In J. P. Ponte, J. F. Matos, J. M. Matos e D. Fernandes (Eds.), *Mathematical problem solving and new information technologies: research in contexts of practice* (pp. 239-254). Berlim: Springer

1992

Cognitive Processes and Social Interactions in Mathematical Investigations

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Abstract: Mathematical investigations may be important educational activities, proving to be useful in the development of mathematical ideas. The principal difficulties may concern students' content knowledge, reasoning processes, and general attitudes and appreciation. This paper refers to a computer based investigation undertaken by three eight-grade students, discussing in special their cognitive processes and social interactions.

Keywords: investigations, computers in mathematics education, cognitive processes, strategies, conjectures, social interactions

In mathematical investigations students are placed in the role of the mathematicians. Given a rich enough and complex situation, object, phenomenon or mechanism, they try to understand it, to find patterns, relationships, similarities, and differences leading to generalizations. Mathematical investigations range from quite elaborated and complex tasks, that may require a considerable amount of time to carry out, to smaller activities that may arise by consideration of a simple variation on a well-established fact or procedure.

Mathematical investigations share common aspects with other kinds of problem solving activities. They involve complex thinking processes and require an high involvement and a creative stand from the student. However, they also involve some distinctive features. While mathematical problems tend to be characterized by well defined givens and goals, investigations are much looser in that respect. The first task of the student is to make them more precise, a common feature that they share with the activity of problem posing.

In the process of carrying out a mathematical investigation it is possible to distinguish activities such as define the objective (what are we trying to know?), set up and conduct experiences (what happens in such or such specific instance?), formulate conjectures (what general rules may we propose?), and test conjectures (what may be critical experiences to ascertain the value of this conjecture? Is it possible to make a proof?).

The realization of mathematical investigations has become a fairly popular curricular orientation [5, 10]. However, little is known about what happens in the course of mathematical investigations, specially if carried out in school settings (a point also made by [5, p. 94]). What kind of profit do students take from them? What difficulties do they find? What constraints do they put on the teachers' role?

The computer, used as a tool, has been proposed as a very useful instrument for carrying out mathematical investigations. It encourages the realization of a large number of experiences, allowing the exploration of quite non-trivial situations and issues. It is also of great interest to know what specific features it may bring to this mathematical activity. These are some of the questions that we set ourselves to respond in this study.

Pedagogical Context and Research Methodology

As in any other educational activity, in carrying out mathematical investigations, it makes a big difference the way things are designed and organised. We need therefore to clarify the general context of the episode, the way the activity was proposed, the role of the teacher (in this case one of the present authors), and the idea that the students made of their own role in this process.

The activity analysed in this study was carried out in an extra classroom setting at a school involved in the MINERVA Project (Pole DEFCUL), during the school year of 1988/89.

The students were 8th graders who voluntarily enrolled to work with computers, in the school computer centre, in a specified weakly time slot of 2 hours. Their relation with the centre lasted for the whole school year, working in Logo activities and projects. These students had previous contact with Logo the year before, in the mathematics classroom and before this episode they developed programming activities for six months, most of them based in projects of that they designed themselves.

One the students, Maria, was a very good achiever in mathematics and in the other subjects. The other two, Nuno and Victor, were medium towards low achievers in most school subjects.

One of the present authors was in the school computer centre during this time slot for the whole school year and was well known of the students. The activity discussed in this study was proposed in one session that took place in 18 April of 1989.

The activity was based in a recursive Logo procedure with three variables which draws peculiar kind of shapes. It shows in an interesting way the effect of recursion in a geometrical procedure [1, 3, 9]. The students were given a sheet of paper with the procedure, a sample screen output, and a general formulation of the task (see figure 1). Besides, the researcher explained the purpose of the investigation and worked out one or two examples with the students.

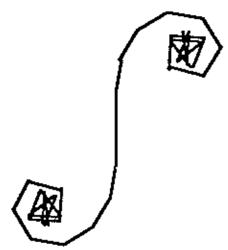
During the activity the researcher had a strong interaction with the group, specially in the beginning and in two or three moments. In the rest of the time the students worked just by themselves. The activity ended with a general discussion of the results between the students and the researcher.

ACTIVITY LEM #15
(File: INSPI)

INSPIRAL

The procedure INSPI allows you to draw a kind of figure that we will call "inspiral".

This is the result of INSPI 10 0 10.



The figure has two "enrollments".

Investigate the nature of the figures that you can obtain with the procedure, trying to elaborate a theory about the number of enrollments and the kind of figure you can get.

Suggestion: At the beginning it will be better to take the first and the third parameters as constants (for example, 10), and give successive integer values to the second parameter ranging from 0 to 20.

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TO INSPI :L :A :I
FD :L
RT :A
INSPI :L :A+:I :I
END
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Figure 1: Working proposal

The involvement of the students varied during the course of the session and, as we will see, was quite different from student to student. Their general attitude was "we are here to try to do whatever the teacher (e.i. the researcher) asks us to do."

In this study data was collected by video-taping the students. The tapes were reviewed a number of times by both of the researchers. Successive analyses of the episodes were produced, including a scheme describing the main stages of the work of the students within the activity. From there new observations of the tapes were made in order to clarify new aspects and produce the final written account.

A Framework to Discuss Mathematical Investigations

Several activities can be identified during the course of an investigation. These can be organised within three main phases of work which will be now discussed in detail: (a) formulation of objectives, (b) definition of strategies, (c) reflection on the experiments carried and formulation and testing of conjectures.

Formulation of Objectives. An investigative task may suggest the setting of a great multiplicity of objectives. Some may be more general, and others refer to aspects of detail. Some may be more precise and others more vague. The ability to formulate precise research objectives is one of the most essential aspects of the ability to undertake investigations.

Significant questions about the setting of the objective of an investigation by the students are:

—How is the research objective initially formulated?

—Are there turning points in the process of conducting an investigation that can be referred to change in the overall objective? What can be said about them?

—Are there more general aspects concerning the way they look at the situation that may change in the process of the investigation?

Professional experience, supported in the literature, asserts that students tend to be not very good on formulating research questions to investigate in a spontaneous way. Even when provided with starting points, they may have difficulty in seeing what more general questions may be asked to extend simple cases already explored [2]. This should be hardly surprising in view of the overwhelming tradition of teaching well organised and formalised knowledge, that students are supposed to acquire, and not introducing them to the process of constructing mathematical knowledge themselves. That is, to teach students "answers" paying no attention to the "questions" they are supposed to correspond nor to the way they were constructed.

In mathematics teaching the tasks are usually given to the students completely formulated. What are sensible or senseless questions to ask, what are interesting or trivial questions, etc, is something to which no attention is usually given. Setting research objectives is therefore one of the aspects in which students show great difficulty.

Definition of strategies. Strategies used in the course of an investigation refer to three aspects. The first concerns the representation of the situation (including the identification of key features and the choice of a suitable notation). The second concerns the key decisions about the sequence of experiences to carry out, indicating a general line of reasoning. The third has to do

with specific tools that are used to construct and interpret the experiences. Significant questions can be asked about these three aspects:

—Is the representation appropriate (in the sense that it describes important aspects of the situation)?

—How is the organization of experiences? Are they relevant for the sought objectives? Are they systematic?

—Is the "technical knowledge" of the students preventing them of devising and organizing a sensible strategy?

Devising appropriate representations and mathematical notations has been widely recognized as an essential element for carrying out mathematical investigations [4, 8, 13]. Not all the representations of a given situation can offer the same insight. Some offer more than others. It is common that students develop more than one kind of representation and fluctuate between them [2].

Investigations are often regarded as good starters for mathematical work. However, it should not be overlooked the fact that "investigational work often rewards mastery of mathematical technique with success, and punishes mathematical inaccuracies heavily" [13, p. 114-115].

Reflecting on the experiences and formulating and testing conjectures. The realization of experiences should lead to a reflection on the situation, gaining insight on it, perhaps revising some aspects of the earlier conceptualization and hopefully to doing some conjecturing.

The results of the experiences performed can be used to better understand the situation and draw up conjectures. The conjectures, once formulated, need to be tested.

The processes of conjecturing and testing form a cycle that may run for several times. Sometimes the students come out the cycle to modify some aspect of the set up of the experiences. Sometimes the students may feel the need to come even earlier and modify the overall research goal.

Testing can also take different forms. It can be test of specific chosen cases, testing of random cases, or attempts to a proof.

Besides our interest in these aspects of the process of conducting investigations, we were also specifically concerned with two further issues: (a) The role of the computer in mathematical investigations and (b) Social interactions.

The role of the computer. These investigations where proposed to the students assuming that the computer would be used to help performing them. If fact, in this activity, it would be difficult to see the work being carried without the computer.

One should therefore ask what are the consequences of using the computer in the working processes of the students. Some of the possible consequences of the computer concerning this kind of mathematical work are well-known:

—It allows a great number of experiences, encouraging strategies where making a good number of experiences is an integral part.

—It allows feedback in different kinds of representations.

—It facilitates the dialogue, since it becomes a new pole of attention. What is done in the computer is not individual property but public.

If the students are programming themselves, as often happens in Logo activities, the act of translating an intuition to a program makes it become more obtrusive and therefore more accessible to reflection [11]. However, in this case the program was already made and the students just interfered with it for changing some of its minor features.

One should note that the computer offers means of representation that are powerful but limited. With the computer it is possible to do many things, some of them quite extraordinary. But computers are limited in what they allow to represent, and they may prove to be unsuitable for some purposes.

Social interactions. One of most common features of the use of computers in mathematics education is a change towards group work. Investigations tend also to be suggested to be performed in groups preferably to individually. However, there are satisfying and less than satisfying situations of group work. Another important partner in the learning process is, of course, the teacher.

Therefore significant questions are for example:

—How does the relationship with the teacher and the colleagues interfere (positively, negatively) with the development of the task?

—How far is carried the process of arguing? Do the students articulate arguments or just statements? Is there listening to others' arguments?

—Is there a search for group consensus or one takes the lead and determines the course of the group?

—What seem to be the implications of the situation of social interaction among students (what seem to be positive effects? negative effects?)

—Why do some students seem to have more initiative than others? Why do some students seem paralysed? Why are some students apparently not able to take profit from the fact that they are in a social interaction situation?

The Investigation on Inspirals

In this activity we may distinguish 8 different segments, in which there was a significant turnover in the course of the events. All of the transitions between segments are characterized by a change in the composition of the group.

Segment 1. The task begun with two students, Maria and Nuno, and the researcher, who handed the sheet with the situation, presented it in general terms, formulated the objective, and gave a suggestion to get them started. This segment lasted for about 2:30 minutes.

The objectives stated in the sheet concerned the nature of the figures that it is possible to get and asked for a theory about the number and the kind of enrollments. These objectives were

rephrased orally by the researcher as "Let us see what happens" and "Try to understand the actual functioning of the procedure".

A first experience was made with the input values of 5 0 5. The students commented on the appearance of the shape: "It looks like a spring!".

Then the researcher introduced one trick: how to slow down the procedure introducing a waiting instruction. He focused the attention of the students in "Why does the turtle seem to turn left?", which was meant as a more specific investigational objective.

The students made several comments about what they were seeing on the screen, specially around the "enrolment points". It appeared that the turtle was drawing "something like a square".

We can say that the intervention of the researcher was dominant in this segment. He stated the objectives, general and specific, made a first experiment, recommended the recording and showed a specific strategy. The students were quite intrigued with the behaviour of the turtle.

Segment 2. In a second segment the students worked for themselves, following the suggestions given. They started exploring the procedure, giving values, and making changes in the first parameter. Having arrived at some conclusion they called the researcher. The segment took 9:30 minutes (1:30 of just waiting time).

The students wanted a larger figure to analyse it better. Following a suggestion of Maria they decided to try out with 10 (therefore introducing the values 10 0 5) and realised that "the figure does not change". Nuno commented that such could be because "they are multiples of 5". At the same time Maria tried to give an explanation for what she was seeing: "The turtle comes back because she does not have a way out". New attempts were made with the inputs 12 0 5, 24 0 5, 28 0 5, 40 0 5. These produced larger figures with the same shape. The students soon realised that modifying the first parameter had an effect on the size but not on the shape. It did not matter if the values were or not multiples of 5. From some point on making the figure larger and larger just became a strategy to see it better and try to understand the behaviour of the turtle. However, at this point the students become much less communicative. They took turns at the keyboard, performed the experiences, registered and carried on with very little or no discussion.

Since apparently the effect of the parameter length was understood and nothing else was happening just by varying it, the students called the researcher. We enter in a third segment of the activity in which he interacts with the students.

In the second segment the students carried out the investigation and successfully discovered the role of one of the parameters of the procedure. At this point they had no idea of what to do further. The discussion that occurred next revealed that they had so far no understanding of the mechanic of the Logo procedure with which they were working.

Segment 3. This segment lasted for about 5:00 minutes. The conclusions reached were briefly presented to the researcher. It should be noted that the students did not address the issue of turning right and turning left. The reason why the turtle comes back still puzzled them.

Maria repeated the former idea: "The turtle comes back because she does not have a way out". The researcher felt that clarifying the working of the procedure was of importance for the pursuing of the investigation and introduced another trick: how to write the successive angles that the turtle was turning, so to give a trace of what it was doing. The students were really surprised to see that such was possible.

An experience was made with 20 0 20. The researcher asked "What is the angle that she is doing when she turns back?" Nuno responded incorrectly taking the increment for the angle "20!" but Maria stated correctly: "180!". This response of 180 seemed to indicate some grasp of the situation but the course of the discussion showed how they were far from a clear understanding.

Maria was intrigued: "Why is it adding up the angles?" The students realised that the angle was varying but did not relate it to the mechanic of the procedure. They were really surprised to see written on the computer that the angle was then becoming larger than 180 degrees. Angles larger than 180 seemed a strange thing to them. Logo was certainly a familiar environment, but in common tasks one gets well along with angles between 0 and 180 degrees, taking both left and right turns. At this point for them RT 200 did not have any meaning.

The researcher made additional questions and comments to try to clarify the role of the increment in the procedure. The point did not come across with the note that "the angle is always increasing". Maria still replied: "And why is it increasing?"

The researcher attempted in another way: "It turns 200 right". He asked Maria to perform a body experience which made her finally understand then that RT 200 is equivalent to some left turn.

The question posed by Maria, "Why is it adding the angles?", prompted the researcher to draw her attention again to the instructions specified in the procedure. At this point the functioning of the increment was apparently finally understood by her. However, she added a strange comment: "The turtle goes by the most difficult side".

The discussion also considered the effects of different increments, realising that increment 20 gives a more pronounced enrolment to the shape than increment 5.

A new objective was then proposed by the researcher:

> "Maintaining the length and the increment, vary the value of the angle, starting with 5 1 5, and see the kind of figure that arises".

Although there were already some discoveries made about the situation, the procedure was still largely not understood. Feeling that, the researcher attempted a clarification. His presence was again quite important, conducting the dialogue, which turned out to be much more intense with Maria than with Nuno.

Segment 4. The students were let to work by themselves, following this suggestion. This constituted a new segment that lasted for 18:00 minutes.

A first experience was made with inputs 5 1 5. The students showed their surprise as they counted 10 enrollments. Nuno introduced the waiting instruction. The idea seemed to be: if it worked while ago let us try it again. Maria commented on the situation of 10 enrollments. She was thinking aloud but she did not articulate any sensible idea. Nuno suggested further "Let us do wait 10 and write the angles". In fact, writing the angles could be of some help, but slowing down the procedure just made it take more time to be performed and Maria did not agree with putting wait 10 ("if 5 is already slow!").

Her attention got concentrated in the initial conditions "I would like to know where it started". She did not recall that such information was given by herself to the computer. "This does not go from 5 to 5... It always appears 1,6,1,6,1,6" (reading the last digit of the numbers written on the screen 11,16,21,...). She looked at the numbers, not at the differences, which constituted a factor for further confusion.

Maria realised that what made a difference was the new initial angle. Nuno took the waiting out and, following a suggestion of Maria, a new experiment was performed with 5 2 5. She commented: "With angle 2 it does not go until there", that is, the figure looks "closed". They counted the enrollments, which turned out to be 10.

Maria asked Nuno to make it smaller "so that we can see" and a new experiment was made with 2 2 2. With 2 they got a much different figure and immediately returned to an increment of 5. That was the increment that they were studying. The idea to use 2 was to make it smaller so that the figure would fit on the screen (avoiding the effect of the wrap mode) in order to see it better. But they realised that changing the increment implied a big change, and come back to were they were before.

An experience was done with 2 2 5 but registered as 5 2 5. They knew well by then that the form was the same.

Nuno described these enrollments as "The turtle coming out in a different way she went in". He advanced a quite complicated (but not thoughtfulness) explanation on why she behaved that way:

> "Look, this is as like the positive and negative numbers. The turtle begins here as if this was the zero... It goes on enrolling, enrolling, gets to 180 and comes back by the same way... And then it is like getting back to zero... 180, 178, 176, etc. And when she finishes she needs to be at 180... 180 and -180 are numbers... how you say it... symmetric".

But Maria, who did not follow very well his explanation conjectured herself: "I think it always will do 10 enrollments. But they will be different".

Nuno suggested a comparison of two different experiences with different inputs on the same screen, but a misplaced command erased the first one before starting the second. Maria said to herself, but loudly "the researcher said to count the number of enrollments", giving an indication of how she was interpreting the objectives of the task, and pursued "she is always doing the same number of enrollments, but they are different". And then she stated with determination "I will discover something!"

Nuno at some point in this segment appeared to have some intuition about the situation, which he tried to explain without much success for two occasions. But he did not take any consequences of his intuition and it was abandoned.

Segment 5. A new student, Victor, arrived to the group. He was given a very brief explanation by his colleagues about what they were registering (but not about the Logo procedure or their former conclusions) and got seated observing with some attention. This fifth segment lasted for 5:30 minutes.

A new experience was carried with inputs 5 3 5. Maria registered 10 enrollments. Them 5 4 5 with the corresponding recording. Them 5 6 5 and Maria commented "It is exactly the same as 4 and 2". (she referred to experiments 4 and 2 not to initial values for angles 4 and 2, what would be a wrong observation.)

Maria conjectured that it should exist a rule for the pairs, giving similar shapes, and the researcher arrived again at the group (now by his initiative).

Victor, the new student who joined the group was not really integrated. By the contrary, his arrival led to a growing distraction of Nuno, who so far had been in second plane but even so was participating in the work along with Maria.

Segment 6. This was a very short segment, lasting for 45 seconds. The researcher arrived and noticed in a glance the experiences already performed by the students. He suggested 7,8,9,10 to be tried for initial angles, and then come back to 5. And he immediately leaved.

Segment 7. In this long segment, that took 35:00 minutes, the students continued their experiences just by themselves.

They begun with 5 7 5. Maria said with confidence "It is going to give the same. I bet they are 10 enrollments. I do not need to say." And in fact the experience confirmed what she had predicted.

A new experience was made with 5 8 5. Maria felt the result strange: "The even numbers should give similar figures and they do not". She showed difficulty in given up her conjecture about the pairs.

The objective of the investigation was them reformulated by Maria: "To me it does not matter the number of enrollments. It matters the way they look like" and she added: "Stupid thing! This breaks all my plans!"

A new experience was made with 5 10 5. Maria became excited again: "I am getting to it now! 2 enrollments."

And new experiments were made in turn with 5 11 5, 5 12 5, 5 13 5, and 5 14 5. It became almost a mechanical activity of experimenting and registering.

They got to try 5 15 5 and Maria commented:

"I bet that 15 is going to give the same as 10... You saw, it did!... When they are multiples of 5 it always gives 2 enrollments".

And she added: "The angles (or increments) do not matter", what in fact is not true; they only had tried increment 5 (and increment 2 with what they regarded as a strange result).

Maria gave finally up of the conjecture on the pairs. She was sure that for multiples of 5 it should give 2 enrollments and that the other numbers have always 10 enrollments in two different families. She announced: "Let us try 16 to see if it is not also like this. 16 and 19". The experiences, of course, confirmed her prediction.

Maria indicated: "The numbers which end in 1,4,6,9 make an enrollments like this". She went on: 20 19 20, with the expected comment "That is the same thing, only it is larger".

In this segment there was an interesting reformulation of the objective, made just by Maria. She was making predictions and testing them eagerly. The other two students became less and less involved in the activity.

Segment 8. In this last segment, that took 32:00 minutes, the researcher joined the group to discuss the activity. In the beginning, a couple of minutes were spent talking about some issues related to the video-taping and then to ecology that were raised by Victor. Then the conversation focused on the activity.

Things were summarized. "Length does not matter. The turtle does not turn left. With the multiples of 5 there are 2 enrollments..." The discussion got tricker as the researcher asked for the justification of these results, the reasons for the different behaviour of the turtle, and other increments than 5 were considered. The students had a class to go, however, and the activity was left with several questions still open.

Discussion

The proposed situation is a quite complex one. There are three parameters to investigate. The role of one of them is fairly simple but the role of the other two is quite complex since their effects are interrelated.

These students had a reasonable previous contact with Logo. But even so they did not understood just by themselves the recursive mechanism of the procedure. They even did not had understood that RT 200 is equivalent to some left turn, a fundamental consequence of the fact that the angle measure works with modules of 360.

Objectives. This situation yields to the formulation of many possible research objectives. Let us regard some of them:

a) Determine the role of each of the parameters (it should be noted that parameters 2 and 3 can not be understood isolated but just together). Students considered this objective as they verified that the first parameter was irrelevant but for size, and then froze the third parameter to study the effects of variations on the second.

b) Understand specific aspects about the working of the procedure: In what point does the turtle turn back? Why does it turn back? Why do enrollments exist? Why does it seem turn left when the procedure just says turn right? What is the relationship between the "coming out angle" (whatever that may be) and the number of enrollments? These aspects were mostly considered by the researcher and did not seem to catch great attention from students, except when he raised specifically that issue.

c) Identify the different kinds of figures that we can get. How can they be classified? What is the reason for each figure? This was just partially pursued by the students for increment 5, and with no search for reasons.

d) In a later stage, when a good grasp of the possible figures begins to emerge, one might also ask: When do figures "tend to infinity"? And when are they "auto-superimposed"? In these, when are 2,3,4,5,6... enrollments? When do we have a figure of kind "open" or of the kind "crossed"? Such questions were raised by the researcher in the final discussion but were left unanswered.

e) State a rule that allows, given a triple of numbers (a,b,c), to say what are the figures that appear, preferably with a proof of such rule. This objective was not considered.

In this activity the first formulation of objectives was made in segment 1 by the researcher.

The general objectives in fact pursued by the students followed his suggestions, except that they disregarded the issues related to the working of the procedure — they were concentrating themselves on the behaviour of the turtle on the screen — and focused more in the influence of all the parameters. The second formulation of objectives was made in segment 3 also by the researcher. These were pursued by the students who, however, made them more precise. And there was a slight precision of the objectives by Maria when she said in segment 7: "To me it does not matter the number of enrollments; it matters the way they look like".

There was general agreement in the interpretation of the task by students and the researcher, although we can see this one much more concerned with aspects of the functioning of the procedure than the students. This was most clearly apparent in the discussion on the equivalence of left turns and right turns.

Strategies and conducting of experiences. By its own nature, and giving the suggestions made, the determination of the sequence of experiments was not a major difficulty in the activity. The students were quite organised in following a natural sequence (they jumped over angle value 5 in segment 5 because of the conjecture about the pairs).

The idea of taking notes in a systematic way was given by the researcher in the beginning and reinforced during the work. It was taken up by Maria. The other students did not get involved in that task although they followed the recording.

Specific strategies used were to make the figure larger in one case and smaller in another. This was done in order to see it better, to figure out what was going on, and was used when the effect of the parameter length was readily understood.

As previously noted, the students showed willingness to try specific strategies (or tricks) that they were shown by the researcher. One of the students, Nuno, was not much critical about

them. Other, Maria, in contrast, appeared much more independent, and accepted them when they seemed to be useful.

Reflections, conjectures and tests. The students were puzzled by a number of things that occurred in this activity. They had real surprises with a number of aspects. Some concerned specific features of the situation: The strange behaviour of the turtle, the resulting figure with 10 enrollments, the accumulation of the angle values. They were amazed with the tricks used by the researcher to make sense of the working of the procedure. But the most striking surprise was with the turtle doing angles larger than 180 degrees. Apparently the usual thing to do is just left turns and right turns with angles smaller than 180 and there was no idea that the angles could be larger.

There were "magical explanations" advanced by Maria about the behaviour of the turtle. She used a metaphor of animal behaviour such as "she come backwards because she does not have a way out" and "does not have another way to follow", etc, which appeared to be a strategy to try to make sense of the situation.

Some discoveries were made by the students in conversation with the researcher: (a) 180 is the angle that the turtle is doing when it returns back, (b) the working of the procedure concerning the angle increments (segment 3), (c) the relation between left turns and right turns (segment 3), and (d) larger values for the increment give a more pronounced enrolment, and change the forms (segment 3).

Other discoveries were made by the students themselves. One concerns the parameter "length forward" that has no effect of the form, other than in its size. Other refers to the fact that there are three main kinds of forms for increment 5:

—multiples of 5 give 2 enrollments
—numbers ended in 1,4,6,9 give 10 enrollments "open"
—numbers ended in 2,3,7,8 give 10 enrollments "crossed"

A conjecture about pairs was made by Maria. Is was dismissed with difficulty, only with the accumulation of contrary evidence. Later on she was able to make predictions about the number of enrollments and kinds of shapes, and verify them. By the end of the activity she was strongly confident in her conjectures.

Students' involvement and interactions. Maria was uniformly highly involved throughout the session. She took the task quite seriously and with determination. As she exclaimed at some point: "I will discover something!". She knew that her role in this activity was to try hard to make discoveries. Nuno was participative, although in second plan, until Victor came. Then both of them had very little participation, becoming more and more distracted.

The interaction between the students was not very productive in this activity. In the beginning Maria and Nuno had several interchanges. As the time went on, most of the work was carried out by Maria or under her direction. The dialogue became less and less effective. For several occasions, we could observe one of the students saying one thing, the other saying a completely unrelated thing, giving rise to no discussion among them, and then they just moving on. Finally, even these interactions became less frequent.

In general, Maria tended to assume that Nuno was making an interpretation similar to her's, or she did not even try to understand his view on the successive results obtained at the computer. Victor had no relevant intervention in the development of the activity.

Maria led the investigative process, making most of the suggestions which were then acted upon as collaborative decisions, or by making the decision on her own.

From segment 4 on, until the end of the activity, Maria took an even more important role, taking decisions and reflecting on their results. Nuno was accompanying her activity, but not really intervening in the decision making. And from time to time he was speaking with Victor of subjects unrelated to the activity.

The absence of collaborative work was identified since the beginning of segment 4. With the progress of the investigation Maria seemed to become more confident about what she was doing and assume that the other students would not be of much help. In this way, the attempt to an explanation of the behaviour of the turtle by Nuno in segment 4 was lost as a discussion opportunity.

Personality factors may constitute the main reason for the nature of the students' interactions. Additional reasons may have to do with their increasing awareness of the complexity of task and also with the fact that the researcher, in his moments with the group, had more interchanges with Maria.

Although absent for most of the time, the researcher (who was supervising the work of other two groups) had an important role on this activity, mostly in the definition of the objectives, in the adoption of general strategies, and in the suggestion of specific strategies (in this case "computer tricks").

Conclusion

One should not underestimate the difficulties of the students in investigating complex situations. We know that making significant discoveries in mathematics is difficult enough for mathematicians [6, 7] and we should not forget that they are strongly motivated for their subject. Rich environments, like this one, entail many complexities and students are likely to find many embarrassments with them and are not necessarily highly motivated for mathematics [12].

But, by the other side, such difficulties have their reverse. They provide good opportunities for discussion and reflection, reveal misconceptions, and promote an awareness of global issues that may become significant for the progress of the students. What happened in this episode with rotations over 180 degrees is quite illustrative in that respect.

Mathematical investigations may be important educational activities. They prove to be useful in the development and consolidation of specific concepts and mathematical ideas. They bear on important thinking skills. They may promote a broader vision of mathematics, much closer to the actual practice of the mathematician.

The development of this activity seems to indicate that two fundamental characteristics appear to be necessary to deal successfully with mathematical investigations: sharpness and flexibility. Sharpness is vital in the formulation of objectives, so that they correspond to essential features of the situation and are amenable to a description in mathematical terms. Flexibility is important in the choice and evaluation of strategies (that is, the ability to set up and modify approaches that do not look promising anymore).

What may be the principal difficulties and obstacles of this kind of activity? There may be problems involving content knowledge, reasoning processes, or general attitudes and appreciation. Students may not be able to figure out any sensible way of starting an investigation. They may do not know relevant background content, or not be able to evaluate a given result.

Many other questions need to be addressed regarding mathematical investigations. What may be the criteria for the assessment of an activity undertaken by the students? Content learning? Development of cognitive skills? Development of appreciation of mathematics? What are good investigation proposals? What is the role of the teacher?

This episode shows that the development of mathematical investigations may involve unexpected difficulties for teachers. Research on this topic, mapping the cognitive processes and the social interactions of the students, is necessary to bring new constructs and provide support for teaching practice.

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