Chapter 7

EXPLORATORY MATHEMATICS TEACHING AND THE DEVELOPMENT OF STUDENTS' USE OF REPRESENTATIONS AND REASONING PROCESSES: AN ILLUSTRATION WITH RATIONAL NUMBERS

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ABSTRACT

This chapter presents a perspective about the exploratory approach as a possible way to enact inquiry based mathematics teaching. In this approach, the teacher, instead of beginning the class by presenting explanations and examples to the students, proposes them to work on tasks that may lead to the construction of new knowledge. We use as illustration the work of a grade 6 class of students solving tasks involving rational numbers. Our aim is to know how students use representations and reasoning processes, seeking to find out how they deal with different representations and how they formulate generalizations and justifications. We follow a qualitative and interpretative approach, with participant observation of a teaching experiment that included five lessons that were integrally videotaped and transcribed. We analyse episodes from the work of the students in two tasks, one involving a complex relationship between fractions and the other involving the use of fractions as operators. The results show that when solving a task that involves rational

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numbers given as fractions, the students mostly use the decimal representation, with which they feel rather comfortable. In another task, involving rational numbers as operators, most students use fractions, but some of them also use of decimal numbers and pictorial representations. In both cases, the students chose the representation that they considered best suit their needs. In their written work, the students justify their choices by presenting the computations done when solving a task, adding explanations in natural language. Just by themselves, they are able to use counterexamples to refute a statement, and, during whole class discussions, prompted by the teacher, they are able to make generalizations and justifications based on definitions.

Keywords: generalization, justification, rational numbers, reasoning, representations

INTRODUCTION

An exploratory approach to mathematics teaching (Ponte and Quaresma 2011; 2016) is a particular enactment of inquiry-based mathematics teaching (Artigue and Blomhøj 2013). In this approach, students work on tasks for which they do not know a method to reach a solution that they can use right away. Therefore, they have to design their own solving strategies, using different mathematical representations. The teacher does not directly teach procedures and algorithms, present explanations, show examples and propose exercises to students to practice, but, instead, suggests the students to work on tasks that lead to the construction of new knowledge. In addition, the teacher promotes opportunities for the students to express themselves in mathematical language, especially through whole class discussions. The aim is to lead the students to develop their ability to use different representations, connecting them in a fluent way, to develop their reasoning and also their understanding of mathematics. We illustrate how to use of this approach for the teaching of rational numbers, a topic that is widely recognized as being difficult for students (Barnett-Clarke et al. 2010; Lamon 2007; Monteiro and Pinto 2005), and in which teaching tends to follow a computational approach, based on rules for the operations and the equivalence of fractions.

The exploratory approach highlights the development of concepts, the use of different representations, the modelling of situations, and also the use of definitions and properties of mathematical objects to arrive at conclusions. In the classroom work, this means that, in addition to the computational procedures, close attention is paid to the conceptual aspects. In the case of rational numbers, the attention to representations, definitions, and properties, allows to bring to the forefront the algebraic aspects of these numbers (Empson, Levi and Carpenter 2010), and creates opportunities to highlight two essential aspects of mathematical reasoning, that is, generalization and justification.

Mathematical representations have a close relationship with reasoning and play a fundamental role in this subject (NCTM 2014). In fact, given the abstract nature of mathematical objects, it is necessary to use representations to reason, that is, to make informed inferences (Duval 2006; Mata-Pereira and Ponte 2012). In addition, when solving a problem or carrying out an investigation that requires reasoning, the choice of the representations to use is often the key for the solver to achieve the sought aim.

In this chapter, we illustrate the exploratory approach and analyse the work of grade 6 students on tasks involving relationships with fractions and the use of fractions as operators. We discuss the representations that the students use to solve tasks involving rational numbers and show how this kind of activity provides opportunities to make generalizations and justifications.

Exploratory Approach, Representations and Reasoning

The three themes that we address in this chapter, the exploratory approach, mathematical representations and mathematical reasoning are getting increasing visibility in mathematics education and provide important consequences for organizing the classroom activities and for promoting mathematics learning. In the sequence, we make a short discussion about them.

Exploratory Approach

This approach (Ponte, Branco and Quaresma 2014) is framed by the nature of the tasks that the teacher proposes, by the organization of students' work and by the kind of communication that occurs in the classroom. Tasks are of key importance because what students learn in the mathematics classroom depends mostly on the kind of activity that they carry out and the reflection that they do on that activity (Christiansen and Walther 1986). Therefore, it is very important to choose appropriate tasks that may be the starting point for a rich and multifaceted students' activity (NCTM 1991; Swan 2014). Tasks should be diversified, including exercises, problems, investigations and explorations (Ponte 2005). Structured tasks such as exercises have a low level of challenge and, therefore, they may be targeted mostly at the consolidation of previously acquired knowledge. Problems, in contrast, have a high level of challenge and aim at the creative application of knowledge that the student already has, by requiring she/he to combine different elements in a coherent problem solving strategy. Explorations are open tasks with a moderate level of challenge, which mostly aim at the construction of new concepts while investigations have a high level of challenge and aim at both the development of new concepts and the creative use of concepts already known. It is the responsibility of the teacher to select the tasks according to the aims settled for each lesson, with attention to the characteristics of the students.

Students' work on exploratory tasks in the classroom may be carried out in different ways. A possibility is whole class work, with the teacher interacting with all students. Another approach is to have students working in groups or in pairs, seeking to provide them opportunities for sharing ideas, resources and experiences. In this way, students may participate in two levels of classroom discourse, the collective, which they share with all the other class members, and the private, that they carry out with their colleagues (Ponte and Santos 1998; Sherin 2002). Still another possibility is individual work, which may foster the development of the students' capacity of concentrating on a task.

Exploratory lessons are usually structured along three phases (Christiansen and Walther 1986; Ponte 2005; Stein et al. 2008): (i) the launching of the task and the way the students interpret it (whole class work); (ii) the development of the work by students (in groups, in pairs or individually); and (iii) the discussion and final synthesis (again in whole class). This last phase is very important since it is a very suitable occasion to expose connections and meanings (Bishop and Goffree 1986), allowing students to relate several themes and to perceive how mathematical ideas are interconnected. Besides, the moments of whole class discussion provide opportunities for negotiating mathematical meanings and constructing new knowledge. Learning with understanding may be improved through the class interactions, as they provide students opportunities to suggest mathematical ideas and conjectures, to develop their mathematical reasoning, and to learn how to evaluate their own and their colleagues' reasoning. Therefore, it is worth that each task always culminates with a whole class discussion as a context to consider different ideas and processes, draw conclusions and reflect on the work done (NCTM 2000).

Classroom communication frames in a fundamental way student learning opportunities. This communication is univocal, when it is dominated by the teacher or dialogical when students' contributions are valued (Brendefur and Frykholm 2000). It is the role of the teacher to define the communication patterns, to propose tasks to undertake and to establish the ways of work in the classroom, bearing in mind the communicational objectives he/she wants to attain. However, this should be done through an ongoing negotiation (very often, a difficult one) with the students. The teacher may assume the role of a single mathematical authority or may prefer to share it with the students (Lampert 1990), seeking to stimulate their reasoning and argumentation ability. In an exploratory lesson, communication is dialogical. A very important aspect of the work of the teacher is to encourage students to present their solutions, seeking that disagreements emerge, leading to situations of argumentation within the class (Wood 1999). Another important aspect is the way the teacher seeks to support students in the appropriation of correct mathematical language, mostly through revoicing actions (Franke, Kazemi and Battey 2007), that is, slowly reformulating students' speech in more correct language.

Representations and Reasoning

Representing a number means to ascribe it a designation. It is important that students understand that a number may have several representations. In addition, we need to distinguish between internal representations, which develop in people's minds, and external representations, that may exist in diverse supports (paper, stone, smoke, sounds, etc.) (Goldin 2008). Bruner (1999) refers three main kinds of representations: active (objects, body movements), iconic (images, diagrams) and symbolic. Duval (2004, 14) also underlines the diversity of possible representations, stating that "numbers, functions, lines, etc. are mathematical objects and decimal or fraction writing, symbols, graphics, etc..., are some of its representations". The same author values the articulation of different registers of representations of a given object as a condition for mathematical understanding. Other authors, like Webb, Boswinkel and Dekker (2008), distinguish between informal, pre-formal and formal representations, arguing that all of them have an important role to play in mathematics learning. In fact, we must note that we can only understand students' thinking and reasoning by observing and analysing their representations (NCTM 2000).

Rational numbers admit a variety of representations: percent, decimal number, fraction, mixed fraction, images, number line and natural language. Students should understand all of them, knowing, for example, that $\frac{1}{4}$, 25%, and 0.25 are just different designations of the same number. Different representations may be used simultaneously and the teacher may encourage students to move from one to another, so that they learn how "to move among equivalent forms, choosing and using an appropriate and convenient form to solve problems and express quantities" (NCTM 2000, 151).

Reasoning means to make inferences based on reasons, that is, inferences based on some justification. Lithner (2008) emphasizes that mathematical reasoning is a basic competence, which underlies the distinction between creative and imitative reasoning, given that the latter is based on memorization and following algorithms.

Mathematics is usually associated to deductive reasoning (Davis and Hersh 1980; Oliveira 2008), the only kind of reasoning that is able to guarantee the mathematical validity of a certain statement. However, several authors also underline the important role of inductive (Pólya 1990) and abductive (Rivera and Becker 2009) reasoning in mathematics, notably in discovery processes. In addition, in many cases, what occurs is a mixture of processes, with inductive, abductive and deductive features. Mata-Pereira and Ponte (2012, 84) present a framework to study students' mathematical reasoning that relates generalization and justification with the formulation of questions and conjectures:

> "Mathematical reasoning […] is supported on representations and articulates with representation and sense making processes. Given the impossibility to directly access students' mathematical reasoning, the representation that they use to communicate that reasoning are fundamental. On the other hand, the sense making processes in articulation with mathematics reasoning are essential for an effective understanding of mathematics […]. Inductive and abductive reasoning occur mainly during the formulation of conjectures, whereas deductive reasoning takes place especially during the test and justification of conjectures."

Mata-Pereira and Ponte (2012) argue that students should be able to reason mathematically using mathematical concepts, procedures, representations and language. In addition, they state that students should learn how to justify their statements, from their early school years, using specific examples. As they move up through the school levels, their justifications should become more general, distinguishing between examples and mathematics arguments that apply to a whole class of objects.

Also, Lannin, Ellis and Elliott (2011) consider that mathematical reasoning involves mostly the production of mathematical generalizations and justifications. For the authors, the "big idea" about mathematical reasoning is that this is a dynamic process of conjecturing, generalizing, investigating why, and developing and evaluating arguments. In their perspective, it is important that students (i) make justifications using logical arguments based on ideas that were previously understood, (ii) justify refutations using the fact that a certain statement is false, (iii) evaluate the validity of arguments used, (iv) have in mind that a mathematical justification is not an argument based on authority, perception, common sense or particular examples, and (v) seek to justify the why of a generalization being true or false by investigating the factors that may bear on that generalization.

Another author, Galbraith (1995), stresses that students show often difficulty in accepting that just one case is enough to refute a statement. He also adds that most students have difficulties in understanding that a counter example of a mathematical statement must satisfy all given conditions and violate its conclusions. Concerning generalizations, this author distinguishes between the inductive (or empirical) approach, in which the students test a few concrete cases, and the deductive approach, in which tests are based on logical arguments. In the inductive approach, he still distinguishes between students that make random tests and those that carefully select the cases based on their understanding of the conjecture that they are testing. In the deductive approach students need to recognize the relevance of a certain external principle, perceive in which way this principle is useful and apply it properly. Finally, Rivera and Becker (2009) underline the importance of abductive reasoning, in which students formulate general conjectures based on clues that may seem unrelated, and refer that, when solving a problem, a person uses a large variety of abductive, inductive and deductive reasoning processes, in a deeply interconnected way.

Davis and Hersh (1980) argue that there are different cognitive styles among mathematicians that have preference for one or another form of reasoning, associated to a certain kind of representation. These authors describe the relationship between cognitive styles and forms of reasoning. They consider, for example: the analytical style that is essentially supported by the work with symbols; the visual style that makes wide use of geometric and iconic representations; and the kinaesthetic style in which the body movement has an important role in the mathematical thinking process.

As the NCTM (2000) points out, “students’ understanding and ability to reason will grow as they represent fractions and decimals with physical materials and on number lines and as they learn to generate equivalent representations of fractions and decimals” (33). According to this document, the representations are important, not only to help students in developing their understanding of mathematical topics, but also as a support for reasoning, allowing to get results and to draw conclusions. When students learn how to use different mathematical representations and know their meaning, they master important tools that strengthen their ability to think mathematically.

To develop students’ reasoning ability is essential to help them to go beyond the simple memorization of facts, rules and procedures. The focus on reasoning may help students to perceive that mathematics has a logical structure that they may understand and use to think, justify, and evaluate. Mathematics classes that value and support reasoning are necessary to achieve these aims.

RESEARCH METHODOLOGY

In this study, we address students' mathematical representations and reasoning. The understanding of these processes requires very close individual interactions with the participants in their natural environment, so we follow a qualitative and interpretative approach (Bogdan and Biklen 1994). We use participant observation, done by both authors, since this methodology allows a very close connection with the object of study in its natural context.

Besides the teacher (the second author of this chapter), the participants in the study are students of a grade 6 class from a public basic school in a deprived rural area at 50 km from Lisbon (all students' names are pseudonyms). Because of the characteristics of the social environment, the school is under a special support program from the Ministry of Education. Families are, in general, of low or low-middle socioeconomic status. As far as school education is concerned, most parents completed grade 6 or grade 9 only, and only a few of them completed secondary school (grade 11 or 12). There are 19 students in the class, 14 boys and 5 girls, with ages between 11 and 17 years old (most of them are 11 years old). It is a class that does show very little commitment and work habits, both when they work individually and when they are asked to work in pairs with a colleague.

The study focuses on a teaching experiment that involves five 90 minutes classes that sought to apply the exploratory approach to several topics involving rational numbers and that are included in the Portuguese grade 6 mathematics curriculum. The first set of tasks proposed to students aimed at diagnosing their knowledge about comparing, ordering, adding and subtracting rational numbers, topics that they had already studied in previous years. The second set of tasks aimed at introducing the multiplication of a fraction by a whole number and the multiplication of two fractions. Finally, the third set aimed at developing the meaning of fraction as operator. According to the principles of the exploratory approach, after the launching of the task, the students worked in pairs during a part of the lesson and, in the other part, there was a whole class discussion. We paid special attention to the launching of the tasks and to whole class discussions carried out by the students.

The lessons were video and audio recorded, and the whole class discussions were fully transcribed. The students' written productions in the different tasks were collected. The analysis of data sought to identify the key segments in solving the tasks, signalling significant events regarding students' representations, generalizations and justifications. Finally, we selected two episodes that we consider relevant concerning the representations, generalizations or justifications done, which we analyse in this chapter. The analysis seeks to interpret the observed reasoning taking into account the representations used, aiming at identifying patterns in the empirical elements.

REPRESENTATIONS AND REASONING IN THE EXPLORATORY CLASSROOM

Next we present two classroom situations, selected to analysing the representations used by the students and their mathematical reasoning, specifically regarding generalization and justification processes. In both situations we analyse several episodes.

Task 1 - True Inequality?

Task 1 asks students to assess the validity of a complex statement, which compares several fractions, and to justify their responses. The task involves the relationship "greater than", and was done in the first part of the teaching experiment, when we sought to diagnose the students' knowledge about comparing rational numbers. Data are presented as fractions and the question context is purely mathematical. As in all lessons in the teaching experiment, the students worked in pairs.

Task 1
$\frac{2}{4}$ is greater than $\frac{1}{3}$, $\frac{4}{5}$ is greater than $\frac{3}{4}$. Can we make the following statement: "If we want to compare two fractions and verify that one of them as the numerator and the denominator greater than the other, can we immediately conclude that such fraction is the larger one? Justify your answer."

Episode 1. The task is presented in writing and the teacher orally comments the question. It has a remarkable complexity for these students, given the quantity and the nature of the information provided and also because this is a very unusual task for them. At the launching stage, the teacher becomes aware of the difficulty of the students in understanding the task, and decides to carry out a collective interpretation of the statement.

The first part of the question does not create any problems, since the students easily understand that $\frac{2}{4} > \frac{1}{3}$ and that $\frac{4}{5} > \frac{3}{4}$. However, the students do not know what to do to figure out if the statement is true or not for all cases. The teacher revoices the statement using other words but giving no clues about the strategy to follow. A strategy is finally formulated by a student, as follows:

"Teacher: The two cases are true. And, what it says next, what is there is… OK, this and this [$\frac{2}{4} > \frac{1}{3}$ and $\frac{4}{5} > \frac{3}{4}$] is true. Can I always say that when the numerator and the denominator of a fraction are larger than the numerator and the denominator of the other fraction, then this one [$\frac{4}{5}$], that has larger numerator and denominator, is always larger than the second [fraction]? This always happens?

Carlos: [Conjecturing] No…

Teacher: How can you try to understand if it always happens or not?

Daniel: Doing more fractions…

Teacher: Finding other examples, isn't it? … It may be a good suggestion from Daniel […].”

The suggestion made by Daniel follows a strategy of using inductive reasoning, experimenting several cases, and expecting to arrive at a conclusion. In this experimentation process one may use mathematical proprieties and operations, which are key elements of deductive reasoning. In her last intervention, the teacher seeks to improve Daniel's language, suggesting that students may verify “other examples”, seeking to find out if they obey to the condition of the question and if the statement is true or not.

Episode 2. After some time of students' autonomous work, several pairs of student have solutions to share. A few of them get correct responses. Concerning representations, we see that almost all students transformed the fractions given in the statement in decimal numbers and solved the task using this representation (figure 1). This preference for the use of decimal numbers to compare two rational numbers, instead of working with fractions transforming them to a common denominator, is related to the fact that decimal numbers, as a kind of representation, have a prominent role in the primary mathematics curriculum in our country.

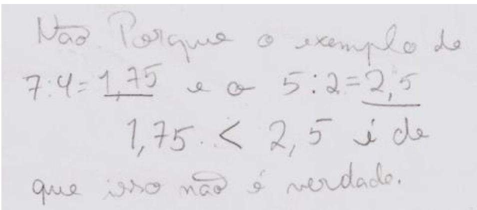
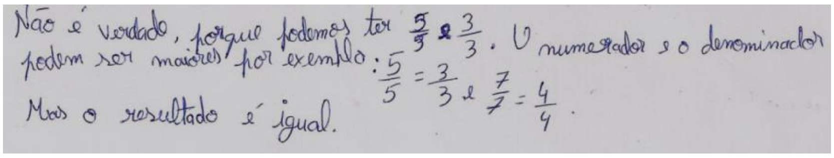
 Não. Porque o exemplo de 7:4=1,75 e o 5:2=2,5 1,75 < 2,5 é de que isso não é verdade.	No because the example of $7:4 = 1.75$ and the $5:2 = 2.5$ $1.75 < 2.5$ that it is not true

Figure1: Answer given by Mara and Ângelo

Students that developed great familiarity with this representation were able to use it at ease. Only a pair of students, Edgar and Juliana, uses the fraction representation (figure 2) and another pair uses the per cent one.



It is not true, since we may have $\frac{5}{5}$ and $\frac{3}{3}$. The numerator and the denominator may be larger. For example $\frac{5}{5}=\frac{3}{3}$ and $\frac{7}{7}=\frac{4}{4}$.
But the result is equal.

Figure 2: Answer from Edgar and Juliana

Several pairs of students are able to answer correctly using only a counter example to justify that this is a false statement. This is the case of the pair Mara and Ângelo (figure 1) that present the rational numbers $\frac{7}{4}$ and $\frac{5}{2}$ (written as quotients and as decimal numbers). It is also the case of Edgar who worked alongside with Juliana and who write $\frac{5}{5}=\frac{3}{3}$ and also $\frac{7}{7}=\frac{4}{4}$ (figure 2). Edgar explains their solution to the class in the following way:

“Teacher: … So… Then? Edgar, what does that mean?
Edgar: Both are (the two fractions) 1. […]. The numerator and the denominator [$\frac{5}{5}$] are greater than that one [$\frac{3}{3}$].
Teacher: Explain it, Edgar, so, how did you arrive to that conclusion? What information do you have from the examples that were given? OK. The examples said… Always when the numerator and the denominator are greater, then such fraction is larger than the other. And you discovered what? Is it true or is it not true?
Edgar: It is not true.
Teacher: Why?
Edgar: Because… The result of this one is greater and the numerator and the denominator are greater. Oh! Yes!
Teacher: The result is not larger. The result is equal. Even it has…
Edgar: Oh, between these two…
Teacher: Between these two. Notwithstanding the terms of the first fraction are larger than the terms of the second, the result is…
Edgar: Equal.”

Edgar shows some difficulty in speaking about fractions, that he seems to regard mainly as quotients, and this is the reason why he says “result” to designate a rational number. However, his reasoning is correct and very interesting. In fact, Edgar and Juliana present a counter example in which the fractions are equal, what is rather surprising since the statement includes the "larger than" relation and not the "equality" relation. Even though these students present computations as the core of their justifications, figures 1 and 2 show that they also provide explanations in natural language. It is noteworthy that the students show difficulties in the interpretation of the statement and in expressing their ideas, but they do not show difficulty in understanding that a counter example of a mathematical statement must satisfy the given conditions and violate its conclusion.

Task 2 - The Candies of Rita

Task 2 asks students to use fractions as operators to find parts of a whole and justify the answer.

Task 2
For her birthday party, Rita brought 250 candies to give to her friends. She decided to give $\frac{1}{5}$ to her swimming colleagues, $\frac{3}{5}$ to give to her school colleagues, and kept $\frac{2}{10}$ to give to the guests of the birthday party. a) How many candies did Rita give to the swimming colleagues? And to the school colleagues? Justify your answer.

It is contextualized in a situation familiar to the students in which the information is given as fractions and whole numbers and the result is sought as whole numbers. It is a task proposed in the third part of the teaching experiment, aiming to develop the meaning of fraction as operator. As in the first task, the students worked in pairs. In previous classes the students had learned the multiplication of a whole number by a fraction and the multiplication of fractions and, in previous grades, they had studied the multiplication of a decimal number by a whole number.

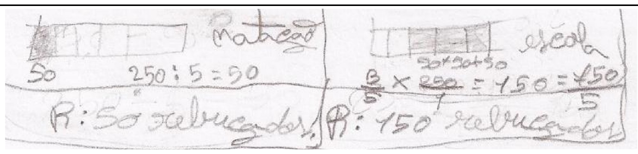
Episode 1. At the launching of this task, given the difficulties that the students showed in presenting justifications, the teacher decided to do a brief discussion with them regarding the way they could justify their answers using different representations:

> "Teacher: So, the following is written in the different questions: "Justify your answer". This justification can be done in several ways […], can be done using…
> Daniel: Drawings…
> Teacher: Schemes, drawings…

Fernando: Computations…
Teacher: Computations, words…
Rui: Graphs…
Teacher: Graphs… If you can do that… With that justification your answer must be very clear and show exactly the way you thought… It needs to be very clear. It must show exactly what you were thinking.”

This initial discussion legitimated the use of different representations to present justifications to the questions proposed in the task. The discussion was also useful to list the representations that the students could consider.

The pictorial representation, which had been widely used in previous lessons, is used in several answers. Therefore, it is not surprising that in this task most students use, at the same time, the fraction and the pictorial representation. One example is the solution of Juliana and Edgar (figure 3). Some students, as Jaime, use the decimal number representation and some other students use only fractions.



swimming	school
R: 50 candies	R: 150 candies

Figure 3: Answer from Juliana and Edgar

Episode 2. The whole class discussion yields interesting moments of interaction and communication among students. Seeking the possibility of emerging disagreements and arguments from students, the teacher begins by asking Daniel, who had worked with Marco and arrived to a wrong answer, to present it to the whole class (figure 4). The students made several computations and sought to explain their reasoning by labelling several quantities.

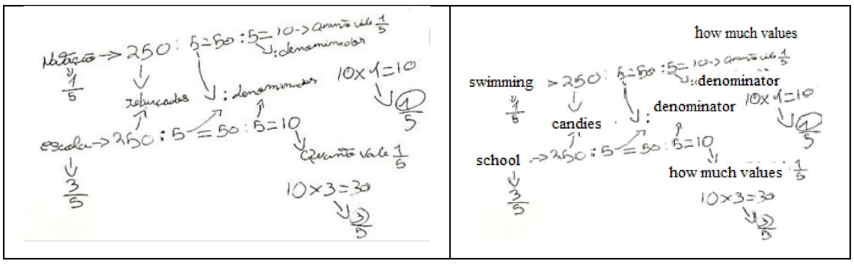


Figure 4: Answer from Daniel and Marco

Daniel begins by saying that what he wrote explains totally the solution they obtained. The teacher asks him to explain it orally and provides directions in order to complete his written explanation:

"Daniel: In swimming 250 is the number of candies to divide by 5 that is the denominator, to see how much valued the… All together…
Teacher: Then, put under swimming, put the fraction, please so that we understand what you are saying… Under the word swimming, put the fraction of the candies that she gave to the children who were in swimming. What was the fraction?
Daniel: $\frac{1}{5}$.
Teacher: That is OK. This is in order that we understand what you are saying… So, when you say that you divided by 5 is because the denominator of that fraction [$\frac{1}{5}$]… Go on…
Daniel: And it gave 50.
Teacher: And what that 50 represents?
Daniel: That 50 means how much this 5 values…. And I did 50 divided by 5 once more because of the denominator to see how much this one would value. It gave 10… So, it gave… I did… 10 times the numerator and it gave 10, I think it gave 10 candies…
Students: I don't think so…"

The last explanation given by Daniel just describes the computations that he and his colleague made. The students make a mistake when they divide twice by five to find $\frac{1}{5}$ of 250. Attending to the conditions of the question, they understand that it is necessary "to divide by 5", but then they erroneously apply that procedure twice paying no attention to what each quantity represents. It is also noteworthy that the students use the equivalence "multiply by $\frac{1}{5}$ is the same as divide by 5" which is a direct application of the definition of unit fraction.

Episode 3. Other students immediately say that they do not agree with this solution. Assuming that it would be difficult to lead Daniel to understand his mistake right away, the teacher decided to promote a contrast with another solution. She asks Jaime to present what he did (figure 5), which is rewritten at the right side:

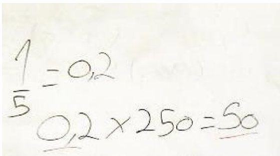
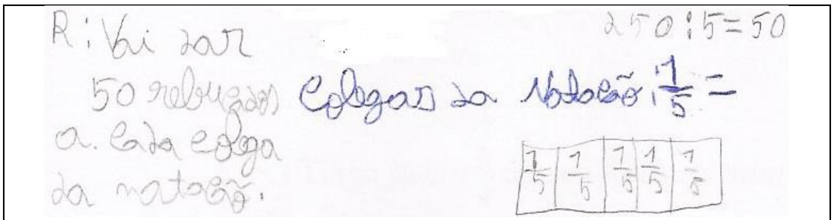
	$\frac{1}{5} = 0{,}2$ $0{,}2 \times 250 = 50$

Figure 5: Answer from Jaime

"Teacher: Let us go on, let us go on, let us explain how Jaime did. Do you want to go to the board? Put your explanation there…
Jaime: Here I put… The result of $\frac{1}{5}$.
Teacher: The result of $\frac{1}{5}$? What is that? The result of $\frac{1}{5}$?
Jaime: The value, the value of $\frac{1}{5}$ as a decimal number.
Teacher: As a decimal number… Of $\frac{1}{5}$… Oh!
Jaime: It gives 0.2 and I did… 0.2 times the number of candies and it gave 50.
Teacher: Why?
Jaime: Because I did the computation.
Teacher: It is all right, it is all right, and why did you the computation? You did this one and not another one …?
Jaime: I did this one because I recalled the exercise that we did in the other day…
Teacher: You recalled the exercise…
Jaime: Of the parliament or something…"

In the same way as Edgar did in the previous task, Jaime also shows to regard the fraction as a quotient that he transforms into a decimal number to do the computation. He is very familiar with the conversion $\frac{1}{5} = 0.2$ and understands that he just has to find out 0.2 of the total quantity of candies, but he has trouble in explaining his procedure, just indicating that he followed a similar procedure to one used in another lesson.

Episode 4. Seeking to have a wider range of solutions to compare, the teacher asked Vasco to present his solution, which uses a pictorial representation as an illustration of the situation (figure 6).



It is going to yield 50 candies to the swimming colleagues

Figure 6: Answer from Vasco

Vasco is very uneasy in explaining his solution, which he does with the support of the teacher:

"Teacher: Oh Vasco, so just say what does that full rectangle represents.
Vasco: The 250 candies.

Teacher: All right… The large rectangle represents the 250 candies, and each slice represents…
Vasco: $\frac{1}{5}$.
Teacher: $\frac{1}{5}$… And, as a fraction, how can we represent the full rectangle? Somebody can help?
Students: It is 5 [slices], $\frac{5}{5}$ or a unit.
Teacher: Or one unit! It is everything, isn't it… All candies. OK, up to that point you were able to arrive. Now what you do in the next step?
Vasco: I did the 250 divided by 5.
Teacher: For what?
Vasco: To know the result of this [each of the five parts].
Teacher: […] OK, and what conclusion did you made?
Vasco: It gave 50 candies […]."

With the support of the teacher, Vasco justifies that he divided 250 by 5 to determine $\frac{1}{5}$ of this quantity, which, again, is a direct consequence of the definition of unit fraction, and stands very clearly from the pictorial representation.

Episode 5. With the different solutions at the board, the teacher asks the students to decide which ones are correct and to justify their choice. The students easily accept that the solutions given by Jaime, Vasco and Juliana and Edgar are correct and identify the mistake in the solution given by Daniel. They say that he should not have divided the second time by 5, since he seeks $\frac{1}{5}$ of 250, and that is one fifth, so it is enough to divide by 5 once. Then, the teacher highlights the relations between the solutions of Vasco and Jaime.

After this discussion, Rui makes a generalization regarding how to multiply a fraction by a whole number:

"Rui: Any time we want to do a computation like this $\left(\text{he points to } \frac{1}{4} \times 250\right)$…
Teacher: Yes.
Rui: We only have to divide the denominator by the thing [the quantity] that is before [he meant "afterwards"] […] that, in this case, the candies.
Teacher: How is that? Explain better… Give more examples…
Rui: For example, $\frac{1}{4}$. If it is, in another example, how many candies? 150 for example… Any time there are such computations, I can do the 4 or the denominator to divide by the number. And it gives the result."

The teacher appreciates this generalization formulated in a very imprecise language by Rui. The generalization expresses in a general way what the students had done for a particular case.

However, the teacher considers that is necessary to clarify the situation and lead the student to adjust his language formulating a more correct generalization that would take into account non-unit fractions. In this way, she asks the student if what he found was also valid for $\frac{2}{4} \times 150$. Rui is quite confused and makes several mistakes, but Guilherme enters the conversation and indicates the difference in determining $\frac{1}{4}$ or $\frac{2}{4}$ of a quantity:

> "Guilherme: I think that… We can do in the same way; we only have to add one thing…
> Teacher: We must follow… Oh, Rui, that is your [statement]… He says that it is not true, you have to support it… Look, Guilherme, go on with your explanation.
> Guilherme: We can do 150 divided by 4… We can do in the same way because we can do… 4 divided by 150 gives 37.50.
> Teacher: All right, 150 divided by 4, so, pay attention…
> Guilherme: 10 divided by 4, after we do the result times the denominator.
> Teacher: The one on the top or the one on the bottom?
> Guilherme: On the top…
> Teacher: Ah, the numerator.
> Guilherme: Numerator…
> Teacher: OK, then we go, let's see, we move on… So we would do… What does it mean…? How much is 150 divided by 4? First I have to know what does it mean 150 divided by 4…? What is this 37.5?
> Guilherme: It is $\frac{1}{4}$ of 150.
> Teacher: So... Here I want… How many quarters?
> Guilherme: 2 quarters! That is why we are going to do times 2."

In this way, Guilherme extends the generalization proposed by Rui, for non-unit fractions, but it is necessary to include another step, the multiplication by the numerator. So, by working from a particular case in the whole class discussion, the students formulate two generalizations that synthetize the work done in this task.

DISCUSSION

Most students had trouble in understanding the statement of task 1. To overcome this difficulty, the teacher promoted a detailed analysis of the statement, with which the students become more involved in the task. Observing their difficulty in defining a strategy, the teacher asked the class for suggestions and the idea of experimenting several cases emerged in a natural way. As a solution strategy, most students transform fractions in decimal numbers, a representation with which they feel quite comfortable.

In this task, the justifications that the students present are, essentially, the computations that they made. But the students show capacity to justify that a statement is false by using counter examples. They accept that only one counter example is enough to refute a statement, what is quite remarkable given the difficulties that many students are used to show as indicated in the literature (Galbraith 1995). In this task there were no generalizations made by the students - they showed that a possible generalization was false.

In task 2, we verify that students use mostly fractions. However, some of them still rely on the decimal number representation to find the solution to the task. They also use the pictorial representation, not as a basis for solving the task but only to illustrate the solution, as, along with this representation, they always use computations with whole numbers and fractions. In addition, to find $\frac{1}{5}$ of 250, the students use the division $250: 5$ based on the definition of unit fraction. A student makes a generalization that is incomplete and yields an interesting moment of discussion leading to the establishment of a broader generalization. We must also note that students were able to detect and correct the mistake of a colleague, showing that they feel that they have the right to make interventions and assess the work of each other. They have internalized the idea that, in an exploratory lesson, their opinion is valued.

Regarding representations, decimal numbers are widely used by the students in the first task. They clearly feel at ease in this representation and it proved to be suitable as it allows an easy way of comparing rational numbers. The main strategy that the students use in comparing, ordering and adding fractions is converting these fractions into decimal numbers. In task 2, they use the pictorial representation in close connection with fractions, showing that this representation may still be useful at this phase to explain reasoning processes (Quaresma and Ponte 2012; Webb, Boswinkel and Dekker 2008). It is not surprising that students use more the fraction representation in solving problems involving multiplications because the rule to multiply fractions is rather easy to apply. Regarding reasoning, despite the difficulties that they face in different moments, these episodes contain fruitful moments of generalization and justification, showing that it is possible to use this kind of tasks to promote students' mathematical reasoning.

In the dialogues, the students show great difficulty in expressing themselves in acceptable mathematical language. This creates problems to their activity of justifying and generalizing and sometimes makes it difficult to understand their reasoning. These difficulties are usual in students and result from the fact that, for them, mathematical language assumes an artificial nature. That is why the progressive appropriation of a more correct and fluent mathematical language is one of the aims of the exploratory lesson in all the three phases, especially during whole class discussion, when the teacher revoices students' answers using correct mathematical language (Christiansen and Walther 1986; Ponte 2005).

CONCLUSION

The exploratory approach (Ponte, Branco and Quaresma 2014) that was followed in the lessons that encompass the episodes analysed in the chapter encouraged the students to construct their own strategies to solve the tasks involving rational numbers, to use different representations in a flexible way, to explain their reasoning and to argue for their positions. That supported students in understanding aspects of the notion of rational number, and also in the development of reasoning processes that are fundamental in mathematics, especially those related to generalization and justification. These situations drawn from a teaching experiment following the exploratory approach, show how it is possible to put into practice, in a normal classroom, curriculum recommendations that underline the importance of supporting the development of students' reasoning (NCTM 2014). This kind of teaching practice depends in the first place of the tasks that the teacher proposes and that should convey some challenge for the students and lead to the emergence of different solution strategies (Ponte and Quaresma 2016). It is also important that, in leading the communication that occurs in the classroom, the teacher explores disagreements (Wood 1999) and uses revoicing, enabling the progressive development of correct mathematical language (Franke, Kazemi and Battey 2007). Besides supporting the students' ability to express their mathematical ideas orally and by writing, the classroom environment encourages students to make generalizations and to produce justifications, thus supporting the development of their' mathematical reasoning (Mata-Pereira and Ponte 2017).

REFERENCES

Barnett-Clarke, Carne, William Fisher, Rick Marks, and Sharon Ross. 2010. *Developing essential understanding of rational numbers: grades 3-5*. Reston, VA: NCTM.

Bishop, Alan, and Fred Goffree. 1986. "Classroom organization and dynamics." In *Perspectives on mathematics education*, edited by Bent Christiansen, A. Geoffrey Howson, and Michael Otte, 309-365. Dordrecht: D. Reidel.

Artigue, Michèlle, and Morten Blomhøj. 2013. "Conceptualising inquiry based education in mathematics." *ZDM* 45(6):797-810. doi.org/10.1007/s11858-013-0506-6

Bogdan, Robert C., and Sari K. Biklen. 1994. *Investigação qualitativa em educação: uma introdução à teoria e aos métodos*. Porto: Porto Editora.

Brendefur, Jonathan L., and Jeffrey Frykholm. 2000. "Promoting mathematical communication in the classroom: two preservice teachers' conceptions and practices." *Journal of Mathematics Teacher Education* 3:125-153.

Bruner, Jerome S. 1999. *Para uma teoria da educação.* Lisboa: Relógio d'Água.

Christiansen, Bent, and G. Walther. 1986. "Task and activity." In *Perspectives on mathematics education*, edited by Bent Christiansen, A. Geoffrey Howson, and Michael Otte, 243-307. Dordrecht: D. Reidel.

Davis, Phillip, and Reuben Hersh. 1980. *The mathematical experience*. Boston, MA: Birkhauser.

Duval, Raymond. 2004. *Semiosis y pensamiento humano*. Cali: Universidad del Valle.

Duval, Raymond. 2006. "A cognitive analysis of problems of comprehension in a learning of mathematics." *Educational Studies in Mathematics* 61:103-131.

Empson, Susan B., Linda Levi, and Thomas P. Carpenter. 2010. "The algebraic nature of fraction: developing relational thinking in elementary school." In *Early algebraization: a global dialogue from multiple perspectives*, edited by Jinfa Cai, and Eric Knuth, 409-428. Heidelberg: Springer.

Franke, Megan Loef, Elham Kazemi, and Daniel Battey. 2007. "Understanding teaching and classroom practice in mathematics." In *Second handbook of mathematics teaching and learning*, edited by Frank Lester, 225-256. Greenwich, CT: Information Age.

Galbraith, Peter. 1995. "Mathematics as reasoning." *The Mathematics Teacher* 88(5):412-417.

Goldin, Gerard. 2008. "Perspectives on representation in mathematical learning and problem solving." In *International research in mathematics education,* 2nd ed, edited by Lyn English, 176-201. New York, NY: Rutledge.

Lampert, Magdalene. 1990. "When the problem is not the question and the solution is not the answer: mathematical knowing and teaching." *American Educational Research Journal* 27(1):29-63.

Lamon, Susan J. 2007. "Rational numbers and proportional reasoning: toward a theoretical framework for research." In *Second handbook of mathematics teaching and learning*, edited by Frank Lester, 629-668. Charlotte, NC: Information Age.

Lannin, John, Amy B. Ellis, and Rebekah Elliot. 2011. *Developing essential* *understanding of mathematical reasoning: pre-K-grade 8*. Reston, VA: NCTM.

Lithner, Johan. 2008. "A research framework for creative and imitative reasoning." *Educational Studies in Mathematics* 67:255-276. doi.org/10.1007/s10649-007-9104-2

Mata-Pereira, Joana, and Ponte, J. Ponte. 2012. "Raciocínio matemático em conjuntos numéricos: uma investigação no 3.º ciclo." *Quadrante*, 21(2):81-110.

Mata-Pereira, Joana, and João P. Ponte. 2017. "Enhancing students' mathematical reasoning in the classroom: teacher actions facilitating generalization and justification." *Educational Studies in Mathematics* 96:169-186. doi.org/10.1007/s10649-017-9773-4

Monteiro, Maria C., and Hélia Pinto. 2005. "A aprendizagem dos números racionais." *Quadrante* 14(1):89-108.

National Council of Teachers of Mathematics [NCTM]. 1991. *Professional standards for teaching mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics [NCTM]. 2000. *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics [NCTM]. 2014. *Principles to actions: ensuring mathematical success for all*. Reston, VA: National Council of Teachers of Mathematics.

Oliveira, Paulo. 2008. "O raciocínio matemático à luz de uma epistemologia soft." *Educação e Matemática* 100:3-9.

Pólya, George. 1990. *Mathematics and plausible reasoning*. Princeton, NJ: Princeton University Press.

Ponte, João P. 2005. "Gestão curricular em matemática." In *O professor e o desenvolvimento curricular*, edited by GTI, 11-34. Lisboa: APM.

Ponte, João P., Neusa Branco, and Marisa Quaresma. 2014. "Exploratory activity in the mathematics classroom." In *Transforming mathematics instruction: multiple approaches and practices,* edited by Yeping Li, Edward Silver, and Shiqi Li, 103-125. Dordrecht: Springer Science+Business Media Dordrecht.

Ponte, João P., and Marisa Quaresma. 2011. "Abordagem exploratória com representações múltiplas na aprendizagem dos números racionais: um estudo de desenvolvimento curricular." *Quadrante* 20(1):55-81.

Ponte, João P., and Marisa Quaresma. 2016. "Teachers' professional practice conducting mathematical discussions." *Educational Studies in Mathematics* 93:51-66. doi.org/10.1007/s10649-016-9681-z

Ponte, João P., and Leonor Santos. 1998. "Práticas lectivas num contexto de

reforma curricular." *Quadrante* 7(1):3-33.
Quaresma, Marisa, and João P. Ponte. 2012. "Compreensão dos números racionais, comparação e ordenação: o caso de Leonor." *Interacções* (8)20:37-69.
Rivera, Ferdinand D., and Joanne R. Becker. 2009. "Algebraic reasoning through patterns: findings, insights, and issues drawn from a three-year study on patterns are intended to help teach prealgebra and algebra." *Mathematics Teaching in the Middle School* 15(4):213-221.
Sherin, Miriam G. 2002. "A balancing act: developing a discourse community in the mathematics classroom." *Journal of Mathematics Teacher Education* 5:205-233.
Stein, Mary K., Randi A. Engle, Margaret Smith, and Elisabeth K. Hughes. 2008. "Orchestrating productive mathematical discussions: five practices for helping teachers move beyond show and tell." *Mathematical Thinking and Learning* 10:313-340.
Swan, Malcolm. 2014. "Designing tasks and lessons that develop conceptual understanding, strategic competence and critical awareness." In *Tarefas matemáticas: livro de atas do encontro de investigação em educação matemática (EIEM2014),* edited by Joana Brocardo, Ana M. Boavida, Catarina Delgado, Elvira Santos, Fátima Mendes, José Duarte, Mário Baía, and Miguel Figueiredo, 9-28. Lisboa: SPIEM.
Webb, David C., Nina Boswrinkel, and Truus Dekker. 2008. "Beneath the tip of the iceberg." *Mathematics Teaching in the Middle School* 14(2):110-113.
Wood, Terry. 1999. "Creating a context for argument in mathematics class." *Journal for Research in Mathematics Education* 30(2):171-191.