcal{R} \in \text{Part}(G \times G)$. We say the pair $(G, \mathcal{R})$ is a **commutative association scheme** over G if:

1. $\Delta(G) = \{(g, g) : g \in G\} \in \mathcal{R}$;
2. If $R \in \mathcal{R}$ then $R' = \{(h, g) : (g, h) \in R\} \in \mathcal{R}$;
3. If $R, S, T \in \mathcal{R}$ and $(g, h) \in T$

$$\mathfrak{c}_{R,S}^{T} := |\{k \in G : (g, k) \in R \text{ and } (k, h) \in S\}| = \mathfrak{c}_{S,R}^{T}.$$

doesn't depend on which element of T was chosen;

Proposition 6.3.2. *If $(\mathcal{K}, \mathcal{X})$ is a supercharacter theory for G then $(G, \mathcal{R}_{\mathcal{S}})$ is an association scheme over G with*

$$\mathcal{R}_{\mathcal{S}} = \{R_{\mathcal{S}}(K) : K \in \mathcal{K}\} \quad \textit{and} \quad R_{\mathcal{S}}(K) = \{(g, gk) : k \in K \textit{ and } g \in G\}$$

for all $K \in \mathcal{K}$.

We also finish this section with some open questions related to this relation between association schemes and supercharacters:

- What can we learn more about supercharacters when studying deeply the relation with association schemes?
- Can we describe the structure constants in terms of supercharacters and with this associate a probability to each supercharacter?

6.4 Wreath product

Since a characterization of all supercharacter theories of cyclic groups has already been made, direct products of cyclic groups and wreath products are families of groups with potentially good combinatorics surrounding supercharacter theories. Note that in the example of section 4.2.1, a direct product of cyclic groups arose when we were studying the adjoint group of the augmentation ideal of the group algebra of a cyclic group. Moreover, the Sylow p-subgroups of the symmetric group, whose representation theory still has many open questions, are wreath products of cyclic groups. The idea of constructing a supercharacter theory for wreath products inspired by what is done with adjoint groups is something that might reveal itself fruitful.

Let F and G be two finite groups. Consider that G acts transitively on a set X and denote by F^X the set of all maps $f : X \to F$, which is a group under pointwise multiplication. The group G acts on F^X in a natural way, by setting

$$(\pi f)(x) = f(\pi^{-1}x)$$

for all $\pi \in G$, for all $f \in F$ and for all $x \in X$.

Definition 6.4.1. The **wreath product** of F by G (on X) consists of the set

$$F \wr G := \{(f;\pi) | f \in F^X, \pi \in G\}$$

together with the multiplication law

$$(f;\pi)(f';\pi') := (f(\pi f');\pi\pi').$$

Let p be a prime and let $\mathbb{F}_p$ be the field with p elements.

The **wreath product of C_p by C_p (on $\{1,\dots,p\}$)**, denoted by $C_p \wr C_p$, is the set of all $\mathbb{F}_p$-valued functions defined on $\{1,\dots,p+1\}$ endowed with the product defined by

$$(g \wr h)(n) := \begin{cases} g(n) + h(n - g(p+1)), & n \neq p+1 \\ g(n) + h(n), & n = p+1. \end{cases}$$

for all $g, h \in C_p \wr C_p$.

We finish with some more open questions:

- When $p = 2$, $C_2 \wr C_2$ is the dihedral group, that is, it is an adjoint group of a finite radical ring. Can we always see the wreath product of cyclic groups as adjoint groups of finite radical rings?
- On the other and, is the wreath product of cyclic groups an involutive adjoint group?

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