Investigação Operacional, Vol. 9 nº1 (Junho/1989) 13 – 21

CUTTING PLANES FROM CONDITIONAL BOUNDS FOR GENERALIZED SET COVERING PROBLEMS

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Abstract. This paper reports on the development of special cutting planes for the generalized set covering problem, GSCP, which is a covering problem where the variables and the right-hand sides are allowed to have any positive integer value. Those inequalities are, actually, a generalization of the cutting planes derived from conditional bounds and originally presented by Balas (1980), for the set covering problem. More recently, Hall & Hochbaum (1985) have extended those results for the multicovering problem. The generalized inequalities that we derive for the GSCP are proved to be of the covering type and, hence, keeping the structure of the problem constraints.

Key Words: cutting planes, conditional bounds, disjunctive inequalities, generalized set covering problem

1. Introduction

In this paper, we deal with the generalized set covering problem (GSCP), which can be stated as the following mathematical program:

$$
\begin{array}{llll}
\text{(GSCP)} & \min & \sum_{j\in N} c_j x_j & \\
& \text{s. to} & \sum_{j\in N} a_{ij} x_j \geq b_i & (i\in M) \qquad (1) \\
& & 0 \leq x_j \leq h_j \text{ and integer} & (j\in N) \qquad (2)
\end{array}
$$

where M and N are the index sets of, respectively, the rows and columns for the problem. The b_i ($i\in M$) and h_j ($j\in N$) are positive integer values. Also, one has $a_{ij}\in\{0,1\}$ ($i\in M, j\in N$) and, in order to avoid a trivial resolution, we assume that $\sum_{j\in N} a_{ij} > 0$. A vector x verifying the constraints (1) and (2) is called a <u>cover</u>. Finally, and for the sake of simplicity of the notation, we define the following two sets: $M_j = \{i\in M : a_{ij} = 1\}$ and $N_i = \{j\in N : a_{ij} = 1\}$. Then, for instance, the covering constraints can be stated as $\sum_{j\in N_i} x_j \geq b_i$ ($i\in M$).

Research for this paper was partially supported by INIC (Instituto Nacional de Investigação Científica) and by JNICT (Junta Nacional de Investigação Científica e Tecnológica) under the Research Contract nb.809.86.148/MIC.

The GSCP has been widely used for several real life situations and a survey for that can be found in Pato (1989). Most of those applications are related to personnel scheduling, particularly the determining of the schedules for drivers in mass transit bus companies [Blais & Rousseau (1982); Bodin, Rosenfield & Kydes (1981); Mitra & Welsh (1981); Paixão et al (1986); Shepardson & Marsten (1980); Yihua (1985)].

As suggested by the designation, the GSCP includes the well known set covering problem (SCP), where all the b_i ($i \in M$) and h_j ($j \in N$) are equal to the unity. It also includes the so called multicovering problem (MCP), where $h_j = 1$ ($j \in N$) but the right-hand side values b_i ($i \in M$) can be any positive integers. From that, it follows straigthforwardly that the GSCP is an NP-hard problem.

Hence, heuristics and LP-based techniques have been the approaches most used for the GSCP. In particular, cutting plane methods have been applied as a way of dealing with instances where a null linear gap occurs or a high number of alternative solutions exists [Geoffrion & Marsten (1972); Wolfe (1984)]. That, actually, is the case of GSCPs related to crew scheduling problems [Pato (1989)], for which lagrangean relaxation and tree-search procedures have been applied too [Paixão & Pato (1989); Shepardson & Marsten (1980)].

In the present paper, we extend to the GSCP a class of cutting planes that have been described by Balas (1980) and Balas & Ho (1980) for the SCP. Those cutting planes are derived from conditional bounds following a general disjunctive approach [Balas (1975), (1979)], and have been extended by Hall & Hochbaum (1985) for the MCP.

The paper is organized as follows. Next, in this section, we introduce the idea of cutting planes from conditional bounds through an example. Some formal notation is stated too. Section 2 is devoted to a single result leading to the obtaining of a valid inequality which is strengthened in the following section. Then, in section 4, the example considered in the introduction is used for the purpose of illustrating the results of the previous sections. Finally, some remarks and conclusions are presented in section 5.

Before giving an example for introducing the idea of conditional bounds and the related cutting planes, let us state some notation. We denote by $\overline{\text{GSCP}}$ the continuous version of the GSCP, and the corresponding dual linear problem, $\overline{\text{DGSCP}}$, has the following formulation:

$$
\begin{array}{llll}
(\overline{\text{DGSCP}}) \quad \max & \sum_{i\in M} b_i u_i - \sum_{i\in N} h_j v_j & & \\
\text{s. to} & \sum_{i\in M_j} u_i - v_j \le c_j & (j\in N) & (3) \\
 & v_j \ge 0 & (j\in N) & (4) \\
 & u_i \ge 0 & (i\in M) & (5)
\end{array}
$$

The constraints (3) can be rewritten as $r_j = c_j - \sum_{i\in M_j} u_i + v_j \ge 0$ ($j\in N$), with r_j being designated as the reduced cost for the variable index j. A vector [u v] satisfying constraints (3)-(5) is said to be a dual feasible solution.

Now, let us consider the following instance for the GSCP:

$$
\begin{array}{llllllllllll}
\min & 2x_1 & + & 2x_2 & + & x_3 & + & 1.5x_4 & + & 2x_5 & & \\
\text{s. to} & & & x_2 & + & x_3 & + & & & x_5 & \ge & 4 \\
 & x_1 & + & & & & & x_4 & + & x_5 & \ge & 2 \\
 & & & x_2 & + & & & x_4 & + & x_5 & \ge & 6 \\
\multicolumn{12}{l}{\qquad x_1 \le 1,\ x_2 \le 2,\ x_3, x_4, x_5 \le 4} \\
\multicolumn{12}{l}{\qquad x_1, \ldots, x_5 \ge 0 \text{ and integers.}}
\end{array}
$$

Also, consider a cover given by $\tilde{x} = [\,0\ \ 2\ \ 2\ \ 4\ \ 0\,]$ and the dual feasible solution defined by $\tilde{u} = [\,0\ \ 0\ \ 1.5\,]$ and $\tilde{v} = [\,0\,0\,0\,0\,0\,]$. Hence, the optimum is a value between 9 and 12.

Suppose that the constraint $x_3 \ge 3$ is added to the covering problem. Then, for this enlarged instance, one may consider the previous dual feasible solution with an additional variable $\tilde{u}_4 = 1$. The optimal value for the enlarged instance is bounded from below by 12. This leads to the conclusion that the constraint $x_3 < 3$ has to be satisfied by any feasible solution for the original problem with an objective value less than 12. Combining this with the first constraint of the covering problem, $x_2 + x_3 + x_5 \geq 4$, one may conclude that $x_2 + x_5 \geq 2$ is a valid inequality for any cover with a better value than the current one. Note, that this new constraint is of the covering type.

2. Weak Inequality

In this section, we derive a first valid inequality for the GSCP following the approach proposed by Balas & Ho (1980) for the set covering problem. For that, a cover and a dual feasible solution must be available. Consequently, an upper bound z_u for the optimal value of the GSCP is known. Then, as suggested in the example, additional constraints are determined in such a way that the corresponding dual linear problem has an optimal value greater than or equal to z_u. Therefore, solutions strictly better than z_u must violate at least one of those additional constraints. Finally, the combining of this last disjunction to some of the covering constraints for the original GSCP leads to a valid inequality for the feasible solutions with a better value than the current one.

The required pair of feasible solutions, $\tilde{x}$ for the GSCP and $[\tilde{u}\ \ \tilde{v}]$ for the dual linear problem ($\overline{\text{DGSCP}}$), must verify the following conditions:

$$(\sum_{j\in N_i} \tilde{x}_j - b_i)\, \tilde{u}_i = 0 \qquad (i\in M) \qquad (6)$$

$$\tilde{v}_j = \max \{ 0, - c_j + \sum_{j\in M_i} \tilde{u}_i \} \qquad (j\in N) \qquad (7)$$

$$(\tilde{x}_j - h_j)\, \tilde{v}_j = 0 \qquad (j\in N). \qquad (8)$$

We denote by $\tilde{z}_u$ and $\tilde{z}_\ell$, respectively, the upper bound on the optimum provided by $\tilde{x}$ and the lower bound given by the associated dual feasible solution $[\tilde{u}\ \tilde{v}]$. Those solutions can be easily obtained through the using of primal-dual greedy heuristics presented for the GSCP by Paixão & Pato (1987).

Now, let us define the set $\tilde{S} = \{j\in N : \tilde{x}_j > 0 \text{ and } \tilde{r}_j > 0\}$ with $\tilde{r}_j$ being the reduced cost produced by $[\tilde{u}\ \tilde{v}]$ for the j^{th} column. And, let $S = \{j(1),\ldots j(p)\}$ be a subset of $\tilde{S}$ and integers $\delta_{j(k)} \geq \tilde{x}_{j(k)}$ $(j(k)\in S)$ such that:

$$\sum_{k=1,\ldots,p} \tilde{r}_{j(k)}\, \delta_{j(k)} \geq z_u - \tilde{z}_\ell \qquad (9)$$

where z_u is the best known upper bound on the optimum for GSCP.

Note that such condition holds for the case where $S = \tilde{S}$, $z_u = \tilde{z}_u$ and $\delta_{j(k)} = \tilde{x}_{j(k)}$.

Theorem 1. Let $\tilde{x}$ and $[\tilde{u}\ \tilde{v}]$ be feasible solutions, respectively, for GSCP and $\overline{\text{DGSCP}}$ verifying the conditions (6)-(8). Also, let $\delta_{j(k)}$ $(k=1,\ldots,p)$ be integer values for which (9) holds with z_u, $\tilde{z}_\ell$ and $\tilde{r}$ defined as above.

Then each feasible solution, x, whose value is less than z_u must satisfy

$$\sum_{j\in W} x_j \geq d \qquad (10)$$

with

$$d = \min_{k=1,\ldots,p} (b_{i(k)} - \delta_{j(k)}) + 1 \qquad (11)$$

$$W = \bigcup_{k=1,\ldots,p} (N_{i(k)} - Q_k) \qquad (12)$$

where, $i(k)$ is any index in M and $Q_k \subseteq N_{i(k)}$ $(k=1,\ldots,p)$ verify

$$\sum_{k:j\in Q_k} \tilde{r}_{j(k)} \leq \tilde{r}_j \qquad (j\in N) \qquad (13)$$

Proof. Consider $S = \{j(1),\ldots,j(p)\}$ and a vector δ, both defined according to our hypothesis. Let $GSCP_A$ denote the GSCP enlarged with the p additional constraints:

$$\sum_{j\in Q_k} x_j \geq \delta_{j(k)} \qquad (k=1,\ldots,p) \qquad (14)$$

The dual linear of this enlarged problem has a feasible solution, given by $[\tilde{u}\ \tilde{v}]$ plus p variables associated to new constraints (14) and, respectively, equal to the reduced costs, $\tilde{r}_{j(1)},\dots,\tilde{r}_{j(p)}$. The definition of the sets Q_k (k=1,...,p) guarantees the dual feasibility for this 'enlarged' solution.

The value of this dual feasible solution for the enlarged problem is given by

$$\tilde{z}_\ell + \sum_{k=1,\dots,p} \tilde{r}_{j(k)}\,\delta_{j(k)}$$

which, according to (9), is greater than or equal to z_u, the upper bound for the optimal value.

From the weak duality theorem applied to the $GSCP_A$, one may conclude that the enlarged problem has no feasible solution better than z_u. Then any feasible solution of GSCP better than z_u must violate at least one of the additional constraints (14). In other words, such cover must satisfy the following disjunction

$$\bigvee_{k=1,\dots,p} \Big(\sum_{j\in Q_k} x_j < \delta_{j(k)} \Big),$$

which implies

$$\bigvee_{k=1,\dots,p} \Big(\sum_{j\in N_{i(k)}-Q_k} x_j \geq b_{i(k)} - \delta_{j(k)} + 1 \Big) \qquad (15)$$

Now, if d and W are defined by (11)-(13), one has that $\sum_{j\in W} x_j \geq d$ is an inequality satisfied by any solution with a value better than z_u. ◊

The existence of the sets Q_k (k = 1,..., p) is guaranteed and, next, we present a procedure (denoted by QKAPA), which produces such a family of sets. There, we assume that Criterium j_r and Criterium i_s have been established, respectively, for selecting a row and a column in each iteration.

Using different criteria for determining j(k) and i(k) in the procedure QKAPA, several cuts can be generated. Naturally, if we intend to obtain a strong cut, it is reasonable to keep the cardinality of W as low as possible and, simultaneously, find a large value for d, the right-hand side of the inequality.

Hence, in order to reduce |W| one should try to add the least possible number of additional constraints. This can be pursued through including the columns in S by decreasing order of the reduced costs (Criterium j_1). Of course, if $z_u = \tilde{z}_u$ and $\delta_{j(k)} = \tilde{x}_{j(k)}$ then p is equal to $|\tilde{S}|$ and any sorting of the reduced costs is irrelevant.

Another way of reducing |W| consists of including as many variables as possible in the additional constraints, while respecting dual feasibility. The reduction of |W| may also be achieved by choosing, accordingly to Criterium i_2, i(k) as the row with minimum cardinality among the ones covered by column j(k).

The determination of a large value d could be attained by taking $\delta_{j(k)} = \tilde{x}_{j(k)}$ and by choosing row i(k) in $M_{j(k)}$ such that $b_{i(k)}$ is the largest one (Criterium i_1).

Therefore, the selection rules for column j(k) and row i(k) may be relevant. Unfortunately these rules are not enough to ensure the elimination of any feasible solution for GSCP, which is the main objective when deducing efficient cutting planes. In the next section we present and discuss a way of achieving that.

```
procedure QKAPA ( input: GSCP, r̃, S ; output: Q_k (k=1,...,p) )
    * initialization *
    r^A ← r̃ ; p ← |S|
    * iterations *
    for k←1,...,p do
        choose j(k) applying Criterium j_r
        Q_k ← {j(k)}
        choose i(k) ∈ M_j(k) applying Criterium i_s
        for j ∈ N_i(k)-S do
            if r^A_j ≥ r̃_j(k) then ( Q_k ← Q_k U {j} ) endif
        enddo
        r^A_j ← r^A_j - r̃_j(k)   (j∈Q_k)
    enddo
end QKAPA
```

3. Strong Inequality

In this section, we define a cutting plane from conditional bounds that eliminates a previously known nonredundant cover (one whose components cannot be reduced without producing unfeasibilities).Also, it will be proved that such cut defines a facet of a GSCP polyhedron.

First, let us define the following sets of variables:

$$S^h = \{j \in N : \tilde{x}_j = h_j\} \tag{16}$$

$$T = \{i \in M : \sum_{j \in N_i} \tilde{x}_j = b_i\} \tag{17}$$

Associated to a given variable j(k) we define a row:

$$i(k) \in T \cap M_{j(k)} \text{ such that } \tilde{x}_j = 0 \text{ for } j \in N_{i(k)} - S^h - \{j(k)\} \tag{18}$$

In order to build up the sets Q_k (k=1,...,p), we shall now consider this last condition in the row selection criterium of the procedure QKAPA.

Theorem 2. Let $\tilde{x}$ be a nonredundant cover for the GSCP, the sets S^h, Q_k (k=1,...,p) and indices i(k) (k=1,...,p) be defined as above. Also, let z_u, $\tilde{z}_u$, p, $[\tilde{u}\ \tilde{v}]$, $\tilde{z}_\ell$, $\tilde{r}$ stand as in Theorem 1. Then every feasible solution to the GSCP whose value is less than z_u verifies the following condition violated by $\tilde{x}$:

$$\sum_{j \in \overline{W}} x_j \geq 1 \tag{19}$$

where

$$\overline{W} = \bigcup_{k=1,\ldots,p} (N_{i(k)} - Q_k - S^h) \tag{20}$$

Proof. Taking into account that $S = \tilde{S}$ and $\delta = \tilde{x}$, the assumptions of Theorem 1 are verified for any $z_u \leq \tilde{z}_u$. Consider that the procedure QKAPA determines Q_k (k = 1,..., p) with the further condition in selecting row i(k), given by (18).

As was seen in the proof of the first theorem, the improved solutions relatively to z_u violate at least one of those additional constraints. That is, they satisfy the condition (15).

Now, let us rearrange the condition (15) in the following form:

$$\bigvee_{k=1,\ldots,p} \Big(\sum_{j \in N_{i(k)}-Q_k-S^h} x_j \geq b_{i(k)} - \tilde{x}_{j(k)} + 1 - \sum_{j \in (N_{i(k)}-Q_k)\cap S^h} h_j \Big) \tag{21}$$

Note that, from the definition, $\tilde{x}_j = h_j$ for all $j \in (N_{i(k)}-Q_k)\cap S^h$.

Since $i(k) \in T$ and (18) is verified for all k=1,...,p, the above condition (21) is equivalent to

$$\bigvee_{k=1,\ldots,p} \Big(\sum_{j \in N_{i(k)}-Q_k-S^h} x_j \geq 1 \Big)$$

Now from the integrality of the GSCP variables, one has the following inequality:

$$\sum_{j \in \bigcup_{k=1,\ldots,p} N_{i(k)}-Q_k-S^h} x_j \geq 1 \tag{22}$$

which is valid for any feasible solution strictly better than z_u.

Since $\tilde{x}_j = 0$ for $j \in \bigcup_{k=1,\ldots,p} (N_{i(k)} - Q_k - S^h)$, it comes straightforwardly that $\tilde{x}$ is eliminated by the deduced cut ◊

Note that the last result states that the cut (19) is a valid inequality for the set of all |N|-dimensional vectors satisfying constraints (1) - (2) amended with $\sum_{j \in N} c_j x_j < \tilde{z}_u$.

Now, one may generalize, for the GSCP, the known facet defining property of the similar cut derived for the SCP [Balas (1980)]. Before stating that, let P be the convex hull of all integer nonnegative |N|-dimensional vectors satisfying (1) and (19). That is,

$$P = \text{conv}\left\{ x \in \mathbb{R}^{|N|} : \sum_{j \in N_i} x_j \geq b_i \ (i \in M),\ \sum_{j \in \overline{W}} x_j \geq 1 \text{ and } x_j \text{ nonnegative integer } (j \in N) \right\} \tag{23}$$

Theorem 3. The cutting plane (19), constructed according to the Theorem 2, defines a facet of the polyhedron P given by (23).

Proof. First let us prove that the index set $\bar{W}$, defined through the hypothesis of Theorem 2, satisfies the following:

$$\nexists\, k \in M : N_k \subseteq \bar{W} \tag{24}$$

Consider $\tilde{x}$, the cover for the GSCP defined in that theorem, and the set of its positive variables, $N_{\tilde{x}} = \{ j \in N : \tilde{x}_j > 0 \}$. Thus, one has $N_k \cap N_{\tilde{x}} \neq \Phi$ for all $k \in M$.

But $\tilde{x}$ violates the inequality (19) and, therefore $N_{\tilde{x}} \cap \bar{W} = \Phi$. Then, the statement (24) is easily seen to be true. This will be used later on.

We already know that (19) is a valid inequality for the polyhedron P. Now, one aims to build up $|N|$ linearly independent $|N|$-dimensional nonnegative integer vectors satisfying (1) and verifying (19) as an equality.

Let us assume without loss of generality that $\bar{W}$ is the set of the first p indexes in N.

Consider the $|N|$ vectors given by the rows of the following matrix:

$$X = \begin{array}{c} \qquad \overleftrightarrow{\;\; p \;\;} \qquad\qquad \overleftrightarrow{\;\; |N| - p \;\;} \\ \left|\begin{array}{cccccc|cccccc} 1 & 0 & 0 & \cdots & 0 & 0 & \ell & \ell & \ell & \cdots & \ell & \ell \\ 0 & 1 & 0 & \cdots & 0 & 0 & \ell & \ell & \ell & \cdots & \ell & \ell \\ 0 & 0 & 1 & \cdots & 0 & 0 & \ell & \ell & \ell & \cdots & \ell & \ell \\ \cdot & & & & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & 0 & \ell & \ell & \ell & \cdots & \ell & \ell \\ 0 & 0 & 0 & \cdots & 0 & 1 & \ell & \ell & \ell & \cdots & \ell & \ell \\ 1 & 0 & 0 & \cdots & 0 & 0 & 2\ell & \ell & \ell & \cdots & \ell & \ell \\ 1 & 0 & 0 & \cdots & 0 & 0 & \ell & 2\ell & \ell & \cdots & \ell & \ell \\ 1 & 0 & 0 & \cdots & 0 & 0 & \ell & \ell & 2\ell & \cdots & \ell & \ell \\ \cdot & & & & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ 1 & 0 & 0 & \cdots & 0 & 0 & \ell & \ell & \ell & \cdots & 2\ell & \ell \\ 1 & 0 & 0 & \cdots & 0 & 0 & \ell & \ell & \ell & \cdots & \ell & 2\ell \end{array}\right| \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\\1\end{array}}\right\updownarrow p \\ \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\\1\end{array}}\right\updownarrow |N| - p \end{array}$$

where $\ell = \max\limits_{i \in M} b_i$.

Taking into account that (24) holds true for our case, each one of these vectors verifies the constraints (1). Moreover, it has nonnegative integer components and it satisfies strictly inequality (19), once there is only one 1 in its first p components.

It now remains to show that X is a nonsingular matrix.

Let us define matrix Z by:

$$\begin{array}{c} \qquad \overleftrightarrow{\;\; p \;\;} \qquad\qquad \overleftrightarrow{\;\; |N| - p \;\;} \\ \left|\begin{array}{cccccc|cccccc} 1+|N|-p & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & -1 & \cdots & -1 & -1 \\ |N|-p & 1 & 0 & \cdots & 0 & 0 & -1 & -1 & -1 & \cdots & -1 & -1 \\ |N|-p & 0 & 1 & \cdots & 0 & 0 & -1 & -1 & -1 & \cdots & -1 & -1 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ |N|-p & 0 & 0 & \cdots & 1 & 0 & -1 & -1 & -1 & \cdots & -1 & -1 \\ |N|-p & 0 & 0 & \cdots & 0 & 1 & -1 & -1 & -1 & \cdots & -1 & -1 \\ -1/\ell & 0 & 0 & \cdots & 0 & 0 & 1/\ell & 0 & 0 & \cdots & 0 & 0 \\ -1/\ell & 0 & 0 & \cdots & 0 & 0 & 0 & 1/\ell & 0 & \cdots & 0 & 0 \\ -1/\ell & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1/\ell & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ -1/\ell & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 1/\ell & 0 \\ -1/\ell & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1/\ell \end{array}\right| \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\\1\end{array}}\right\updownarrow p \\ \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\\1\end{array}}\right\updownarrow |N| - p \end{array}$$

and see that $X\,Z = I_{|N|}$, where $I_{|N|}$ stands for the $|N| \times |N|$ identity matrix.

Hence, the $|N|$ vectors given above are linearly independent and the result comes straightforwardly from the definition of facet of a $|N|$-dimensional polyhedron. $\Diamond$

In the next page we describe the procedure CUT which, when successful, produces a cutting plane under the conditions stated above.

However, the procedure CUT may fail in obtaining the 'strong' cut that one aims for. This is due to the fact that, for most of the instances, the hypothesis of Theorem 2, namely the one expressed by the condition (18), can hardly be verified. This is completely different from the case of the SCP and MCP where the variable upper bounds are all equal to the unity. For those cases, the condition (18) is easily satisfied and the cuts from conditional bounds can be more effective [Balas & Ho (1980); Fernandez (1985); Hall & Hochbaum (1985)].

procedure CUT (**input:** GSCP, z_u, $\tilde{x}$, $\tilde{z}_u$, $[\tilde{u}\ \tilde{v}]$, $\tilde{z}_\ell$ and $\tilde{r}$ satisfying the hypothesis of Theorem 2 ;
output: should the answer be **true**, cut $\sum_{i \in \bar{W}} x_j \geq 1$)

* initialization *

$D \leftarrow \tilde{z}_\ell$; $k \leftarrow 1$; $\bar{W} \leftarrow \Phi$; $\tilde{S} \leftarrow \{j \in N : \tilde{x}_j \tilde{r}_j > 0\}$; $S^0 \leftarrow \{j \in N : \tilde{x}_j = 0\}$
$S^h \leftarrow \{j \in N : \tilde{x}_j = h_j\}$; $T \leftarrow \{i \in M : \sum_{i \in N_i} \tilde{x}_j = b_i\}$; $r_j^A \leftarrow \tilde{r}_j$ $(j \in N)$; answer $\leftarrow$ **true**

* iterations *

label 1: $j(k) \leftarrow \arg\max_{j \in \tilde{S}} \tilde{r}_j$ * Criterium j_1 *

$\bar{O}_k \leftarrow \Phi$; $\bar{T} \leftarrow \{i \in T \cap M_{j(k)} : (N_i - S^h - \{j(k)\}) \cap S^0 = (N_i - S^h - \{j(k)\})\}$

if $\bar{T} \neq \Phi$

then choose $i(k) \in \bar{T}$ with Criterium i_s * select row index *
else answer $\leftarrow$ **false** ; **stop** * cutting plane not determined *
endif

for $j \in (N_{i(k)} - \tilde{S} - \bar{W}) \cup \{j(k)\}$ **do**
if $r_j^A < \tilde{r}_{j(k)}$
then (**if** $\tilde{x}_j = 0$ **then** $\bar{O}_k \leftarrow \bar{O}_k \cup \{j\}$ **endif**) * variables for the cut *
else $r_j^A \leftarrow r_j^A - r_{j(k)}$ * variables for the new additional constraint *
endif
enddo

$D \leftarrow D + \tilde{r}_{j(k)} \tilde{x}_{j(k)}$; $\bar{W} \leftarrow \bar{W} \cup \bar{O}_k$ * verify validity of the cut *
if $D < z_u$ **then** ($\tilde{S} \leftarrow \tilde{S} - \{j(k)\}$; $k \leftarrow k+1$; **goto label 1**) **endif**

if $\bar{W} = \Phi$ **then** answer $\leftarrow$ **false** **endif** * optimal solution for GSCP found *
end CUT

A further difficulty, in practical terms, comes from the fact that a valid inequality obained from Theorem 2 eliminates the current feasible solution, but does not necessarily cut off other solutions with the same value. And this is very frequent for the GSCP.

Thus the procedure without improvement turns out to be very slow. In fact, our personal experience in using this approach for real life GSCPs proved to be very poor. The procedure CUT was tried out for 34 instances related to a real life scheduling application, all of them with 36 rows, 100 or 865 columns and high density - more than 50% of 1s in the covering constraint matrix [Paixão & Pato (1989); Pato (1989)]. The primal and dual solutions needed for generating the cuts were found by means of a combined primal-dual greedy and improving heuristic procedure [see Paixão & Pato (1987)]. However, we were not successful with any one of the above test problems.

4. Example

To illustrate the results of the previous sections we consider the example given in section 1 assuming that the same pair of feasible solutions is available and also that $z_u = \tilde{z}_u = 12$. In this case, one has that $\tilde{r} = [\,2\ \ 0.5\ \ 1\ \ 0\ \ 0.5\,]$ and $\tilde{z}_\ell = 9$. Since $\tilde{r}_3\, \delta_3 = 1 \times 3$ and $\tilde{z}_u - \tilde{z}_\ell = 12 - 9$, we may easily verify that the hypothesis of Theorem 1 is true, with $S=\{3\}$ and $p=1$.

Let us take $Q_1 = 3$, $i(1)=1$ and, then, calculate $W = N_{i(1)} - Q_1 = N_1 - Q_1 = \{2,5\}$ and $d = b_{i(1)} - \delta_{j(1)} + 1 = b_1 - \delta_3 + 1 = 4 - 3 + 1 = 2$.

The previously constructed valid inequality, $x_2 + x_5 \geq 2$, may now be derived in accordance with the first theorem.

Bearing in mind the same cover and the same dual feasible solution, along with the fact that condition (9) is fulfilled with $S = \tilde{S}$ and $\delta_{j(k)} = \tilde{x}_{j(k)}$, we can see that $S = \{2,3\}$ and so p=2, in the hypothesis of Theorem 2.

Firstly, the column $j(1) = 2$ is chosen and any row selection criterium picks up $i(1) = 3$, because it is the only one from set $M_2 = \{1,3\}$ satisfying condition (18). Thus $Q_1 = \{2,5\}$ and the reduced costs are updated: $r^A = [\,2\ \ 0\ \ 1\ \ 0\ \ 0\,]$. At last $j(2) = 3$ and now $i(2) = 1$, the only possible choice, leading to $Q_2 = \{3\}$. The reduced costs for the enlarged problem become $r^A = [\,2\ \ 0\ \ 0\ \ 0\ \ 0\,]$.

As the set of variables equal to their upper bounds is $S^h = \{2,4\}$, the variables with positive coefficient in the cut belong to the set $\bar{W} = \{N_3 - Q_1 - S^h\} \cup \{N_1 - Q_2 - S^h\} = \{5\}$.

Thus, from Theorem 2, the valid cutting plane is $x_5 \geq 1$, which is clearly violated by the current cover and is verified by every strictly less-than-12 cover. As may be observed, the three optimal alternative solutions with value 11 ($x_4 = 2, x_5 = 4$; $x_2 = 1, x_4 = 2, x_5 = 3$; $x_2 = x_4 = x_5 = 2$) satisfy this new valid inequality.

5. Remarks

In this paper, we have characterized two families of valid inequalities for the GSCP. These results correspond to a generalization of Balas's cuts for the SCP. Those cuts keep the covering structure.

The first type, studied throughout section 2, can be defined from conditional bounds for every GSCP, but does not necessarily eliminate one single feasible solution.

The other kind of cutting plane, also derived from conditional bounds (Theorem 2), eliminates the current cover. However, very tight conditions are required and, thus, one gets a harder process to generate the cuts for genuine GSCP instances. In fact, our computational experience attested this difficulty.

The study of a less restrictive hypothesis in Theorem 2 could be a promising field of research, and we hope that an algorithm combining these cuts with several heuristic or other bounding tools will enable us to efficiently tackle the generalized set covering problem.

Acknowledgments

The authors are grateful to Profs. Jaume Barceló and Elena Fernandez of Facultat d'Informàtica of Universitat Politècnica de Catalunya for their suggestions at the beginning of this study.

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