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FACULDADE DE CIÊNCIAS
DEPARTAMENTO DE MATEMÁTICA

# An Introduction to Graphs on Groups

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**Mestrado em Matemática**

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2024

# Acknowledgements

After the process of creating this thesis, several names need to be mention in order to express my gratitude.

First of all, I want to thank the first person that introduced me to mathematics: my brother Luis Andrade; and the person who made me love them: Ana Paula Jardim, who not only taught me math, but about life and kindness. Next, I want to thank the people at Universidade da Madeira that helped me consolidate my love for mathematics: Prof. Maurício Reis, Prof. Maribel Gordon, Prof. Elias Rodrigues, and, of course, my co-advisor: Prof. Jorge Nélio Marques Ferreira, without whom this essay would not have been possible. Prof. Jorge Nélio, with his constant guidance throughout the years, taught me invaluable lessons on mathematics, the writing of a scientific document and the philosophy of math and life. Words can not fully express my gratitude.

Second, I would like to thank Faculdade de Ciências da Universidade de Lisboa, for everything they did and still do for me throughout these three years; for everything they taught me and for welcoming me. A especial thanks to investigation center CMAFcIO for giving me the opportunity to have a BI and do research in this area. I would like to thank my seminar advisor, Marija Dodig, who allowed me to give my firsts steps in creating a mathematical document and who guided me to see the simplest case when doing proofs. Especially, I am grateful for Prof. Carlota Rebelo Gonçalves and Prof. Maria Manuel Torres, whose kindness made a difference in my path in Ciências. Furthermore, I am incredibly grateful for my advisor, Prof. Maria Manuel Torres, once again. Since my first year in Ciências, she taught me valuable lessons in mathematics and helped me with kindness and dedication throughout these three year. Without her guidance and lessons, this essay would not have been possible.

Third, I must thank my family and friends. My mom, dad and aunt Rita. My friends Melissa and Caroline, that helped me in very different ways, with a different view of life. And, of course, the people that made Lisbon more beautiful: Diana Ferreira, Catarina Pacheco and my brother Jorge Diaz, who gave me shelter, laughs and a home.

A especial gratitude to Diana Ferreira, once again, as she taught me so much throughout the masters degree and is always willing to help me clear out my doubts.

Finally, I want to show my appreciation to those who helped me indirectly in this process: those who helped me live. Carla Vale Lucas, Luciana Ferreira and Dra. Carla Spínola. Without them, this essay would not have been possible. Moreover, I would like to show my gratitude to Professor Sónia Martins, who provided me with joy and life when it was difficult for me to find it, and who inspired me in more ways than I can count.

As an African said reads: ”if you want to go fast, go alone. If you wish to get far, go with friends”.

# Abstract

The present dissertation, titled **An Introduction to Graphs on Groups**, aims to study connections between two fundamental areas of mathematics: groups and graphs.

In the first chapter we introduce the basic definitions and results that will be needed throughout this essay. It has four sections. The first one is dedicated to Graph Theory, and it is mainly composed by basic definitions and examples. The rest of the chapter focuses on Group Theory. In this part of the essay, we introduce the different classes of groups with which we will be working on.

Chapter 2 is divided into four sections, one for each graph on group we will be working on: **the commuting graph**, **the power graph**, **the enhanced power graph** and **the non-generating graph**. Here we study for which groups each of these graphs is complete or null, we deduce some results regarding maximal cliques and present a few results regarding diameter and connectivity. Several examples are given.

The next chapter has as objective to study some differences (in the sense of subtraction) between the graphs presented in **Chapter 2**. Hence, we begin **Chapter 3** with the definition of **graph hierarchy** and we show that it is well defined. Next, we study when two graphs in said hierarchy are equal. We then compute the difference between pairs of graphs in the hierarchy. In particular, we study the difference between the enhanced power graph and the power graph and the difference between the non-generating graph and the commuting graph. For each of these new classes of graphs, we determine their isolated vertices, study connectivity, diameter, and give several examples.

We finish the main body of this dissertation with **Chapter 4**. Here we define and study two differences of graphs that have not yet, to the best of our knowledge, been considered.

KEYWORDS: graphs on groups, graph hierarchy, difference of graphs on groups, connectivity, diameter

# Resumo

Esta dissertação, intitulada *An introduction to graphs on groups* (que se traduz para *Uma introdução à teoria de grafos em grupos*), tem como objetivo estudar relações entre duas áreas fundamentais da matemática: Teoria dos grafos e Teoria dos grupos. Para isso, conta com quatro capítulos e dois anexos.

Em 1878, Arthur Cayley, no seu artigo “The theory of groups: graphical representation” [18], propõe uma representação das relações entre os elementos de um grupo através de um grafo. Anos depois, em 1955, os grafos foram usados na Teoria dos Grupos sob outra perspetiva. O ***commuting graph*** surgiu de forma um pouco camuflada. Com o objetivo de estudar propriedades de grupos, Brauer e Fowler definiram em [9] o conceito de distância no *proper commuting graph*, sem usar a palavra grafo. Com resultados fundamentais para a prova da Classificação dos Grupos Finitos Simples, os autores inadvertidamente criaram uma nova ponte entre grafos e grupos ao considerar um grafo que era preservado pelo grupo de automorfismo do grupo que o gerou. Esta ponte foi designada **grafos definidos em grupos** ou **grafos em grupos**.

Grafos em grupos é uma área relativamente recente de Teoria dos Grafos e Teoria dos Grupos. Embora o *commuting graph* tenha sido definido há quase setenta anos, o segundo grafo deste tipo, tanto quanto sabemos, foi considerado em 1975, quando o ***non-commuting graph*** apareceu. Este grafo é definido como o complemento do *commuting graph*. Nos anos 2000, o ***power graph*** foi introduzido por Kelatev et. al em [35] com o objetivo de estudar propriedades combinatórias de grupos. Em particular, os autores decreveram a estutura do *power graph* para qualquer grupo abeliano finito.

O *power graph* consolidou as potencialidades desta área da Matemática, potencialidades que o *commuting graph* já tinha começado a sugerir. Desde então, os grafos em grupos têm sido estudados abundantemente por matemáticos como: Peter J. Cameron, Paul S. Freedman, Aalipour G., Bera S., Kelarev, entre outros.

Um dos autores que mais tem contribuído para esta área é Peter Cameron, que considerou, juntamente com outros autores, o ***enhanced power graph*** com o objetivo de investigar semelhanças entre o *power graph* e o *commuting graph* de determinados grupos. É de realçar que este grafo fora definido em 2009 por Abdollahi e Hassanabadi como o complemento do *non-cyclic graph*, o seu objeto de estudo. No entanto, este autores removeram o *cyclicizer* do conjunto dos vértices, algo que Cameron e coautores não fizeram.

Em 2020, Cameron considerou o ***non-generating graph*** e o ***deep-commuting graph***, sendo que este último não vai ser abordado nesta dissertação. Para além destes grafos, o autor considerou o *commuting graph*, o *power graph*, o *enhanced power graph*, o grafo nulo e o grafo completo para descrever o conceito que designou por **hierarquia de grafos em grupos**.

Assim como o *enhanced power graph* surgiu para estudar semelhanças entre dois grafos de um mesmo grupo, a hierarquia surgiu como uma cadeia de *spanning subgraphs* para estudar a diferença, no

sentido de subtração, entre dois grafos que a ela pertencem. Este estudo deu origem a dois novos grafos: o ***non-commuting non-generating graph***, que resulta da diferença entre o *non-generating graph* e o *commuting graph*; e o ***enhanced non-power graph***, resultante da diferença entre o *enhanced power graph* e o *power graph*.

Este trabalho tem como objetivo introduzir a hierarquia de grafos em grupos definida em [10] e estudar os grafos que resultam de considerar a diferença entre dois grafos que a ela pertencem. Para isso, começamos esta dissertação com um capítulo que visa introduzir os conceitos e resultados de Teoria de Grupos e Teoria dos Grafos que serão necessários ao longo do trabalho. É no **capítulo 1** que são definidas as classes de grupos que mais vão ser trabalhadas na dissertação. Em particular, são introduzidos os $p$-grupos e EPPO grupos, nos quais a maior parte dos resultados são aplicados. É ainda apresentada uma série de propriedades dos grupos cíclicos, com especial ênfase nos seus subgrupos. Por outro lado, é feita uma construção para os grupos nilpotentes seguindo [42] e são determinadas algumas classes de grupos nilpotentes.

No segundo capítulo, são explorados quatro tipos de grafos em grupos: o *commuting graph*, o *power graph*, o *enhanced power graph* e o *non-generating graph*, classificando os grupos para os quais cada um destes grafos é completo ou nulo, abordando também questões de conectividade. São ainda demonstradas algumas propriedades dos cliques maximais e, quando conexos, apresentamos limites superiores para o diâmetro de alguns destes grafos em certos grupos, em particular nos $p$-grupos, grupos simétricos e grupos alternados.

Nos últimos anos, a diferença (no sentido da subtração) entre grafos na hierarquia tem tido particular atenção, existindo resultados foram publicados em 2024. Por isso, o capítulo 3 é dedicado a este estudo. Até ao momento, tanto quanto sabemos, duas diferenças não triviais têm sido consideradas. Uma destas diferenças é a que resulta entre o *non-generating graph* e o *commuting graph*, que dá origem ao ***non-commuting non-generating graph***. Este grafo foi estudado por Peter J. Cameron para grupos nilpotentes em [14] e por Paul S. Freedman para grupos finitos simples em [25] e para grupos finitos não simples em [26]. Uma outra diferença é aquela entre o *enhanced power graph* e o *power graph*, que dá origem ao ***enhanced non-power graph***. Este grafo foi definido e estudado em [8]. Nesta parte do trabalho vamos explorar estas duas diferenças. Para isso, começamos por determinar quais são os vértices isolados do grafo resultante. Isto é necessário para estudar a conectividade e o diâmetro do grafo de uma forma não trivial. Para além disso, classificamos para que grupos cada um destes grafos é, se possível, completo. É de salientar que são também classificados os grupos para os quais alguns dos grafos na hierarquia são iguais. Ao determinarmos estes grupos, determinamos também aqueles para os quais o *enhanced non-power graph* e o *non-commuting non-generating graph* são nulos. Nomeadamente, são classificados os grupos para os quais o *power graph* e o *enhanced power graph* são iguais; provamos que o *commuting graph* e o *enhanced power graph* de um grupo $G$ são iguais se e só se o grupo $\mathbb{Z}_p \times \mathbb{Z}_p$ não é subgrupo de $G$. Determinamos ainda os grupos para os quais o *non-generating graph* e o *commuting graph* são iguais, sendo estes os grupos abelianos minimais. Finalmente, apresentamos uma nova, tanto quanto sabemos, igualdade ao determinarmos em que condições o *power graph* e o *non-generating graph* são iguais. Concluímos este capítulo com alguns resultados sobre a connectividade do *enhanced non-power graph* e do *non-commuting non-generating graph* de grupos específicos, nomeadamente: grupos simétricos, alternados e diedrais.

Finalizamos o corpo desta dissertação com um quarto capítulo que consta de alguns resultados sobre

duas outras diferenças de grafos em grupos que, tanto quanto sabemos, ainda não foram consideradas. Em particular, estudamos a diferença entre o *non-generating graph* e o *enhanced power graph*, e a diferença entre o *commuting graph* e o *power graph*. O grafo resultante da primera diferença será designado ***non-generating non-enhanced power graph***. Na secção dedicada a este grafo determinamos todas as classes de grupos para as quais é completo ou nulo. O grafo resultante da diferença entre o *commuting graph* e o *power graph* é designado ***commuting non-power graph***. Na secção dedicada ao seu estudo fornecemos alguns resultados para grupos diedrais, grupos simétricos e grupos alternados. Determinamos ainda em que condições é completo e algumas classes para as quais é não nulo.

Terminamos esta dissertação com dois anexos onde são expostos conceitos básicos e notações de Teoria dos Grupos. São também apresentadas duas tabelas com todos os grupos, abelianos e não abelianos, até certa ordem, que foram particularmente importantes para o estudo realizado no quarto capítulo. Nessas tabelas estão também indicados os seus subgrupos.

É de notar que, de momento, não existem traduções para português de certas terminologias usadas nesta disseração, como os nomes dos grafos definidos em grupos. Atualmente, o estudo das diferenças em grafos tem abrangido uma outra dimensão ao considerar os *super graphs*. Este tópico vai para além dos propósitos desta dissertação.

**PALAVRAS-CHAVE**: grafos em grupos, hierarquia de grafos, diferença na hierarquia, conectividade, diâmetro

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$\boldsymbol{|G|}$ order of a group $G$.

$\boldsymbol{H \leqslant G}$ the group $H$ is a subgroup of $G$.

$\langle X \rangle$ group generated by the set $X$.

$\boldsymbol{G/H}$ quotient of $G$ by $H$.

$\boldsymbol{G \times H}$ direct product of groups $H$ and $G$.

$\boldsymbol{o(g)}$ order of the element $g$ of a given group $G$.

$\boldsymbol{|G:H|}$ number of left cosets of $H$ in $G$.

$\mathrm{gcd}(\mathrm{x},\mathrm{y})$ greatest common divisor of $x$ and $y$.

$\boldsymbol{G \cong H}$ the groups $G$ and $H$ are isomorphic.

$\boldsymbol{x \sim y}$ the vertices $x$ and $y$ are adjacent.

$\boldsymbol{x \nsim y}$ the vertices $x$ and $y$ are not adjacent.

$\boldsymbol{d(x,y)}$ distance between two vertices of $\Gamma$.

$\boldsymbol{diam}(\Gamma)$ diameter of $\Gamma$.

$\boldsymbol{\kappa}(\Gamma)$ conectivity number of $\Gamma$.

$\Xi(\Gamma)$ chromatic number of $\Gamma$.

$\Gamma_{p}(\boldsymbol{G})$ the power graph of a group $G$.

$\Gamma_{p}^{*}(\boldsymbol{G})$ the proper power graph of a group $G$.

$\Gamma_{ep}(\boldsymbol{G})$ the enhanced power graph of a group $G$.

$\Gamma_{ep}^{*}(\boldsymbol{G})$ the proper enhanced power graph of a group $G$.

$\Gamma_{c}(\boldsymbol{G})$ the commuting graph of a group $G$.

$\Gamma_{c}^{*}(\boldsymbol{G})$ the proper commuting graph of a group $G$.

$\Gamma_{ng}(\boldsymbol{G})$ the non-generating graph of a group $G$.

$\Gamma^*_{ng}(G)$ the proper non-generating graph of a group $G$.

$\Gamma^-_{ng}(G)$ the non-generating graph of a group $G$ after removing dominant vertices.

$\Gamma_d(G)$ the enhanced non-power graph of a group $G$.

$\Xi(G)$ the non-commuting non-generating graph of a group $G$.

$\Gamma_{cnp}(G)$ the commuting non-power graph of a group $G$.

$\Gamma^*_{cnp}(G)$ the proper commuting non-power graph of a group $G$.

$\Upsilon(G)$ the non-generating non-enhanced power graph of a group $G$.

**$GK$-graph of $G$** the Gruenberg-Kegel graph of a group $G$.

# Introduction

The present dissertation was made with the goal of obtaining a Master's degree in Mathematics from the Faculty of Sciences of the University of Lisbon (FCUL). Additionally, because of the author's interest in the relation between graphs and groups, the present work has as objectives to understand the basis of graphs defined on groups, to introduce the graph hierarchy and to compute some differences between two graphs in said hierarchy. Summarizing, this work is

An Introduction to Graphs on Groups

## What are Graphs on Groups?

In 1878, Arthur Cayley created a bridge between Graph Theory and Group Theory when he suggested to represent a group with a graph. With a fairly simple construction, these graphs have the elements of a group $G$ as vertex set and its edges reflect the structure of $G$ through its generators. However, **Cayley graphs** are not usually preserved by the automorphism group of $G$. Graphs on groups *must* satisfy this additional condition.

The **commuting graph** is the first graph of this type and was considered in 1955. With it, a new bridge in the connection between Graph Theory and Group Theory was built. It had a camouflaged introduction, as Brauer and Fowler (see [9]) defined the concept of distance in the proper commuting graph without ever using the word *graph*. This distance was used by the authors to show that only finitely many simple groups of even order have the centralizer of an involution isomorphic to a given group. This work is credited (by some) to be the first step into the classification of finite simple groups.

*Graphs on groups* are those with vertex set the elements of a group $G$ where adjacency is defined based on a group property in such way that it is preserved by the automorphism group of $G$.

## From "then" to "now"

In the years that followed 1955, several other graphs on groups were defined. To the best of our knowledge, the second graph of this type to be considered was in 1975: the complement of the commuting graph.

Twenty five years later, A. V. Keralev and S. J. Quinn defined in [35], 45 years after the commuting graph, a new graph defined on groups. In their article, the authors used the **power graph** to study combinatorial properties of groups, much like Brauer and Fowler did with the proper commuting graph. In this article, they described as well the structure of the power graph for any finite abelian group.

As an in-between of the commuting graph and the power graph, it was defined in 2017 (see [1]) **the enhanced power graph**. This graph was considered by Aalipout et al. when they noticed that, for some groups, the commuting graph and the power graph were equal, which led them to seek, for the cases where they were not equal, an answer to the question: how different are they?

In the years that followed, several other graphs on groups were defined, like the **non-generating graph**, the **deep commuting graph** or the **nilpotent graph**.

This dissertation does not aim to be a survey of each of these graphs, as such work would be unattainable. Rather, we will focus on doing comparisons between some of them. For this, we will consider the graph hierarchy, introduced in 2021 by Peter J. Cameron (see [10]). This hierarchy is a chain of spanning subgraphs containing the null graph, the power graph, the enhanced power graph, the deep commuting graph, the non-generating graph (under certain conditions) and the complete graph. Both the non-generating graph and the deep commuting graph were first considered with the goal of defining the hierarchy. It's important to note that we will not be studying the deep commuting graph, as it goes beyond the scope of this essay.

With this hierarchy, we will determine for which groups two of these graphs are equal. From this study we will notice that for most groups equality does not hold. Saul Freedman and coauthors noticing this, studied the difference (in the sense of subtraction) between the non-generating graph and the commuting graph. This defined a new graph on groups for which some invariants like connectivity and diameter were studied. More recently, Aalipour et al. (see [8]) considered a new difference, doing a similar study as it was done for other graphs on groups.

Currently, the study of these differences is one of the main focuses of graphs defined on group, with results as recent as May of 2024 being published. As of when this dissertation was made, and to the best of our knowledge, only two non-trivial differences between graphs in the hierarchy have been considered. It is our goal to give a few of the most recent results and to carry this study further.

## The Dissertation

In order to achieve the goals presented in the previous paragraphs, this essay counts with four chapters and two appendix.

**Chapter 1** is structured in four sections. As a whole, it has the goal of providing the necessary background to understand the rest of this essay. Throughout the chapter, most proofs were omitted, but examples were included in order to clarify each definition and result.

The second Chapter is structured in four different sections: *the commuting graph*, *the power graph*, *the enhanced power graph* and *the non-generating graph*. In each of these sections we introduce each of these four graphs defined on groups and show some results regarding their properties. We mainly classify for which groups they are complete or null; we study connectivity and diameter for some particular classes of groups; and we study maximal cliques. The general goal in each section is to use the structure of the graph and its invariants to identify different properties on the group that generated it.

**Chapter 3** is divided into two sections. In **Section 3.1**, we introduce the graph hierarchy and show that it is well defined. Next, we determine for which group classes are two graphs in the hierarchy equal. In **Section 3.2**, we study two new graphs. These graphs result from the difference between the enhanced power graph and the power graph, as well as the difference between the non-generating graph and the commuting graph. The same study as in **Chapter 2** is made for these graphs, with a particular focus on diameter and connectivity.

We conclude the primary part of this dissertation with **Chapter 4**. Here, we explore two other differences between graphs in the hierarchy. Particularly, we consider the difference between the non-generating graph and the enhanced power graph, as well as the difference between the commuting graph and the power graph. To the best of our knowledge, these differences have not yet been considered. This chapter is divided into two sections, one for each difference. In **Section 4.1**, we show when the commuting non-power graph of dihedral groups is null and, when not null, we compute its chromatic number. We also determine other groups for which the difference graph is not null. In **Section 4.2**, we classify the groups for which the non-generating non-enhanced power graph is complete or null.

Finally, this work counts with two **Appendix** where basic definitions and notations regarding Group Theory are given and two tables with all groups (abelian and non-abelian) up to a certain order are given. In these tables, the proper subgroups of each group are also indicated.

# Chapter 1

# Preliminary Notions

In this chapter we provide the foundation for the rest of this essay. With this aim, we divided the chapter into four different sections. The first section is dedicated to Graph Theory. Here, we introduce the concepts and results that will be necessary throughout the thesis. Proofs are omitted and several examples are given. We based this section mainly on [4, 46].

In the second section of **Chapter 1**, we mainly focus on five classes of groups: cyclic groups, symmetric and alternating groups, dihedral groups and quaternion groups, presenting important and elementary results regarding these classes of groups. Next, we define the prime power order groups, namely the $p$-groups and the EPPO groups, presenting some fundamental results on this subject.

Finally, the last section is dedicated to nilpotent groups, where we do a simple construction and introduce some classes of groups which are also nilpotent. For the most part, we use as references [10, 24, 31, 34, 42].

## 1.1 Graph Theory

In this section we give some important definitions and results. Proofs are omitted but several examples are given.

**Definition 1.1.1**
A **graph** is an ordered pair $\Gamma = (V, E)$, where $V$ is a non-empty set and $E \subseteq V \times V$. The elements of $V$ are called **vertices** and the elements of $E$ are called **edges**. The set of all vertices of $\Gamma$ is denoted by $V(\Gamma)$ while the set of all the edges of $\Gamma$ is denoted by $E(\Gamma)$.

**Example 1.1.2**
*Let* $\Gamma = (V, E)$ *be a graph with vertex set*

$$V(\Gamma) = \{e, x, x^2, x^3, x^4, x^5\}$$

*and edge set*

$$E(\Gamma) = \{(e,x); (e,x^2); (e,x^3); (e,x^4); (e,x^5); (x,x^2); (x,x^3); (x,x^4); (x^2,x^4); (x^2,x^5); (x^4,x^2)\}$$

All throughout literature, graphs are represented using diagrams. In order to do this, we need to state some assumptions about the edges. On one hand, if we consider that $x$ being adjacent to $y$ is different from $y$ being adjacent to $x$, then the graph is **directed** and we represent its edges by arrows. As an example, consider the graph defined above. A possible diagram for $\Gamma$ is:

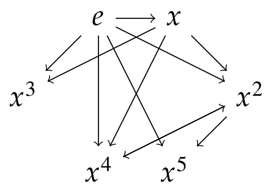

Figure 1.1: $\Gamma$ as a directed graph

Edges in a directed graph are usually called **arcs**. If a graph has two arcs of the form $(x,y)$ and $(y,x)$, we say the graph has a **bidirected arc**. In the diagram above, we have that $(x^2,x^4)$ is a bidirected arc.

On the other hand, if we consider that $x$ being adjacent to $y$ is the same as $y$ being adjacent to $x$, then the resulting graph is **undirected**. In this case, the graph $\Gamma$ defined in **Example 1.1.2** would be represented by the following diagram:

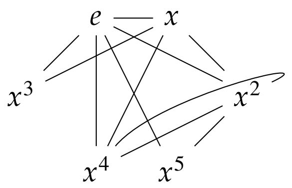

Figure 1.2: Example of a undirected graph

Throughout this essay, we have the following restrictions in $\Gamma(V,E)$:

i) Take $x \in V(\Gamma)$. Then, $(x,x) \notin E(\Gamma)$. This type of edges are called **loops**.

ii) If there is an edge $(x,y)$ in $\Gamma$, then $\Gamma$ does not have another edge $(x,y)$.

**Remark 1.1.3**
Undirected graphs that satisfy both of the restrictions above are called **simple graphs**.

The undirected graph $\Gamma$ of **Example 1.1.2** with these restrictions is the following:

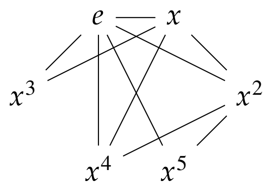

Figure 1.3: The graph $\Gamma$ as a simple undirected graph

Considering the vertices of a graph, we have the following definitions.

**Definition 1.1.4**
Let $\Gamma$ be a graph and $k \in \mathbb{N}$. We say that $\Gamma$ has order $k$ if the vertex set of $\Gamma$ has $k$ elements. The order of $\Gamma$ is represented by $|V(\Gamma)|$.

**Example 1.1.5** (**Null graph and complete graph**)
*Let* $\Gamma_1 = (V_1, E_1)$ *and* $\Gamma_2 = (V_2, E_2)$ *be graphs such that:*

*i)* $V(\Gamma_1) = \{1,2,3,4,5\}$ *and* $E(\Gamma_1) = \emptyset$. *Then,* $|V(\Gamma_1)| = 5$. *We also see that* $E(\Gamma_1) = \emptyset$. *When this happens, we say that* $\Gamma_1$ *is a* ***null*** *graph.*

*ii)* $V(\Gamma_2) = \{x,y,z\}$ *and* $E(\Gamma_2) = \{(x,y);(x,z);(y,z)\}$. *In this case, the order of* $\Gamma_2$ *is* 3. *Notice that* $E(\Gamma_2) = V(\Gamma_2) \times V(\Gamma_2)$. *In this case* $\Gamma_2$ *is said to be a* ***complete*** *graph on* $|V(\Gamma_2)|$ *vertices and it's denoted by* $K_{|V(\Gamma_2)|}$.

**Definition 1.1.6**
Let $\Gamma = (V, E)$ be a graph and $x, y \in V(\Gamma)$. We say that $x$ is **adjacent** to $y$, denoted by $x \sim y$, if $(x,y) \in E(\Gamma)$. Otherwise, we say they are not adjacent and we write $x \nsim y$.

**Example 1.1.7**
*In the following representation of a graph* $\Gamma$, *we have that* $v_1 \sim v_2$. *However,* $v_1 \nsim v_3$.

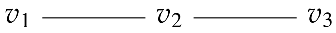

Figure 1.4: Example of graph with non-adjacent vertices

**Definition 1.1.8**
Let $\Gamma = (V, E)$ be a graph. The **degree** of $x \in V(\Gamma)$, denoted by $\deg(x)$, is the number of vertices adjacent to $x$. Particularly, when $\deg(x) = 0$ we say that $x$ is an **isolated vertex**. If $\deg(x) = |V(\Gamma)| - 1$ then $x$ is called **dominant vertex**.

**Example 1.1.9**
*Let* $\Gamma$ *be the following:*

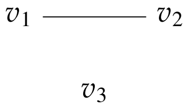

Figure 1.5: Graph with an isolated vertex

*We see that* $v_3$ *is an isolated vertex and* $\deg(v_1) = \deg(v_2) = 1 \neq 2 = |V(\Gamma)| - 1$. *Therefore, this graph has no dominant vertices.*

**Remark 1.1.10**
A graph can not have, simultaneously, dominant and isolated vertices.

Next, we define two of the most important invariants of a graph.

**Definition 1.1.11**
Let $\Gamma$ be a graph. The **maximum/minimum degree** of $\Gamma$ is the maximum/minimum degree among all vertices. The maximum degree of $\Gamma$ is denoted by $\Delta(\Gamma)$ and the minimum degree of $\Gamma$ is denoted by $\delta(\Gamma)$.

**Example 1.1.12**
*Consider the following diagram of a graph* $\Gamma$.

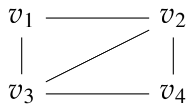

Figure 1.6: Graph with different maximum and minimum degree

*It's easy to see that* $\delta(\Gamma) = 2$ *and* $\Delta(\Gamma) = 3$.

Note that if $v_3$ was not adjacent to $v_2$, then $deg(v) = 2, \forall v \in V(\Gamma)$. In this case we would have that $\Delta(\Gamma) = \delta(\Gamma) = 2$. This kind of graphs are called regular graphs.

**Definition 1.1.13**
Let $\Gamma$ be a graph and $k \in \mathbb{N}$. We say that $\Gamma$ is **k-regular** when $\deg(v) = k, \forall v \in V(\Gamma)$.

**Example 1.1.14**

1. *The complete graph of order* $n$ *and the null graph of the same order are* $n$*-regular and* $0$*-regular graphs, respectively. These are the trivial regular graphs.*
2. *The graph from* ***Example 1.1.12*** *is not regular since* $\deg(v_3) = 3 \neq 2 = \deg(v_1)$.

Subsets of the vertex and edge set of a graph $\Gamma$ can generate a new graph.

**Definition 1.1.15**
Let $\Gamma = (V, E)$ be a graph. We say that $\Gamma_1 = (V_1, E_1)$ is a:

1. **subgraph** of $\Gamma$ when $V_1 \subseteq V$ and $E_1 \subseteq E$.
2. **subgraph induced by** $V_1$ when $V_1 \subseteq V$ and if $x, y \in V_1$ and $(x, y) \in E$, then $(x, y) \in E_1$.
3. **spanning subgraph** of $\Gamma$ if $V_1 = V$ and $E_1 \subseteq E$.

For the rest of this essay, we will denote the subgraph induced by $V_1$ as **induced subgraph**.

**Example 1.1.16**
*Let* $\Gamma$ *be the graph represented by the following diagram*

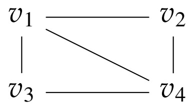

Figure 1.7: Example of a graph with four vertices

*Consider the subgraph of* $\Gamma$

$$\Gamma_1 = (V_1, E_1) = (\{v_1, v_2, v_4\}, \{(v_1, v_2), (v_1, v_4)\}).$$

*Since* $v_2, v_4 \in V_1(\Gamma_1)$, *but* $(v_2, v_4) \notin E_1(\Gamma_1)$ *and* $(v_2, v_4) \in E(\Gamma)$, *we have that* $\Gamma_1$ *it's not an induced subgraph of* $\Gamma$.

*Consider now the following graph* $\Gamma_2$

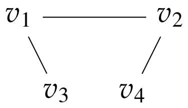

Figure 1.8: Example of a spanning subgraph of **Figure 1.7**

*By definition,* $\Gamma_2$ *is a spanning subgraph of* $\Gamma$.

**Remark 1.1.17**

i) For a graph $\Gamma_1$ to be a spanning subgraph of a graph $\Gamma$, it most have *the same number of vertices* as $\Gamma$.

ii) Every spanning subgraph of a given graph $\Gamma$ is a subgraph of $\Gamma$. However, the only spanning subgraph that is an induced subgraph is the original graph.

With the concept of subgraph, we can describe more classes of graphs.

**Definition 1.1.18**
Let $\Gamma = (V, E)$.

1) A subgraph $\Gamma_1 = (V_1, E_1)$ of $\Gamma$ is a **path** when we can rearrange $\{v_1, \ldots, v_n\} = V_1$ such that $v_i \sim v_{i+1}$, for $i = 1, \ldots, n$. If this happens, we denote $\Gamma_1$ by $P_n$. Moreover, we say that $P_n$ has length $n - 1$.

2) A **cycle** is a path that starts and ends at the same vertex. If it has $n$ elements, we denote it by $C_n$.

**Example 1.1.19**
*Let* $\Gamma$ *be the following graph:*

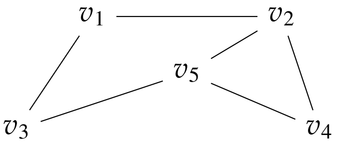

Figure 1.9: Example of a graph with five vertices

*Consider* $\Gamma_1 = (V_1, E_1)$*, subgraph of* $\Gamma$*, with vertex set* $V_1 = \{v_1, v_2, v_4, v_5\}$ *and edge set* $E_1 = \{(v_1, v_2), (v_2, v_4), (v_4, v_5)\}$*. Then* $\Gamma_1 = P_4$*. The path* $P_4$ *is*

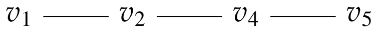

Figure 1.10: Example of a path of $\Gamma$

*Now, we can consider the subgraph of* $\Gamma$ *generated by* $\{(v_1,v_3),(v_1,v_2),(v_2,v_4),(v_4,v_5),(v_3,v_5)\}$. *This graph is the cycle* $C_5$ *and its diagram is*

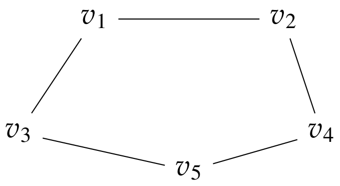

Figure 1.11: Example of a cycle of $\Gamma$

**Remark 1.1.20**
Note that every cycle is a 2-regular graph.

Recall the concept of isolated vertex. An equivalent definition is the following:

**Definition 1.1.21**
Let $\Gamma = (V, E)$ and take $v \in V(\Gamma)$. If for all $v' \in V(\Gamma) \setminus \{v\}$ there is no path connecting $v$ to $v'$ we say that $v$ is an **isolated vertex** of $\Gamma$.

More generally, a new class of graphs is defined when any pair of vertices has a path connecting them.

**Definition 1.1.22**
Let $\Gamma$ be a graph. The graph $\Gamma$ is said to be **connected** if for any pair of distinct vertices, $u$ and $v$, there exists a path from $u$ to $v$. Otherwise, $\Gamma$ is **disconnected**.

**Example 1.1.23**
*Let* $\Gamma$ *be the following:*

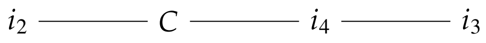

Figure 1.12: Example of a connected graph with four vertices

*In this case,* $\Gamma$ *is connected.*

*Now, consider the following graph where, for example, there is no path connecting* $i_4$ *to* $i_2$.

Figure 1.13: Example of a disconnected graph with four vertices

*In this case, the graph represented above is disconnected.*

When working with a disconnected graph, we may ask: how disconnected is it? Equivalently, we ask: how many connected parts does it have?

**Definition 1.1.24**
A **connected component** of a graph $\Gamma$ is a maximal connected induced subgraph.

**Example 1.1.25**

*Consider the following graph:*

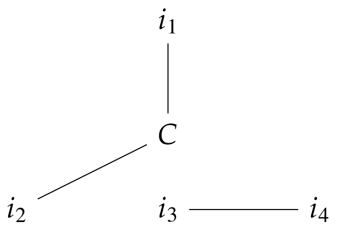

Figure 1.14: Example of a disconnected graph

*This graph has two connected components which we identify bellow*

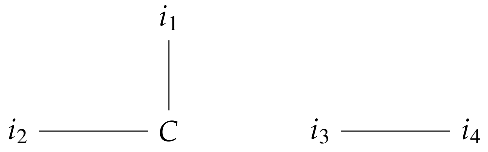

Figure 1.15: Two connected components of **Figure 1.14**

*Now, consider the following representation of a methane molecule:*

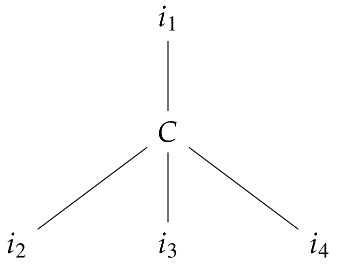

Figure 1.16: Example of a graph with one connected component

*Clearly, the graph that represents this molecule is connected. Moreover, it has only one connected component. This happens for any connected graph.*

**Proposition 1.1.26** ([4], **page 9**)
A graph is connected if and only if it has exactly one connected component.

**Remark 1.1.27**
The connected components of a graph $\Gamma$ form a partition of $\Gamma$.

In a connected graph, sometimes it's necessary to know the minimal number of edges needed to go from one vertex to another.

**Definition 1.1.28**
Let $\Gamma$ be a graph. The **distance** between two vertices, $u$ and $v$, denoted by $d(u,v)$, is the number of edges of the shortest path from $u$ to $v$. If there is no path connecting $u$ to $v$, then the distance is infinite.

With the distance we can define another graph invariant.

**Definition 1.1.29**
The **diameter** of a connected graph $\Gamma$ is the greatest distance between all vertices pairwise. It is denoted by $\mathrm{diam}(\Gamma)$.

**Example 1.1.30**
*Consider the following graph* $\Gamma$

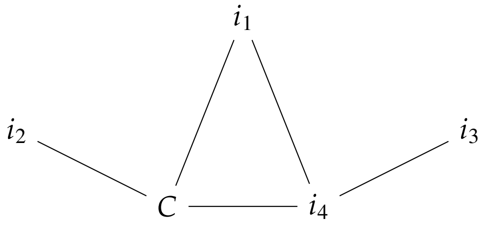

Figure 1.17: Example of distance and diameter

*It's not difficult to see that*

$$d(C,i_1) = d(C,i_2) = d(C,i_4) = 1$$

*We also note that*

$$d(i_2,i_3) = 3$$

*With similar calculations we can determine the distance for every pair of vertices. Therefore we are able to say that* $\mathrm{diam}(\Gamma) = 3$.

*Consider now the following graph* $\Gamma'$

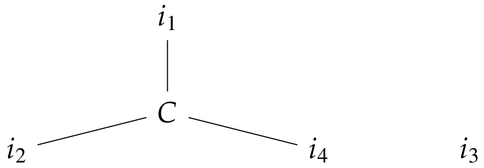

Figure 1.18: Example of a graph with two connected components

*Since there is no path connecting* $C$ *and* $i_3$*, we say that distance between them is infinite, i.e.,*

$$d(C,i_3) = \infty$$

*For the diameter is analogous*

$$\mathrm{diam}(\Gamma) = \infty$$

In the same way that we study ”how connected” a graph $\Gamma$ is, we can ask ”how complete” it is.

**Definition 1.1.31**
Let $\Gamma = (V,E)$ be a graph and $W \subset V(\Gamma)$ be a subset of the vertex set of $\Gamma$. We say that $W$ is a **clique** of $\Gamma$ if $W$ induces a complete graph, i.e., every vertex in $W$ is adjacent to all other vertices in $W$ in the original graph. The **clique number** of $\Gamma$, denoted by $\omega(\Gamma)$, corresponds to the greatest possible number of elements in a clique .

When talking about cliques it is useful to think about maximal cliques.

**Definition 1.1.32**
Let $\Gamma$ be a graph and $W$ a clique of $\Gamma$. We say that $W$ is a **maximal clique** if $W \not\subset C$, for any clique $C \neq W$ of $\Gamma$.

**Example 1.1.33**
*Consider the following graph* $\Gamma$*:*

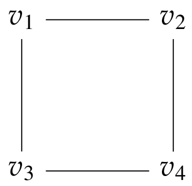

Figure 1.19: Example of a graph with cliques with the same order

*This graph has four maximal cliques, all of order* 2*. These cliques are*

$$\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_4\} \text{ and } \{v_3, v_4\}$$

Perfect graphs are especial graphs whose definition depends on the clique number, but also on the coloring properties.

**Definition 1.1.34**
Let $\Gamma = (V, E)$ be a graph and let $k$ be a positive integer. A $k$**-coloring** of the graph $\Gamma$ is a function $f : V(\Gamma) \longrightarrow \{1, 2, \ldots, k\}$ such that if $v_i$ and $v_j$ are adjacent in $\Gamma$, then $f(v_i) \neq f(v_j)$. If $\Gamma$ has a $k$-coloring we say that $\Gamma$ is $k$**-colorable**.

**Example 1.1.35**
*Consider the graph from the previous example. We may define several colorings for distinct* $k$ *as follows*

$$\begin{gathered} f_2 : V(\Gamma) \longrightarrow \{1, 2\} \\ f_2(v_1) = f_2(v_4) = 1 \\ f_2(v_3) = f_2(v_2) = 2 \end{gathered}$$

$$\begin{gathered} f_3 : V(\Gamma) \longrightarrow \{1, 2, 3\} \\ f_3(v_1) = f_3(v_4) = 1 \\ f_3(v_2) = 2 \\ f_3(v_3) = 3 \end{gathered}$$

$$\begin{gathered} f_4 : V(\Gamma) \longrightarrow \{1, 2, 3, 4\} \\ f_4(v_1) = 1 \\ f_4(v_4) = 2 \\ f_4(v_2) = 3 \\ f_4(v_3) = 4 \end{gathered}$$

*In this case, $f_k$ is a $k$-coloring. Note that $\Gamma$ does not have a $1$-coloring.*

**Definition 1.1.36**
Let $\Gamma$ be a graph. The **chromatic number** of $\Gamma$ is the smallest possible integer $k$ such that $\Gamma$ is $k$-colorable. It is denoted by $\chi(\Gamma)$.

**Example 1.1.37**
*The graph from **Example 1.1.35** has chromatic number equal to* 2.

Now we are ready to define perfect graphs.

**Definition 1.1.38**
A graph $\Gamma$ is **perfect** if any induced subgraph $\Gamma'$ of $\Gamma$ has chromatic number equal to the clique number, that is

$$\chi(\Gamma') = \omega(\Gamma'), \text{ for all induced subgraph } \Gamma' \text{ of } \Gamma$$

**Example 1.1.39**
*It follows that $C_5$ is not perfect. In fact, $C_5$ has clique number* 2*, but chromatic number equal to* 3.

We finish this section with the definitions of complement graph and isomorphic graphs.

**Definition 1.1.40**
Let $\Gamma = (V, E)$ be a graph. We say that $\overline{\Gamma} = (V, \overline{E})$ is the **complement** of $\Gamma$ when $(x, y) \in \overline{E}$ if and only if $(x, y) \notin E$.

**Example 1.1.41**
*Consider the following graph:*

Figure 1.20: Example of a disconnected graph with four vertices

*It's easy to see that $\overline{\Gamma}$ is*

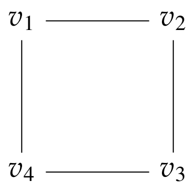

Figure 1.21: Complement graph of **Figure 1.20**

**Remark 1.1.42**

1. Regard that in the previous example $\Gamma$ is disconnected and its complement, $\overline{\Gamma}$, is connected. In general, at least one of $\Gamma$ or $\overline{\Gamma}$ are connected.
2. Observe that $\overline{\overline{\Gamma}} = \Gamma$.

**Definition 1.1.43**
Two graphs $\Gamma_1$ and $\Gamma_2$ are **isomorphic** if there is a one-one correspondence between the vertices of $\Gamma_1$ and those of $\Gamma_2$ such that the number of edges joining any two vertices of $\Gamma_1$ is equal to the number of edges joining the corresponding vertices of $\Gamma_2$.

**Example 1.1.44**
*Consider the two graphs from **Example 1.1.41**. These graphs are not isomorphic since* $\overline{\Gamma}$ *is a* 2*-regular graph but* $\Gamma$ *is not.*

## 1.2 Some classes of groups

In this section we give some fundamental definitions and results. It is divided into several subsections. In each of them, we introduce a specific class of groups and some results regarding it. We begin by recalling the concept of cyclic group, finishing with the classification of finite abelian groups. Then, we give the concept of permutation in order to define the symmetric and alternating groups. Next, we briefly define the dihedral groups and the generalized quaternion groups. A very important theorem is given.

For the purpose of this essay, all groups are assumed to be finite unless stated otherwise.

### 1.2.1 Cyclic groups

Cyclic groups will be fundamental for this essay. With this line of thought, recall that a group is called **cyclic** if it is generated by one element.

**Example 1.2.1**

*i) It is well known that the group* $\mathbb{Z}_n$ *with the additive operation is a cyclic group for any* $n \in \mathbb{N}$*. In fact, we have* $(\mathbb{Z}_n, +) = \langle 1 \rangle$.

*ii) The finite direct product of* $k$ *copies of* $\mathbb{Z}_n$ *is not cyclic, as it is generated by* $k$ *elements.*

As we saw in item $i)$ of the previous example, $\mathbb{Z}_n$ is generated by 1. However, this is not the only generator of this group. In general, we have:

**Proposition 1.2.2** ([23], **Proposition 6**)
Let $G = \langle a \rangle$ be a cyclic group of order $n$. Then $a^k$, with $k \in \mathbb{Z}$, is also a generator of $G$ if and only if $\gcd(n, k) = 1$.

The subgroups of a cyclic subgroup also belong to this class, as the next theorem states.

**Theorem 1.2.3** ([34], **Theorem 3.1.1**)
Let $G$ be a cyclic group. If $H$ is a subgroup of $G$, then $H$ is cyclic.

The subgroups from the previous example are unique for each order.

**Proposition 1.2.4** ([42], **Lemma 2.15**)
If $G$ is a cyclic group of order $n$, then there exists a unique subgroup $G_d$ of order $d$, for every divisor $d$ of $n$. Moreover, these subgroups form the chain

$$\{e\} = G_{d_1} \subseteq G_{d_2} \subseteq \cdots \subseteq G_{d_m} = G$$

where $d_i$ is a divisor of $n$ and $d_i < d_j$ when $i < j$.

We use cyclic groups to present the **classification of finite abelian groups**. For this, recall that any cyclic group of order $n$ is isomorphic to $\mathbb{Z}_n$.

**Theorem 1.2.5** ([23], **Theorem 5**)
Let $G$ be an abelian group of order $n > 1$ and let the unique factorization of $n$ into distinct prime powers be

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}, \text{where } \alpha_i \text{ are positive integers}$$

Then

1) $G \cong A_1 \times A_2 \times \cdots \times A_k$, where $A_i$ is abelian and $|A_i| = p_i^{\alpha_i}$;

2) For each $A \in \{A_1, \dots, A_k\}$ with $|A| = p^{\alpha}$

$$A \cong \mathbb{Z}_{p^{\beta_1}} \times \mathbb{Z}_{p^{\beta_2}} \times \cdots \times \mathbb{Z}_{p^{\beta_t}}, \text{where } \beta_i \text{ are positive integers}$$

with $\beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant \beta_t$ and $\beta_1 + \beta_2 + \cdots + \beta_t = \alpha$ (where $t$ and $\beta_1, \beta_2, \dots, \beta_t$ depend on $i$).

3) The decomposition in 1) and 2) is unique.

### 1.2.2 Symmetric and alternating groups

**Definition 1.2.6**
Let $X$ be a nonempty set. A **permutation** of $X$ is a bijection $\alpha : X \longrightarrow X$. We denote the set of all permutations of $X$ by $S_X$. In the case where $X = \{1, 2, 3, \dots, n\}$, we write $S_n$ instead of $S_X$.

**Remark 1.2.7**
We will be working with the case where $X = \{1, 2, \dots, n\}$.

**Definition 1.2.8**
Let $x \in \{1, \dots, n\}$ and $\alpha \in S_n$. Then $\alpha$ **fixes** $x$ if $\alpha(x) = x$. Otherwise, $\alpha$ **moves** $x$.

**Example 1.2.9**
*Consider the permutation $\alpha \in S_4$ defined as follows:*

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \alpha(1) & \alpha(2) & \alpha(3) & \alpha(4) \end{pmatrix}$$

*with $\alpha(1) = 3$, $\alpha(2) = 2$, $\alpha(3) = 1$ and $\alpha(4) = 4$. In this case we have that $\alpha$ fixes $2$ and $4$, while it moves $1$ and $3$.*

**Definition 1.2.10**
Let $i_1$, $i_2$, $\dots$, $i_k$ be distinct integers between 1 and $k$. If $\alpha \in S_n$ fixes the remaining $n-k$ integers and

$$\alpha(i_1) = i_2,\ \alpha(i_2) = i_3,\ \dots\ \alpha(i_{k-1}) = i_k \text{ and } \alpha(i_k) = i_1$$

then $\alpha$ is an $k$**-cycle** and it is denoted by $(i_1 i_2 \dots i_k)$.

**Remark 1.2.11**
We will use cycles to represent permutations.

**Example 1.2.12**
*Recall the permutation $\alpha \in S_4$ defined in **Example 1.2.9**:*

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

*Based in the two previous definitions, we may write $\alpha$ as follows,*

$$\alpha = (13)$$

The set $S_n$ is a group and it is called the **symmetric group of order n**. The group operation is the composition of permutations, which we will refer to as *product of permutations*.

**Example 1.2.13**
*Consider the following two permutations in $S_3$*

$$\alpha = (123) \text{ and } \beta = (12)$$

*The product $\alpha\beta$ is determine as follows*

$$\alpha(\beta(1)) = \alpha(2) = 3$$
$$\alpha(\beta(2)) = \alpha(1) = 2$$
$$\alpha(\beta(3)) = \alpha(3) = 1$$

*Hence*

$$(123)(12) = (13)$$

*Similarly we see that*

$$(12)(123) = (23)$$

From the previous example we conclude that $S_3$ is not abelian. This is true for any $S_n$ with $n \geqslant 3$.

Now, we will define a very important subgroup of the symmetric group. With this goal in mind, we will look at permutations further.

**Definition 1.2.14**
A permutation is called a **transposition** if it is a 2-cycle.

**Example 1.2.15**
*Consider symmetric group $S_5$. The cycle $(15) \in S_5$ is a transposition. In fact, $S_5$ has $10$ transpositions.*

**Definition 1.2.16**
A permutation is said to be an **even permutation** if it is the product of an even number of transpositions. Otherwise, the permutation is said to be **odd**.

**Proposition 1.2.17** ([27], **Theorem 5.6**)
The set of all even permutations on $n$ vertices is a subgroup of $S_n$ called the **alternating group**. It is denoted by $A_n$.

**Example 1.2.18**
*Consider $S_4$. Then*

$$A_4 = \{e,(123),(134),(143),(124),(142),(132),(234),(243),(12)(34),(13)(24),(14)(23)\}$$

Note that we may rewrite the permutation (123) from the previous example as follows

$$(123) = (13)(12)$$

We see in an analogous way that all permutations in $A_4$ are even. We can also see very easily that $|A_4| = 12 = |S_4|/2$. In general, the order of $A_n$ is half the order of $S_n$.

### 1.2.3 Dihedral groups

The set of symmetries of a regular polygon is proven to be a group. It can be represented using two elements called generators.

**Definition 1.2.19**
The **dihedral group** $D_{2n}$, for $2n \geqslant 4$, is a group of order $2n$ which is generated by two elements $r$ and $s$ such that

$$r^n = e,\ s^2 = e \text{ and } srs = r^{-1}$$

In other words,

$$D_{2n} = \langle r,s : r^n = e,\ s^2 = e \text{ and } srs = r^{-1}\rangle$$

Considering the geometric interpretations of dihedral groups, we easily check that $D_{2n}$ is a subgroup of $S_n$. In general, these groups are not isomorphic. Nonetheless, next theorem tells us that, when $n \geqslant 3$, the symmetric group $S_n$ always has a dihedral group as a subgroup.

**Theorem 1.2.20** ([42], **Theorem 3.32**)
Let $G$ be a finite group. If $a,b \in G$ have order 2, then $\langle a,b\rangle \cong D_{2n}$ for some $n$.

Next result it's not difficult to prove.

**Proposition 1.2.21**
The dihedral group $D_{2n}$ is not abelian for all $n \geqslant 3$.

We now classify the subgroups of a dihedral group.

**Theorem 1.2.22** ([21])
The proper nontrivial subgroups of $D_{2n}$ are either cyclic or dihedral groups.

*Proof.* Let $H$ be a subgroup of $D_{2n}$ with the representation given in **Definition 1.2.19**. We have two cases:

i) If $H \subset \langle r\rangle$, then it follows form **Theorem 1.2.3** that $H$ is cyclic.

ii) If $H \not\subset \langle r\rangle$, then there is an element of the form $sr^i$ that belongs to $H$, for some integer $i$.

    Now note

$$H \cap \langle r\rangle = \langle r^d\rangle$$

for some integer $d \leqslant n$, divisor of $n$. We may choose $d$ to be the smallest possible.

We will show that $H = \langle r^d, sr^i \rangle$. First we note that $\langle r^d, sr^i \rangle \subseteq H$. For the other inclusion, take $h \in H$. If $h \in \langle r \rangle$, then $h \in \langle r^d, sr^i \rangle$. Hence suppose that $h = sr^j$, for some integer $0 < j \leqslant n-1$ different from $i$. We have two cases:

1) If $i = 0$, then $s \in \langle r^d, s \rangle$ and we write

$$r^j = sh \in \langle r^d \rangle$$

Hence $j = dk$, for some integer $k$. Thus $h = sr^{dk} \in \langle r^d, s \rangle$. We conclude that $H = \langle r^d, s \rangle$.

2) Suppose now that $i \neq 0$. We know that any automorphism in $D_{2n}$ is determined by where it sends its generators. Consider the automorphism $f : D_{2n} \longrightarrow D_{2n}$ such that $f(r) = r$ and $f(sr^i) = s$. Since $f$ is an automorphism, it is easy to show that $f(H)$ is a subgroup of $D_{2n}$ that contains $s$. It follows from the first part of the proof that $f(H) = \langle r^d, s \rangle$, for some integer $d$. Thus

$$H = f^{-1}(H) = \langle f^{-1}(r^d), f^{-1}(s) \rangle = \langle r^d, sr^i \rangle$$

Now it's left to show that $\langle r^d, sr^i \rangle$ is a dihedral group of order $2o(r^d)$. For this, consider the representation

$$D_{2o(r^d)} = \langle r^d, sr^i : (r^d)^{o(r^d)} = e,\ (sr^i)^2 = e,\ sr^i r^d sr^i = r^{-d} \rangle$$

We will prove that this representation is well defined. First it's clear that $(r^d)^{o(r^d)} = e$. Now let us see that $(sr^i)^2 = e$:

$$sr^i sr^i = r^{-i} ssr^i = r^{-i} r^i = e$$

Finally, let us check the third condition:

$$sr^i r^d sr^i = sr^{i+d} sr^i = r^{-i-d} ssr^i = r^d$$

We conclude that $\langle r^d, sr^i \rangle$ is a dihedral group of order $o(r^d)$.

Therefore $H$ is either cyclic or dihedral. □

### 1.2.4 The quaternion group

**Definition 1.2.23**
The group with representation

$$Q_8 = \langle a, b : a^4 = e,\ b^2 = a^2, \text{ and } bab^{-1} = a^{-1} \rangle$$

is called the **quaternion group**.

The quaternion group is also usually defined as follows

$$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$$

The diagram of the group operation is as follows

$$\begin{tikzcd}
i \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & j \arrow[dl, leftrightarrow] \\
& k &
\end{tikzcd}$$

Figure 1.22: Product diagram of the quaternion group

where $i \cdot j = k$ and $j \cdot i = -k$. The other products are defined in a similar way.

Next definition generalizes the notion of quaternion groups.

**Definition 1.2.24**
If $n \geqslant 3$, a **generalized quaternion group** is a group $Q_n$ of order $2^n$ generated by elements $a$ and $b$ such that

$$a^{2^{n-1}} = e,\ b^2 = a^{2^{n-2}},\ \text{and}\ bab^{-1} = a^{-1}$$

In the previous notation, the quaternion group $Q_8$ is represented by $Q_3$.

## 1.3 Prime power order groups and miscellaneous results

In this section we introduce the $p$-groups and the EPPO groups, differentiating them with some examples. A few properties of prime power order groups are given. In the same line of thought, we present some results regarding the prime power divisors of the order of $G$. It is also in this subsection where we present miscellaneous results of Group Theory that will be highly used throughout this essay.

**Definition 1.3.1**
A group $G$ is said to be **2-generated** if there is a generating set of $G$ that has *at most* 2 elements.

**Example 1.3.2**

1. *Every dihedral group is* 2*-generated.*
2. *Every cyclic group is* 2*-generated.*

The previous example draws attention to the fact that a group generated by one element is also 2-generated.

As we will see later on this section, elements of prime order are particularly relevant. As a special case, elements of order 2 have their own name.

**Definition 1.3.3**
In any group $G$ the elements of order 2 are called **involutions**.

**Example 1.3.4**
*In the dihedral group $D_{2n}$, every reflection is an involution.*

We now present an elementary definition that will be necessary for the rest of this essay.

**Definition 1.3.5**
Let $(G,*)$ be a group. We define:

i) The **center** of $G$, denoted by $Z(G)$, as

$$Z(G) = \{g \in G : g * a = a * g,\ \forall a \in G\}$$

ii) The **centralizer** of $g \in G$, denoted by $C_G(g)$, as

$$C_G(g) = \{a \in G : g * a = a * g\}$$

In other words, the center of a group $G$ is the set of elements of $G$ that commute with all others. Naturally, $Z(G)$ is never empty since it always contains the identity. We note that if $G$ is abelian, then $Z(G) = G$ and $C_G(g) = G$, $\forall g \in G$. Therefore, the definitions above become interesting for non-abelian groups.

**Example 1.3.6**
*Let us consider the dihedral group* $D_8$*, which is the group of the symmetries of a square, that is*

$$D_8 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$$

*where* $r$ *is the* 90° *rotation and* $s$ *is the vertical reflection. Recall that the group operation satisfies the following*

$$srs^{-1} = r^{-1}$$

*Taking this into consideration, it is not difficult to prove that*

$$Z(D_8) = \{e, r^2\}$$

*We are also able to determine the centralizer of any element in the group. For this example consider the reflection* $s$*. It's important to note that the only reflection that commutes with* $s$ *is the one perpendicular to it. In general, it's true that two reflections by perpendicular lines commute. Based on this, we obtain:*

$$C_{D_8}(s) = \{e, r^2, s, sr^2\}$$

*Finding the centralizer for the other three reflections is analogous. As for rotations, we have that* $r$ *generates an abelian group. Thus the centralizer of any rotation (except* $r^2$*) consists on all rotations and the identity. As for* $r^2$*, recall that it belongs to the center of* $D_8$*. Consequently,* $C_{D_8}(r^2) = D_8$*.*

We now present a result that will be needed for **Chapter 3**.

**Theorem 1.3.7** ([14], **Proposition 10**)
Suppose that $G$ contains a normal non-abelian maximal subgroup $M$, with $Z(G) < Z(M)$. Then each maximal subgroup of $G$ is non-abelian.

**Remark 1.3.8**
It is easy to check that the center of a group is always a subgroup. In particular, it is a normal subgroup.

By the previous remark, we can quotient a group $G$ with its center.

**Proposition 1.3.9**
Let $Z(G)$ be the center of $G$. If $G/Z(G)$ is cyclic, then $G$ is abelian.

*Proof.* Suppose that $G/Z(G) = \langle xZ(G)\rangle$ for some $x \in G$. Take distinct nonidentity elements $g_1, g_2 \in G$. We can write

$$g_1 = x^t Z(G) \text{ and } g_2 = x^k Z(G) \text{ for some distinct integers } t \text{ and } k$$

Then

$$g_1 g_2 = x^t Z(G) x^k Z(G) = x^{t+k} Z(G) = x^k Z(G) x^t Z(G) = g_2 g_1$$

Therefore $G$ is abelian. □

The negation of the previous Proposition can be useful, as demonstrated bellow.

**Example 1.3.10**
*The group $D_8/Z(D_8)$ is not cyclic, since $D_8$ is not an abelian group.*

One class of groups that is of interest for this essay is defined through the order of the elements of a group.

**Definition 1.3.11**
Let $G$ be a group and $p$ a prime number. $G$ is a **$p$-group** if the order of each element of $G \setminus \{e\}$ is a power of $p$.

**Example 1.3.12**

1) *Any group of prime order is a $p$-group. This can be verified applying **Theorem A.0.12**. Additionally, groups of prime order are always cyclic and any non-identity element is a generator.*
2) *Consider the smallest non-cyclic abelian group $K_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$. This group, known as the Klein-four group, has all elements of order 2. In particular, this is an elementary abelian 2-group.*

An equivalent definition for finite $p$-groups is presented bellow.

**Definition 1.3.13**
A finite group is a $p$-group if and only if it is of prime power order, $p^m$.

**Example 1.3.14**

i) *The quaternion group $Q_8$ is a non-abelian 2-group with three subgroups isomorphic to $\mathbb{Z}_2$;*
ii) *$\mathbb{Z}_3 \times \mathbb{Z}_3$ is an abelian 3-group with four subgroups isomorphic to $\mathbb{Z}_3$.*

On contrary to the previous example, some $p$-groups may have only one subgroup of order $p$. Next theorem classifies them.

**Theorem 1.3.15** ([34], **Theorem 12.5.2**)
A $p$-group which contains only one subgroup of order $p$ is either cyclic or a generalized quaternion group.

**Example 1.3.16**
*Consider the cyclic group* $\mathbb{Z}_{27}$*, which is a* $3$*-group. By* ***Theorem 1.2.4****, we have that it has only one subgroup of order* $3$*. Moreover, the subgroups of* $\mathbb{Z}_{27}$ *form the chain*

$$\{e\} \subseteq \mathbb{Z}_3 \subseteq \mathbb{Z}_9 \subseteq \mathbb{Z}_{27}$$

An important fact about nontrivial $p$-groups is that they don't have trivial center.

**Proposition 1.3.17** ([24], **Proposition 5.2.2**)
Let $G$ be a group and $p$ a prime number. If $|G| = p^m$, then $|Z(G)| = p^k$ with $k \geqslant 1$.

We now generalize the notion of $p$-groups.

**Definition 1.3.18**
A group $G$ is an **EPPO group** (element prime power order) if every element in $G \setminus \{e\}$ has prime power order.

Two distinct elements in an EPPO group can have orders the power of distinct primes. Therefore it becomes clear that every $p$-group is an EPPO group, but the converse is not true.

**Example 1.3.19**
*Consider the symmetric group* $S_3$*. Take* $(123), (12) \in S_3$*. We have*

$$(123)(123) = (132) \textit{ and } (132)(123) = (1)(2)(3).$$

*However*

$$(12)(12) = (1)(2)(3.)$$

*Thus* $(123)$ *has order* $3$ *while* $(12)$ *has order* $2$*. Similar calculations show that any nonidentity element in* $S_3$ *has either order* $2$ *or order* $3$*. Therefore* $S_3$ *is an EPPO group, but not a* $p$*-group.*

Now, we present the **Cauchy Theorem**.

**Theorem 1.3.20** ([42], **Theorem 4.2**)
If $G$ is a group whose order is divisible by a prime $p$, then $G$ contains an element of order $p$.

**Example 1.3.21**
*Consider the dihedral group* $D_{14}$*. By the previous Theorem,* $D_{14}$ *has at least one element of order* $2$ *and at least one of order* $7$*. In fact, all seven rotations have order* $7$ *and all seven reflections have order* $2$*.*

The previous example shows that the elements in **Theorem 1.3.20** are not necessarily unique.

We have seen that we can determine some properties of a group based on its order, particularly when the order is the power of a prime. Now, we will introduce one of the Sylow Theorems which extends this study and generalizes **Theorem 1.3.20**.

**Theorem 1.3.22** ([24], **Theorem 5.2.5**)
Let $G$ be a finite group, $p$ a prime number and $k \in \mathbb{N}$. If $p^k$ is an element in the factorization of the order of $G$, then $G$ has a subgroup $H$ of order $p^k$.

**Example 1.3.23**

- *i)* *Consider, once again,* $D_8$. *By the previous Theorem, there exists a subgroup* $H_i$ *of* $D_8$ *such that* $|H_i| = i$, *with* $i \in \{2,4,8\}$. *If* $i = 8$, *then* $H_8$ *is the whole group. If* $i = 4$, *it's easy to see that* $H_4$ *is the subgroup of all rotations, i.e.,* $H_4 = \{e,r,r^2,r^3\} \cong \mathbb{Z}_4$. *Finally, when* $i = 2$ *we have* 4 *subgroups* $H_2$ *which are the cyclic subgroups generated by one reflection. This subgroups are all isomorphic to* $\mathbb{Z}_2$.
- *ii)* *Take the dihedral group* $D_{12}$. *This group has order* $12 = 2^2 \times 3$. *Thus, by* ***Theorem 1.3.22****,* $D_{12}$ *has subgroups of orders* 2, 3 *and* 4.

The dihedral group $D_{12}$, has two non-isomorphic 2-subgroups. However, we are generally more interested in the greatest $k$ given in **Theorem 1.3.22**.

**Definition 1.3.24**
Let $G$ be a group and $p$ a prime number. A **Sylow** $p$**-subgroup of** $G$ is a subgroup of order $p^m$, where $p^m$ is the largest power of $p$ that divides the order of $G$.

**Example 1.3.25**
*Consider the finite group* $A_4 \times \mathbb{Z}_2$. *It's clear that* $|A_4 \times \mathbb{Z}_2| = 3 \times 2^3$. *It follows from* ***Theorem 1.3.22*** *that* $A_4 \times \mathbb{Z}_2$ *has subgroups of orders* 3, 2, 4 *and* 8. *In particular, the subgroups of orders* 3 *and* 8 *are the Sylow* 3*-subgroup and Sylow* 2*-subgroup of* $A_4 \times \mathbb{Z}_2$, *respectively. Finally, we conclude from* ***Theorem 1.3.20*** *that* $A_4 \times \mathbb{Z}_2$ *has at least one element of order three and one of order two, which generate two cyclic subgroups of* $A_4 \times \mathbb{Z}_2$.

## 1.4 Nilpotent groups

In this section we introduce the nilpotent groups. We begin by defining the commutator of a group. Next, we arrive to the definition of lower central series, from which the nilpotent groups arrive. Examples of these concepts are given. Finally, we present, without proof, a few classes of groups which are also nilpotent.

**Definition 1.4.1**
Let $G$ be a group and consider $g$, $h \in G$. The **commutator** of $g$ and $h$ is the element of the form $g^{-1}h^{-1}gh \in G$. It's denoted by $[g,h]$.

**Example 1.4.2**

- *i)* *Let* $\mathbb{Z}_n$ *be the cyclic group of order* $n$ *and* $g,h \in \mathbb{Z}_n$. *It's clear that*

$$[g,h] = g^{-1}h^{-1}gh = g^{-1}gh^{-1}h = e.$$

*This happens for any two elements in an abelian group.*

- *ii)* *Consider the non-abelian group* $D_8$. *Take* $s,r \in D_8$ *where* $s$ *and* $r$ *are as in* ***Example 1.3.6***. *Then*

$$[r,s] = r^3srs = r^2.$$

We obtain a new type of group by considering the commutators of two subgroups of a given group.

**Definition 1.4.3**
Let $S$ and $H$ be two subgroups of $G$. We denote by $[S,H]$ the group generated by the commutators of the form $[s,h]$, with $s \in S$ and $h \in H$.

**Example 1.4.4**
*Consider the quaternion group* $Q_8$ *and two of its subgroups* $H = \langle -1 \rangle$ *and* $S = \langle i \rangle$. *Since* $-1 \in \langle i \rangle$ *we have that* $[S,H] = \{1\}$. *Now, if we consider* $S = \langle j \rangle$ *and* $H = \langle i \rangle$, *then* $[H,S] = \{1,-1\}$ *because* $ij(-i)(-j) = k^2 = -1$.

If we take $S = H$ in **Definition 1.4.3**, we have the following:

**Definition 1.4.5**
The group of the form $[G,G]$ is called **commutator group**.

We will see that the groups in **Definition 1.4.3** allow us to determine, in a sense, how commutative a group is. This is done using lower central series.

**Definition 1.4.6**
Let $G$ be a group. The **lower central series** of $G$ is defined by

$$Z^0(G) \supseteq Z^1(G) \supseteq \cdots \supseteq Z^n(G)$$

where

$$Z^0(G) = G \text{ and } Z^{k+1}(G) = [G, Z^k(G)],\ k \geqslant 0.$$

**Example 1.4.7**

i) *Let G be an abelian group. By definition,* $Z^0(G) = G$. *Now, it follows from* ***Example 1.4.2*** *that* $Z^1(G) = [G,G] = \{e\}$. *Similarly, we conclude that* $Z^2(G) = [G,\{e\}] = \{e\}$. *This happens for any* $k \geqslant 2$. *Therefore the lower central series of any abelian group ends with the trivial group in one step.*

ii) *Take the symmetric group* $S_3$. *By definition,* $Z^0(G) = S_3$. *Also, we check the following*

$$[(132),(23)] = (123)(23)(132)(23) = (123)(23)(31) = (123)(123) = (132)$$

*Now, we compute*

$$[(12),(23)] = (12)(23)(12)(23) = (231)(231) = (123)$$

*Doing similar calculations, we verify that* $Z^1(G) = [G,G] = \langle (123) \rangle$. *Analogously, we obtain that* $Z^2(G) = [G, \langle (123) \rangle] = \{e, (132), (123)\} = Z^1(G)$. *Thus the lower central series of* $S_3$ *never ends with the trivial group.*

In the particular case where the lower central series ends with the trivial group, we obtain a new class of groups.

**Definition 1.4.8**
A group $G$ is **nilpotent** if for some $k \in \mathbb{N}$ we have that $Z^k(G) = \{e\}$, i.e., the lower central series of $G$ ends at the trivial group after finitely many steps. The smallest $k$ such that $Z^k(G) = \{e\}$ is known as the **nilpotency class of** $G$.

**Example 1.4.9**

*i) Looking at* ***Example 1.4.7****,* $S_3$ *is* ***not*** *a nilpotent group as there is no* $k \in \mathbb{N}$ *such that* $Z^k(G) = \{e\}$.

*ii) Let us determine if the quaternion group is nilpotent. By definition,* $Z^0(Q_8) = Q_8$*. Following the same procedure as* ***Example 1.4.7****, we can determine* $Z^1(Q_8) = [Q_8, Q_8]$*. It is not difficult to see that if* $g$ *and* $h$ *are are such that* $g \notin \langle h \rangle$ *and* $h \notin \langle g \rangle$*, then* $[g,h] = -1$*. However, if* $g$ *and* $h$ *belong to the same cyclic subgroup of* $Q_8$*, then* $[g,h] = 1$*. Thus* $[Q_8, Q_8] = \{1, -1\} = Z(Q_8)$*. Now, to determine* $Z^2(Q_8) = [Q_8, Z(Q_8)]$ *we notice that for any* $g \in Q_8$ *and* $h \in Z(Q_8)$ *it follows that* $h \in \langle g \rangle$ *or* $g \in \langle h \rangle$*. Thus* $Z^2(Q_8) = \{e\}$*. In conclusion,* $Q_8$ *is nilpotent with nilpotency class 2.*

The next theorem shows some important classes of groups that are nilpotent.

**Theorem 1.4.10** ([42], **Theorem 5.33**)

1. Every finite $p$-group is nilpotent.
2. Every abelian group is nilpotent.

More generally we have the following result.

**Proposition 1.4.11** ([8], **Proposition 4.1**)
A finite group $G$ is nilpotent if and only if for all $x, y \in G$ with $\gcd(o(x), o(y)) = 1$ we have $xy = yx$.

As we saw for $p$-groups, the next theorem shows that nilpotent groups do not have trivial center.

**Theorem 1.4.12** ([42], **Theorem 5.34**)
If $G \neq \{e\}$ is nilpotent, then $Z(G) \neq \{e\}$.

We finish this section with two theorems that will be useful for **Chapter 2**.

**Theorem 1.4.13** ([42], **Theorem 5.37**)
If $G$ and $H$ are nilpotent groups, then $G \times H$ is also nilpotent.

**Theorem 1.4.14** ([42], **Theorem 5.39**)
Let $G$ be a group. The group $G$ is nilpotent if and only if it can be decomposed as a direct product of its Sylow subgroups.

# Chapter 2

# Graphs defined on groups

In this chapter we introduce the *power graph*, the *enhanced power graph*, the *commuting graph* and the *non-generating graph*. Throughout the chapter we study some properties of these graphs. Particularly, we classify, when possible, the groups for which each of these graphs are complete or null. We then look at graph connectivity and establish some bounds for the diameter of some types of graph of some particular group. Finally, we show some results regarding maximal cliques.

The goal of this chapter is to give the background to define the graph hierarchy (see [10]) and study the difference between some of these graphs. This will be done in the next chapters.

This chapter is based on several articles. For the first section we reference [15, 16, 19, 22, 39]. The second section is based on [1, 6, 7, 10]. The third section was written taking into account [10, 29, 30]. Finally, the last section takes into consideration [37].

## 2.1 The power graph

In this section we are going to introduce the **power graph**. This graph was first considered in [35] to study combinatorial properties of groups. In the article, the power graph was defined as a directed graph. Nowadays, it is studied as an undirected graph. For the most part, we will also follow this line of thought.

The beginning of this section aims to classify the classes of groups for which the power graph is complete. In order to do this,we determine the number of edges in the power graph of a given group and give a condition for which two vertices are non-adjacent.

After considering complete power graphs, we look at groups without complete power graphs. Our goal is to study connectivity for the proper power graph. Next, we determine the types of groups for which the maximal cliques of the power graph correspond to maximal cyclic subgroups.

Finally, we end this section with a result regarding the isomorphism of the power graph of two different groups.

**Definition 2.1.1**
Let $G$ be a group. The **directed power graph** of $G$, denoted by $\vec{\Gamma}_p(G)$, is the directed graph with vertex set the elements of $G$ where an arc $(x, y)$ exists if and only if $x \neq y$ and $x = y^n$, for some $n \in \mathbb{Z}$. Loops are removed.

**Example 2.1.2**

*Consider the symmetric group* $S_3$*. Note that* $(e,(12))$ *is an arc in the directed power graph since* $(12)^2 = e$*. However,* $((12),e)$ *is not an arc in this graph because* $e^k \neq (12)$ *for any* $k \in \mathbb{Z}$*. Doing some similar calculations for other pairs of elements, we conclude that* $\vec{\Gamma}_p(S_3)$ *is*

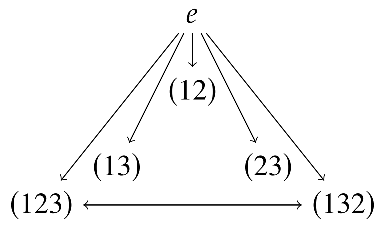

Figure 2.1: Directed power graph of the symmetric group $S_3$

*Note that* $((123),(132))$ *and* $((132),(123))$ *are arcs. These are* ***bidirected arcs****.*

Usually, we will not be interested in making any distinction regarding direction. In this case, we obtain a undirected simple graph known as the **power graph** of a group $G$, which we will denote by $\Gamma_p(G)$.

**Example 2.1.3**

*Let us look at a diagram of the power graph of the symmetric group* $S_3$*.*

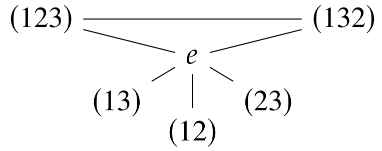

Figure 2.2: Power graph of the symmetric group $S_3$

Note that the power graph of any finite group always has a dominant vertex, the identity. As a direct consequence, it follows that the power graph of any group is never null.

We will now determine when the set of dominant vertices of the power graph equals the set of elements of the group.

**Definition 2.1.4**

The **Euler's totient function**, represented by $\phi$, is defined in the following way:

$$\begin{gathered}\phi(1) = 1 \\ \text{if } n > 1, \text{ then } \phi(n) = |\{k : 1 \leqslant k < n \text{ and } \gcd(k,n) = 1\}|\end{gathered} \tag{2.1}$$

**Example 2.1.5**

*Let us consider the number* 12*. We can quickly check that* $\phi(12) = 4$ *since the set of numbers coprime to* 12 *is* $\{1,5,7,11\}$*.*

**Proposition 2.1.6** ([5], **Theorem 2.5**)

For Euler's totient function it is verified that

$$\phi(p^m) = p^m - p^{m-1} \tag{2.2}$$

where $p$ is a prime number and $m > 1$.

The previous proposition enables us to determine the number of generators in a cyclic group of prime power order.

**Corollary 2.1.7**
A cyclic group of order $p^m$ (with $p$ a prime number) has

$$\phi(p^m) = p^m - p^{m-1}$$

generators.

*Proof.* Let $G = \langle a \rangle$. We know from **Proposition 1.2.2** that $a^k \in G$, for $k \in \mathbb{Z}$, is a generator of $G$ if and only if $k$ is coprime to the order of $G$. Hence the number of generators is given by **Expression 2.2**. □

**Theorem 2.1.8** ([19], **Theorem 4.2**)
Let $G$ be a group. The number of edges in $\Gamma_p(G)$ is given by

$$|E(\Gamma_p(G))| = \frac{1}{2}\sum_{g \in G}(2o(g) - \phi(o(g)) - 1) \tag{2.3}$$

where the function $\phi$ is **the Euler totient function**.

*Proof.* We will show that **Expression 2.3** results from removing bidirected arcs in the directed power graph.

Let $g \in \vec{\Gamma}_p(G)$. Consider the outdegree of $g$ to be the number of arcs of the form $(g, x)$, for any $x \in G$. It's not difficult to see that this number is exactly $o(g) - 1$. The sum of the outdegrees of all vertices gives the number of arcs in the directed power graph. Now, in order to obtain the edges of the power graph we need to associate bidirected arcs to only one edge, i.e, for any bidirected arc, we need to count it only once. Note that we have a bidirected arc between two vertices $g$ and $h$ in the power graph if and only if $\langle g \rangle = \langle h \rangle$. Therefore the number of bidirected arcs associated to $g$ equals the number of generators of $\langle g \rangle$ except $g$ itself. This number is given by $\phi(o(g)) - 1$. Finally, since we are counting bidirected arcs twice, we must divide by two, obtaining the following:

$$|E(\Gamma_p(G))| = \sum_{a \in G}\left((o(g) - 1) - \frac{\phi(o(g)) - 1}{2}\right)$$

With some algebraic manipulation, we obtain the desired result. □

**Example 2.1.9**
*Consider the direct product $\mathbb{Z}_2 \times \mathbb{Z}_2$. From the previous Theorem we may determine the number of edges of its power graph. We begin by computing $2o(g) - \phi(o(g)) - 1$ for the identity element.*

$$2o((0,0)) - \phi(o(0,0)) - 1$$
$$= 2 - 1 - 1 = 0$$

*We now compute the expression for all other elements. First, we notice that any non-identity element* *has order* 2*. Consider* $H = \{(1,0);(0,1);(1,1)\}$*. For any* $x \in H$ *we have*

$$\begin{gathered} 2o(x) - \phi(o(x)) - 1 \\ = 4 - 1 - 1 = 2 \end{gathered}$$

*Computing **Expression 2.3** we obtain*

$$|E(\Gamma_p(\mathbb{Z}_2 \times \mathbb{Z}_2))| = \frac{0}{2} + \frac{2}{2} + \frac{2}{2} + \frac{2}{2} = 3$$

*Hence non-identity vertices are only adjacent to the identity vertex.*
*Consider now the cyclic group* $\mathbb{Z}_4$*. Doing similar calculations as we did for* $\mathbb{Z}_2 \times \mathbb{Z}_2$*, we obtain*

$$|E(\Gamma_p(\mathbb{Z}_4)| = \frac{0}{2} + \frac{2}{2} + \frac{5}{2} + \frac{5}{2} = 6$$

*As six is the maximum number of edges in a graph with four vertices, we conclude that the power graph of* $\mathbb{Z}_4$ *is complete.*

As we saw in the previous example, the power graph of $\mathbb{Z}_4$ is complete. We may ask: is this true for all cyclic groups? The answer is no. To see this, we draw $\Gamma_p(\mathbb{Z}_6)$ bellow.

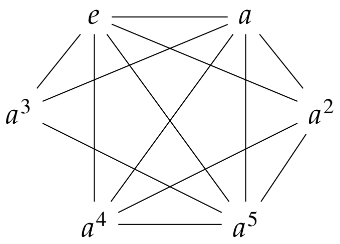

Figure 2.3: Power graph of the cyclic group $\mathbb{Z}_6$

Clearly, this graph is not complete. In general, the following proposition holds.

**Proposition 2.1.10**
Let $G$ be a group and take $x, y \in G \setminus \{e\}$. If $\gcd(o(x), o(y)) = 1$, then $x$ and $y$ are not adjacent in the power graph.

*Proof.* Let $x, y \in G$ be non-identity elements such that $\gcd(o(x), o(y)) = 1$. Suppose, in order to get a contradiction, that $x$ and $y$ are adjacent in the power graph. Without loss of generality, consider $x = y^k$. We may assume that $x$ has order $t$ and $y$ has order $n$, for some positive integers $t$ and $n$. If follows that $\langle y \rangle$ has a subgroup of order $t$. From **Theorem A.0.12** we have that $t$ is in the decomposition of $n$, a contradiction. □

Note that the previous proposition tells us that only $p$-groups are candidate to have complete power graphs. In [19], the authors classified the groups for which the power graph is complete.

**Theorem 2.1.11** ([19], **Theorem 2.12**)
A nontrivial group $G$ has complete power graph if and only if $G$ is a cyclic group of prime power order.

*Proof.* First, note that **Proposition 2.1.10** implies that if $\Gamma_p(G)$ is complete, then $G$ must have prime power order. We will now show by contrapositive that if $\Gamma_p(G)$ has all edges, then $G$ must be cyclic. Suppose that G is a non-cyclic group. Then there exist non-identity elements $x$ and $y$ such that $x \notin \langle y \rangle$ and $y \notin \langle x \rangle$. This implies that $x$ and $y$ are not adjacent. Therefore the power graph of G is not complete.

Conversely, let G be a cyclic group such that $|G| = p^m$. It follows from **Proposition 1.2.4** that G has exactly one cyclic subgroup of order $p^i$ $(0 < i \leq m\ )$ and that they form an ordered chain

$$G_0 \subseteq G_p \subseteq G_{p^2} \subseteq \cdots \subseteq G_{p^m}$$

Thus for any $x \in G_{p^i}$ there exists $y \in G_{p^j}$, with $i < j$, such that $x = y^k$.

□

Recall that, in the power graph, the identity is always a dominant vertex, i.e., it is connected to all other vertices. Hence the power graph of any group is never disconnected. If we remove the identity from the vertex set of the power graph, studying connectivity is no longer trivial. We will denote by $\Gamma_p^*(G)$ the power graph generated by non-identity elements of G and we will call it **proper power graph of** $G$. We will adopt a similar notation for other graphs in this essay.

In [15], the authors showed a necessary condition for the proper power graph of a group to be connected. For this we will consider G to be infinite.

**Definition 2.1.12**
We say that G is a **torsion group** (or a periodic group) if every element has finite order.

**Example 2.1.13**
*The group* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *is a torsion group in which every element has order* 2*. Actually, the infinite direct product of* $\mathbb{Z}_2$ *is a torsion group.*

**Definition 2.1.14**
Let G be a group. $G$ is a **torsion-free group** if the only element with finite order is the identity.

**Example 2.1.15**

*i)* $(\mathbb{R}, +)$ *is a torsion-free group.*

*ii)* $\forall n \in \mathbb{Z} \setminus \{0\}$*, we have that* $\langle n \rangle$*, subgroup of* $(\mathbb{Z}, +)$*, is a torsion-free group.*

A group can be, simultaneously, not torsion and not torsion-free.

**Example 2.1.16**
*Consider the group* $(\mathbb{Q} \setminus \{0\}, \times)$*. The element* $-1$ *has order two. Nonetheless, the element* 2 *has infinite order. Thus* $\mathbb{Q} \setminus \{0\}$ *is neither torsion nor torsion-free.*

We are now ready to present the result.

**Proposition 2.1.17** ([15], **Lemma 1**)
Let $G$ be a group. If $\Gamma_p^*(G)$ is connected, then $G$ is either a torsion or a torsion-free group.

*Proof.* We will show that the contrapositive is true. Suppose that G is neither a torsion nor a torsion-free group. Then there exists elements $g$ and $h$ in $G \setminus \{e\}$ such that $g$ has infinite order and $h$ has finite order.

It follows that $g$ is not a power of $h$ as $h$ cannot have a subgroup of infinite order. Now, if $h = g^k$, then $g^k$ has finite order. Thus $g$ has finite order, a contradiction. Therefore $g$ and $h$ are non-adjacent in the power graph and the proof is complete. □

Sufficient conditions for connectivity of finite or infinite groups are trickier. Regarding the finite case, some advances have been made for $p$-groups and other specific groups. However, to the best of our knowledge, a complete characterization has not been made.

From now on, we consider $G$ to be finite. We will look at the maximal cliques of the power graph of $G$ and their relation with the subgroups of $G$.

**Proposition 2.1.18** ([39], **Lemma 14**)
$G$ is an EPPO group if and only if the vertices of every maximal clique of $\Gamma_p(G)$ form a maximal cyclic subgroup of $G$.

*Proof.* We will first assume that the *if part* holds. Let $C$ be a maximal clique of $\Gamma_p(G)$. It follows from **Theorem 2.1.11** that $C$ is a cyclic group of order $p^k$, for some prime $p$ and $k \in \mathbb{N}$. From the maximality of $C$ we conclude that $C$ is a maximal subgroup of $G$.

Conversely, take $g \in G$. It's clear that $g$ is contained in a maximal clique $C$ of $\Gamma_p(G)$. By definition we know that $\Gamma_p(C)$ is complete. Then the result follows immediately from **Theorem 2.1.11**. □

**Example 2.1.19**
*Take the group of symmetries of an hexagon, $D_{12}$. We know from **Proposition 1.2.2** that the elements $r$ and $r^5$ are generators of $\langle r \rangle$. It's also clear that $r^2$ and $r^4$ are adjacent to each other, but not to $r^3$. As for reflections, each one is clearly adjacent only to the identity. Hence the power graph of $D_{12}$ is*

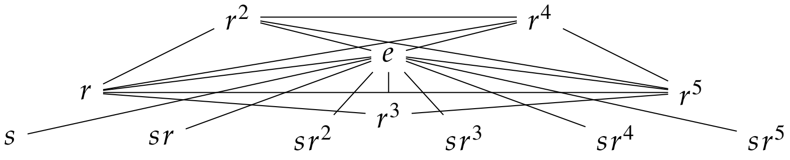

Figure 2.4: Power graph of the dihedral group $D_{12}$

*Looking at this graph we have that a maximal clique is*

$$\{e, r, r^2, r^4, r^5\}$$

*The elements that are in this clique clearly don't form a subgroup of $D_{12}$ since it is not closed under group operation. Therefore we have from **Proposition 2.1.18** that $D_{12}$ is not an EPPO group.*

Note that if $G$ and $H$ are two isomorphic groups, then their respective power graphs are also isomorphic. Naturally, we ask if the converse is also true. This was first showed true for the directed power graph. Later on, the case of the power graph was proven false for arbitrary finite groups, as it was shown by Cameron and Shamik in [16]. However, in the same article the authors also showed that the equivalence is true for finite abelian groups.

**Theorem 2.1.20** ([16], **Theorem 1**)
Let $G$ and $H$ be two abelian groups such that $\Gamma_p(G) \cong \Gamma_p(H)$, then $G \cong H$.

Beside this result, it was proved in [17] that if two groups have isomorphic power graphs, then the groups have the same number of elements of each order. This follows partly from the fact that if two groups have isomorphic power graphs, then they have isomorphic directed power graphs, also showed in [17].

## 2.2 The enhanced power graph

In this section we study the **enhanced power graph** of a group. This graph was first considered by Abdollahi and Hassanabadi (see [3]) as the complement of the **noncyclic graph** of a group. It was named **cyclic graph**. The authors removed some elements of $G$ from the vertex set of the cyclic graph. Later on, Aalipour et al. independently defined a graph similar to the cyclic graph, but without restrictions on the vertex set. They named this graph *enhanced power graph* in [1]. This is the definition we use for this section.

Our goal in this section is to give a brief exposition on the enhanced power graph of a group. We begin by determining for which classes of groups it is a complete or a null graph. Afterwards, we conclude that, for any group, the enhanced power graph is connected. This allows us to, given a group $G$, study how many vertices need to be removed so that the resulting graph is disconnected. Finally, we end this section with a result on maximal cliques.

**Definition 2.2.1**
Let $G$ be a group. The **enhanced power graph** of $G$, denoted by $\Gamma_{ep}(G)$, is the graph with vertex set the elements of $G$ and such that two vertices, $x$ and $y$, are adjacent if and only if the group generated by them is cyclic. Loops are removed.

**Remark 2.2.2**
An equivalent condition for adjacency is that two vertices, $x$ and $y$, are adjacent in the enhanced power graph if and only if $x, y \in \langle z \rangle$, for some $z \in G$.

**Example 2.2.3**
*Let us determine the proper enhanced power graph of $D_{12}$. We know from **Theorem 1.2.22** that the subgroups of $D_{12}$ are either cyclic or dihedral. We have that $\langle r^i, r^j \rangle$ is cyclic since $\langle r \rangle$ is cyclic. Now, as reflections have order two, subgroups generated by two reflections are not cyclic. Similarly, any subgroup generated by a reflection and a rotation is also non-cyclic. With this information, we draw:*

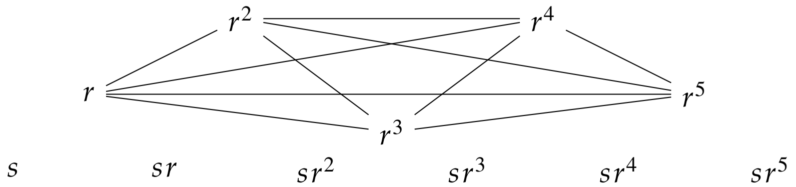

Figure 2.5: Proper enhanced power graph of the dihedral group $D_{12}$

Taking into account **Example 2.1.19**, we construct the proper power graph of $D_{12}$.

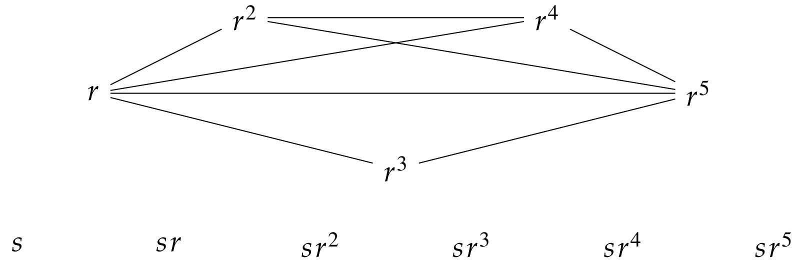

Figure 2.6: Proper power graph of the dihedral group $D_{12}$

Note that $\Gamma^*_p(D_{12})$ is a spanning subgraph of $\Gamma^*_{ep}(D_{12})$. We will see in the next chapter that this is true for all finite groups, while also doing a similar analysis for other graphs defined on groups.

The next proposition shows a distinction in adjacency between the power graph and the enhanced power graph of a group.

**Proposition 2.2.4** ([6], **Lemma 2.5**, [8], **Proposition 4.2**)
Let $G$ be a finite group and $x, y \in G \setminus \{e\}$ be such that $\gcd(o(x), o(y)) = 1$, with $xy = yx$. Then $x$ is adjacent to $y$ in $\Gamma_{ep}(G)$.

*Proof.* Take $x, y \in G \setminus \{e\}$ such that $\gcd(o(x), o(y)) = 1$ and $xy = yx$. As $\langle x, y\rangle$ is abelian and $\gcd(o(x), o(y)) = 1$ we have that $\langle x, y\rangle = \langle xy\rangle$. Therefore $x$ is adjacent to $y$ in the enhanced power graph. □

More generally, we have:

**Lemma 2.2.5** ([6], **Lemma 2.6**)
Let $G$ be a $p$-group and $x$, $y$ be two elements of $G$ of order $p$, $p^i$ $(i \geqslant 1)$, respectively. If there is a path between $x$ and $y$ in $\Gamma^*_{ep}(G)$, then $\langle x\rangle \subseteq \langle y\rangle$. In particular, if both $x$ and $y$ have order $p$, then $\langle x\rangle = \langle y\rangle$.

*Proof.* Take $x, y \in G$ such that $o(x) = p$ and $o(y) = p^i$ $(i \geqslant 1)$. Suppose that there is a path between $x$ and $y$ in the proper enhanced power graph. Let this path be

$$x \sim a_1 \sim a_2 \sim \cdots \sim a_n \sim y$$

Since $x \sim a_1$, then there exists $g \in G$ such that $x, a_1 \in \langle g\rangle$. Therefore $\langle x\rangle$ and $\langle a_1\rangle$ are subgroups of a cyclic group. It follows from **Proposition 1.2.4** that $x \in \langle a_1\rangle$. Similarly, $a_1$ is adjacent to $a_2$. With an analogous analysis we conclude that either $a_1 \in \langle a_2\rangle$ or $a_2 \in \langle a_1\rangle$. In either case it follows that $x \in \langle a_2\rangle$. Repeating this process we conclude that $x \in \langle y\rangle$. In particular, if $y$ has order $p$ then $\langle x\rangle = \langle y\rangle$. □

As done in the previous section, we will study for which groups the enhanced power graph is null or complete. Like the power graph, it trivially follows that the graph $\Gamma_{ep}(G)$ is never a null graph, since the identity is a dominant vertex. As for completeness, we have the following result:

**Proposition 2.2.6** ([7], **Theorem 2.4**)
$\Gamma_{ep}(G)$ is complete if and only if $G$ is cyclic.

*Proof.* The *only if* part follows directly from **Theorem 1.2.3** and the definition of enhanced power graph.

We will prove the converse statement by contradiction. Suppose that $G$ is non-cyclic and take a non-identity element $x \in G$ such that it has maximum order. Let $H$ be the subgroup generated by $x$. Since $H \neq G$, there is an element $y \in G \setminus H$. Consider the subgroup $\langle x, y \rangle$. Since the enhanced power graph of $G$ is complete, it follows that there is $z \in G$ such that $\langle x, y \rangle = \langle z \rangle$. As the order of $x$ is maximum and $\langle z \rangle \neq G$, it follows that $\langle x \rangle = H = \langle z \rangle$. But this contradicts the fact of $y$ not being in $H$. Therefore $G$ must be cyclic. □

We previously saw that the enhanced power graph is always connected. Hence we can determine its diameter. Next proposition shows that, for any group, it never exceeds 2.

**Proposition 2.2.7**
The diameter of the enhanced power graph of a given group $G$ is at most 2.

*Proof.* Consider non-identity elements $x$ and $y$ of $G$ such that $x \neq y$. Then, there is a path between $x$ and $y$ defined as follows

$$x \sim e \sim y$$

In conclusion, the diameter of the enhanced power graph is at most two. □

In a connected graph, we have the following definition:

**Definition 2.2.8**
Let $\Gamma$ be a graph. The **connectivity number** of $\Gamma$ is the minimum number of vertices that have to be removed from $V(\Gamma)$ so that the induced subgraph of the remaining vertices is disconnected. We denote it by $\kappa(\Gamma)$.

Since the enhanced power graph of $G$ is connected, we may determine its connectivity number. It turns out it varies depending on $G$.

**Proposition 2.2.9** ([6], **Theorem 1.2**)
Let $G$ be a non-cyclic abelian group. Then, $\kappa(\Gamma_{ep}(G)) = 1$ if and only if $G$ is a $p$-group.

*Proof.* We will first show that if $G$ is a $p$-group, then the connectivity number equals one. Since $G$ is non-cyclic and abelian, it follows from **Theorem 1.2.5** that $G$ has at least two distinct cyclic subgroups of order $p$. Suppose that those subgroups are $H_1$ and $H_2$. As a consequence of **Lemma 2.2.5**, there is no path between $x \in H_1$ and $y \in H_2$ in the proper enhanced power graph. Therefore if we remove the identity of $G$, the enhanced power graph induced by all other vertices is disconnected. In conclusion, $\kappa(\Gamma_{ep}(G)) = 1$.

We will show the converse by contrapositive. For this part of the proof we use the additive notation. Assume that $G$ is not a $p$-group. Then we may write $|G| = p_1^{r_1} \dots p_n^{r_n}$, for distinct prime powers $p_i^{r_i}$. Take $x, y \in G \setminus \{e\}$ such that they do not have order the power of the same prime. Suppose that $o(x) = p_1^{k_1} \dots p_n^{k_n}$ and $o(y) = p_1^{s_1} \dots p_n^{s_n}$. We can now consider the elements $a = p_1^{k_1} \dots p_{i-1}^{k_{i-1}} p_{i+1}^{k_{i+1}} \dots p_n^{k_n} x$ and $b = p_1^{s_1} \dots p_{j-1}^{s_{j-1}} p_{j+1}^{s_{j+1}} \dots p_n^{s_n} y$, of orders $p_i^{k_i}$ and $p_j^{s_j}$, respectively, where $p_i \neq p_j$. It follows from **Proposition 2.2.4**, that $a$ is adjacent to $b$. Now, as $a$ is a multiple of $x$, they are adjacent in the enhanced power graph. Similarly we conclude that $b$ is adjacent to $y$. Therefore there is the path

$$x \sim a \sim b \sim y$$

in the enhanced power graph of $G$. Thus $\Gamma^*_{ep}(G)$ is connected. In conclusion, $\kappa(\Gamma_{ep}(G)) > 1$.

If $x$ and $y$ have order a power of the same prime $p$, then we can consider a non-identity element $z \in G$ with order $q \neq p$. It follows from **Proposition 2.2.4** that there is a path $x \sim z \sim y$ in the proper enhanced power graph. We conclude that $\Gamma^*_{ep}(G)$ is connected, completing the proof. □

Now, we will focus our attention on the maximal cliques of the enhanced power graph.

**Remark 2.2.10**
Recall that a finite abelian group with $n$ elements is isomorphic to exactly one group (up to a permutation of factors) of the form

$$(\mathbb{Z}/p_1^{\alpha_1}\mathbb{Z}) \times (\mathbb{Z}/p_2^{\alpha_2}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_k^{\alpha_k}\mathbb{Z})$$

where $p_i$ are primes (not necessarily distinct) and $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$.

**Lemma 2.2.11** ([1], **Lemma 32**)
Let $x, y, z$ be elements of a group, $G$, such that $\langle x, y\rangle, \langle x, z\rangle$ and $\langle z, y\rangle$ are cyclic. Then, $\langle x, y, z\rangle$ is cyclic.

**Proposition 2.2.12**
If $G$ is a finite non-cyclic group and all proper subgroups generated by 2 elements are cyclic, then $G$ is a minimal non-cyclic group.

*Proof.* Since every subgroup of a finite group is finitely generated, we will show the result by induction in the number of generators of the proper non-trivial subgroup $H$. Suppose that $\langle x_1, \dots, x_k\rangle = \langle r\rangle$. We want to show that $H = \langle x_1, \dots, x_k, x_{k+1}\rangle \neq G$ is cyclic. If $k = 2$, the result follows by assumption. Thus, suppose that $k > 2$. Note the following,

$$\langle x_1, \dots, x_k, x_{k+1}\rangle = \langle \langle x_1, x_2\rangle \cup \langle x_1, x_3\rangle \cup \cdots \cup \langle x_1, x_{k+1}\rangle\rangle$$

By hypothesis:

$$\langle x_1, \dots, x_k\rangle = \langle \langle x_1, x_2\rangle \cup \langle x_1, x_3\rangle \cup \cdots \cup \langle x_1, x_k\rangle\rangle = \langle r\rangle$$

Thus

$$\langle x_1, \dots, x_k, x_{k+1}\rangle = \langle \langle r\rangle \cup \langle x_1, x_{k+1}\rangle\rangle$$

From assumption it follows that $\langle x_1, x_{k+1}\rangle = \langle z\rangle$, for some $z \in G$. Therefore

$$\langle \langle r\rangle \cup \langle x_1, x_{k+1}\rangle\rangle = \langle \langle r\rangle \cup \langle z\rangle\rangle = \langle r, z\rangle = \langle q\rangle$$

for some $q \in G$.

We conclude that $G$ is a minimal non-cyclic group. □

**Proposition 2.2.13** ([1], **Lemma 33**)
A maximal clique in $\Gamma_{ep}(G)$ corresponds to a cyclic subgroup of $G$.

*Proof.* Let $C$ be a maximal clique in the enhanced power graph of $G$. Take $x, y \in C$. Then by **Lemma 2.2.11**, any element in $\langle x, y\rangle$ is adjacent to any $z \in C$. Thus $C \cup \langle x, y\rangle$ is a clique. It follows from maximality of $C$ that $\langle x, y\rangle \subseteq C$. Therefore for any $x, y \in C$ we have that $xy \in C$. We show in an analogous way that $C$ is closed under inverse. Hence $C$ is a subgroup of $G$. Now, we may show by induction that any finite subset of $G$ generates a cyclic group. Since $C$ is finite, it follows that it is cyclic. □

The converse of the previous Proposition is not true.

**Example 2.2.14**
*Consider the generalized quaternion group with* $16$ *elements,* $Q_4$*. By* ***Theorem 1.3.20****,* $Q_4$ *has an element of order* $2$*. Consider the representation of* $Q_4$*:*

$$Q_4 = \langle a, b : a^8 = e, b^2 = a^4, bab^{-1} = a^{-1}\rangle$$

*The element* $a^4$ *is an involution in* $Q_4$*. We have that* $a^4$ *generates a cyclic subgroup of order* $2$*. However, the group* $\langle a^4\rangle$ *does not correspond to a maximal clique in* $\Gamma_{ep}(Q_4)$ *since it is contained in* $\langle a\rangle$*.*

We may ask for which groups does a cyclic subgroup corresponds to a maximal clique in the enhanced power graph. To the best of our knowledge, the answer is unknown.

## 2.3 The commuting graph

Chronologically, the **commuting graph** was the first graph on groups to be defined. It was introduced in 1955 by Brauer and Fowler in [9] to study the normalizer and centralizer (indirectly) of involutions. This article has been considered by some to be one of the firsts steps towards the classification of finite simple groups.

Although the commuting graph is the first attributed work in graphs defined on groups, the authors of [9] did not use the word graph in their paper. We will follow the current definition.

In this section, we determine for which groups the commuting graph is complete. Next, we study the cliques of the commuting graph of a given group. Then, we consider the commuting graph without the center of $G$ as part of the vertex set. We do this to study connectivity, as well as isolated edges and vertices in a non-trivial way. Finally, we bound the diameter of the commuting graph of nilpotent groups.

**Definition 2.3.1**
Let $G$ be a group. The **commuting graph** of $G$, denoted by $\Gamma_c(G)$, is the graph with vertex set $G$ in which two vertices, $g_1$ and $g_2$, are adjacent if and only if $g_1 g_2 = g_2 g_1$ in $G$. Loops are removed.

**Example 2.3.2**
*Consider the symmetric group* $S_3$*. The commuting graph of this group has as vertex set*

$$V(\Gamma_c(S_3)) = \{e, (12), (13), (23), (123), (132)\}$$

*and the following edge set*

$$E(\Gamma_c(S_3)) = \{(e, (12)); (e, (13)); (e, (23)); (e, (123)); (e, (132)); ((123), (132))\}$$

*The diagram of* $\Gamma_c(S_3)$ *is*

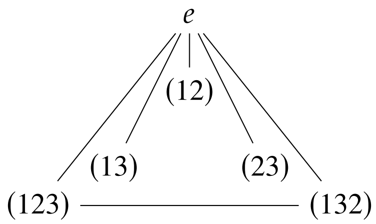

Figure 2.7: Commuting graph of the symmetric group $S_3$

Note from the previous example that the identity of $S_3$ is the only dominant vertex in the commuting graph. Thus $S_3$ has trivial center. This follows from the fact that any element in the center of $S_3$ must be a dominant vertex in the commuting graph.

**Example 2.3.3**
*Take the cyclic group* $\mathbb{Z}_4$*. Since* $\mathbb{Z}_4$ *is abelian, every vertex in its commuting graph is dominant. Hence* $\Gamma_c(\mathbb{Z}_4)$ *is complete.*

More generally, it follows:

**Proposition 2.3.4**
If $G$ is an abelian group then its commuting graph is complete.

It's not difficult to see that the converse of the previous Proposition is true.

Now we will study some properties on the commuting graph that allow us to obtain some information on the corresponding group.

**Proposition 2.3.5** ([10], **Proposition 2.1**)
Let $G$ be a group and $\Gamma_c(G)$ be its commuting graph. The maximal cliques in $\Gamma_c(G)$ correspond to the maximal abelian subgroups of $G$.

*Proof.* Let $S$ be a a maximal clique of $\Gamma_c(G)$. It is sufficient to show that $S$ is a subgroup of $G$. First, note that $e \in S$ since $S$ is maximal and $e$ is a dominant vertex. We will now show that $S$ is closed under the group operation. Take $g_1, g_2 \in S$. By definition, $g_1g_2 = g_2g_1$. Consider any $g \in S$. We have

$$gg_1g_2 = g_1gg_2 = g_1g_2g$$

By maximality of $S$, we conclude that $g_1g_2 \in S$.

Finally, it remains to show that $S$ is closed under inverse. For this consider $g \in S$ and $g^{-1} \in G$. Clearly, $g \sim g^{-1}$ in $\Gamma_c(G)$. Consider any $h \in S$. We have

$$g^{-1}h = g^{-1}hg^{-1}g = g^{-1}ghg^{-1} = hg^{-1}$$

It follows from maximality of $S$ that $g^{-1} \in S$. Therefore $S$ is a maximal abelian subgroup of $G$.

□

**Example 2.3.6**

*Consider the dihedral group $D_8$. We know that all rotations belong to the subgroup $\langle r \rangle$. Therefore rotations commute with one another. Also, we have that $r^2$ is in the center of $D_8$. Finally, it's not difficult to see that $s$ and $sr^2$ commute. Similarly, $sr$ and $sr^3$ also commute with each other. No other two reflections commute. We conclude that $\Gamma_c(D_8)$ is*

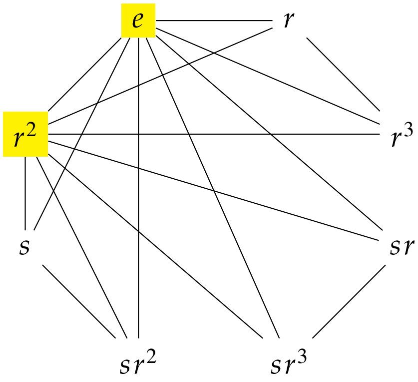

Figure 2.8: Commuting graph of $D_8$

From the previous example we see that $\Gamma_c(D_8)$ is not complete and, as expected, we conclude that $D_8$ is not abelian. Note as well that all vertices (except $e$ and $r^2$) have degree 3. Therefore maximal cliques have, *at most*, order 4. We can easily see that the commuting graph of $D_8$ has three maximal cliques. Thus, by **Proposition 2.3.5**, the dihedral group of the square has, *at least*, three maximal abelian subgroups. One of these subgroups is $H = \{e, s, r^2, sr^2\}$. Other cliques can be found by observation.

**Remark 2.3.7**

It's not difficult to see that if $H$ is a maximal abelian subgroup of a group $G$, then the elements of $H$ would form a maximal clique in $\Gamma_c(G)$. Therefore the converse **Proposition 2.3.5** is true.

**Remark 2.3.8**

i) In order to study certain graph properties in a non-trivial way for the commuting graph, we must exclude the center of the group $G$ from the vertex set of $\Gamma_c(G)$.

ii) Some authors define the commuting graph considering $V(\Gamma_c(G)) = G \setminus Z(G)$.

From now on, we assume that $V(\Gamma_c(G)) = G \setminus Z(G)$.

**Definition 2.3.9**

Let $H$ be a subgroup of the quotient $G/Z(G)$. Let $\overline{H}$ be the set:

$$\overline{H} := \bigcup_{hZ(G) \in H \setminus \{Z(G)\}} hZ(G)$$

That is, $\overline{H}$ is the set of all non-central elements of $H$.

Note that the previous definition is only relevant when $G$ does not have trivial center.

**Lemma 2.3.10** ([30], **Lemma 3.1**)
Let $G$ be a group and take $x, y \in G \setminus Z(G)$. If $x \sim y$, then $\overline{\langle xZ(G), yZ(G)\rangle}$ is a clique in $\Gamma_c(G)$. In particular, $\overline{\langle xZ(G)\rangle}$ is a clique in $\Gamma_c(G)$.

*Proof.* Consider $x, y \in G \setminus Z(G)$ and let $H = \langle xZ(G), yZ(G)\rangle$. Since $x \sim y$, we have that

$$H = \langle xZ(G), yZ(G)\rangle = \{x^i y^j Z(G) : i, j \in \mathbb{Z}\}$$

Take $x^{i_1}y^{j_1}Z(G),\ x^{i_2}y^{j_2}Z(G) \in H$. We have:

$$x^{i_1}y^{j_1}Z(G)x^{i_2}y^{j_2}Z(G) = x^{i_1}y^{j_1}x^{i_2}y^{j_2}Z(G) = x^{i_2}y^{j_2}Z(G)x^{i_1}y^{j_1}Z(G)$$

Therefore $\overline{H}$ is a clique in $\Gamma_c(G)$. □

**Lemma 2.3.11** ([29], **Lemma 3.1**)
Let $G$ be a finite group with center $Z(G)$. Then, the order of $Z(G)$ is a common divisor of the set of integers $\{\deg(v) + 1 : v \in V(\Gamma_c(G))\}$. In particular, $Z(G)$ has at most $\delta(\Gamma_c(G)) + 1$ elements, where $\delta(\Gamma)$ is the minimum among all degrees of vertices in $\Gamma$.

*Proof.* Consider $v \in V(\Gamma_c(G))$ and its centralizer. Clearly, $Z(G) \subseteq C_G(v)$. Thus, we may consider $C_G(v) \setminus Z(G)$. Note that $C_G(v) \setminus Z(G)$ has $v$ and all vertices adjacent to it in $\Gamma_c(G)$ as elements. Therefore, $|C_G(v) \setminus Z(G)| = \deg(v) + 1$. Finally, by **Theorem A.0.12** and knowing that $Z(G) \leq C_G(v)$, it follows that $|Z(G)|$ divides $|C_G(v)|$. Consequently, $|Z(G)|$ divides $|C_G(v) \setminus Z(G)| = \deg(v) + 1$.

Particularly, $Z(G)$ has at most $\min\{\deg(v) : v \in V(\Gamma_c(G))\} + 1$ elements, i.e., $\delta(\Gamma_c(G)) + 1$ elements. □

The next corollary is immediate.

**Corollary 2.3.12** ([29], **Lemma 3.11**)
Let $G$ be a group whose commuting graph contains an isolated edge, $(x, y)$. Then $Z(G)$ contains at must two elements.

**Example 2.3.13**
*Consider the commuting graph of $S_3$ without its center:*

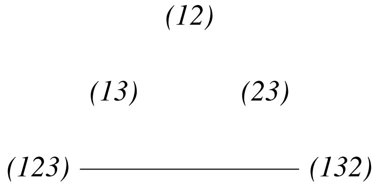

Figure 2.9: Proper commuting graph of $S_3$

*Note that* (12), (23) *and* (13) *are isolated vertices. From* ***Lemma 2.3.11*** *it follows that $Z(S_3)$ has at most one element. In general, if $\Gamma_c(G)$ has an isolated vertex, then the center of $G$ is trivial.*

**Remark 2.3.14**
It follows from the previous example and **Theorem 1.4.12** that groups whose commuting graph has an isolated vertex are not nilpotent.

**Theorem 2.3.15** ([30], **Theorem 1.5**)
If the index of the center of a group $G$ is finite and it is the product of **at least** two and **at most** three prime numbers, not necessarily distinct, then $\Gamma_c(G)$ is disconnected.

*Proof.* Suppose $[G : Z(G)] = pq$ or $[G : Z(G)] = pqr$, where $p, q, r$ are prime numbers not necessarily distinct.

Let $[G : Z(G)] = pq$. Suppose, in order to get a contradiction, that $\Gamma_c(G)$ is connected. We know from **Theorem 1.3.20** that $G/Z(G)$ admits an element of prime order. Let this element be $xZ(G)$, with $x \in G \setminus Z(G)$. Define $A := \langle xZ(G) \rangle$. It follows from **Lemma 2.3.10** that $\overline{A}$, as defined in **Definition 2.3.9**, is a clique in $\Gamma_c(G)$. Now, since $\Gamma_c(G)$ is connected, there exists some $a \in G \setminus (\overline{A} \cup Z(G))$ and $b \in \overline{A}$ such that $a \sim b$. Since $|A| = p$, for prime $p$, and $bZ(G) \in A \setminus \{Z(G)\}$,then $\langle bZ(G) \rangle = \langle xZ(G) \rangle$. By **Lemma 2.3.10** $a \sim b$ implies that $a \sim x$ and $\overline{B}$ is a clique in $\Gamma_c(G)$, where $B := \langle aZ(G), xZ(G) \rangle$. By definition, we have that $|A| < |B|$. Hence we may assume that $|B| = pq$ for primes $p$ and $q$, not necessarily distinct. Also, either $B = G/Z(G)$ or $[G : Z(G)] = pqr$ for not necessarily distinct primes $p, q$ and $r$.

Suppose that $[G : Z(G)] = pqr$. Again, since $\Gamma_c(G)$ is connected, it follows that there is $b \in \overline{B}$ and $y \in G \setminus (\overline{B} \cup Z(G))$ such that $y \sim b$. Hence we obtain a clique in $\Gamma_c(G)$ defined by $\overline{C}$, with $C := \langle aZ(G), xZ(G), yZ(G) \rangle$. Now, since $|B| < |C|$ and $Z(G)$ has index the product of at most three primes, then $C = G/Z(G)$. Hence $\overline{C}$ has all of the elements of $\Gamma_c(G)$. It follows that since $\overline{B}$ is a clique, then $b$ commutes with $x$ and $a$. Now, $b$ also commutes with $y$. Thus $b$ commutes with any coset leader in $C$. It follows from **Lemma 2.3.10** that $b$ commutes with every element in those cosets. Therefore $b \in Z(G)$. This is a contradiction. We conclude that $\Gamma_c(G)$ must be disconnected.

If we suppose that $B = G/Z(G)$, it's analogous. □

Sometimes we are able to construct subgroups of a given group based on its commuting graph.

**Lemma 2.3.16** ([29], **Lemma 3.2**)
Let $\Gamma_1$ be a connected component of $\Gamma_c(G)$ with diameter at most 2. Then, $\Gamma_1 \cup Z(G)$ is a subgroup of $G$.

*Proof.* It's sufficient to show that $\Gamma_1 \cup Z(G)$ is a group. We have two cases:

1. $\operatorname{diam}(\Gamma_1) = 1$.

   In this case, it's clear that $\Gamma_1$ is a complete graph. Therefore, by **Proposition 2.3.5**, $\Gamma_1 \cup Z(G)$ is a subgroup of $G$.

2. $\operatorname{diam}(\Gamma_1) = 2$.

   First note that $e \in Z(G)$. We will now show that $\Gamma_1 \cup Z(G)$ is closed under group operation and inverse. Take $g, h \in \Gamma_1$. If $g$ and $h$ commute, the proof is analogous to that of **Proposition 2.3.5**, so assume they don't commute. By hypothesis, it follows that there is a path

$$g \sim x \sim h$$

   for some $x \in \Gamma_1$. Let's suppose, without loss of generality, that $gh = s$. If $s \in Z(G)$, there's nothing to prove. If not, then

$$xs = xgh = gxh = ghx = sx$$

Therefore $x \sim s$. Since $\Gamma_1$ is a connected component, it follows that $s = gh \in \Gamma_1$. Similarly we conclude that $hg \in \Gamma_1$. It remains to show that $\Gamma_1$ is closed under inverse. By definition of connected component, it follows that for any $g \in \Gamma_1$ we have that $g^{-1} \in \Gamma_1$. The proof is complete.

$\square$

**Example 2.3.17**
*Take the quaternion group* $Q_8$*. We have seen that* $Z(Q_8) = \{1, -1\}$*. Hence* $\Gamma_c(Q_8)$ *is represented by the following graph:*

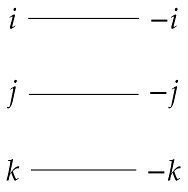

Figure 2.10: Commuting graph of the quaternion group

*It's immediate that all connected components have order* 2*. It follows from* ***Lemma 2.3.16*** *that* $H = \{1, -1, i, -i\}$ *is a subgroup of* $Q_8$*. In fact, this subgroup is one of the three maximal abelian subgroups of* $Q_8$*. The remaining two are obtained in a similar way.*

The previous lemma does not hold when a connected component has diameter greater than two.

**Example 2.3.18** ([29], **Example 3.3**)
*Consider the symmetric group* $S_4$*. By doing some calculations, we can see that the elements in* $S_4$ *of order* 2 *or* 4 *are those of the form:* $(ij)$*,* $(ij)(kl)$ *or* $(ijkl)$*, where* $i, j, k, l \in \{1, 2, 3, 4\}$*. There are* 15 *of these elements in* $S_4$*. It was shown in [29] that these elements form a connected component of* $\Gamma_c(S_4)$ *with diameter* 3*. However,* $S_4$ *has no subgroup of order* 16*.*

We will now find a bound for the diameter of the commuting graph of a nilpotent group and determine when this bound is not verified.

**Definition 2.3.19**
We say that $G$ is a **central product** of two subgroups $H$ and $K$ if:

i) $G$ is generated by $H$ and $K$.

ii) Every element of $H$ commutes with every element of $K$, and vice-versa.

iii) $H \cap K \subseteq Z(G)$.

**Remark 2.3.20**
Note that if $H \cap K = \{e\}$, then the central product of $H$ and $K$ is the **direct product** of $H$ and $K$.

**Example 2.3.21** ([40], **Section 2.1.3**)
*We define the* ***Pauli matrices*** *as the four* $2 \times 2$ *matrices defined bellow:*

$$I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \textit{and} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

*It can proved that the matrices $X, Y$ and $Z$ generate a group of order* 16. *Such group is known as the Pauli group and it is the central product between $D_8$ and $\mathbb{Z}_4$, [20, 33].*

**Theorem 2.3.22** ([30], **Theorem 1.2**)
Suppose that a non-abelian group $G$ is the central product of two of its subgroups $H$ and $K$. If $H$ and $K$ are both non-abelian, then

$$\operatorname{diam}(\Gamma_c(G)) \leqslant \min\{3, \operatorname{diam}(\Gamma_c(H)), \operatorname{diam}(\Gamma_c(K))\}$$

If, otherwise, exactly one of $H$ or $K$ is abelian, say $K$, then

$$\operatorname{diam}(\Gamma_c(G)) = \operatorname{diam}(\Gamma_c(H))$$

*Proof.* Let $G$ be a group such that it is the central product of two of its non-abelian subgroups, $H$ and $K$. Take $v, w \in G \setminus Z(G)$. Since $G$ is equal to the central product of $H$ and $K$, then $v = h_1k_1$ and $w = h_2k_2$, for some $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Now, because $v, w \notin Z(G)$ and since $Z(G) = Z(H)Z(K)$, we have three distinct cases:

1) None of the $h_1$, $h_2$, $k_1$, $k_2$ are central.
2) Without loss of generality, $h_1$, $h_2$ are central.
3) Without loss of generality, $h_1$ is central and $h_2$ is not central.

We will show that in any of these cases, the inequality holds. 1) Suppose that none of the $h_1$, $h_2$, $k_1$, $k_2$ are central. Then,

$$v = h_1k_1 \sim k_1 \sim h_2 \sim h_2k_2 = w$$

since every element in $H$ commutes with every elements on $K$, and vice-versa. Hence there is a path of length three connecting $v$ and $w$ in the commuting graph. Now, if $\Gamma_c(K)$ has diameter two, then $\exists\, x \in K \setminus Z(K)$ such that $k_1 \sim x \sim k_2$; and similarly if $\Gamma_c(H)$ has diameter two. We conclude that, in either case, we can find a path between $v$ and $w$ of length 2.

2) Suppose now, without loss of generality, that $h_1$ and $h_2$ are central. Then,

$$v = h_1k_1 \sim h' \sim h_2k_2 = w$$

where $h'$ is a non-central element. This element is guarantee to exist because $H$ is non-abelian.

For the third and final case suppose, without loss of generality, that $h_1$ is central and $h_2$ is not. Then,

$$v = h_1k_1 \sim h_2 \sim h_2k_2 = w$$

Thus, for any $v$, $w$, non-central elements, there exists a path of length less or equal than 3 of non-central elements connecting them. Therefore

$$\operatorname{diam}(\Gamma_c(G)) \leqslant \min\{3, \operatorname{diam}(\Gamma_c(H)), \operatorname{diam}(\Gamma_c(N))\}$$

We will now study the particular case when exactly one of $H$ or $K$ is abelian. For this suppose, without loss of generality, that $H$ is abelian. Let $v$ and $w$ be as the first part of the proof. It follows from the hypothesis that $k_1, k_2$ are not central. Suppose that $\Gamma_c(K)$ is connected and that it has diameter $n$. As a consequence, there exists a path between $v$ and $w$ in $\Gamma_c(K)$ of length at most $n$. Let this path be

$$v = h_1k_1 \sim x_1 \sim \cdots \sim x_{n-1} \sim h_2k_2 = w$$

with $x_i \in K \setminus Z(K)$. Hence $\mathrm{diam}(\Gamma_c(G)) \leqslant \mathrm{diam}(\Gamma_c(K))$.

Conversely, suppose that there exists a path in $\Gamma_c(G)$ of length $m$. Let this path be

$$h_1k_1 \sim a_1b_1 \ldots a_nb_n \sim \cdots \sim a_{m-1}b_{m-1} \sim h_2k_2$$

Then, since $H$ is abelian and $a_ib_i$ is not central, this means that $b_i$ is not central for all $i$. Thus, there exists a path

$$k_1 \sim b_1 \ldots b_n \sim \cdots \sim b_{m-1} \sim k_2$$

in $\Gamma(K)$, meaning that $m \geqslant n$. We conclude that

$$\mathrm{diam}(\Gamma_c(G)) = \mathrm{diam}(\Gamma_c(K))$$

Particularly, if $\Gamma_c(K)$ is disconnected, then $\Gamma_c(G)$ is also disconnected. □

We now have the particular case when $G$ is nilpotent.

**Corollary 2.3.23** ([30], **Corollary 1.3**)
Let $G$ be a finite nilpotent group. Then, two things can happen: either $\mathrm{diam}(\Gamma_c(G)) \leqslant 3$ or $G = A \times P$, where $A$ is an abelian group and $P$ is a non-abelian $p$-group.

*Proof.* It follows from **Theorem 1.4.14** and from **Theorem 2.3.22**. □

**Example 2.3.24**
*Consider the group* $G = Q_8 \times \mathbb{Z}_2$*. By item ii) of* ***Example 1.4.9****, together with* ***Theorem 1.4.10*** *and* ***Theorem 1.4.13****, we have that* $Q_8 \times \mathbb{Z}_2$ *is nilpotent. Therefore, it follows from the previous Theorem that* $\mathrm{diam}(\Gamma_c(G)) = \mathrm{diam}(\Gamma_c(Q_8))$*. As we saw in* ***Example 2.3.17****, the commuting graph of the quaternion group is disconnected. Hence* $\Gamma_c(G)$ *must also be disconnected.*

## 2.4 The non-generating graph

In this last section of **Chapter 2** we define the **non-generating graph**. As far as we know, this graph was only considered in 2022, separately, by Peter Cameron and Lucchini and Nemmi (2020 if we consider the unofficial publications). Cameron considered it for its definition of **graph hierarchy** (see [10]), which we will define in **Chapter 3**. On their part, Lucchini and Nemmi studied the connectivity of the non-generating graph (see [37]). To the best of our knowledge, it is the only article solely focused on the non-generating graph.

In this section we determine for which groups the non-generating graph is complete. Next, as there is no characterization, to the best of our knowledge, for when it is a null graph, we explore when it has isolated vertices.

**Definition 2.4.1**
Let $G$ be a group. The **non-generating graph** of $G$, which will be denoted by $\Gamma_{ng}(G)$, is the graph with vertex set the elements of $G$ where two vertices $x$ and $y$ are adjacent if and only if $\langle x,y\rangle \neq G$, i.e., $x$ and $y$ are adjacent if and only if they do not generate the whole group. Loops are removed.

**Remark 2.4.2**
Defined in 1996, the **generating graph** is the complement of the non-generating graph, and vice-versa.

Let's look at an example to see both the non-generating graph and the generating graph of a group.

**Example 2.4.3**
*Consider the direct product $\mathbb{Z}_3 \times \mathbb{Z}_3$. We first determine with some simple calculations all subgroups of $\mathbb{Z}_3 \times \mathbb{Z}_3$ generated by one element:*

$$\begin{aligned}\langle(0,0)\rangle &= \{(0,0)\}\\ \langle(1,0)\rangle &= \{(1,0),(2,0),(0,0)\} = \langle(2,0)\rangle\\ \langle(0,1)\rangle &= \{(0,1),(0,2),(0,0)\} = \langle(0,2)\rangle\\ \langle(1,1)\rangle &= \{(1,1),(2,2),(0,0)\} = \langle(2,2)\rangle\\ \langle(1,2)\rangle &= \{(1,2),(2,1),(0,0)\} = \langle(2,1)\rangle\end{aligned}$$

*Using this information and the fact that if $y \in \langle x\rangle$ then $\langle x,y\rangle \neq \mathbb{Z}_3 \times \mathbb{Z}_3$, we conclude that the generating graph of $\mathbb{Z}_3 \times \mathbb{Z}_3$ is*

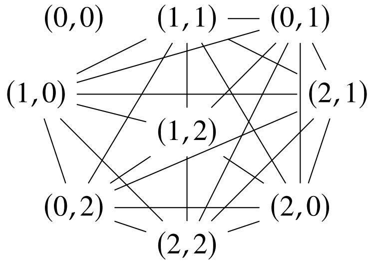

Figure 2.11: Generating graph of $\mathbb{Z}_3 \times \mathbb{Z}_3$

*We now compute the complement of the previous graph and obtain:*

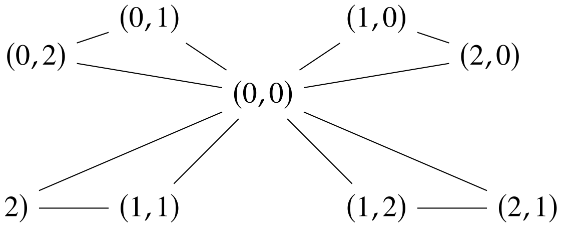

Figure 2.12: Non-generating graph of $\mathbb{Z}_3 \times \mathbb{Z}_3$

Note that the non-generating graph is connected, while the generating graph is disconnected. In general, at least one of these graphs must be connected.

We will now study the extreme cases, edgewise. We start by determining for which groups the non-generating graph is complete.

**Proposition 2.4.4**
Let $G$ be a group. The graph $\Gamma_{ng}(G)$ is complete if and only if $G$ is not 2-generated.

*Proof.* We will show the only if part by contrapositive. For this, suppose that $G$ is 2-generated. It follows that its generating graph has at least one edge. Hence $\Gamma_{ng}(G)$ can not be complete. Conversely, suppose that $\Gamma_{ng}(G)$ is a complete graph. Then the generating graph of $G$ doesn't have any edges. Therefore $G$ is not 2-generated. ◻

**Example 2.4.5**

i) *By **Example 1.3.2**, the non-generating graph of any dihedral group is not complete. Likewise for the cyclic groups.*

ii) *Let G be a group generated by three or more elements. Then, clearly, the non-generating graph of G is complete.*

**Remark 2.4.6**
Note that the previous Proposition also classifies the groups whose generating graph has no edges.

We have seen conditions for the non-generating graph to be complete. Conversely, we can ask: when is it null? The answer to this question, to the best of our knowledge, is not known. Therefore, we will begin by searching for conditions for the graph $\Gamma_{ng}(G)$ to have an isolated vertex.

For the next theorem we consider $\Gamma_{ng}(G)$ after removing dominant vertices. To avoid any confusion, we will denote the resulting graph by $\Gamma^{-}_{ng}(G)$.

**Theorem 2.4.7** ([37], **Proposition 2**)
Let $G$ be a 2-generated finite group. Then $\Gamma^{-}_{ng}(G)$ has an isolated vertex if and only if one of the following happens:

1. $G$ is cyclic;
2. $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$;
3. $G \cong D_{2p}$, for $p$ odd prime.

*Proof.* Let $G$ be a 2-generated finite group.

For the only if part we note that the generators of a cyclic group and the involutions of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $D_{2p}$ are isolated vertices in its non-generating graph. Conversely, suppose that $\Gamma^{-}_{ng}(G)$ has an isolated vertex, $g$. We have two cases:

a) $G$ is cyclic. In this case, the result follows immediately.

b) $G$ is not cyclic. It follows that $g = g^{-1}$. Otherwise they would be adjacent in the non-generating graph, which is a contradiction. Take $h \in V(\Gamma^{-}_{ng}(G))$. We will first show that if $\langle g, h\rangle = G$, then $\langle g, h^k\rangle = G$, for any power of $h$.

   Let $k$ be an integer. We first notice that $\langle gh^k, h\rangle = \langle g, h\rangle$. It follows that $gh^k$ is a vertex in the $\Gamma^{-}_{ng}(G)$. Therefore $h^k = e$ or $\langle g, gh^k\rangle = G$ $(i \in \mathbb{Z})$. If $h^k = e$, then $G$ would be cyclic, which is a contradiction. Therefore the second condition holds. We see that $\langle g, gh^k\rangle = \langle g, h^k\rangle$.

In conclusion, if $g$ and $h$ generate $G$, then $g$ and any non-trivial power of $h$ also generate $G$. Now we are ready to finish the if part of the proof.

First, suppose that $G$ is abelian. It follows from the previous paragraph that $\langle g, h^k\rangle = G$ for $k \in \mathbb{Z}$. Since $G$ is not cyclic, we have that $\mathrm{gcd}(\mathrm{o(g)}, \mathrm{o(h^k)}) \neq 1$. Hence $o(h^k) = o(h) = 2^t s$, for some $s, t \in \mathbb{Z}$. Next, note that $h \in \langle g, h^k\rangle$, thus

$$h = g^a (h^k)^b, \text{ for some } b \in \mathbb{Z}$$

Since $g$ has order two, then $a \in \{0, 1\}$.

If $a = 1$ it follows that

$$\begin{aligned} h &= gh^{kb} \\ \Leftrightarrow gh &= h^{kb} \\ \Leftrightarrow g &= h^{kb-1} \end{aligned}$$

Hence $g$ is a power of $h^k$ which contradicts the fact of $G$ not being cyclic. Therefore $a = 0$. From this we obtain

$$h = h^{kb}$$

We conclude that

$$kb - 1 = o(h)t$$

It follows that $\mathrm{gcd(k, o(h))} = 1$. Since this is true for any $k$, then $h$ must have prime order. Combining this with $\mathrm{gcd}(\mathrm{o(g)}, \mathrm{o(h^k)}) \neq 1$ and $o(h^k) = o(h)$, we have that $o(h) = 2$. Consequently, $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Suppose now that $G$ is non-abelian. Take $g^h$ to be the conjugation by $h$, i.e., $h^{-1}gh$. It follows that $\langle g, h\rangle = \langle h, h^{-1}gh\rangle = G$. Hence $g^h$ is a vertex in $\Gamma^{-}_{ng}(G)$. Since $g$ is an isolated vertex, then $\langle g, h^{-1}gh\rangle = G$. Consider the following representation for the dihedral group of order $2n$:

$$D_{2n} = \langle s, t | s^2 = e, t^2 = e, (st)^n = e\rangle$$

We may do the following distinction: $s = g$, $t = g^h$ and $st = gg^h$. In this case, we have that $\langle g, g^h\rangle = \langle g, gg^h\rangle \cong D_{2n}$. By the first part of the proof, $g$ generates $G$ for any non-trivial power of $gg^h$. Now, suppose that $gg^h$ does not have prime order. Take $o(gg^h) = p_1 p_2 p_3 \ldots p_k$., where $p_i$ is a prime number (not necessarily distinct). Consider

$$\langle g, (gg^h)^{p_1}\rangle = G$$

As $(gg^h)^{p_1}$ has order $p_2 p_3 \ldots p_k \neq o(gg^h)$, then it will never generate all rotations. Hence $\langle g, (gg^h)^{p_1}\rangle \neq G$, which is a contradiction.

□

**Example 2.4.8**

*Let us determine the proper non-generating graph of $D_{12}$. Studying the subgroups generated by each pair of elements in the dihedral group $D_{12}$, we obtain:*

*i) For any* $x, y \in \langle r \rangle$ *it's clear that* $\langle x, y \rangle \neq D_{12}$. *Thus any two rotations are adjacent;*

*ii) The element* $r^3$ *is central to all other elements. Thus any 2-generated subgroup, with* $r^3$ *as one of the generators, is commutative;*

*iii) Note that* $s$ *is adjacent to* $sr^3$, $sr$ *is adjacent to* $sr^4$, *and* $sr^2$ *is adjacent with* $sr^5$ *(because each pair commutes);*

*iv) Finally, the sets* $\{r^2, r^4, s, sr^2, s^4\}$ *and* $\{r^2, r^4, sr, sr^3, sr^5\}$ *are cliques in* $D_{12}$. *These, join with the identity, are the dihedral proper subgroups of* $D_{12}$.

*In conclusion, the proper non-generating graph of* $D_{12}$ *is the following*

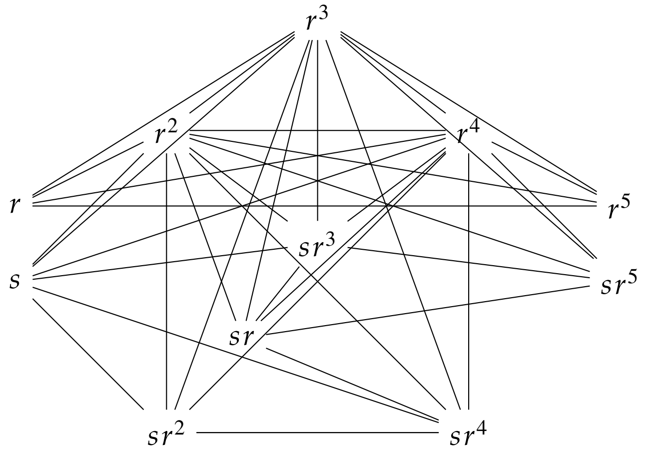

Figure 2.13: Proper non-generating graph of $D_{12}$

# Chapter 3

# Difference of graphs in the hierarchy

In this chapter we introduce the concept of **graph hierarchy** defined by Peter Cameron in [10] and we show that it is well defined. Next, we classify the groups for which some of the graphs in the hierarchy are equal. Particularly, we study equality between the power graph and the enhanced power graph; the commuting graph and the enhanced power graph; the non-generating graph and the commuting graph; and the non-generating graph and the power graph. In **subsection 3.2.1** we study the difference between the enhanced power graph and the power graph. We show which are its isolated vertices and use this fact to study connectivity and the diameter of some classes of groups. Next, we see for which groups it is null and that it is never complete. We finish with **subsection 3.2.2** where we study the difference between the non-generating and the commuting graph. We study its isolated vertices and show that no connected component in this graph is complete. Then, we show that if $G$ is a group with every maximal subgroup normal and if its non-generating non-commuting graph has one edge, then, after removing isolated vertices, it has diameter at most three. We complete this study with a result for alternating groups and dihedral groups.

This chapter uses [1, 8, 10, 14, 25] as main references.

As it was previously stated, we consider $G$ to be a **finite** group, unless stated otherwise.

## 3.1 The graph hierarchy and equalities

**Definition 3.1.1**
Let $G$ be a group. We call **hierarchy of graphs** to the following ordered chain of spanning subgraphs, from top to bottom.

1. Null graph.
2. Power graph.
3. Enhanced power graph.
4. Commuting graph.
5. Complete graph.

Moreover, if $G$ is a non-abelian or a non 2-generated group, we can add the non-generating graph between the commuting graph and the complete graph.

**Remark 3.1.2**
In the hierarchy presented in [10] the author includes the deep commuting graph. This graph is between the commuting and the enhanced power graph. The deep commuting graph was not included in the previous definition as it exceeds the scope of this essay.

We begin by showing that the hierarchy is well-defined.

**Theorem 3.1.3** ([10], **Proposition 2.6**)
Let $G$ be a group. The following hold:

i) $E(\Gamma_p(G)) \subseteq E(\Gamma_{ep}(G)) \subseteq E(\Gamma_c(G))$.

ii) If $G$ is non-abelian or not 2-generated, then $E(\Gamma_c(G)) \subseteq E(\Gamma_{ng}(G))$.

*Proof.* The inclusions in $i)$ follow immediately from the definitions of each respective graph.

Now, in order to show that $E(\Gamma_c(G)) \subseteq E(\Gamma_{ng}(G))$, suppose that $G$ is a non-abelian group and let $(x,y) \in E(\Gamma_c(G))$. We will prove that $(x,y) \in E(\Gamma_{ng}(G))$. We know that $xy = yx$. Since $G$ is non-abelian, then $x$ and $y$ can not generate $G$. Consequently, $(x,y) \in E(\Gamma_{ng}(G))$. Now, if $G$ is not 2-generated, then $\Gamma_{ng}(G)$ is complete and the inclusion follows. □

If $G$ is abelian and 2-generated, then the commuting graph may not be a spanning subgraph of the non-generating graph.

**Example 3.1.4**
*Consider the group* $\mathbb{Z}_2 \times \mathbb{Z}_2$. *Since this group is abelian, its commuting graph is complete. Also, it is generated by* $(1,0)$ *and* $(0,1)$. *Therefore these elements are not adjacent in the non-generating graph. We conclude that the commuting graph of* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *is not an spanning subgraph of the non-generating graph of the same group.*

Before we study equality of graphs in the hierarchy, we need a definition.

**Definition 3.1.5**
Let $G$ be a group of order $n$. The $GK$**-graph** of $G$ is the graph with vertex set the prime divisors of n where two vertices, $p$ and $q$, are adjacent if there exists an element in $G$ of order $pq$.

The graph from the previous definition is frequently called **prime graph** of $G$. This graph was first defined in an unpublished manuscript by Gruenberg and Kegel. Because of this, some authors call this graph the $GK$-graph, [10]. This is the name we will adopt.

**Example 3.1.6**
*Consider the group representation of the dihedral group* $D_{12}$ *presented in* ***Chapter 1.4***. *We know that* $D_{12}$ *has twelve elements and that the only prime divisors of* $12$ *are* $2$ *and* $3$. *Also, the group* $D_{12}$ *has an element of order* $6$, *which is the rotation* $r$. *Thus the GK-graph of* $D_{12}$ *is*

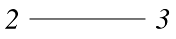

Figure 3.1: $GK$-graph of $D_{12}$

Now, we introduce some of the most important theorems of this essay.

**Theorem 3.1.7** ([1], **Theorem 28**)
Let $G$ be a group. Then the following assertions are equivalent:

(1) The power graph of $G$ is equal to the enhanced power graph of $G$.

(2) Every cyclic subgroup of $G$ has prime power order.

(3) The $GK$-graph of $G$ is a null graph.

*Proof.* We will first show that if the power graph of $G$ is equal to its enhanced power graph, then every cyclic subgroup of $G$ has prime power order. Let $C$ be a cyclic subgroup of $G$. It follows that $C$ induces a complete subgraph of $\Gamma_{ep}(G)$. Since the power graph is equal to the enhanced power graph, then $C$ also induces a complete subgraph of $\Gamma_p(G)$. It's immediate from **Theorem 2.1.11** that $C$ must have prime power order.

Now we will show that (2) implies (3). Hence suppose that (2) holds. Then every element of $G$ has prime power order. The result follows.

Next, we prove that (3) implies (1). Let $(x, y) \in E(\Gamma_{ep}(G))$. By definition, there exists $z \in G$ such that $x, y \in \langle z \rangle$. From the assumption we know that $z$ has prime power order. Therefore the power graph of $\langle z \rangle$ is complete. Consequently, $(x, y) \in \Gamma_p(G)$. We conclude that the enhanced power graph of $G$ is equal to the power graph of $G$. □

As an immediate consequence we have:

**Corollary 3.1.8**
For any finite $p$-group $G$, the power graph of $G$ is equal to the enhanced power graph of $G$.

**Remark 3.1.9**
It is possible to classify the groups that satisfy **Theorem 3.1.7**. Nonetheless, this was omitted because it was beyond the scope of this essay.

We may also ask: for which classes of groups is the enhanced power graph equal to the commuting graph? Next theorem answers this question.

**Theorem 3.1.10** ([1], **Theorem 30**)
Let $G$ be a group and $p$ a prime number. The following conditions are equivalent:

(1) $\Gamma_c(G) = \Gamma_{ep}(G)$.

(2) $G$ has no subgroup $\mathbb{Z}_p \times \mathbb{Z}_p$.

(3) The Sylow subgroups of $G$ are cyclic or (for $p = 2$) a generalized quaternion.

*Proof.* We will first show that (1) is equivalent to (2). For this, we begin by showing the contrapositive of the if part. It is easy to check that there are at least two elements in $\mathbb{Z}_p \times \mathbb{Z}_p$ that are adjacent in $\Gamma_c(G)$, but not in $\Gamma_{ep}(G)$. Therefore the commuting graph of $G$ is not equal to the enhanced power graph of $G$. In conclusion, we have that (1) implies (2). Conversely, assume that (2) holds. Let $g_1, g_2 \in G$ such that $g_1 \sim g_2$ in the commuting graph. Consider the group $H = \langle g_1, g_2 \rangle$. We know from **Theorem 1.2.5** that $H \cong \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_n}$, where $n_k$ is a positive integer. Now, it's clear that $\mathbb{Z}_p \times \mathbb{Z}_p$ is not a subgroup of $H$

since $\mathbb{Z}_p \times \mathbb{Z}_p$ is not a subgroup of $G$. Thus $H \cong \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_t}$, where $p_1, \ldots, p_t$ are distinct primes. It not difficult to see that $H \cong \mathbb{Z}_{p_1 \times \cdots \times p_t}$. Therefore $\langle g_1, g_2 \rangle$ is cyclic and $g_1 \sim g_2$ in $\Gamma_{ep}(G)$.

To finish the proof, we will show that (2) is equivalent to (3). Suppose that $G$ has no subgroups isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. It's clear that for any Sylow subgroup $P$ of $G$, the group $\mathbb{Z}_p \times \mathbb{Z}_p$ is not a subgroup of $P$. Therefore it follows from **Theorem 1.3.15** that $P$ is either cyclic or (when $p = 2$) a generalized quarternion. Conversely, let condition (3) hold. Then **Theorem 1.3.15** implies that no Sylow subgroup of $G$ has $\mathbb{Z}_p \times \mathbb{Z}_p$ as subgroup. Therefore $\mathbb{Z}_p \times \mathbb{Z}_p$ is not a subgroup of $G$.

□

**Example 3.1.11**
*Let G be a p-group that contains only one subgroup of order p. It follows from* ***Theorem 3.1.10*** *that the enhanced power graph and the commuting graph of G are equal. We know from* ***Theorem 1.3.15*** *that G is either cyclic or a generalized quaternion group.*

**Remark 3.1.12**
It is possible to classify the groups that satisfy **Theorem 3.1.10**. Nonetheless, this was omitted because it was beyond the scope of this essay.

For the next equality between graphs defined on groups we need the following definition.

**Definition 3.1.13**
Let $G$ be a non-abelian group. A group $G$ is **minimal non-abelian** when every proper subgroup of $G$ is abelian.

**Example 3.1.14**

1) *$S_3$ and $Q_8$ are minimal non-abelian groups because all of their proper subgroups are cyclic.*

2) *$D_8$ and $D_{10}$ are minimal non-abelian groups. However, the group $D_{12}$ is not. In general, the group $D_{2n}$ is not minimal non-abelian when $n$ is not a prime number and it is greater that* 4.

**Theorem 3.1.15** ([10], **Proposition 3.1**)
Let $G$ be a non-abelian group. The non-generating graph of $G$ is equal to the commuting graph of $G$ if and only if $G$ is a minimal non-abelian group.

*Proof.* We first show the only if part. It's sufficient to show that $V(\Gamma_{ng}(G)) \subseteq V(\Gamma_c(G))$ because of the hierarchy. Let $(x, y) \in E(\Gamma_{ng}(G))$. By **Definition 3.1.13** we have that $\langle x, y \rangle$ is abelian. Hence $(x, y) \in E(\Gamma_c(G))$.

Conversely, let $H$ be a proper subgroup of $G$ and take $x, y \in H$. Then $\langle x, y \rangle \neq G$. It follows from hypothesis that $\langle x, y \rangle$ is abelian. Consequently, $H$ is abelian. □

**Example 3.1.16**
*By* ***Example 3.1.14*** *we have that the commuting graph equals the non-generating graph for $S_3$, $D_8$, $D_{10}$ and $Q_8$.*

We finish this section with the following equality which, to the best of our knowledge, has not been considered.

**Proposition 3.1.17**
Let $G$ be a 2-generated non-abelian group. The non-generating graph of $G$ is equal to the power graph of $G$ if and only if all proper 2-generated subgroups of $G$ are cyclic with prime power order.

*Proof.* We will first show the *if part*. Let $(x, y)$ be an edge in the non-generating graph. Then, without loss of generality, it follows from hypothesis that $\langle x, y\rangle = \langle x\rangle$. Hence we conclude by **Theorem 2.1.11** that $\langle x\rangle$ has prime power order.

Conversely, assume that all proper 2-generated subgroups of $G$ are cyclic and have prime power order. Let $(x, y)$ be an edge in the non-generating graph. Then $\langle x, y\rangle$ is cyclic with prime power order. Thus $\langle x, y\rangle$ induces a complete power graph, so $(x, y) \in E(\Gamma_p)$. □

## 3.2 Differences between two graphs in the hierarchy

Looking at the results from **Section 3.1**, it's clear that, in general, two graphs of a given group are not equal. Hence it's natural to wonder about the difference between these graphs and what it can tell us about the structure of the group that generated it.

In some cases, the answer is simple, like in the top and bottom of the hierarchy. On the middle of the hierarchy things get trickier and, thus, more interesting.

### 3.2.1 The enhanced non-power graph

The difference between the power graph and the enhanced power graph was first defined by Biswas et al. in [8]. In the article, the authors called the graph resulting from this operation by **Difference graph**. However, since we will be studying several other differences between graphs in the hierarchy, we won't be using this terminology. Rather, we will name it the **enhanced non-power graph**.

We begin this section by identifying the isolated vertices of the enhanced non-power graph. This identification is needed to study connectivity in a non-trivial way, which is the main focus of this section. Next, we bound the diameter of the enhanced non-power graph of some particular groups, mainly: the dihedral, symmetric and alternating groups. To help with this study, we introduce a few conditions that guarantee adjacency in the enhanced non-power graph.

**Definition 3.2.1**
Let $G$ be a group. The **enhanced non-power graph of** $G$ is the graph with vertex set the elements of $G$ such that two vertices are adjacent if and only if they are adjacent in $\Gamma_{ep}(G)$ but not in $\Gamma_p(G)$. We denote this graph by $\Gamma_d(G)$.

**Example 3.2.2**
*Consider the abelian group $\mathbb{Z}_6$. We already saw the diagram for $\Gamma_p(\mathbb{Z}_6)$ in **Figure 2.3**. Now, it follows from **Proposition 2.2.6** that the enhanced power graph of $\mathbb{Z}_6$ is a complete graph. Therefore the enhanced non-power graph of $\mathbb{Z}_6$ is*

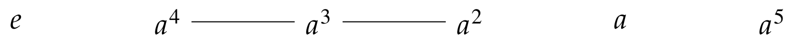

Figure 3.2: Enhanced non-power graph of $\mathbb{Z}_6$

**Remark 3.2.3**
When studying certain graph's invariants, like diameter and connectivity, it becomes necessary to remove isolated vertices from the enhanced non-power graph.

The enhanced non-power graph of $\mathbb{Z}_6$ after removing isolated vertices is

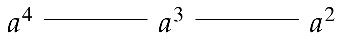

Figure 3.3: Enhanced non-power graph of $\mathbb{Z}_6$ after removing isolated vertices

Taking into account **Remark 3.2.3**, we need to identify which vertices are going to be isolated in order to study connectivity. Next theorem tells us precisely this.

**Proposition 3.2.4** ([8], **Proposition 2.1**)
Let $G$ be a group with order greater than 1 and consider $g \in G \setminus \{e\}$. We have that $g$ is an isolated vertex in $\Gamma_d(G)$ if and only if either $\langle g \rangle$ is a maximal cyclic subgroup of $G$ or if every cyclic subgroup of $G$ containing g has prime power order. Moreover, if $G$ is cyclic, then $G$ must have prime power order.

*Proof.* We will first show the only if part by contrapositive. Let $g \in G \setminus \{e\}$ be a non-isolated vertex of $\Gamma_d(G)$, i.e., there is $h \in G \setminus \{e\}$ such that $g \sim h$ in $\Gamma_{ep}(G)$ and $g \nsim h$ in $\Gamma_p(G)$. Then

$$(\langle g, h \rangle = \langle z \rangle, \text{for some } z \in G) \text{ and } (g \neq h^k \text{and } h \neq g^r \ \forall k, r \in \mathbb{Z})$$

Consequently, $\langle g \rangle \subsetneq \langle z \rangle$. Hence $\langle g \rangle$ is not maximal. Now, since $g$ and $h$ are not adjacent in the power graph we conclude by **Theorem 2.1.11** that $z$ does not have prime power order.

Conversely, let $g \in G \setminus \{e\}$ be an isolated vertex of $\Gamma_d(G)$ that does not generate a maximal cyclic subgroup of $G$. Thus there exists $h \in G \setminus \{e, g\}$ such that $\langle g \rangle \subsetneq \langle h \rangle$. Suppose, in order to get a contradiction, that $o(h)$ is not a prime power number. We may suppose that $g$ has prime power order, say $p^k$. Since $h$ does not have prime power order, it follows from **Theorem 1.3.20** that there is $x \in \langle h \rangle$ such that $x$ has order $q$ for some prime number different than $p$. On one hand, it's clear that $g$ and $h$ are adjacent in the enhanced power graph. On the other hand, under these conditions, **Proposition 2.1.10** guarantees that $g$ and $h$ are not adjacent in the power graph, which contradicts the assumption of $g$ being isolated in the enhanced non-power graph. Hence suppose that $g$ does not have prime power order. It follows from **Theorem A.0.12** that the order of $g$ divides the order of $h$. If the prime decomposition of $g$ does not share the same primes as $h$, it's analogous to the previous case. Hence we may assume that there is a prime $p$ in the prime decomposition of the order of $g$ and $h$ such that the exponent of $p$ in $o(h)$ is greater than that of $o(g)$. Let $y \in \langle h \rangle$ be such that the order of $y$ is the $p$-th part of the order of $h$. It follows from **Theorem 1.3.22** that such $y$ does exists. Thus $g$ and $y$ are adjacent in the enhanced power graph because they generate a subgroup of $\langle h \rangle$. However, $g \notin \langle y \rangle$ and vice-versa. Therefore $g$ and $y$ are adjacent in the enhanced non-power graph, a contradiction.

Finally, suppose that $G = \langle g \rangle$ is a cyclic group that does not have prime power order. We quickly check that $g$ is an isolated vertex in the enhanced non-power graph. It is clear that $\langle g \rangle$ is not a maximal cyclic subgroup of $G$. We also note that $g \in G$ and $G$ does not have prime power order. Therefore $G$ must have prime power order.

□

**Example 3.2.5**

*Recall the difference between the enhanced power graph and the power graph of $\mathbb{Z}_6$ before removing isolated vertices*

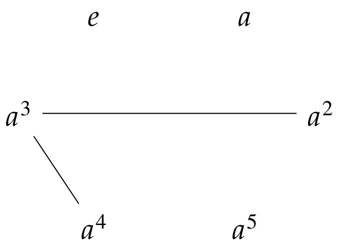

Figure 3.4: Difference between the enhanced power graph and the power graph of $\mathbb{Z}_6$

*As $\mathbb{Z}_6$ does not have prime power order, then its generators are isolated vertices that do not generate a maximal subgroup and are contained in a cyclic subgroup that does not have prime power order.*

**Example 3.2.6**

*Consider the difference between the proper enhanced power graph and the proper power graph of the dihedral group $D_{12}$*

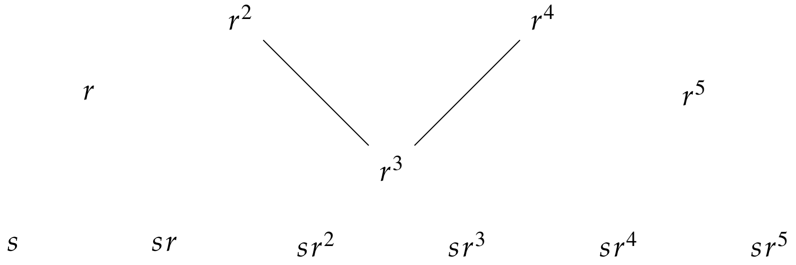

Figure 3.5: Difference between the proper enhanced power graph and the proper power graph of $D_{12}$

*Note that for any isolated vertex $x \in G$ we have that either $\langle x \rangle \cong \mathbb{Z}_6$ or $\langle x \rangle \cong \mathbb{Z}_2$. Both of these groups are maximal cyclic subgroups of $G$. Moreover, if $x$ is a reflection, then any cyclic group that contains $x$ has order 2. Hence reflections satisfy both conditions of* ***Proposition 3.2.4****.*

**Remark 3.2.7**

It follows from **Theorem A.0.12** that every finite group has a maximal cyclic subgroup. This means that the isolated vertex set of $\Gamma_d(G)$ has order greater that 1.

We now relate the isolated vertices of the enhanced non-power graph with those of the $GK$-graph.

**Theorem 3.2.8** ([8], **Theorem 3.2**)

Let $G$ be a group and $p$ a prime number. If the order of $G$ is divisible by $p$, then the following conditions are equivalent:

1. Every element of prime power order is an isolated vertex in $\Gamma_d(G)$.
2. $p$ is an isolated vertex in the $GK$-graph.
3. The centralizer of every element of order $p$ is a $p$-group.

*Proof.* We will first show that 1. implies 2. by contrapositive.

Assume that $p$ is not an isolated vertex in the $GK$-graph. This implies that for some prime $q$ divisor of $|G|$ there is $g \in G$ such that $o(g) = pq$. By **Theorem 1.3.20**, there exist elements $v$ and $w$ in $\langle g \rangle$ with orders $p$ and $q$, respectively. Since $\langle v, w \rangle = \langle vw \rangle$, it follows that $v \sim w$ in $\Gamma_{ep}(G)$. However, by **Proposition 2.1.10** we conclude that $v \nsim w$ in $\Gamma_p(G)$. Hence there are elements of prime power order that are not isolated in the enhanced non-power graph.

Now we will show that 2. implies 3. by contrapositive. Assume that there exists $h \in G$ with $o(h) = p$ whose centralizer does not have prime power order. Now, it follows from **Theorem 1.3.20** that there exists $t \in C_G(h)$ such that $o(t) = q$, for some prime divisor $q$ of $|G|$ different than $p$. Hence $\langle t, h \rangle = \langle th \rangle$. Therefore $th$ has order $pq$, which implies that $p \sim q$ in the $GK$-graph. We conclude that $p$ is not an isolated vertex of the $GK$-graph of $G$.

Finally, we will show that 3. implies 1. by contrapositive. Let $p$ be a prime number. Suppose that there exists $g \in G$ with order $p^k$ such that $g \sim h$ in $\Gamma_d(G)$ for some $h \in G$. Then there exists $x \in G$ such that $\langle g, h \rangle = \langle x \rangle$. Another consequence is that $g$ and $h$ are not a power of each other. If $\langle x \rangle$ has prime power order, then the power graph of the group generated by $x$ would be complete, a contradiction. Therefore it follows that $\langle x \rangle$ is not a $p$-group. Thus there exist elements $g', h' \in \langle x \rangle$ such that $o(g') = p_i$ and $o(h') = p_j$, for distinct primes $p_i$ and $p_j$. Suppose, without loss of generality, that $p_j \neq p$. It's clear that $h' \in C_G(g')$. Hence the centralizer of $g'$ does not have prime power order.

□

**Example 3.2.9**

*As we saw in* ***Example 3.2.2****, the enhanced non-power graph of the cyclic group* $\mathbb{Z}_6$ *has two non-identity isolated vertices. However, as there are elements with prime power order that are not isolated in* $\Gamma_d(\mathbb{Z}_6)$*, we conclude from the previous Theorem that at least one of the centralizers of* $a$, $a^2$, $a^3$, $a^4$ *and* $a^5$ *is not* $p$*-group.*

**Example 3.2.10**

*Recall the enhanced non-power graph of the dihedral group* $D_{12}$ *presented in* ***Example 3.2.6****. Since* $r^3$ *is an element of prime power order, but is not an isolated vertex in* $\Gamma_d(D_{12})$*, then the primes* 2 *and* 3 *are not isolated vertices in the* $GK$*-graph of* $D_{12}$*. In fact, the dihedral group on six vertices has two elements of order six, mainly* $r$ *and* $r^5$*.*

We saw that the set of isolated vertices of the enhanced non-power graph is never empty. A natural question is when is it equal to the whole group.

**Proposition 3.2.11**

$G$ is an EPPO group if and only if $\Gamma_d(G)$ has no edges.

*Proof.* Follows immediately from **Theorem 3.1.7** and the definition of EPPO group. □

**Example 3.2.12**

*Consider the symmetric group* $S_3$*. We saw in* ***Chapter 2*** *that it is an EPPO group. Thus it follows from the previous Proposition that the enhanced non-power graph of* $S_3$ *is a null graph with six vertices.*

**Remark 3.2.13**

As a direct consequence of the previous Proposition and definition of $p$-group, we have that the enhanced non-power graph of a $p$-group is a null graph.

Now that we know when the enhanced non-power graph has no edges, we may ask when it is complete. The answer is obvious when we consider the identity in the vertex set. Now, if we remove the identity, although not as trivial, the answer is the same.

**Proposition 3.2.14**
The proper enhanced non-power graph is never complete.

*Proof.* Let $G$ be a group. For the proper enhanced non-power graph of $G$ to be complete, two things must happen:

1) The proper enhanced power graph of $G$ must be complete;

2) The proper power graph of $G$ must be null.

On one hand, condition 1) is satisfied whenever $G$ is cyclic, as seen in **Proposition 2.2.6**. On the other hand, if $G$ is cyclic then the power graph of $G$ has at least one non-identity dominant vertex, the generator of $G$. Therefore the enhanced non-power graph is never complete. □

**Proposition 3.2.15**
If $\Gamma_{ep}(G)$ is complete, then $\Gamma_d(G)$ is the complement of $\Gamma_p(G)$. In this case, $G$ is cyclic.

*Proof.* It follows directly from the definition of enhanced non-power graph and from **Theorem 2.2.6** □

Now, we will determine some conditions that guarantee adjacency in the enhanced non-power graph.

**Proposition 3.2.16** ([6], **Lemma 2.5**, [8], **Proposition 4.2**)
Let $G$ be a finite group and $x, y \in G \setminus \{e\}$ be such that $\gcd(o(x), o(y)) = 1$ and $xy = yx$. Then $x$ and $y$ are adjacent in $\Gamma_d(G)$.

*Proof.* Take $x, y \in G$ such that $\gcd(o(x), o(y)) = 1$ and $xy = yx$. We know from **Proposition 2.1.10** that $x$ and $y$ are not adjacent in the power graph. Now, it follows by hypothesis that $\langle x, y\rangle = \langle xy\rangle$. Therefore $x$ and $y$ are adjacent in the enhanced non-power graph. □

**Corollary 3.2.17** ([8], **Corollary 4.3**)
Let $G$ be a finite nilpotent group and take $x, y \in G$. If $\gcd(o(x), o(y)) = 1$, then $x \sim y$ in the enhanced non-power graph of $G$.

*Proof.* It follows from the previous Proposition and **Proposition 1.4.11**. □

**Example 3.2.18**
*Consider the abelian group $\mathbb{Z}_{12}$. As a consequence of* ***Theorem 1.4.9****, we have that $\mathbb{Z}_{12}$ is nilpotent. Now, $(2,3)$ is an edge in the enhanced non-power graph. Note that $\gcd(o(2), o(3)) = 2$. Therefore the converse of* ***Proposition 3.2.17*** *is* ***not*** *true.*

The next lemma will be necessary to study connectivity and, if connected, the diameter of the enhanced non-power graph of some particular groups.

**Lemma 3.2.19** ([8], **Lemma 4.6**)
Let $G$ be a group and $\Gamma_d(G)$ be its enhanced non-power graph. Take $x$ to be a non-isolated vertex of $\Gamma_d(G)$. We have the following:

1) If $o(x) = p^{\alpha}$ for some prime $p$, then there exist a prime $q(\neq p)$ and $y \in G$ such that $o(y) = q^{\beta}$ and $x \sim y$ in $\Gamma_d(G)$.

2) If $o(x)$ is not a prime power, then there exists a prime $p$ and $y \in G$ such that $o(y) = p^{\alpha}$ and $x \sim y$ in $\Gamma_d(G)$.

*Proof.* Since $x$ is a non-isolated vertex, it follows from **Proposition 3.2.4** that $\langle x \rangle$ is not a maximal cyclic subgroup and $x$ is contained in a cyclic subgroup $H$ that does not have prime power order. Hence there exists an element $y \in H$ such that $o(y) = p^{\alpha}$ and $x \sim y$ in the enhanced non-power graph. This proves item 2). Now, if $o(x)$ is a prime power $p^{\alpha}$, there clearly exists an element $y \in G$ of prime power order $q^{\beta}$, with $q \neq p$. Hence there exists an element of prime power order adjacent to $x$. □

**Theorem 3.2.20** ([8], **Theorem 5.1**)
Let $G$ be a not $p$-group with non-trivial center. Then, $\Gamma_d(G)$ is connected and diam($\Gamma_d(G)$)$\leqslant 6$.

*Proof.* Since $Z(G) \neq \{1\}$, **Theorem 1.3.20** ensures there exists an element $z \in Z(G)$ such that $o(z) = p$. We will first show that for any vertex $x$ in the enhanced non-power graph, there is a path connecting $x$ to $z$. Take $x \in G$. According to **Lemma 3.2.19**, there exists an element $y \in G$ with prime power order $q^{\beta}$ that is adjacent to $x$. We have two cases:

1. If $q \neq p$, then $z \nsim y$ in the power graph. Now, we know that $\langle z, y \rangle = \langle zy \rangle$ because $z \in Z(G)$ and $p \neq q$. Therefore $y$ and $z$ are adjacent in the enhanced non-power graph. Hence there is a path $x \sim y \sim z$ connecting $x$ to $z$ in $\Gamma_d(G)$.
2. If $q = p$, then by **Lemma 3.2.19** there exists $w \in G$ with $o(w) = r^{\theta}$, $r$ prime ($r \neq q$) and positive integer $\theta$, such that $w \sim y$. We also see that $w$ is adjacent to $z$ in the enhanced non-power graph. Thus, there is a path $x \sim y \sim w \sim z$.

In either case, we conclude that any vertex in $\Gamma_d(G)$ has a path connecting it to $z$. Therefore the enhanced non-power graph is connected. Now we will show that $x$ is connected to any vertex $x'$ by a path with at most six edges. For this first note that $x$ is connected to $z$ by a path of at most three edges. Repeating the same procedure as before, we can also show that there is a path from $x'$ to $z$ with length at most three. Thus, for any $x$ and $x'$ in $G$, we have that $d(x, x') \leqslant 6$ and the proof is complete. □

The bound of the previous Theorem is tight. This was proven by the authors in [8] (see **Remark 5.2**) using GAP (see [28]).

Now we will show that if we consider $G$ in $\Gamma_d(G)$ to be nilpotent and not a $p$-group, then the bound for the diameter of the graph can be improved.

**Theorem 3.2.21** ([8], **Theorem 5.3**)
Let $G$ be a nilpotent group that is not a $p$-group. Then, $\Gamma_d(G)$ is connected and diam($\Gamma_d(G)$) $\leqslant 4$.

*Proof.* It follows from **Theorem 1.4.14** that

$$G \cong P_1 \times P_2 \times \cdots \times P_n$$

where $P_i$ is the Sylow $p_i$-subgroup of $G$. We also note that

$$Z(G) \cong Z(P_1) \times Z(P_2) \times \cdots \times Z(P_n)$$

Now, since $G$ is not a $p$-group, then it has at least two Sylow subgroups. It follows that $|Z(G)|$ can be divided by at least two different primes $p$ and $q$. By **Theorem 1.3.20**, there exist at least two elements, $x$ and $x'$, in $Z(G)$ of order $p$ and $q$, respectively. Next, it follows from **Theorem 3.2.19** that there exist $y, y' \in V(\Gamma_d(G))$ such that $x \sim y$ and $x' \sim y'$ in $\Gamma_d(G)$ with $o(y) = q^t$, $o(y') = p^k$, for some prime numbers $p$ and $q$ and positive integers $k$ and $t$. If $y = y'$, the result follows, so suppose that they are distinct. If $p \neq q$, then it follows from **Corollary 3.2.17** that $y \sim y'$ in $\Gamma_d(G)$ and $\operatorname{diam}(\Gamma_d(G)) \leqslant 3$. So suppose that $q = p$. Then we have from **Theorem 3.2.19** that there exists $w \in V(\Gamma_d(G))$, with $o(w) = r^s$, for some prime $r(\neq q)$, such that $y \sim w \sim y'$ is a path in the enhanced non-power graph. In this case, $\operatorname{diam}(\Gamma_d(G)) \leqslant 4$.

□

In [8], the authors showed using GAP that the diameter of the enhanced non-power graph of the group $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_6$ is exactly four. This shows that the bound in the previous Theorem is tight.

We will finish this section with some results regarding the enhanced non-power graph of dihedral, symmetric and alternating groups.

**Proposition 3.2.22** ([8], **Theorem 5.5**)
Let $D_{2n}$ be the dihedral group of order $2n$. Then:

i) $\Gamma_d(D_{2n})$ is a null graph if $n$ is a prime power.

ii) $\Gamma_d(D_{2n})$ is connected if n is not a prime power.

*Proof.* As we know, in the enhanced non-power graph of any dihedral group, reflections are isolated vertices. Thus we only need to study rotations.

i) Suppose that $n = p^\alpha$ for some prime number $p$. We have that the subgroup generated by two rotations $r_i$ and $r_j$ is a cyclic subgroup of prime power order. It follows from this and **Theorem 2.1.11**, that $r_i$ is non-adjacent to $r_j$ in the enhanced non-power graph, concluding the proof.

ii) Suppose that $n$ is not a prime power. We know that any finite set of rotations generates a cyclic subgroup. From **Theorem 1.4.9** it follows that these subgroups are nilpotent. Therefore, it follows from **Theorem 3.2.21** that $\Gamma_d(G)$ is connected.

□

Let us construct the enhanced non-power graph of some symmetric groups.

**Example 3.2.23**

*i) The groups $S_3$ and $S_4$ are EPPO groups, as a consequence neither $\Gamma_d(S_3)$ nor $\Gamma_d(S_4)$ have edges.*

*ii) Consider the symmetric group $S_5$. Since this group has at least one element with order $6$, it is not an EPPO-group. It was shown in [8] that there is no path connecting $(12)$ and $(13)$. Using computational analysis, in particular GAP, they also showed disconnectivity for $S_6$ and $S_7$.*

Disconnectivity for the enhanced non-power graph of a symmetric group $S_n$ is verified up until $n = 8$. In general, we have:

**Theorem 3.2.24** ([8], **Theorem 6.1**)
The graph $\Gamma_d(S_n)$ is connected if and only if $n \geqslant 8$.

For the enhanced non-power graph of alternating groups, the authors in [8] showed that it is connected from a certain order. We present the result without proof.

**Theorem 3.2.25** ([8], **Theorem 6.10**)
If $n \geqslant 10$, then $\Gamma_d(A_n)$ is connected.

### 3.2.2 The non-commuting non-generating graph

In this section we study the difference between the non-generating graph and the commuting graph of a group. We begin the section by determining which are the isolated vertices resulting from this difference. With this information we are ready to study connectivity and diameter. Next, we conclude that a connected component in the non-commuting non-generating graph never has diameter equal to one. This shows that the diameter of this type of graph is at least two for any group. Particularly, we show that if a group has all maximal subgroups normal and the non-commuting non-generating graph has at least one edge, then the graph generated by non-isolated vertices is connected with diameter at most three. Lastly, we study connectivity and diameter of the non-commuting non-generating graph of alternating and dihedral groups.

This section is based on the results given in [14] and [25].

**Definition 3.2.26**
Let $G$ be a group. The **non-commuting non-generating graph** of $G$ is the graph with vertex set $G \setminus Z(G)$ where two vertices are adjacent if and only if they are adjacent in $\Gamma_{ng}(G)$ but not in $\Gamma_c(G)$. We will denote this graph by $\Xi(G)$.

**Example 3.2.27**
*Let us look at the dihedral group $D_{12}$. We have that all elements in $\langle r \rangle$ commute with one another. Now, the element $r^3$ belongs to the center of $D_{12}$ and the reflections $s$ and $sr^3$ commute with one another. The same happens between $sr$ and $sr^4$, and between $sr^2$ and $sr^5$. In conclusion, we have*

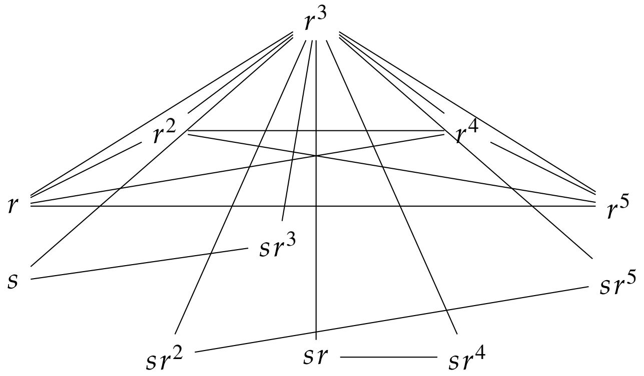

Figure 3.6: Proper commuting graph of $D_{12}$

*Recall the non-generating graph of $D_{12}$ presented in **Example 2.4.8**. Hence the non-commuting non-generating graph of the group is*

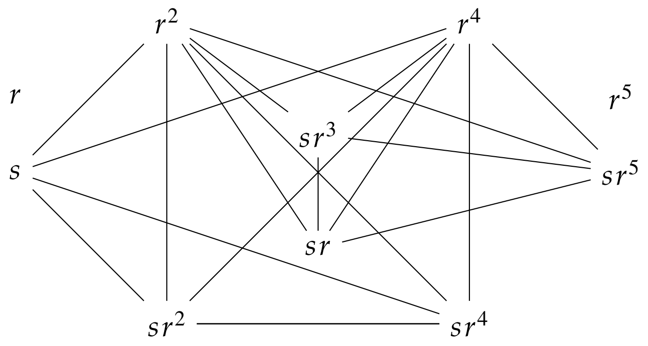

Figure 3.7: Non-commuting non-generating graph of $D_{12}$

The non-commuting non-generating graph can also be a null graph, as next example shows.

**Example 3.2.28**
*Consider the quarternion group $Q_8$. Since for any $g$ and $h$ in the quarternion group such that $gh \neq hg$ we have that $\langle g, h \rangle = Q_8$, the non-commuting non-generating graph of $Q_8$ is given by*

$$-i \qquad i \qquad -j \qquad j \qquad -k \qquad k$$

Figure 3.8: Null non-commuting non-generating graph

In fact, it follows from **Theorem 3.1.15** that the non-commuting non-generating graph of a group $G$ has no edges when $G$ is a minimal non-abelian group. In this case, the set of isolated vertices is the whole group. However, it's clear that this is not usually the case. Thus we will study which are the isolated vertices of the non-commuting non-generating graph of a given group.

**Theorem 3.2.29** ([14], **Proposition 9**)
Let $G$ be a non-abelian group that is 2-generated and consider $g \in G \setminus Z(G)$. Then $g$ is an isolated vertex in $\Xi(G)$ if and only if:

1. There exists a unique maximal subgroup $M$ of $G$ containing $g$;
2. $g \in Z(M)$.

Moreover, if $g$ is not isolated in $\Xi(G)$, then there exists a maximal subgroup $L$ of $G$ such that $g \in L \setminus Z(L)$.

*Proof.* Assume that 1. and 2. hold and that, in order to get a contradiction, $g$ is not an isolated vertex. Take $x$ to be a vertex adjacent to $g$ in $\Xi(G)$. Since $g$ commutes with all the elements in $M$, it follows that $x \notin M$. As a consequence, we have that $G \neq \langle x, g \rangle \not\subset M$. Therefore there exists another maximal subgroup of $G$ containing $g$, which is a contradiction. Thus $g$ is an isolated vertex in $\Xi(G)$.

Conversely, let $g \in G \setminus Z(G)$ be an isolated vertex in $\Xi(G)$ and take $M$ to be a maximal subgroup of $G$ that contains $g$. It follows from the assumption and the fact that $M$ is a proper subgroup of $G$ that $g \in Z(M)$. Therefore $M \leqslant C_G(g) \neq G$, since $g$ is not in the center of $G$. Since $M$ is maximal, it follows that $M = C_G(g)$. We conclude that there is only one maximal group that contains $g$ and centralizes it.

In particular, if $g$ is non-isolated, then there exists at least one maximal subgroup $L$ such that $g \in L \setminus Z(L)$.

□

**Example 3.2.30**
*Consider, once again, the quarternion group $Q_8$. We know that $\forall g \in G \setminus Z(Q_8)$ the subgroup $\langle g \rangle$ is maximal. Clearly, $g \in Z(M)$. Thus, like we previously saw, $g$ is an isolated vertex.*

We can find some classes of groups for which the non-generating non-commuting graph is connected. For these classes, we also have a bound for the diameter.

**Proposition 3.2.31** ([14], **Proposition 4**)
No connected component of $\Xi(G)$ has diameter 1.

*Proof.* We will show the result by contradiction. Let $\Gamma$ be a connected component of $\Xi(G)$ with diameter 1. We will first show that every element in $\Gamma$ is an involution. First, note that the elements $g$ and $g^{-1}$ are adjacent to the same vertices in $\Xi(G)$. However, since they commute, they are not adjacent to each other. Therefore any $g' \in \Gamma$ is an involution. Consider $x, y \in \Gamma$. It's clear that $\langle x, y \rangle = \langle x, xy \rangle$. As a consequence, $xy \in \Gamma$. Hence, by the first part of the proof, we conclude that $xy$ has order 2. Thus

$$\begin{aligned} &xyxy = 1 \\ \Leftrightarrow\ &xy = yx \end{aligned} \tag{3.1}$$

It follows that $x \sim y$ in the commuting graph. Therefore $x \nsim y$ in $\Gamma$, which contradicts the fact of $\Gamma$ being complete. □

Since the non-commuting non-generating graph is never complete, we may ask if it has bounded diameter. We answer this question for some classes of groups.

**Theorem 3.2.32** ([14], **Corollary 7**; [2], **Proposition 2.1**)
If $G$ is non-abelian and not 2-generated, then $\Xi(G)$ is connected with diameter 2.

*Proof.* By definition, it's clear that $\Gamma_{ng}(G)$ is complete. Thus $\Xi(G) = \overline{\Gamma}_c(G)$. Take $x, y \in \overline{\Gamma}_c(G)$. If $x \sim y$, it follows that $d(x, y) = 1$. Thus suppose that $x$ and $y$ are not adjacent. It follows that $xy = yx$. Since $G$ is non-abelian and $x, y \notin Z(G)$ there must exist $x'$ and $y'$ in $G$ such that $(x, x')$ and $(y, y')$ are edges in the non-commuting graph. If $(x', y)$ is an edge in $\overline{\Gamma}_c(G)$, then $d(x, y) = 2$. It's analogous for the case of $(y', x)$ being an edge. For the remaining cases, $x'y'$ is adjacent to $x$ since

$$x'y'x = x'xy' \neq xx'y'$$

In a similar way we conclude that $x'y'$ is adjacent to $y$. Hence $d(x, y) = 2$.

Now, suppose that $\Xi(G) = \overline{\Gamma}_c(G)$ has diameter 1 and let $a$ be a non-central element of $G$. It follows that $a$ is an involution and if $b$ is a central element, which by definition does not belong to the vertex set of $\overline{\Gamma}_c(G)$, then $ab$ is non-central and $(ab)^2 = b^2 = 1$. Thus, $G$ is abelian, which is a contradiction.

In conclusion, the graph $\Xi(G)$ is connected with diameter 2. □

**Proposition 3.2.33** ([14], **Lemma 12**)
Suppose that $G$ is 2-generated. In addition, let $(x, L, y, M)$ be an ordered 4-tuple such that $L$ and $M$ are normal, non-abelian maximal subgroups of $G$, with $x \in L \setminus Z(L)$ and $y \in M \setminus Z(M)$. Then, $d(x, y) \leqslant 3$. Moreover, $d(x, y) = 3$ if and only if either:

i) $x \in Z(M)$, $y \notin L$, and $M$ is the only maximal subgroup of $G$ containing but not centralizing $y$; or

ii) $y \in Z(L)$, $x \notin M$, and $L$ is the only maximal subgroup of $G$ containing but not centralizing $x$.

**Notation 3.2.34**
We will represent by $\Xi^+(G)$ the subgraph induced by all non-isolated vertices of $\Xi(G)$.

**Theorem 3.2.32** allows us to show the following statement.

**Theorem 3.2.35** ([14], **Theorem 13**)
Let $G$ be a group with every maximal subgroup normal. If $\Xi(G)$ has an edge, then $E^+(G)$ is connected with diameter 2 or 3. Moreover, if $\operatorname{diam}(\Xi^+(G)) = 3$, then $\Xi(G) = \Xi^+(G)$.

*Proof.* Suppose that the graph $\Xi(G)$ has an edge. As a direct consequence of this and **Theorem 3.1.15**, $G$ is non-abelian and not minimal non-abelian. If $G$ is not 2-generated, it follows from **Theorem 3.2.32** that $\Xi(G)$ is connected with diameter two. Therefore suppose that $G$ is 2-generated. Let $x$ and $y$ be two non-isolated vertices. It follows from **Proposition 3.2.29** and assumption that there are maximal normal subgroups $L$ and $M$ such that $x \in L \setminus Z(L)$ and $y \in M \setminus Z(M)$. Hence by **Proposition 3.2.33**, the graph $\Xi(G)$ has diameter less or equal to 3. Now, by **Proposition 3.2.31** we conclude that the diameter of $\Xi(G)$ is either 2 or 3. If $\Xi(G) \neq \Xi^+(G)$ then there is at least one isolated vertex $g$ in $\Xi(G)$. Then **Proposition 3.2.29** ensures that there is a unique a maximal subgroup $K$ of $G$ such that $g \in Z(K)$. Take $h \in G \setminus K$. We have that $\langle g, h\rangle$ is not contained in any maximal subgroup of $G$ because $g$ is contained in a unique maximal subgroup of $G$. Now, since $G$ is 2-generated, it follows that $\langle g, h\rangle$ doesn't lie in any proper subgroup of $G$. Therefore, $\langle Z(K), h\rangle = G$. This implies that $G/Z(K)$ is cyclic. Hence $K/Z(K)$ is also cyclic and it follows from **Proposition 1.3.9** that $K$ is abelian. Therefore **Proposition 1.3.7** shows that for any maximal non-abelian subgroup $M$, we have $Z(M) \leqslant Z(G)$. Finally, in these conditions, the requirements for **Proposition 3.2.33** can not be satisfied. Thus the diameter of the non-commuting non-generating graph is two.

□

We finish this chapter with some results for the non-commuting non-generating graph of dihedral and alternating groups.

**Theorem 3.2.36** ([25], **Theorem 3.1**)
The graph $\Xi(G)$ of the alternating group $G$ of degree $n \geqslant 5$ is connected with diameter at most 4, or at most 3 if $n$ is even.

The author of [25] brings attention to the fact that the previous Theorem and proof can be modified to show that the non-commuting non-generating graph of symmetric groups are connected.

The next result follows immediately from the definition of non-commuting non-generating graph and from **Theorem 1.2.22**.

**Proposition 3.2.37**
Let $(x, y)$ be an edge on $\Xi(D_{2n})$. Then, $\langle x, y\rangle$ generates a proper dihedral subgroup of $D_{2n}$.

In **Chapter 4** we continue the study on differences of graphs. In particular, the difference between the non-generating graph and the enhanced power graph, and the difference between the commuting graph and the power graph. To the best of our knowledge, these differences have not yet been considered.

# Chapter 4

# Further Differences

We introduced in **Chapter 3** the differences between the enhanced power graph and the power graph as well as between the non-generating graph and the commuting graph. In 2022, Peter J. Cameron wondered in [10] what other differences between graphs in the hierarchy could tell us. This Chapter is dedicated to the difference between the commuting and the power graph, and to the difference between the non-generating graph and the enhanced power graph.

## 4.1 The commuting non-power Graph

In this section we introduce a new graph by computing the difference between the commuting graph and the power graph of a group $G$. We present some results for particular groups and we determine when the proper graph resulting from the difference is complete.

**Definition 4.1.1**
Let $G$ be a finite group. We denote by $\Gamma_{cnp}(G)$ the graph with vertex set $G$ such that $(x, y) \in E(\Gamma_{cnp}(G))$ if and only if $(x, y) \in E(\Gamma_c(G))$ and $(x, y) \notin E(\Gamma_p(G))$. This graph is called the **commuting non-power graph** of $G$.

**Example 4.1.2**
*Consider the abelian group $\mathbb{Z}_6$. It's immediate that $\Gamma_c(\mathbb{Z}_6)$ is a complete graph. Recall from* ***Figure 2.3*** *that the power graph of $\mathbb{Z}_6$ has all but two edges. These edges are $(a^3, a^2)$ and $(a^3, a^4)$. Thus the commuting non-power graph of $\mathbb{Z}_6$ is equal to the enhanced non-power graph of $\mathbb{Z}_6$ (before removing isolated vertices).*

In general, it follows from **Theorem 3.1.10** and the definition of the commuting non-power graph that $\Gamma_{cnp}(G) = \Gamma_d(G)$ whenever $G$ has no subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, with $p$ prime.

It's immediate that $\Gamma_{cnp}(G)$ is never complete since the identity of $G$ is always an isolated vertex. However, next theorem classifies for which groups $\Gamma^*_{cnp}$ is complete.

**Theorem 4.1.3**
Let $G$ be a finite group. Then $\Gamma^*_{cnp}(G)$ is a complete graph if and only if $G$ is an elementary abelian 2-group.

*Proof.* We first show the only if part. It's clear that in these conditions, the graph $\Gamma^*_c(G)$ is complete. Next, it follows from the definition of elementary abelian 2-group that $\forall g \in G \setminus \{e\}$, $o(g) = 2$. Hence, the edge set of the proper power graph, $E(\Gamma^*_p(G))$, is empty. Consequently, $\Gamma^*_{cnp}(G)$ is complete.

Conversely, assume that $\Gamma^*_{cnp}(G)$ is a complete graph. It follows that $\Gamma^*_c(G)$ is a complete graph and that $\Gamma^*_p(G)$ is a null graph. From this we conclude that $G$ is abelian and that no element in $G$ is a power of another. Since no element is a power of another, then every non-identity element has order 2. Therefore $G$ is an elementary abelian 2-group. □

For the rest of this Section, we will look at some classes of groups for which $\Gamma_{cnp}$ is null. First, as an immediate consequence of **Theorem 2.1.11**, we get the following result.

**Theorem 4.1.4**
If $G$ is cyclic group of prime power order then $\Gamma^*_{cnp}$ is the null graph.

A natural question is if the converse of the previous Theorem holds. The answer is no.

**Example 4.1.5**
*Consider the dihedral group $D_{10}$. It follows from $n$ being prime and* ***Theorem A.0.12*** *that every rotation has order 5. Hence rotations form a maximal clique in the proper power graph of $D_{10}$ and, consequently, in $\Gamma^*_c(D_{10})$. Now, the product of any two distinct reflections is a rotation of order $n$. Consequently, any two reflections generate the whole group. Therefore no two reflections commute. Similarly, we see that no rotation can commute with a reflection. Thus both the proper commuting graph and the proper power graph are*

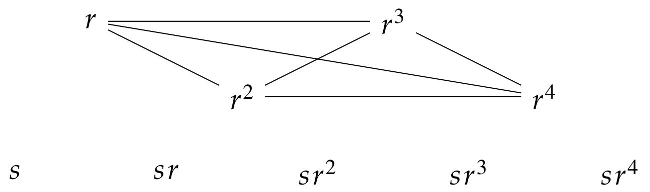

Figure 4.1: Proper power graph of $D_{10}$ and proper commuting graph of $D_{10}$

*It follows that $\Gamma^*_{cnp}(D_{10})$ is a null graph.*

In general, we have the following result.

**Theorem 4.1.6**
Consider the dihedral group $D_{2n}$ with $n > 2$. If $n$ is a prime number, then $\Gamma^*_{cnp}(D_{2n})$ is the null graph.

*Proof.* Recall the usual representation for the dihedral group

$$D_{2n} = \langle r, s | s^2 = e, r^n = e, srs^{-1} = r^{-1} \rangle$$

We have that $\langle r \rangle$ is a cyclic group whose elements are all the rotations of $D_{2n}$. Consequently, all rotations commute. Now, since $n$ is prime, every non-identity rotation generates $\langle r \rangle$. It follows that no two rotations are adjacent in the commuting non-power graph. Next, without loss of generality, consider the rotation $r$. By **Theorem A.0.12** we have that $\langle r, h \rangle = D_{2n}$, where $h$ is any reflection. Hence $r \nsim h$ in the proper commuting graph. As $n$ is an odd prime and any product of two distinct reflections $x$ and $y$ is a rotation, it follows that $\langle x, y \rangle = D_{2n}$. Thus no two distinct reflections commute. Therefore, $\Gamma^*_{cnp}(D_{2n})$. □

The converse of the previous Theorem is not true.

**Example 4.1.7**

*Consider the dihedral group $D_{18}$. With simple computations, we see that the center of $D_{18}$ is the identity. Since 9 is a prime power, the proper commuting graph and the proper power graph generated by all rotations are complete. It follows from **Theorem 1.2.22** that no two reflections commute. In conclusion, the dihedral group $D_{18}$ has null proper commuting non-power graph.*

Next example shows that **Theorem 4.1.6** does not hold for all $n$.

**Example 4.1.8**

*Consider the dihedral group $D_8$. Take $g, h \in D_8 \setminus \{e\}$ and suppose that $g^k = h$ for some $k \in \mathbb{Z}$. Since any power of a rotation is a rotation and reflections have order 2, it follows that $g$ and $h$ must be rotations. Therefore the proper power graph of $D_8$ is given by*

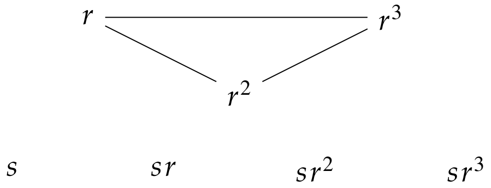

Figure 4.2: Proper power graph of $D_8$

*Now, we know that $r^2$ commutes with any reflection. We can also see that two reflections with straight angle commute, and no other pair of reflections commute. Therefore the graph $\Gamma_c^*(D_8)$ is*

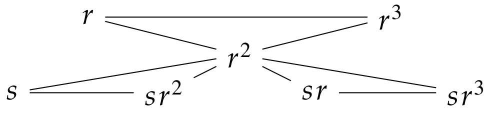

Figure 4.3: Proper commuting graph of $D_8$

*Thus $\Gamma_{cnp}^*(G)$ is given by*

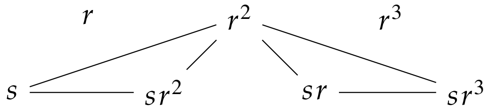

Figure 4.4: Commuting non-power graph of $D_8$

By analyzing the proper commuting non-power graph of the dihedral group on different $n$, we deduce the following result.

**Theorem 4.1.9**

Consider $G$ to be the dihedral group $D_{2n}$ with $n > 2$. Then:

i) $\Gamma_{cnp}^*(G)$ is the null graph if and only if $n$ is an odd prime power.

ii) If $n = 2^k$, with $k \geqslant 2$, then $\Gamma_{cnp}^*(G)$ has exactly $\frac{n}{2}$ maximal cliques of order 3.

*Proof.* Consider the usual representation for the dihedral group

$$D_{2n} = \langle r, s | s^2 = e,\ r^n = e,\ srs^{-1} = r^{-1} \rangle$$

i) Suppose that $n$ is an odd prime power. If $n$ is prime, the result follows from **Theorem 4.1.6**. Hence assume that $n$ is not prime. If follows that rotations form a maximal clique in $\Gamma_c^*(D_{2n})$ and in $\Gamma_p^*(D_{2n})$. Now, since $\frac{2\pi k}{n}$ is never 180°, for any $k$. There is no rotation of amplitude $\pi$. It follows from **Theorem 1.2.22** that any rotation with a reflection generates a dihedral group. Now, we know that the product of two distinct reflections is a rotation. Thus no two different reflections commute. Therefore the only edges in $\Gamma_c^*(D_{2n})$ are those in $\Gamma_p^*(D_{2n})$.

   Conversely, if the proper commuting graph equals the proper power graph, then the proper power graph of all rotations is complete. It follows from **Theorem 2.1.11** that $n$ is a prime power. If $n$ has order a power of 2, then the center of $D_{2n}$ is not trivial. Thus, $n$ must be odd and the result follows.

ii) If $n$ is a power of 2, the proper power graph of the subgroup generated by the rotations is complete. Now, the rotation of order $\frac{n}{2}$ commutes with every reflection. We have that the product of two distinct reflections is a rotation. Consequently, and by **Theorem 1.2.22**, each reflection $g$ commutes with the reflection $h$ if and only if $gh = r^{n/2} = hg$. Thus, $\Gamma_{cnp}^*(D_{2n})$ has exactly $\frac{n}{2}$ maximal cliques of order 3.

□

**Remark 4.1.10**

i) Item $i)$ from the previous Theorem does not hold for any $n$ odd. For example, it does not verify for $n = 15$ since the proper power graph generated by the rotations of $D_{30}$ is not complete, but its proper commuting graph is.

ii) The converse of item $ii)$ of the previous Theorem is not true. Take $G = D_{24}$ generated by the rotation $r$ and reflection $s$. The maximal cliques of the proper power graph of $\langle r \rangle$ have order 2. Hence the graph $\Gamma_{cnp}^*(G)$ has exactly 6 maximal cliques of order 3. Note also that the converse of **Theorem 4.1.9** does not hold for all $n$ even. Take $n = 30$. The elements $r^2, r^3$ and $r^5$ in $D_{60}$ form a maximal clique of order 3 in $\Gamma_{cnp}^*(D_{60})$.

   It appears that it only holds for $n$ whose prime power decomposition has, *at most*, two primes.

From **Theorem 4.1.9**, we immediately deduce the following corollary.

**Corollary 4.1.11**
Consider the dihedral group $D_{2^{k+1}}$, with $k > 1$. Then

$$\chi(\Gamma_{cnp}^*(G)) = 3 \text{ and } \Gamma_{cnp}^*(G) \text{ is perfect.} \tag{4.1}$$

As we have seen, determining for which classes of groups the power graph is equal to the commuting graph is not an easy task. Nonetheless, we have the following results:

**Lemma 4.1.12**
The commuting non-power graph of $\mathbb{Z}_p \times \mathbb{Z}_p$ is not null.

*Proof.* Clearly, $\Gamma_c(\mathbb{Z}_p \times \mathbb{Z}_p)$ is complete. However, $(1,0)$ and $(0,1)$ are not adjacent in the power graph of $\mathbb{Z}_p \times \mathbb{Z}_p$. Hence the graph $\Gamma_{cnp}(\mathbb{Z}_p \times \mathbb{Z}_p)$ has at least one edge. □

**Proposition 4.1.13**
Let $G$ be a group. If the commuting graph of $G$ equals the power graph of $G$, then $\mathbb{Z}_p \times \mathbb{Z}_p$ is not a subgroup of $G$.

*Proof.* Follows from **Lemma 4.1.12**. □

The converse of the previous Proposition is false, as we saw in **Example 4.1.2**. However, we have the following results.

**Theorem 4.1.14**
The alternating group $A_n$ does not have a null commuting non-power graph for $n \geqslant 4$.

*Proof.* We have that $A_4$ has a subgroup $H$ defined as follows

$$H = \{e, (12)(34), (13)(24), (14)(23)\}$$

This group is commutative and since each non-identity element has order two, we see that $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. It follows from **Proposition 4.1.13** that $\Gamma_{cnp}(A_4)$ is not a null graph. We can easily check that $A_4 \subseteq A_n$, for all $n \geqslant 4$. The result follows. □

**Theorem 4.1.15**
The symmetric group $S_n$ does not have a null commuting non-power graph for $n \geqslant 4$.

*Proof.* It follows from **Theorem 4.1.14** and from the fact that $A_4$ is a subgroup of $S_n$, for all $n \geqslant 4$. □

**Theorem 4.1.16**
The commuting non-power graph of $Q_8$ is null.

*Proof.* Recall the diagram for the group operation of $Q_8$:

$$\begin{tikzcd}
i \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & j \arrow[dl, leftrightarrow] \\
& k &
\end{tikzcd}$$

Figure 4.5: Product diagram of the quaternion group

We see that the element $-1$ is contained in all cyclic subgroups of non-identity elements. Hence the elements $-1$ and $1$ are isolated vertices in $\Gamma_{cnp}(Q_8)$. We note that $i$ is adjacent to $-i$ in the power and commuting graphs of $G$. We conclude the same for $j$ and $k$. Hence the graph $\Gamma_{cnp}(Q_8)$ has no edges. □

**Corollary 4.1.17**
Let $p$ be a prime number. If $G \cong Q_8 \times \mathbb{Z}_p$, it follows that $\Gamma^*_{cnp}(G)$ is the null graph.

*Proof.* Follows directly from **Theorem 4.1.16**. □

## 4.2 The non-generating non-enhanced power graph

In this section we introduce a new graph by computing the difference between the non-generating graph and the enhanced power graph of a group $G$. We will classify the groups for which this graph is complete or null.

**Definition 4.2.1**
Let $G$ be a finite group. We denote by $\Upsilon(G)$ the graph with vertex set $G \setminus \{e\}$ such that $(x, y) \in E(\Upsilon(G))$ if and only if $(x, y) \in E(\Gamma_{ng}(G))$ and $(x, y) \notin E(\Gamma_{ep}(G))$. We will call this graph **the non-generating non-enhanced power graph**.

**Remark 4.2.2**
We do not consider the identity in the vertex set because it is a dominant vertex for both the non-generating and the enhanced power graph.

**Example 4.2.3**
*Consider the dihedral group $D_{12}$. We have seen in* ***Example 2.4.8*** *the non-generating graph of $D_{12}$. Likewise, we saw in* ***Example 2.2.3*** *the enhanced power graph of $D_{12} \setminus \{e\}$. Taking both of these examples into account, we conclude that the graph $\Upsilon(D_{12})$ is*

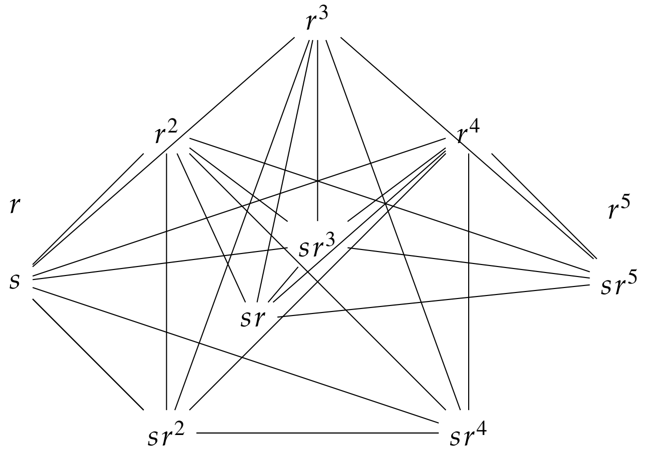

Figure 4.6: Non-generating non-enhanced power graph of $D_{12}$

Now, we will study for which groups $\Upsilon(G)$ is complete.

**Theorem 4.2.4**
Let $G$ be a group. The graph $\Upsilon(G)$ is complete if and only if $G$ is an elementary abelian 2-group that is not 2-generated.

*Proof.* For the *if part* of the proof, assume that $\Upsilon(G)$ is complete. It follows that $\Gamma_{ng}(G)$ is complete and $\Gamma_{ep}(G)$ is empty. Hence, $G$ is not 2-generated and every element has order 2.

The converse follows by definition of elementary abelian 2-group that is not 2-generated. □

Now we will classify all finite groups for which $\Upsilon(G)$ is the null graph.

**Theorem 4.2.5**
Let $G$ be a finite group. Then, $\Gamma_{ep}(G) = \Gamma_{ng}(G)$ if and only if $G$ is a minimal non-cyclic group.

*Proof.* The *only if part* of the equivalence follows from definition. Thus, let us show the converse. Since $\Gamma_{ep}(G) = \Gamma_{ng}(G)$, it follows that every two generated proper subgroup of $G$ is cyclic. By **Theorem 2.2.12**, all other finitely generated proper subgroups are cyclic. Therefore, $G$ is a minimal non-cyclic group. □

**Remark 4.2.6**

Note that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a minimal non-cyclic group that is abelian. As we saw in **Example 3.1.4**, the non-generating graph of $\mathbb{Z}_2 \times \mathbb{Z}_2$ can not belong to the graph hierarchy simultaneously with the commuting graph of the same group. Thus, if we are not forcing the conditions for both the non-generating graph and the commuting graph to be in the hierarchy, then the fact that the enhanced power graph equals the non-generating graph does not mean that the commuting graph equals the non-generating graph nor the enhanced power graph.

# Conclusion

Writing this thesis has been a long journey that has nearly come to an end. Although we only explored the surface of **Graphs defined on Groups**, this work has provided us with the necessary knowledge to continue our work in this area.

In this dissertation we defined what are graphs on groups and study four of such graphs. In particular, we determined for which groups each of these graphs is null or complete. Also, we looked at the connection between maximal cliques and subgroups of $G$. For the commuting graph we also explored relations between the degree of the graph and the center of the group.

In **Chapter 3** we used the graphs previously defined to introduce the graph hierarchy and explore the difference between these graphs. Firstly, we classified the groups for which two given graphs in the hierarchy are equal and their difference (in the sense of subtraction) is the null graph. More specifically, we study the equality between: the power graph and the enhanced power graph; the enhanced power graph and the commuting graph; the non-generating graph and the commuting graph; and the power graph and the non-generating graph. Next, we used the fact that for most groups these graphs are not equal to motivate the study of difference (in the sense of subtraction), were we looked at properties like isolated vertices, connectivity and diameter.

In **Chapter 4** we started our attempt to investigate the difference between other pairs of graphs in the hierarchy. We defined two new differences and studied what groups gave rise to a null graph or a complete graph. In particular, we gave some results for dihedral groups, symmetric and alternating groups.

## Future Studies

There is no doubt that this dissertation was the first step into our future work in this particular connection between group theory and graph theory. We wish to continue our study of the differences defined in **Chapter 4**. In particular, we wish to study isolated vertices and connectivity. The open question (see [32]): *Let $G$ and $H$ be finite groups such that their graphs of a certain type are isomorphic. If $G$ is nilpotent, does this imply that $H$ is also nilpotent?*, also wishes to be explored.

We conclude this essay with more questions than answers, which carries hope and enthusiasm for the future.

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# Appendix A

# Basic notions

Here we present some theorems and definitions, as well as their respective notations, that are necessary to fully understand the main part of this essay. Since these concepts are rather elementary, we will omit proofs and examples for most cases.

We start by giving the definition of group and recognizing when it is abelian. Then we use this to define group homomorphism and subgroups. We later present the definition and notation of left cosets and group index which are necessary for **Lagrange's theorem**. We finish with the concept of quotient group and direct product.

This section uses [42] as main reference.

**Definition A.0.1**
A **group** $(G, *)$ is a nonempty set $G$ equipped with an associative operation $*$, and an element $e$ such that:

i) $e * a = a = a * e$ for all $a \in G$.

ii) for every $a \in G$, there is an element $b \in G$ with

$$a * b = e = b * a$$

The element $b$ is called **inverse of a**.

The element $e$ from the previous definition is called the **identity of** $G$.

**Definition A.0.2**
A group is **abelian** if $a * b = b * a$ for every pair $a, b \in G$.

We now present a definition that is fundamental for Group Theory.

**Definition A.0.3**
Let $G, *)$ and $(H, \circ)$ be groups. A function $f : G \longrightarrow H$ is a **homomorphism** if, for all $a, b \in G$,

$$f(a * b) = f(a) \circ f(b)$$

An **isomorphism** is a homomorphism that is also a bijection. We say that $G$ is isomorphic to $H$ if there is an isomorphism $f : G \longrightarrow H$. In this case we write $G \cong H$.

**Definition A.0.4**
A nonempty subset $S$ of a group $G$ is a **subgroup** of $G$ if $s \in G$ implies $s^{-1} \in G$ and $s, t \in G$ imply $st \in G$. If this verifies, we write $S \leqslant G$.

We may construct a group from a given set.

**Definition A.0.5**
If $X$ is a subset of a group $G$, then the smallest subgroup containing $X$, denoted by $\langle X \rangle$, is called the **subgroup generated by** $X$. We also say that $X$ generates $G$.

A especial case of the previous definition is given bellow.

**Definition A.0.6**
If $G$ is a group and $a \in G$, then the **cyclic subgroup generated by a**, denoted by $\langle a \rangle$, is the set of all powers of $a$.

**Definition A.0.7**
A non-cyclic group is **minimal non-cyclic** if all of its proper subgroups are cyclic.

Now we present some definitions which will be fundamental in the main part of the dissertation.

**Definition A.0.8**
The **order** of a group $G$ is the number of elements in $G$. It is denoted by $|G|$.

**Definition A.0.9**
If $G$ is a group and $a \in G$, then the **order** of $a$ is the number of elements $\langle a \rangle$. It will be denoted by $o(a)$.

In order to present one of the most important theorem's in Group Theory, we need a few definitions.

**Definition A.0.10**
If $S$ is a subgroup of $G$ and if $t \in G$, then the **right coset** of $S$ in $G$ is the subset of $G$

$$St = \{st : s \in S\}.$$

The left coset is defined in an analogous way.

**Definition A.0.11**
If $S$ is a subgroup of $G$, then the **index** of $S$ in $G$ is the number of right cosets of $S$ in $G$. It is denoted by $[G : S]$.

We are now ready to give one of the most important results in Group Theory: **Lagrange's Theorem**.

**Theorem A.0.12** ([42], **Theorem 2.11**)
Let $G$ be a finite group with a subgroup $H$. Then the order of $H$ divides the order of $G$ and

$$|G| = [G : H]|H|$$

**Example A.0.13**
*Consider the symmetric group $S_3$ and let $H = \{(1), (123), (132)\}$. Since $|H| = 3$ and $|G| = 6$, it follows from* ***Lagrange's Theorem*** *that there are two left cosets of H on G.*

**Remark A.0.14**
As an immediate consequence of the previous Theorem we have that the order of an element $g$ of a group $G$ divides the order of $G$. We quickly see this follows from the fact that $|\langle g\rangle| = o(g)$.

The converse of Lagrange's Theorem is not true in general.

**Example A.0.15**
*Consider the alternating group $A_4$. It is known that $A_4$ has $12$ elements and that $6 \mid |A_4|$. However, $A_4$ has no subgroup of order six.*

Sometimes a subgroup of a group $G$ can have additional properties.

**Definition A.0.16**
A subgroup $S$ of $G$ is a **normal subgroup** if $gSg^{-1} = S$, for every $g \in G$. It is denoted by $S \triangleleft G$.

**Theorem A.0.17**
If $S \triangleleft G$, then the cosets of $S$ in $G$ form a group, denoted by $G/S$, of order $[G : S]$. To this group we call **quotient group**.

**Definition A.0.18**
If $x \in G$, then a conjugate of $x$ is an element of the form $axa^{-1}$, for some $a \in G$. We will represent it by $x^a$.

We finish with the construction of a new group based on two possibly independent groups.

**Definition A.0.19**
If $H$ and $K$ are groups, then their **direct product**, represented by $H \times K$, is the group with elements all ordered pairs $(h,k)$, where $h \in H$ and $k \in K$, and with operation

$$(h,k)(h',k') = (hh',kk')$$

**Theorem A.0.20**
Let $G$ be a group with normal subgroups $K$ and $H$. If $HK = G$ and $H \cap K = \{e\}$, where

$$HK = \{hk : h \in H \text{ and } k \in K\}$$

then $G \cong H \times K$.

# Appendix B

# Groups of small order

In order to better understand the results on **Chapter 4** and some proofs given in **Chapter 3**, we need to introduced some of the small groups.

Bellow, we present two tables: the first one has the nontrivial abelian groups of order less or equal to 8. Next, we have a table of the non-abelian groups of order between 6 and 14. Note that the numbers beside the subgroups in parenthesis indicate how many copies of that group are there.

Table B.1: List of small abelian groups up until order 8

| Small abelian groups | | |
|---|---|---|
| order | group | proper non-trivial subgroups |
| 2 | $\mathbb{Z}_2$ | - |
| 3 | $\mathbb{Z}_3 = A_3$ | - |
| 4 | $\mathbb{Z}_4$ | $\mathbb{Z}_2$ |
| 4 | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\mathbb{Z}_2(3)$ |
| 5 | $\mathbb{Z}_5$ | - |
| 6 | $\mathbb{Z}_6 = \mathbb{Z}_3 \times \mathbb{Z}_2$ | $\mathbb{Z}_3, \mathbb{Z}_2$ |
| 7 | $\mathbb{Z}_7$ | - |
| 8 | $\mathbb{Z}_8$ | $\mathbb{Z}_4, \mathbb{Z}_2$ |
| 8 | $\mathbb{Z}_4 \times \mathbb{Z}_2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4(2)$, $\mathbb{Z}_2(3)$ |
| 8 | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2 \times \mathbb{Z}_2(7)$, $\mathbb{Z}_2(7)$ |

Table B.2: List of small non-abelian groups up until order 14

| Small non-abelian groups | | |
|---|---|---|
| order | group | proper non-trivial subgroups |
| 6 | $D_6 = S_3$ | $\mathbb{Z}_3$, $\mathbb{Z}_2(3)$ |
| 8 | $D_8$ | $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2(2)$, $\mathbb{Z}_2(5)$ |
| 8 | $Q_8$ | $\mathbb{Z}_4(3)$, $\mathbb{Z}_2$ |
| 10 | $D_{10}$ | $\mathbb{Z}_5$, $\mathbb{Z}_2(5)$ |
| 12 | $Q_{12}$ | $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4(3)$, $\mathbb{Z}_6$ |
| 12 | $A_4$ | $\mathbb{Z}_3(4)$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2(3)$ |
| 12 | $D_{12} = D_6 \times \mathbb{Z}_2$ | $\mathbb{Z}_6$, $D_6(2)$, $\mathbb{Z}_2 \times \mathbb{Z}_2(3)$, $\mathbb{Z}_3, \mathbb{Z}_2(7)$ |
| 14 | $D_{14}$ | $\mathbb{Z}_7, \mathbb{Z}_2(7)$ |