

Investigating mathematical investigations¹

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There are good reasons why we may involve the students in doing mathematics investigations. Recent curricula encourage this sort of activity but we notice that its application in the classroom is not a simple matter. We discuss the issues that arise when students are presented with investigative tasks, with special interest in the dynamics of the classroom and in the role of the teacher. Our aim is to derive suggestions for classroom practice as well as for further research and teacher development.

Why mathematical investigations in the classroom?

New perspectives on the philosophy of mathematics, stressing the consideration of mathematical activity, brought attention to the investigative processes involved in the creation of new mathematical knowledge (Davis and Hersh, 1980; Ernest, 1991; Lakatos, 1976; Tymoczko, 1986). However, the idea is not new. It may be seen behind Pólya's proposal to make problem solving an essential element of students' mathematical experience: "mathematics has two faces; it is the rigorous science of Euclid but it is also something else (...) mathematics in the making appears as an experimental, inductive science" (1957, p. vii). This idea may also be traced in the writings of the Portuguese mathematician Bento Carvalhal who wrote: "we may find hesitations, doubts, contradictions, that only a long reflection and refinement period may overcome, so that other hesitations, other doubts, and other contradictions soon emerge" (1958, p. xiii). Today, there is general agreement among educators that learning mathematics involves, in a fundamental way, making mathematics. And

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making mathematics is, first of all, doing mathematical investigations (Poincaré, 1908/1996).

A mathematical investigation begins with a situation which must be understood or a set of data which must be organized and explained in mathematical terms. It is necessary to start asking reasonable and fruitful questions. Then, it is essential to run the risk of proposing conjectures. Testing these conjectures and collecting more data may support them or lead to new conjectures. Plausible arguments and formal proofs may provide further arguments to confirm or reject our standing conjectures. During this process new questions may also arise for further research.

Mathematical investigations are also important from the educational point of view. In our perspective (Oliveira et al., 1997), they: (a) stimulate the sort of student involvement required for significant learning; (b) provide multiple entry points for students at different ability levels; (c) stimulate a holistic mode of thought, relating many topics, a basic condition for significant mathematical reasoning; and (d) are indispensable to provide a complete view of mathematics, since they are an essential part of mathematical activity.

Mathematical investigations provide a good context for making students understand the need to justify their assertions, expressing their reasoning before teacher and colleagues. When confronting their different conjectures and justifications, the members of a class establish themselves as a small mathematical community, constantly interacting, where mathematical knowledge develops as a common endeavor. Investigations are good starting points for an inquiry mathematical class (Wood, 1994), helping to create a new and dynamic learning environment. However, they pose new requirements to teacher's competencies.

Mathematical investigations are closely related to other sorts of tasks used in mathematics education, such as problems, modelling and projects. All of them have much in common, referring to complex mathematical processes and requiring at least some creativity from the student. In problem solving, the problem (usually) consists of a well-defined mathematical (even if contextualized) question proposed by the teacher. Modelling tasks (in general) refer to real-world situations that require the construction, refinement and validation of some sort of mathematical model. Projects (very often) involve an extended work originated by some general problem and end up with the production of a report or some sort of product or prototype. Mathematical investigations bring the student to an activity close to that of the research mathematician. They have two distinctive features: (a) the concern lies more in the mathematical ideas and relationships than in its relation to the context and (b) the student has a critical role in setting up the questions to investigate, as well in designing strategies and carrying them out and in validating the results.

Despite the emphasis on the curriculum in several countries, in general, mathematics teaching pays little attention to the more advanced aspects of mathematical activity. The reference in official documents seems to have little effect in classroom practice (see, e.g., Lerman, 1989; Mason, 1991; Silver, 1993). There is clearly a need to understand better what is the real potential of this sort of activity and the factors that may hamper its successful realization in mathematics classrooms.

MPT — Mathematics for all: A collaborative project

The interest in understanding what is involved in doing mathematical investigations brought together several researchers and classroom teachers. We decided to set up a project to experiment tasks involving students exploring and investigating around mathematical ideas, concepts, and processes. Our work includes producing, experimenting, and evaluating such tasks and studying the competencies needed for using them successfully in the mathematics classroom.

This project aims to establish a strong connection between research and classroom practice. Its team includes 11 teacher educators and researchers from universities and teacher education colleges and 9 teachers from grades 5 to 12. There are three main groups, devoted to specific mathematical topics (functions, geometry, and numbers and patterns), as we want to give great attention to the existing curriculum and ensure a clear suitability of the tasks for the classroom. Several months ago, our group was also created, with the main objective of studying classroom processes and interactions in mathematical investigations.

The work of the project involves several domains. In terms of *curriculum development* we are concerned with issues such as: Are investigations worthwhile in themselves or should they support the learning of mathematical “content”? Should the tasks have more or less structure? Should we provide more or less suggestions for teachers? What kind of suggestions?

Another aspect of concern includes *students’ learning*: What is the influence of these activities in students’ conceptions regarding school mathematics? And what is the influence of students’ conceptions in the way they work on mathematical investigations? What are the relationships between basic mathematical knowledge and advanced reasoning processes such as conjecturing and proving?

We also consider *classroom dynamics*: What issues arise when students work in groups, in pairs, and individually in this kind of activity ? What new patterns in classroom interactions are associated with doing mathematical investigations? What are the changes in students’ and teachers’ roles?

And finally, we look at the *teachers* and, specially, at the decisions they face: How to integrate these tasks in the curriculum? How much time to spend on investigations? How to present them to students? What sort of support to offer students? How to promote classroom discussions? How to assess students? What other competencies are required in conducting this kind of activity?

The task

In several 7th and 9th grade classes we proposed to students the following task:

Explorations with numbers				
<ul style="list-style-type: none"> • Try to discover relationships among the following numbers 				
0	1	2	3	
4	5	6	7	
8	9	10	11	
12	13	14	15	
16	17	18	19	
...	
<ul style="list-style-type: none"> • Make a record of the conclusions you reach. 				

Classes dedicated to mathematical investigations usually involve (Christiansen and Walther, 1986): (a) *introduction to the task* by the teacher and its beginning by the students; (b) *accomplishment of the task*, during which the teacher interacts with students individually or in small groups; and (c) *presentation of the students' results and debate*, including the analysis of the used strategies and the outcome and eventually issues for future investigation. That was the basic structure of all the classes we consider in this paper.

The episodes presented refer to classes conducted by teachers in the MPT project that were observed by members of our group and video-recorded. Transcripts were made and then analyzed considering how interactions develop between teacher-student and student-student. We consider, in turn, the main steps involved in doing an investigation: (1) proposing questions and establishing conjectures; (2) testing and refining the conjectures, and (3) arguing and proving the conjectures.

Questions and conjectures

A mathematical conjecture is a statement that answers a certain question and that is considered to be true. It may refer to a single object or a whole class of objects.

Obviously, not all conjectures have the same mathematical reach. Some, refer to relatively simple and self-evident facts. Others, involve a strong element of the unexpected. Still others, are particularly important because from them it is possible to derive many conclusions.

Students do not formulate their questions precisely neither do they discuss them in detail among each other. Any conjecture presupposes one or more questions, but students only formulate them implicitly. This confirms results from previous research, indicating that the elaboration of questions to investigate is one of the students' weak points (Ponte and Matos, 1992/1996; Silver, 1993).

Despite the task's strong arithmetical mark, there are several conjectures with a clearly geometrical inspiration. Some students pay attention to what happens in the various columns or in the diagonals, observing, for example, the location of 16-13-10-7, 3-6-9-12 and 0-5-10-15. These conjectures are mostly based on observation.

Other students look for links between numbers of the table using arithmetic operations. Some add the elements of the columns, or the elements of the lines, while others search for ways to transform numbers into each other through simple operations. The idea of having to manipulate numbers shows a strong influence of the notion that mathematics consists essentially of computation procedures.

In this task, perhaps motivated by the enunciate, many students identify conjecture with "conclusion". This is a possible and probably acceptable approach towards the idea of conjecture, but it has its dangers. A conclusion can normally be reached through a relatively simple process, which does not involve either the test or the proof. Certain students seem to feel great pressure to reach as many "conclusions" as they can, elaborating many trivial conjectures (as one student said: "it doesn't matter, we'll just put them all!"). This type of irrelevant conjectures seems to have to do mostly with a certain difficulty in understanding the general meaning of the task as the discovery and validation of interesting mathematical relationships.

Interaction between teacher and students has an important role in understanding the conjectures which are already formulated and in the emergence of new conjectures. Let us focus our attention in some key moments of this interaction. In a specific example in which a group of students present their work to the class, the teacher starts adopting the role of mediator and guide. She directs the students' attention to key concepts:

Teacher: *So, you discovered perfect squares, which are?*

This calling of attention is very important, in some cases, in the way students carry about their work.

The teacher points to connections with other activities:

Teacher: *1, 4, 9, 16... So, perfect squares, which we previously saw in the match activity, remember?*

From this moment on she assumes a more active role and behaves as is she was working alongside the students, in a large group. While a student presents his results, the others are called to participate.

The teacher starts by directing attention towards the “path”, trying to provide students information about strategies they can use to find a conjecture:

Teacher: *Look for a while to see if you can discover the path that “it” follows.*

Students refer their localization. This is an instance of reasoning through geometrical patterns:

Students: *It's in “zigzags”.*

During the dialogue between teacher and students new conjectures may arise or, as in this case, there are refinements and variations of conjectures which had been brought forth previously. This only happens if the teacher believes that students themselves can have interesting ideas.

Testing conjectures

Once formulated, a conjecture must go through several tests. A single counter-example is enough to reject it. However, the fault may lie in the counter-example. It may also happen that a conjecture which is considered false, becomes true just by introducing a small change.

Another situation refers to a group of students that establish a conjecture as a result of manipulating numbers in the table. After having formulated a conjecture concerning the sum of the elements of the columns, they decided to add each line and observe what would happen. The students begin by analyzing several cases, which helps them to refine their conjecture. Next there is a “clarifying” phase for a student (Josefa) who has not yet understood her colleagues. In the last phase, driven by the need to explain things to Josefa, they end up testing the conjecture in a few more cases. Here we see an instance of reasoning focused in arithmetical operations:

Marcelo: *Here we add 6. 4 plus 5, 9. 15 plus 7, 22.*

By analogy with a conjecture previously studied by them, the students try to establish a conjecture that involves the difference between the sums of the lines:

Marcelo: *54. Let's see what we can say about this. It increases...*

This student recalls what the task is all about:

Marcelo: *Wait, we have to think about the conclusions.*

We must note that during interaction among students many things seem to be thought but not said out loud. Students do not say all they think, even when thinking aloud, and this often hampers communication. Between each other, a great part of their communication is non-oral. Students sometimes have interesting ideas that they have difficulty in expressing clearly and correctly. This is a problem concerning the mastery of basic cognitive tools. Once conclusions are reached, students have great difficulty in elaborating them in writing. During communication inadequate mathematical terms are used successfully. However, this does not prevent students from correct reasoning:

Rosa: *So, "adding the lines..."*

The teacher is always struggling to try to promote the use of an adequate mathematical language in students. But this leads to dilemmas: imposing correct language suddenly and too soon may break self-confidence and fluency in students; to let them use incorrect language for too long may further hinder the process of adopting a correct language by students... Therefore, the teacher needs to have a reasonable tolerance concerning the use of inadequate terms, but must know when to intervene.

The fact that there is one student, Josefa, who does not understand the conjecture, forces Marcelo, who formulated it, to test it, allowing Josefa to understand it and be persuaded. On the other hand, it makes Marcelo express his reasoning more clearly. Thus, group work, where students who find it easier to formulate conjectures are confronted with other students, promotes carrying out tests. What starts as a need to discuss in order to explain and convince others, may later also become a logical need for the person himself.

However, most students are not particularly concerned with testing their conjectures, except when the teacher specifically asks them to. In order to make the process of testing conjectures easier for students, the teacher may: (a) direct their attention to the need of testing conjectures; (b) help remember/clarify concepts that are relevant for a particular conjecture; and (c) indicate (or help students identify) some basic information which is important to understand the situation. Several aspects of the teacher's role, are seen in the following examples:

Teacher: *Do you know what prime numbers are?* (Aiming to clarify the concept).

Teacher: *The 3, 7 and 11 are missing, as well as 23.* (Calling attention to basic

- data, that must be correct for someone to formulate a valid conjecture).
- Teacher: *But... Does anyone want to comment this?* (Encouraging students to give their opinions on the conjecture, after clarifying concepts and correcting data).
- Teacher: *Isn't finding out that there is no rule a conclusion?* (Trying to stretch the concept of conjecture, to “negative cases”, thus not confining itself only to positive cases).

Validation

Conjectures, even when they survive various tests, still do not achieve the *status* of mathematical truths. In order to be considered mathematically valid their justification must be based on a logical or at least plausible rationale. Let's see how students face this need of validate results and the influence the teacher may have on this aspect of mathematical activity.

When students are working in group and discover a conjecture that may be true, their most common attitude is not to try and prove it, but to communicate it as quickly as possible to the teacher. They probably do this for two reasons: (a) to “show work done”, gaining credit for the discovery; (b) to gain its validation, that is, a confirmation that the conjecture “is right”. Some students show that they are greatly dependent on the teacher and have little confidence in themselves, as the source for validating knowledge. Besides, this lack of students' confidence concerning their opinions and reasoning is a strong feature of their relationship with mathematics. For example, in a group work situation, when the teacher refutes a student's statement, his colleagues follow her immediately. It does not even cross their minds for one moment that their colleague might be right...

Interaction between students and the teacher occurs in two distinct situations. In one, the teacher runs through the several groups, sees the work they are doing, tries to solve difficulties that may arise and inspire students for the need to justify the conjectures that they believe to be true. In another, the class is in a situation of collective work where students present their results, which are discussed by everyone. In this case, part of the teacher's role is to call their attention to the need to justify the conjectures they express. To do so, the teacher must choose conjectures that are to be proved. He needs to pick conjectures that are striking, that are models and not too repetitive in relation to others.

This following episode is a rather interesting case of presentation and justification of a conjecture. It occurs in a moment of collective work when a 9th grade student presents the result of one of his group's investigations:

Ramiro: *Suppose we make a small square, anywhere. If we follow the arrows, here it adds 1, 4, and then it subtracts 1 and 4.*

This is an instance of combination of geometrical and arithmetical strategies. The teacher ends the presentation stage and asks the remaining students to react to the conjecture which has been presented. She tries to engage all students in discussion. Rudolfo agrees with the conjecture but argues that his colleague is presenting nothing new:

Rudolfo: *Good thinking! But what's new about that? What does that conjecture add to what we've already discovered?*

The teacher asks once again students to comment and encourages all of them to join the discussion:

Teacher: *Who wants to reply ?*

The proponent of the conjecture gives his reasons, “defending” his discovery. He generalizes, indicating that not only is it valid for squares but also for any other polygons. He proposes to give an example of an extreme case (“the biggest polygon possible”):

Ramiro: *Besides the square, that is true for any polygon, just as long as it fits in the table, I'm going to make the biggest one possible!*

The student who questioned the conjecture's interest presents several points that clarify it and, in some way, justify its validity. He does so using arithmetical and combined arguments:

Rudolfo: *If we divide a polygon in two we'll get two equal “faces”. If we add numbers on one side, we will have to subtract them on the other side.*

Thus, the justification emerges naturally in the context of a discussion about the mathematical range of a conjecture. Initially the conjecture refers to a simple case but because of a colleague's criticism it is generalized. Its validity is confirmed for extreme examples, which is an interesting form of validation.

In situations of collective work, the teacher asks students to present their results, which are open to everyone for debate. Diversity of interventions and the validation of reasoning are typical features of a genuine debate, whilst the lack of confrontation, or even the lack of debate, are typical of a situation where students wait for the teacher to take the initiative.

This example is interesting, yet it is not truly representative of the way students understand the need to prove conjectures. Actually, students' performance in this aspect of mathematical activity shows how they belittle the demonstrative side. They don't feel

the need to find a justification, neither do they seem to understand what elements they can depend on to present justifications for a conjecture.

The diminished perception most students have of needing to prove conjectures is understandable given the lack of importance that carrying out investigations, reasoning and proof have in mathematical activity in the school routine. In this sense both students' conceptions and their difficulties in mathematical fluency interfere. Situations of group work are quite poor concerning the demonstration of conjectures, even when the teacher interacts with students. In contrast, moments of collective work may be powerful, when the teacher pays attention to this aspect.

Exploration of the task

Conjectures formulated by students were reached through observation, either from the initial table or from a table to which more lines were added, or reached through the manipulation either of isolated numbers or of lines or columns. Let us first take a look at the ones reached through observation.

Students identified even numbers (columns 0 and 2) and odd numbers (columns 1 and 3). Also, the multiples of 2 and 4, which are in columns, were easily identified. A few groups discovered multiples of 3, 5, 7 and 9 in various types of diagonals. They identified numerical patterns: (a) the last figures of the numbers in column 0 are successively 0, 4, 8, 2, 6, 0, ... and (b) the last figures of the numbers in the diagonals also obey a pattern (formed by two figures).

Students also formulated conjectures concerning prime numbers, perfect squares and powers of 2. Prime numbers are found in columns 1 and 3, except 2, and their arrangement does not obey any regularity. Perfect squares are found alternately in columns 1 and 0, and their distribution in each of the columns obeys a regularity. Powers of 2, except 2^0 and 2^1 , are in column 0. From 2^2 onwards, their distribution in column 0 is also related to the powers of 2.

Let us also look at the conjectures reached by manipulation. The most frequent are related to the structure of the table, concerning lines, columns and diagonals. Horizontally, we go from one number to the next, adding 1 and vertically adding 4. In the diagonals from top to bottom and right to left, we go from one number to the next, adding 3; in the diagonals from top to bottom but from left to right, we add 5 to the previous number.

Other conjectures also arose that involved operations such as (a) the difference between the sum of the numbers in consecutive lines (16) and the difference between the sum of numbers in each column (depends on the number of lines used); (b) if we multiply the numbers of a column by 2 we will obtain values that belong to the column

0 (and that, from number 4 onwards, alternate); (c) the sums of the numbers that form each line are in column 2 (the sum of the first line (6) is in the second line and the sums of the following lines are sequentially found every 4 lines); and (d) the remainders of the division by 4 of the numbers in each column are respectively 0, 1, 2 and 3.

Students also formulated a conjecture concerning the construction of polygons which vertexes and sides are numbers of the table. By forming a square with, for example, the numbers 17, 18, 21 and 22, and algebraically adding the variation of each number to the next, we get zero. This conjecture was expanded for any other polygon.

A group of 7th grade students identified properties of column operations. If we add a number in column 0 to another in column 1 we get a number in column 1; if we add a number in column 0 to another in column 2 we get the result in column 2, and so on; therefore, column 0 is the identity of the addition of columns. If we multiply a number in column 0 by numbers in columns 1, 2 and 3 we get results in column 0; that is, column 0 is the absorbing element of the multiplication of columns.

Some of the ideas presented by students may be examined more deeply. We do not intend to do an extensive study, only to give an idea of the potential of the task.

Where can we find the multiples of several numbers? We can distinguish 4 situations:

<i>Multiples of numbers of the form</i>	<i>Are found...</i>
$4n$, where $n \geq 1$	in column 0
$4n+1$, where $n \geq 0$	in all columns, in diagonals of the form \ with $n-1$ lines of interval
$4n+2$, where $n \geq 0$	alternately in columns 0 and 2, in zigzag
$4n+3$, where $n \geq 0$	all columns, in diagonals of the form / with n lines of interval

Students saw where the powers of 2 are. We may also ask: where can we find powers of other numbers? It is also possible to propose a generalization. We can distinguish the same 4 cases.

<i>Multiples of numbers of the form</i>	<i>Are found...</i>
$4n$, where $n \geq 1$	in column 0, except $(4n)^0 = 1$
$4n+1$, where $n \geq 0$	in column 1
$4n+2$, where $n \geq 0$	in column 0, except $(4n)^0 = 1$ and $(4n+2)^1 = 4n+2$
$4n+3$, where $n \geq 0$	in columns 1 and 3

Students discovered the localization of perfect squares. A more general question concerns the powers of any order, that is, the sequence of the cubes of natural numbers, fourth order powers, etc. Here too we propose a generalization. Looking at a few cases

we perceive the existence of a regularity in the disposition of powers in terms of columns. If the exponent is even (n^{2p}) then the sequence of columns is 1, 0, 1, 0, 1, 0,... If the exponent is odd (n^{2p+1}) then the sequence of columns is 1, 0, 3, 1, 0, 3,... A deeper analysis shows that, in the first case, the power is found in column 1 if the base is odd, and in column 0 if the base is even. In the second case, the potency is found in column 0 if the base is even, and in columns 1 and 3 alternately, if the base is an odd number.

Is it possible to define an algebraic structure for the table? One of the conjectures elaborated by a group of 7th grade students, which we mentioned earlier, referred to the fact that column 0 is the identity element of the addition of columns and the absorbing element of the multiplication of columns. This conjecture suggests a deeper analysis. Let's consider $C = \{0,1,2,3\}$ the set of columns. Whatever the elements of columns X and Y are, their sum belongs to a single column Z . This property allows us to define the addition of columns: $X \oplus Y = Z$. Also, whatever the elements of X and Y are, their product belongs to a single column Z . Therefore, we may define $X \otimes Y = Z$. We easily verify that the (C, \oplus, \otimes) is a commutative ring. This structure is not a field because there is one element (column 2) that has no multiplicative inverse.

Implications

Students do not have a very clear notion that the investigative process begins with questions and that it is essential to begin by asking good questions. They also seem not to have a very elaborate idea on what a conjecture is, associating it with "conclusion". They seem to understand reasonably the role of tests and counter-examples, although they do not always derive the respective consequences from them. Generally, they have little perception of the need to justify their conjectures, so that these assume the status of mathematical truths. But if we take into account the scarce importance that the teaching of mathematics gives to investigative tasks, we must consider the students' performance very positive. Actually, how can we be surprised that 7th or 9th graders have difficulty with the notion of conjecture, if this notion is sometimes not known by students in university mathematics teacher education courses after 3 years' specialized studies in this area?

Students' difficulties are related to their conceptions, knowledge and competencies and values. But students do not only show difficulties. They also demonstrate abilities (a) use of geometrical strategies and integration of geometrical and arithmetical strategies to reach conjectures; (b) use of variation and generalization strategies to reach conjectures as well as building evidence by analyzing extreme cases; (c) ability to change and adapt conjectures prompted by counter-examples; and (d) use of reasoning

strategies that indicate a remarkable intellectual flexibility. This suggests that if more attention were given to this type of work in class, we could expect meaningful progress in carrying out mathematical investigations.

It is not possible to establish any comparison between this type of class and a class where, as it often happens, students are not expressly given a task, merely participating with no need to engage in meaningful mathematical thought (Wood, 1994). Neither can this type of class be compared to another in which students simply perform routine tasks such as solving exercises (Christiansen and Walther, 1986).

Work group and collective group emerge, in a class dedicated to carrying out investigations, as something natural and complementary. Group work encourages communication between students and promotes a better explicitness in conjectures and carrying out tests. Collective work imposes a greater formalization of reasoning and urges students toward a more mature performance in arguing with the teacher and colleagues. It is in this kind of work that the need of justification most reveals itself. This way the best is made of the potential of interaction between peers (emphasized by researchers in the Piagetian tradition) and of collaboration with a more competent partner (emphasized by researchers in the Vygotskian tradition) (JŠrvelŠ, 1996; Wood, 1994).

The teacher's role is essential in task selection, structuring the class, conducting it and negotiating meanings. In special, negotiating meanings is essential for students' learning. The teacher has to give heed to the communication process, acknowledging the role of non-oral language, but trying to promote mathematical language — without imposing it prematurely. The teacher must be aware that by act or omission, he or she is constantly transmitting information, through oral as well as non-oral language — and so that intervening and not intervening are, basically, two forms of intervening.

Difficulties shown by students in understanding the meaning of the task suggest that the teacher must pay attention to the way to present it. Giving each student a sheet of paper with a written enunciation is a practical way to start a class but it might not be the most convenient way to provide a good understanding of the task. Dedicating time, in the final discussion, to reflect on the nature of the activity, may be equally important, so that students develop another notion of what it means to do a mathematical investigation.

Such final reflection is a fundamental element of an investigation class. Carrying out investigative work and not reflecting on it is to waste most of the activity's learning potential. As Bishop and Goffree (1986) highlight, learning does not result only from activity but from reflecting on the activity. For the student, it is essential to be able to confront his opinions with those of others, feel the public appraisal of his and others' work, appreciate the attitude of doubt, criticism and the need to justify exemplified by

the teacher. The moment of final reflection is also particularly adequate to establish links between this type of mathematical work and other ideas that students find in learning this subject. All these aspects may mark the student decisively, helping him understand the meaning of mathematical investigations and the strategies that he may use to carry them out.

We are constantly seeing strong recommendations to conduct group work in the mathematics class. Inversely, collective work tends to be presently underrated in the field of curricular innovation. However, it is essential to share meanings, to promote a more correct mathematical language, to understand the importance of demonstrations — in short, to advance formalization. It is important that collective work is valued again — granting the student the leading role, alongside the teacher.

A class where mathematical investigations are carried out has a certain resemblance to a class where problems are solved, where people work on modelling tasks, or where a project is developed. But it is certainly a very different class from those where exercises are solved or topics are “laid out”. The type of participation expected from students is very different, as well as the cognitive processes that are used. Students’ requests are also very different during this type of class, as well as the competencies the teacher must have in order to conduct it.

Therefore, we should not underestimate the requirements that the conduction of investigative classes demand of teachers. They must: (a) view mathematics not as being about memorizing definitions and obtaining correct answers, but as questioning, thinking, correcting and confirming; (b) have a good competency in carrying out mathematical investigations, feeling at ease when confronted with complex and unpredictable situations; (c) value a different type of curricular objectives, as a full range of competencies, much beyond the mastery of computation and basic mathematical facts; (d) develop their curricular creativity in order to conceive and adapt adequate tasks for students; (e) assume a perspective on students’ learning based in activity, interaction and reflection; and (f) be able to conduct a class with a very different dynamic from the usual class, without leading students either too much or too little (Mason, 1991), offering them a more autonomous but more interactive learning experience (both in group work and in collective discussions).

The work we have been developing shows that all this is well within teachers’ reach so long as two conditions are met: that the teacher assumes this type of teaching as a personal goal and that he or she gets support. This support must come from several sources, from inside the actual school to educational authorities, without forgetting the contributions of mathematics education research. In particular, this research can help to: (a) better understand the cognitive processes which are involved; (b) identify the phenomena going on in the dynamics of this type of class; (c) detect special difficulties

that teachers have in adjusting this kind of task and identifying adequate strategies of professional development in this area; and (d) enhance these types of curricular objectives in the eyes of society, the school and the mathematical community itself.

Conclusion

A successful mathematics class is necessarily based on valid and involving mathematical tasks. The teacher needs to be able to build a stimulating learning environment and create multiple opportunities of debate and reflection among students, but these will not be enough if the proposed mathematical tasks are not a good ground for a rich mathematical exploration and sufficiently challenging. The curriculum may mandate investigations as part of students' mathematical experience and the shelves of educational materials may be full of suggestions. Research may figure out many patterns of what happens during this sort of activity. However, the teacher will always be a central actor in this process and part of his or her work cannot be formalized. It needs to rest in his or her mathematical and educational sense, deciding what is important in each moment, listening a lot, showing flexibility, and trying to figure out what may be the most adequate next move. It is this combination of intentional activity, professional knowledge and artistry that makes teaching a unique profession and mathematics teaching a most challenging activity.

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